Quandles and Symmetric Spaces 2022

Osaka Central Advanced Mathematical Institute (OCAMI)
Osaka Metropolitan University
MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics

OCAMI Reports Vol. 9 (2022)
doi: 10.24544/omu.20230322-001

Quandles and Symmetric Spaces 2022

Organized by

Seiichi Kamada
Akira Kubo
Takayuki Okuda
Kanako Oshiro
Hiroshi Tamaru
Makiko Sumi Tanaka
Hiroyuki Tasaki

December 8–9, 2022

Abstract

The workshop "Quandles and Symmetric Spaces" has been held annually since 2018. This volume records the abstracts and the slides of talks presented in this workshop on 2022.

2020 *Mathematics Subject Classification.*
53C35, 57K12

Key words and Phrases.
quandles, symmetric space

OCAMI. Quandles and Symmetric Spaces 2022. OCAMI Reports. Vol. 9, Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University. 2022, 160 pp. doi: 10.24544/omu.20230322-001

Preface

The workshop “Quandles and Symmetric Spaces” has been held annually since 2018 in order to encourage the cross-pollination among topology (knot theory), differential geometry (symmetric spaces), and other areas through quandles. The series of workshops was organized by experts of knot theory (Kamada and Oshiro) and symmetric spaces (Kubo, Okuda, Tamaru, Tanaka and Tasaki). There have been many presenters and participants from various fields, not only topology and differential geometry but also algebraic geometry and combinatorics, etc.

On the conference “Quandles and Symmetric Spaces 2022”, the talks consisted of presentations by young researchers. Some of their topics are as below:

- a category equivalence between a certain category of faithful quandles and that of groups with certain generators,
- groupoid racks defining colorings for spatial surface diagrams and the universality,
- homogeneous quandles with commutative inner automorphism groups and the number of isomorphism classes for small orders,
- generalized s-manifolds, which is a generalization of Riemann symmetric spaces, and a construction of examples using compact symmetric triads.

All of the talks are very interesting, and after the talks, the participants exchanged their ideas and information, and discussed possible perspectives actively.

In this volume the abstracts and the slides of the talks in the conference are collected. For the talks in 2019–2021, one can refer to the previous volumes (OCAMI Reports Vol. 4 and Vol. 9). The organizers are convinced that the workshops and the volumes would disseminate quandles, and be effective for further developments of the theory of quandles.

February 2023

On behalf of the organizers:
Akira Kubo
Takayuki Okuda
Hiroshi Tamaru

Organizers

Seiichi Kamada

Department of Mathematics, Graduate School of Science, Osaka University,
1-1 Machikaneyama, Toyonaka, Osaka 560-0043, Japan

Email address: kamada@math.sci.osaka-u.ac.jp

Akira Kubo

Department of Food Sciences and Biotechnology, Hiroshima Institute of Technology,
2-1-1 Miyake, Saeki-ku, Hiroshima, 731-5193, Japan

Email address: a.kubo.3r@cc.it-hiroshima.ac.jp

Takayuki Okuda

Graduate School of Advanced Science and Engineering,
Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526, Japan

Email address: okudatak@hiroshima-u.ac.jp

Kanako Oshiro

Deparment of Information and Communication Sciences, Sophia University,
7-1 Kioicho, Chiyoda-ku Tokyo, 102-8554, Japan

Email address: oshirok@sophia.ac.jp

Hiroshi Tamaru

Department of Mathematics, Graduate School of Science, Osaka Metropolitan University,
3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan

Email address: tamaru@omu.ac.jp

Makiko Sumi Tanaka

Department of Mathematics, Faculty of Science and Technology,
Tokyo University of Science, Noda, Chiba, 278-8510, Japan

Email address: tanaka_makiko@rs.tus.ac.jp

Hiroyuki Tasaki

Department of Mathematics, Faculty of Pure and Applied Sciences,
University of Tsukuba, Tsukuba, Ibaraki, 305-8571, Japan

Email address: tasaki@math.tsukuba.ac.jp

Contents

On categories of faithful quandles with quandle homomorphisms

Yasuki Tada

1 Quandles and inner automorphism groups of quandles

The concept of quandles was introduced by Joyce ([3]). A quandle is a set with a binary operator, whose axioms are corresponding to Reidemeister moves of classical knots. Quandles have been studied actively from various viewpoints ([1]). From the view point of differential geometry, quandles can be regarded as a generalization of symmetric spaces. There have already been several studies of quandles that transfer notations and ideas in the theory of symmetric spaces to that of quandles ([2], [5]).

In this paper, we employ a formulation of quandles in terms of symmetries as [3].

Definition 1.1. Let Q be a set. We consider a map $s : Q \to \mathrm{Map}(Q, Q) : x \mapsto s_x$. Then the pair (Q, s) is a *quandle* if
(Q1) $\forall x \in Q, s_x(x) = x$,
(Q2) $\forall x \in Q, s_x$ is bijective,
(Q3) $\forall x, y \in Q, s_x \circ s_y = s_{s_x(y)} \circ s_x$.
For each $x \in Q$, the map s_x is called a *symmetry* at x on Q.

We denote by $\mathrm{Aut}(Q)$ the group of quandle automorphisms of Q.

Definition 1.2. Let (Q, s) be a quandle and Q' a subquandle of Q. We use the symbol $\mathrm{Inn}(Q, Q')$ for the group of $\mathrm{Aut}(Q)$ generated by the set $s(Q') = \{s_x : Q \to Q \mid x \in Q'\}$. The group $\mathrm{Inn}Q := \mathrm{Inn}(Q, Q)$ is called the *inner automorphism group* of (Q, s).

The inner automorphism groups play important roles in the structure theory of quandles.

2 A question

Definition 2.1. Let us denote several categories as below:

- $\mathbf{Grp}$: the category of groups and group homomorphisms.
- $\mathbf{Grp^{gen}}$: the category of groups with generators, whose morphisms are group homomorphisms inducing maps between fixed generators.
- $\mathbf{Q}$: the category of quandles and quandle homomorphisms.
- $\mathbf{Q_{surj}}$: the category of quandles and surjective quandle homomorphisms.
- $\mathbf{Q^f}$: the category consists of faithful quandles and quandle homomorphisms.
- $\mathbf{Q^f_{surj}}$: the subcategory of $\mathbf{Q^f}$ with surjective quandle homomorphisms.
- $\mathbf{Q^f_{inj}}$: the subcategory of $\mathbf{Q^f}$ with injective quandle homomorphisms.

Here, a quandle (Q, s) is said to be *faithful* if $s : Q \to \mathrm{Inn}(Q)$ is injective.

By Definition 1.2, we have the correspondence:

$$\mathrm{Inn} : Q \mapsto \mathrm{Inn}(Q).$$

We consider the following Question:

Question 2.2. Can "Inn" be expanded into a "good" functor $\mathbf{Q} \to \mathbf{Grp}$ or $\mathbf{Q^f} \to \mathbf{Grp}$?

3 For surjective quandle homomorphisms

First, we focus on $\mathbf{Q_{surj}}$ and $\mathbf{Q^{f}_{surj}}$. In this case, "Inn" is naturally expanded into a functor as in [1]. Actually, the following holds.

Theorem 3.1 (cf. [1])**.** *The correspondence* $\mathrm{Inn} : Q \mapsto \mathrm{Inn}(Q)$ *is expanded to a functor* $\mathbf{Q_{surj}} \to \mathbf{Grp}$. *Furthermore, the functor is faithful on* $\mathbf{Q^{f}_{surj}}$.

We shall remark that the functor in Theorem 3.1 is not a category equivalence. In this paper, we focus on the correspondence $Q \mapsto (\mathrm{Inn}(Q), s(Q))$ that conisder not only the inner automorphism group $\mathrm{Inn}(Q)$ but also its generator $s(Q)$ for each quandle Q.

We define a group theoretic category $\mathbf{Grp^{g.c.f}_{surj}}$ as follows.

Definition 3.2 ($\mathbf{Grp^{g.c.f}_{surj}}$)**.** An object (G, Ω) of $\mathbf{Grp^{g.c.f}_{surj}}$ is a pair of a group G and its conjugation-stable faithful generator Ω. Here, a generator Ω of a group G is said to be *conjugation-stable* if $g\Omega g^{-1} \subset \Omega$ for any g in G, and is said to be *faithful* if the following action $G \curvearrowright \Omega$ is faithful:

$$g.\omega = g\omega g^{-1} \quad (g \in G, \omega \in \Omega).$$

A morphism $\varphi : (G_1, \Omega_1) \to (G_2, \Omega_2)$ of $\mathbf{Grp^{g.c.f}_{surj}}$ is a group homomorphism $\varphi : G_1 \to G_2$ such that $\varphi(\Omega_1) \subset \Omega_2$ and the restriction $\varphi|_{\Omega_1} : \Omega_1 \to \Omega_2$ is surjective.

We have the following theorem as one of our main results.

Theorem 3.3 (see [4, Theorem 3.9])**.** *There exists an equivalence* $\mathcal{F}_{\mathrm{surj}} : \mathbf{Q^{f}_{surj}} \to \mathbf{Grp^{g.c.f}_{surj}}$ *such that* $\mathcal{F}_{\mathrm{surj}}(Q, s) = (\mathrm{Inn}Q, s(Q))$ *for each faithful quandle* (Q, s).

4 For injective quandle homomorphisms

Next, we focus on $\mathbf{Q^{f}_{inj}}$. We should remark that the correspondence $Q \mapsto (\mathrm{Inn}(Q), s(Q))$ can not be expanded into any faithful functor $\mathbf{Q^{f}_{inj}} \to \mathbf{Grp^{gen}}$. Then we consider a question below.

Question 4.1. Find a category D such that objects of D are pairs of groups and their generators, and $\mathbf{Q^{f}_{inj}} \cong D$.

Because of the following proposition, we define a group theoretic category $\mathbf{Grp^{g.c.f}_{\star}}$ as in Definition 4.3.

Proposition 4.2. *For a morphism* $f : Q_1 \to Q_2$ *of* $\mathbf{Q^{f}_{inj}}$, *the following* $\pi(f)$ *is well-defined surjective group homomorphism :*

$$\pi(f) : \mathrm{Inn}(Q_2, f(Q_1)) \to \mathrm{Inn}(Q_1) : s_{f(x_1)} \mapsto s_{x_1},$$

where $s(f(Q_1)) := \{s_{f(x_1)} : Q_2 \to Q_2 \mid x_1 \in Q_1\}$ *and* $\mathrm{Inn}(Q_2, f(Q_1)) := \langle s(f(Q_1))\rangle <$ $\mathrm{Inn}(Q_2)$.

$$\begin{tikzcd} Q_1 \arrow[r, hook, "f"] & Q_2 \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Inn}(Q_1) & \mathrm{Inn}(Q_2) \\ & \mathrm{Inn}(Q_2, f(Q_1)) \arrow[ul, two heads, "\pi(f)"] \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Definition 4.3 ($\mathbf{Grp}_\star^{\mathbf{g.c.f}}$)**.** We define a category $\mathbf{Grp}_\star^{\mathbf{g.c.f}}$ as follows. Let us put $\mathrm{Obj}(\mathbf{Grp}_\star^{\mathbf{g.c.f}}) := \mathrm{Obj}(\mathbf{Grp}_{\mathbf{surj}}^{\mathbf{g.c.f}})$. For objects $(G_1, \Omega_1), (G_2, \Omega_2) \in \mathrm{Obj}(\mathbf{Grp}_\star^{\mathbf{g.c.f}})$, we define the set of morphisms $\mathrm{Hom}_{\mathbf{Grp}_\star^{\mathbf{g.c.f}}}((G_1, \Omega_1), (G_2, \Omega_2))$ from (G_1, Ω_1) to (G_2, Ω_2) in $\mathbf{Grp}_\star^{\mathbf{g.c.f}}$ as follows.

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{Grp}_\star^{\mathbf{g.c.f}}}((G_1, \Omega_1), (G_2, \Omega_2)) \\ &:= \left\{ ((H, \Gamma), \pi) \;\middle|\; \begin{array}{c} H : \text{a subgroup of } G_2, \\ \Gamma : \text{a subset of } \Omega_2, \\ \Gamma : \text{a conjugation-stable generator of } H, \\ \pi(\Gamma) \subset \Omega_1 \text{ and } \pi|_\Gamma : \Gamma \to \Omega_1 \text{ is bijective} \end{array} \right\}.\end{aligned}$$

We remark that each morphism is an opposite directional partial map, and a diagram of a morphism can be written as Figure 1. We define composition of morphisms in $\mathbf{Grp}_\star^{\mathbf{g.c.f}}$

$$\begin{tikzcd}
(G_1,\Omega_1) \arrow[r, dashed, "\Phi"] & (G_2,\Omega_2) \arrow[d, phantom, "\cup"] \\
& (H,\Gamma) \arrow[ul, two heads, "\pi"]
\end{tikzcd}$$

Figure 1: $\Phi = ((H, \Gamma), \pi) \in \mathrm{Hom}_{\mathbf{Grp}_\star^{\mathbf{g.c.f}}}((G_1, \Omega_1), (G_2, \Omega_2))$.

by using "pullback".

We also have the following theorem.

Theorem 4.4 (see [4, Theorem 4.17])**.** *There exists an equivalence $\mathcal{F}_{\mathrm{inj}} : \mathbf{Q}_{\mathbf{inj}}^{\mathbf{f}} \to \mathbf{Grp}_\star^{\mathbf{g.c.f}}$ such that $\mathcal{F}_{\mathrm{inj}}(Q, s) = (\mathrm{Inn}Q, s(Q))$ for each faithful quandle (Q, s).*

By Theorem 3.3 and 4.4, for each pair of faithful quandles (Q_1, Q_2), we have bijections:

$$\mathrm{Hom}_{\mathbf{Q}_{\mathbf{surj}}^{\mathbf{f}}}(Q_1, Q_2) \overset{1:1}{\leftrightarrow} \mathrm{Hom}_{\mathbf{Grp}_{\mathbf{surj}}^{\mathbf{g.c.f}}}((\mathrm{Inn}(Q_1), s(Q_1)), (\mathrm{Inn}(Q_2), s(Q_2))),$$

$$\mathrm{Hom}_{\mathbf{Q}_{\mathbf{inj}}^{\mathbf{f}}}(Q_1, Q_2) \overset{1:1}{\leftrightarrow} \mathrm{Hom}_{\mathbf{Grp}_\star^{\mathbf{g.c.f}}}((\mathrm{Inn}(Q_1), s(Q_1)), (\mathrm{Inn}(Q_2), s(Q_2))).$$

As an easy application of Theorem 4.4, we also study the set of all injective quandle homomorphisms from the dihedral quandle R_3 of order 3 to the dihedral quandle R_9 of order 9 by group theoretic approach.

References

[1] E. Bunch, P. Lofgren, A. Rapp and D. N. Yetter, *On quotients of quandles*, J. Knot Theory Ramifications, **19** (2010), 1145–1156.

[2] Y. Ishihara and H. Tamaru, *Flat connected finite quandles*, Proc. Amer. Math. Soc., **144** (2016), 4959–4971.

[3] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra, **23** (1982), 37–65.

[4] Y. Tada, *On categories of faithful quandles with surjective or injective quandle homomorphisms*, to appear in Hiroshima Mathematical Journal (available at arXiv:2211.15014).

[5] H. Tamaru, *Two-point homogeneous quandles with prime cardinality*, J. Math. Soc. Japan, **65** (2013), 1117–1134.

(Y. Tada) Graduate School of Advanced Science and Engineering Hiroshima University Higashi-Hiroshima, 739-8526, JAPAN

Email address: `tada-yasu@hiroshima-u.ac.jp`

On categories of faithful quandles with surjective or injective quandle homomorphisms

Graduate School of Advanced Science and Engineering,
Hiroshima University
TADA Yasuki

Quandles and symmetric spaces 2022

Dec.08-09.2022

Outline

(1)Background

$Q \mapsto \mathrm{Inn}(Q)$ is not a faithful functor : $\mathbf{Q^f} \to \mathbf{Grp}$.

(2)Our idea

Focus on $\mathbf{Q^f_{inj}}$: the category of faithful quandles and injective hom.
Construct $\mathbf{Grp}^{\mathrm{g.c.f.}}_{\star}$: $\mathrm{Obj} = \{(G, \Omega) \mid G$: group, Ω : generator, $+\alpha\}$.

(3)Result (Main Thm)

$\exists \mathcal{F} : \mathbf{Q^f_{inj}} \to \mathbf{Grp}^{\mathrm{g.c.f.}}_{\star} : Q \overset{\text{obj}}{\mapsto} (\mathrm{Inn}Q, s(Q))$: category equivalence i.e.

$$\mathrm{Hom}_{\mathbf{Q^f_{inj}}}(Q_1, Q_2) \overset{1:1}{\leftrightarrow} \mathrm{Hom}_{\mathbf{Grp}^{\mathrm{g.c.f.}}_{\star}}(\mathcal{F}Q_1, \mathcal{F}Q_2).$$

(1)-1. Definition of quandle

Q : set. $s : Q \to \mathrm{Map}(Q, Q) : x \mapsto s_x$.

Definition

(Q, s) : **quandle**

$:\Leftrightarrow$ (Q1) $\forall x \in Q, s_x(x) = x$.

(Q2) $\forall x \in Q, s_x$: bijective.

(Q3) $\forall x, y \in Q, s_x \circ s_y = s_{s_x(y)} \circ s_x$.

s : **quandle structure**, s_x : **point symmetry at** x.

Example (Trivial quandle)

Any set T is a quandle : $s_t = \mathrm{id}_T$ $(\forall t \in T)$.

T_n : trivial quandle of order n.

(1)-2. Faithful quandle

Q : quandle.

Definition (faithful)

Q is **faithful** if, for any $x, y \in Q$:

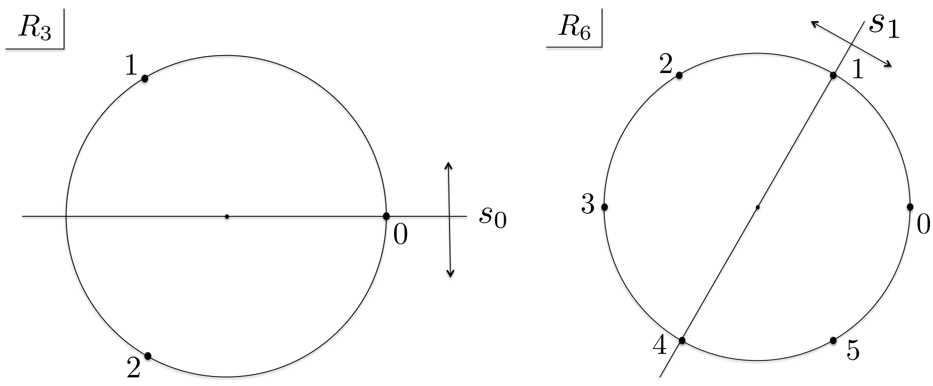
$$s_x = s_y \Leftrightarrow x = y.$$

Q : faithful $\Leftrightarrow s : Q \to \mathrm{Map}(Q, Q)$: injective.

(1)-3. Dihedral quandle

Example

(R_n, s) : the **dihedral quandle**, where $R_n := \mathbb{Z}/n\mathbb{Z}$, $s_x(y) := 2x - y$.



Proposition

R_n : faithful $\Leftrightarrow$ n : odd.

(1)-4. Conjugate quandle

G : a group, $\Omega \subset G$: a union of some conjugacy classes.

Example ($\mathrm{Conj}(\Omega)$)

$\mathrm{Conj}_G(\Omega)$ (or $\mathrm{Conj}(\Omega)$) : the **conjugate quandle**, where $\mathrm{Conj}(\Omega) = \Omega$,

$$s_\omega(\omega') = \omega\omega'\omega^{-1} \quad (\omega, \omega' \in \Omega).$$

Proposition

The centralizer of Ω in G is trivial $\Rightarrow$ $\mathrm{Conj}(\Omega)$ is faithful.

Example

$\mathfrak{S}_n$: the n-symmetric group,
$\Omega_n = \{(ij) \mid 1 \le i < j \le n\}$: all transpositions.
$\mathrm{Conj}_{\mathfrak{S}_n}(\Omega_n)$: faithful.

(1)-5. Inner automorphism group

Q : a quandle.
$\mathrm{Aut}(Q) := \{g \in \mathrm{Bij}(Q) \mid g \circ s_x = s_{g(x)} \circ g \quad (\forall x \in Q)\}$: the automorphism group of Q.

Definition ($\mathrm{Inn}(Q)$)

$s(Q) := \{s_x \mid x \in Q\} \subset \mathrm{Aut}(Q)$.
$\mathrm{Inn}(Q) := \langle s(Q) \rangle < \mathrm{Aut}(Q)$: the **inner automorphism group** of Q.

(1)-6. Example of inner automorphism groups

Example

For the dihedral quandle R_n,

$$\mathrm{Inn}(R_n) \cong \begin{cases} D_{2n} & (n : \text{odd}) \\ D_n & (n : \text{even}) \end{cases}$$

where D_{2k} is the dihedral group with $|D_{2k}| = 2k$.

Example

$n \geq 3$, $\Omega_n = \{(ij) \mid 1 \leq i < j \leq n\} \subset \mathfrak{S}_n$.
For the quandle $\mathrm{Conj}_{\mathfrak{S}_n}(\Omega_n)$,

$$\mathrm{Inn}(\mathrm{Conj}_{\mathfrak{S}_n}(\Omega_n)) \cong \mathrm{Inn}_{\mathbf{Grp}}(\mathfrak{S}_n) \cong \mathfrak{S}_n.$$

(1)-7. In order to consider Inn as a functor

$\mathbf{Q}^{\mathbf{f}}$: the category of **faithful** quandles and quandle homomorphisms.
$\mathbf{Grp}$: the category of groups and group homomorphisms.
We have a correspondence :

$$\mathrm{Inn} : \mathrm{Obj}(\mathbf{Q}^{\mathbf{f}}) \to \mathrm{Obj}(\mathbf{Grp}) : Q \mapsto \mathrm{Inn}(Q).$$

Question

Does "Inn" become a faithful functor $\mathbf{Q}^{\mathbf{f}} \to \mathbf{Grp}$?

For surjective quandle hom., the answer is YES.

Theorem (Bunch, Lofgren, Rapp and Yetter (2010) $+\alpha$)

$\mathbf{Q}^{\mathbf{f}}_{\mathbf{surj}}$: the category of **faithful** quandles and **surjective** homomorphisms.
Then $\mathrm{Inn} : \mathbf{Q}^{\mathbf{f}}_{\mathbf{surj}} \to \mathbf{Grp}$ becomes a faithful functor.

$$\begin{tikzcd} Q_1 \arrow[r, two heads, "f"] \arrow[d, "s"'] & Q_2 \arrow[d, "s"] \\ \mathrm{Inn}(Q_1) \arrow[r, two heads, "\mathrm{Inn}(f)"'] & \mathrm{Inn}(Q_2) \end{tikzcd}$$

(1)-8. For NOT surjective hom.

Question

Does "Inn" become a faithful functor $\mathbf{Q}^{\mathbf{f}} \to \mathbf{Grp}$?

For $\mathbf{Q}^{\mathbf{f}}$ (or $\mathbf{Q}^{\mathbf{f}}_{\mathbf{inj}}$), the answer is NO.

Remark

T_1 : the trivial quandle of order 1, R_3 : the dihedral quandle of order 3.
$|\mathrm{Hom}_{\mathbf{Q}}(T_1, R_3)| = 3$.
$|\mathrm{Hom}_{\mathbf{Grp}}(\mathrm{Inn}(T_1), \mathrm{Inn}(R_3))| = 1. \quad (\mathrm{Inn}(T_1) \cong 1, \mathrm{Inn}(R_3) \cong D_6)$

$\mathbf{Q}^{\mathbf{f}}_{\mathbf{inj}}$: the category of **faithful** quandles and **injective** homomorphisms.

Outline

(1)Background

$Q \mapsto \mathrm{Inn}(Q)$ is not a faithful functor : $\mathbf{Q^f} \to \mathbf{Grp}$.

(2)Our idea

Focus on $\mathbf{Q^f_{inj}}$: the category of faithful quandles and injective hom.
Construct $\mathbf{Grp}_\star^{\mathrm{g.c.f.}}$: $\mathrm{Obj} = \{(G, \Omega) \mid G :$ group$, \Omega :$ generator$, +\alpha\}$.

(3)Result (Main Thm)

$\exists \mathcal{F} : \mathbf{Q^f_{inj}} \to \mathbf{Grp}_\star^{\mathrm{g.c.f.}} : Q \overset{\text{obj}}{\mapsto} (\mathrm{Inn}Q, s(Q))$: category equivalence i.e.

$$\mathrm{Hom}_{\mathbf{Q^f_{inj}}}(Q_1, Q_2) \overset{1:1}{\leftrightarrow} \mathrm{Hom}_{\mathbf{Grp}_\star^{\mathrm{g.c.f.}}}(\mathcal{F}Q_1, \mathcal{F}Q_2).$$

(2)-1. Focus on $\mathbf{Q^f_{inj}}$

$\mathbf{Q^f_{inj}}$: the category of **faithful** quandles and **injective** quandle homomorphisms.
Consider a correspondence $Q \mapsto (\mathrm{Inn}(Q), s(Q))$.

$(\mathrm{Inn}(Q), s(Q))$ is a pair of a group and its generator.

Problem

Find a category D s.t.
$\mathrm{Obj} = \{(G, \Omega) \mid G :$ a group$, \Omega \subset G :$ a generator of $G, +\alpha\}$ and
$\mathbf{Q^f_{inj}} \cong D$.

(2)-2. Propositions on $\mathbf{Q}^{\mathrm{f}}_{\mathrm{inj}}$

Proposition (A)

For a morphism $f : Q_1 \to Q_2$ of $\mathbf{Q}^{\mathrm{f}}_{\mathrm{inj}}$, the following $\pi(f)$ is well-defined surjective group hom. :

$$\pi(f) : \mathrm{Inn}(Q_2, f(Q_1)) \to \mathrm{Inn}(Q_1) : s_{f(x_1)} \mapsto s_{x_1},$$

where $s(f(Q_1)) := \{s_{f(x_1)} : Q_2 \to Q_2 \mid x_1 \in Q_1\}$ and $\mathrm{Inn}(Q_2, f(Q_1)) := \langle s(f(Q_1)) \rangle < \mathrm{Inn}(Q_2)$.

$$Q_1 \xhookrightarrow{f} Q_2 \qquad \Bigg| \qquad \begin{tikzcd} \mathrm{Inn}(Q_1) & \mathrm{Inn}(Q_2) \arrow[d, phantom, "\cup"] \\ & \mathrm{Inn}(Q_2, f(Q_1)) \arrow[ul, two heads, "\pi(f)"] \end{tikzcd}$$

(2)-3. Propositions on $\mathbf{Q}^{\mathrm{f}}_{\mathrm{inj}}$

Proposition (B)

For $Q \in \mathrm{Obj}(\mathbf{Q}^{\mathrm{f}}_{\mathrm{inj}})$, $(\mathrm{Inn}(Q), s(Q))$ satisfies following (g), (c) and (f) :

(g) $s(Q)$ generates $\mathrm{Inn}(Q)$.

(c) $s(Q)$ is a union of conjugacy classes of $\mathrm{Inn}(Q)$.
i.e. $s(Q)$ is $\mathrm{Inn}(Q)$-stable w.r.t. the following left action $\mathrm{Inn}(Q) \curvearrowright s(Q)$:

$$g.s_x = g s_x g^{-1} \quad (g \in \mathrm{Inn}(Q), x \in Q).$$

(f) The above left action $\mathrm{Inn}(Q) \curvearrowright s(Q)$ is faithful.
i.e. the centralizer of $s(Q)$ is free.

Based on Proposition(A) and (B), we construct $\mathbf{Grp}^{\mathbf{g.c.f.}}_{\star}$ as below.

(2)-4. Definition of $\mathbf{Grp}_\star^{\text{g.c.f.}}$

Let G : group, $\Omega \subset G$. We define conditions (g), (c) and (f) of (G, Ω) :

- (g) Ω generates G.
- (c) Ω is a union of conjugacy classes of G.
  i.e. Ω is G-stable w.r.t. the following left action $G \curvearrowright \Omega$:
  $$g.\omega = g\omega g^{-1} \quad (g \in G, \omega \in \Omega).$$
- (f) The above left action $G \curvearrowright \Omega$ is faithful.
  i.e. the centralizer of Ω is free.

Definition

$\mathbf{Grp}_\star^{\text{g.c.f.}}$: $\mathrm{Obj} = \{(G, \Omega) \mid$ (g), (c) and (f) hold $\}$.

$$\mathrm{Hom}_{\mathbf{Grp}_\star^{\text{g.c.f.}}}((G_1,\Omega_1),(G_2,\Omega_2))$$
$$:= \left\{ \Phi = ((H,\Gamma),\pi) \;\middle|\; \begin{array}{c} H < G_2, \Gamma \subset \Omega_2, \\ (H,\Gamma) \text{ satisfies (g) and (c)}, \\ \pi : H \to G_1 : \text{a group hom.}, \\ \pi(\Gamma) \subset \Omega_1 \text{ and } \pi|_\Gamma : \Gamma \to \Omega_1 : \text{bijective} \end{array} \right\}.$$

(2)-5. Morphisms of $\mathbf{Grp}_\star^{\text{g.c.f.}}$

$$\mathrm{Hom}_{\mathbf{Grp}_\star^{\text{g.c.f.}}}((G_1,\Omega_1),(G_2,\Omega_2))$$
$$:= \left\{ \Phi = ((H,\Gamma),\pi) \;\middle|\; \begin{array}{c} H < G_2, \Gamma \subset \Omega_2, \\ (H,\Gamma) \text{ satisfies (g) and (c)}, \\ \pi : H \to G_1 : \text{a group hom.}, \\ \pi(\Gamma) \subset \Omega_1 \text{ and } \pi|_\Gamma : \Gamma \to \Omega_1 : \text{bijective} \end{array} \right\}.$$

$$\begin{tikzcd}
(G_1,\Omega_1) \arrow[r, dashed, "\Phi"] & (G_2,\Omega_2) \\
 & (H,\Gamma) \arrow[ul, two heads, "\pi"] \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

(2)-6. Example of morphisms of $\mathbf{Grp}_\star^{\text{g.c.f.}}$

$\Omega_n = \{(ij) \mid 1 \le i < j \le n\} \subset \mathfrak{S}_n$.

Example

The following $\Phi = ((H, \Gamma), \pi)$ is a morphism $(\mathfrak{S}_3, \Omega_3) \to (\mathfrak{S}_6, \mathfrak{S}_6)$ in $\mathbf{Grp}_\star^{\text{g.c.f.}}$:

$$H = \mathfrak{S}_3 \times \{\mathrm{id}, (456), (465)\}, \Gamma = \Omega_3 \times \{(456)\},$$

$$\pi : \mathfrak{S}_3 \times \{\mathrm{id}, (456), (465)\} \to \mathfrak{S}_3 : (g, a) \mapsto g.$$

Furthermore, $\pi : H \to \mathfrak{S}_3$ is NOT injective.

$$\begin{tikzcd}
(\mathfrak{S}_3, \Omega_3) \arrow[r, dashed, "\Phi"] & (\mathfrak{S}_6, \mathfrak{S}_6) \arrow[d, phantom, "\cup"] \\
& (H, \Gamma) \arrow[ul, two heads, "\pi"]
\end{tikzcd}$$

(2)-7. Composition of morphisms on $\mathbf{Grp}_\star^{\text{g.c.f.}}$

Proposition

$\Phi_1 = ((H_2, \Gamma_2), \pi_2) : (G_1, \Omega_1) \to (G_2, \Omega_2)$,
$\Phi_2 = ((H_3, \Gamma_3), \pi_3) : (G_2, \Omega_2) \to (G_3, \Omega_3)$: morphisms of $\mathbf{Grp}_\star^{\text{g.c.f.}}$.
Then the following is a morphism $(G_1, \Omega_1) \to (G_3, \Omega_3)$ of $\mathbf{Grp}_\star^{\text{g.c.f.}}$:

$$\Phi_2 \circ \Phi_1 := ((\langle \pi_3|_{\Gamma_3}^{-1}(\Gamma_2)\rangle, \pi_3|_{\Gamma_3}^{-1}(\Gamma_2)), \pi_2 \circ \pi_3|_{\langle \pi_3|_{\Gamma_3}^{-1}(\Gamma_2)\rangle}).$$

$$\begin{tikzcd}
(G_1, \Omega_1) \arrow[r, dashed, "\Phi_1"] & (G_2, \Omega_2) \arrow[d, phantom, "\cup"] \arrow[r, dashed, "\Phi_2"] & (G_3, \Omega_3) \arrow[d, phantom, "\cup"] \\
& (H_2, \Gamma_2) \arrow[ul, two heads, "\pi_2"] \arrow[r, phantom, "\text{PB}"] & (H_3, \Gamma_3) \arrow[ul, two heads, "\pi_3"'] \arrow[d, phantom, "\cup"] \\
& & (\langle \pi_3|_{\Gamma_3}^{-1}(\Gamma_2)\rangle, \pi_3|_{\Gamma_3}^{-1}(\Gamma_2)) \arrow[ul, two heads, "\pi_3|_{\langle \pi_3|_{\Gamma_3}^{-1}(\Gamma_2)\rangle}"]
\end{tikzcd}$$

This composition is associative, since there is pullback on $\mathbf{Grp}^{\mathbf{gen}}$.

Outline

(1)Background

$Q \mapsto \mathrm{Inn}(Q)$ is not a faithful functor : $\mathbf{Q}^{\mathbf{f}} \to \mathbf{Grp}$.

(2)Our idea

Focus on $\mathbf{Q}^{\mathbf{f}}_{\mathbf{inj}}$: the category of faithful quandles and injective hom.
Construct $\mathbf{Grp}^{\mathbf{g.c.f.}}_{\star}$: $\mathrm{Obj} = \{(G, \Omega) \mid G$: group, Ω : generator, $+\alpha\}$.

(3)Result (Main Thm)

$\exists \mathcal{F} : \mathbf{Q}^{\mathbf{f}}_{\mathbf{inj}} \to \mathbf{Grp}^{\mathbf{g.c.f.}}_{\star} : Q \overset{\text{obj}}{\mapsto} (\mathrm{Inn}Q, s(Q))$: category equivalence i.e.

$$\mathrm{Hom}_{\mathbf{Q}^{\mathbf{f}}_{\mathbf{inj}}}(Q_1, Q_2) \overset{1:1}{\leftrightarrow} \mathrm{Hom}_{\mathbf{Grp}^{\mathbf{g.c.f.}}_{\star}}(\mathcal{F}Q_1, \mathcal{F}Q_2).$$

(3)-1. Definition of category equivalence

C, D : categories.

Definition

$\mathcal{F} : C \to D$: a functor.
$\mathcal{F}$ is a **category equivalence** if the followings hold:

- $\forall d \in \mathrm{Obj}(D), \exists c \in \mathrm{Obj}(C)$ s.t. $\mathcal{F}c \cong_D d$. (essentially surjective)
- $\mathcal{F} : \mathrm{Hom}_C(c_1, c_2) \to \mathrm{Hom}_D(\mathcal{F}c_1, \mathcal{F}c_2)$: bijective. (faithful full)

(3)-2. Main Theorem

Theorem (Main Thm)

The following $\mathcal{F} : \mathbf{Q}^{\mathbf{f}}_{\mathbf{inj}} \to \mathbf{Grp}^{\mathrm{g.c.f.}}_{\star}$ is a category equivalence :

- For $Q \in \mathrm{Obj}(\mathbf{Q}^{\mathbf{f}}_{\mathbf{inj}})$, $\mathcal{F}Q := (\mathrm{Inn}(Q), s(Q))$.
- For $f : Q_1 \to Q_2 \in \mathbf{Q}^{\mathbf{f}}_{\mathbf{inj}}$, $\mathcal{F}f := ((H, \Gamma), \pi(f))$, where
  $\Gamma := s(f(Q_1)) = \{s_{f(x_1)} : Q_2 \to Q_2 \mid x_1 \in Q_1\} \subset \mathrm{Inn}Q_2$,
  $H := \mathrm{Inn}(Q_2, f(Q_1)) = \langle s(f(Q_1)) \rangle < \mathrm{Inn}Q_2$,
  $\pi(f) : (\mathrm{Inn}(Q_2, f(Q_1)), s(f(Q_1))) \to (\mathrm{Inn}Q_1, s(Q_1)) : s_{f(x_1)} \mapsto s_{x_1}$.

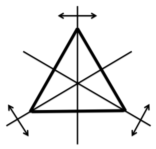
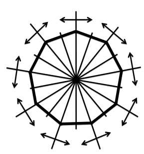
$$Q_1 \xhookrightarrow{f} Q_2$$

$$\begin{tikzcd}
(\mathrm{Inn}(Q_1), s(Q_1)) \arrow[r, dashed, "\mathcal{F}f"] & (\mathrm{Inn}(Q_2), s(Q_2)) \arrow[d, phantom, "\cup"] \\
 & (\mathrm{Inn}(Q_2, f(Q_1)), s(f(Q_1))) \arrow[ul, two heads, "\pi(f)"]
\end{tikzcd}$$

(3)-3. Application of Main Theorem

$$|\mathrm{Hom}_{\mathbf{Q}^{\mathbf{f}}_{\mathbf{inj}}}(R_3, R_9)| = |\mathrm{Hom}_{\mathbf{Grp}^{\mathrm{g.c.f.}}_{\star}}(\mathcal{F}R_3, \mathcal{F}R_9)| = 18.$$

$$R_3 \hookrightarrow R_9$$

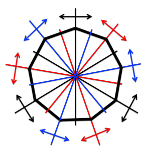
$$\mathcal{F}R_3 = (\mathrm{Inn}(R_3), s(R_3)) = (D_6,$$

$$) \dashrightarrow (D_{18},$$

$$)$$

$$\cup$$

$$\mathcal{F}R_9 = (\mathrm{Inn}(R_9), s(R_9))$$

π

(H_1, Γ_1)
(H_2, Γ_2)
(H_3, Γ_3)



Thank you for your attention.

Minimum numbers of Dehn colors of knots

Gaishi Yamagishi

1 Introduction

This presentation is based on the author's collaboration with Kanako Oshiro and Eri Matsudo.

In knot theory, minimum numbers of colors for arc colorings have been studied in many papers. Fox p-coloring is the *dihedral quandle* coloring. We denote the number by $\mathrm{mincol}_p^{\mathrm{Fox}}(K)$ for a Fox p-colorable knot K. For $\mathrm{mincol}_p^{\mathrm{Fox}}(K)$, the following result is known.

- For each odd prime number p with $p \leq 19$,

$$K : \text{Fox } p\text{-colorable knot } \Longrightarrow \mathrm{mincol}_p^{\mathrm{Fox}}(K) = \lfloor \log_2 p \rfloor + 2$$

  (see [1, 2, 3, 4, 5, 6, 7]).

Dehn colorings are one of region colorings which are known to be corresponding to Fox colorings.

2 Preliminary

2.1 Dehn coloring

In this paper, for a prime number p, we denote by $\mathbb{Z}_p$ the cyclic group $\mathbb{Z}/p\mathbb{Z}$.

Let p be an odd prime number. Let D be a diagram of a knot K and $\mathcal{R}(D)$ the set of regions of D. A Dehn *p-coloring* of D is a map $C : \mathcal{R}(D) \to \mathbb{Z}_p$ satisfying the following condition:

- for each crossing c with regions x_1, x_2, x_3, and x_4 as depicted in Figure 1,

$$C(x_1) + C(x_3) = C(x_2) + C(x_4)$$

  holds, where the region x_2 is adjacent to x_1 by an under-arc and x_3 is adjacent to x_1 by the over-arc.

We call $C(x)$ the *color* of a region x by C. We mean by (D, C) a diagram D given a Dehn p-coloring C, and call it a Dehn *p-colored diagram*. We denote by $\mathcal{C}(D, C)$ the set of colors assigned to a region of D by C, that is $\mathcal{C}(D, C) = \mathrm{Im}C$. The set of Dehn p-colorings of D is denoted by $\mathrm{Col}_p(D)$. We remark that the number $\sharp\mathrm{Col}_p(D)$ is an invariant of the knot K.

Let χ be a crossing of D with regions x_1, x_2, x_3, and x_4 as depicted in Figure 1. We say that χ of (D, C) is *trivially colored* if

$$C(x_1) = C(x_4), \text{ and } C(x_3) = C(x_2)$$

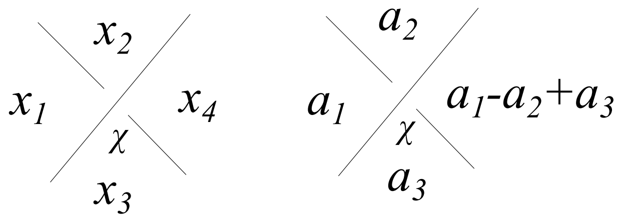


Figure 1: A crossing on D with regions x_1, x_2, x_3, and x_4

hold, and *nontrivially colored* otherwise. A Dehn p-coloring C of D is *trivial* if each crossing of (D, C) is trivially colored (and $\sharp\mathcal{C}(D, C) \leq 2$), and *nontrivial* otherwise. We denote by $\mathrm{Col}_p^{\mathrm{NT}}(D, C)$ the set of nontrivial Dehn p-colorings of D. A knot K is Dehn *p-colorable* if K has a Dehn p-colored diagram (D, C) such that C is nontrivial.

Lemma 2.1. *1. Let $C, C' \in \mathrm{Col}_p(D)$. Then we have*

$$C \sim C' \Longrightarrow \mathcal{C}(D, C) \sim \mathcal{C}(D, C').$$

Hence we have

$$C \sim C' \Longrightarrow \sharp\mathcal{C}(D, C) = \sharp\mathcal{C}(D, C').$$

2. Let $S, S' \subset \mathbb{Z}_p$, and we assume that $S \sim S'$. Then there exists $C \in \mathrm{Col}_p(D)$ such that $\mathcal{C}(D, C) = S$ if and only if there exists $C' \in \mathrm{Col}_p(D)$ such that $C(D, C') = S'$.

Here $C \sim C'$(or $S \sim S'$) means that there exists $s \in \mathbb{Z}_p^\times$ and $t \in \mathbb{Z}_p$ such that $C = sC' + t$(or $S' = sS + t$).

Definition 2.2. The *minimum number of colors* of a knot K for Dehn p-colorings is the minimum number of distinct elements of $\mathbb{Z}_p$ which produce a nontrivially Dehn p-colored diagram of K, that is,

$$\min\{\sharp\mathcal{C}(D, C) \mid (D, C) \in \begin{Bmatrix} \text{nontrivially Dehn } p\text{-colored} \\ \text{diagrams of } K \end{Bmatrix}\}.$$

We denote it by $\mathrm{mincol}_p^{\mathrm{Dehn}}(K)$.

2.2 $\mathcal{R}$-palette graph

Let p be an odd prime number, and let $S \subset \mathbb{Z}_p$ with $S \neq \emptyset$. Set $\mu(S) = \{\{a_1, a_2\} \mid a_1, a_2 \in S\}$, where $\{a_1, a_2\}$ is regarded as the multiset $\{a_1, a_1\}$ when $a_1 = a_2$.

For $\{a_1, a_2\}, \{a_3, a_4\} \in \mu(S)$, we set an equivalence relation $\sim$ on $\mu(S)$ by

$$\{a_1, a_2\} \sim \{a_3, a_4\} \text{ if } a_1 + a_2 = a_3 + a_4 \text{ in } \mathbb{Z}_p.$$

We denote by $\overline{a_1 + a_2}$ the equivalence class of $\{a_1, a_2\} \in \mu(S)$.

Definition 2.3. The *$\mathcal{R}$-palette graph* of S is the simple graph $G_S = (V_S, E_S)$ composed of the vertex set $V_S = \mu(S)/\sim = \{\overline{a_1 + a_2} \mid \{a_1, a_2\} \in \mu(S)\}$ and the edge set E_S satisfying that

$$e = \overline{b_1}\ \overline{b_2} \in E_S \Longleftrightarrow \begin{array}{l} \text{there exist } \{a_1, a_2\} \in \overline{b_1} \text{ and } \{a_3, a_4\} \in \overline{b_2} \text{ such} \\ \text{that } a_1 + a_3 = a_2 + a_4 \text{ or } a_1 + a_4 = a_2 + a_3 \text{ in } \mathbb{Z}_p, \end{array}$$

where $e = \overline{b_1}\ \overline{b_2}$ means that e is an edge connecting the vertices $\overline{b_1}$ and $\overline{b_2}$. We attach the label $\overline{2^{-1}(b_1 + b_2)} \in \mathbb{Z}_p$ to the edge e between $\overline{b_1}$ and $\overline{b_2}$ (see Figure 2 for example).

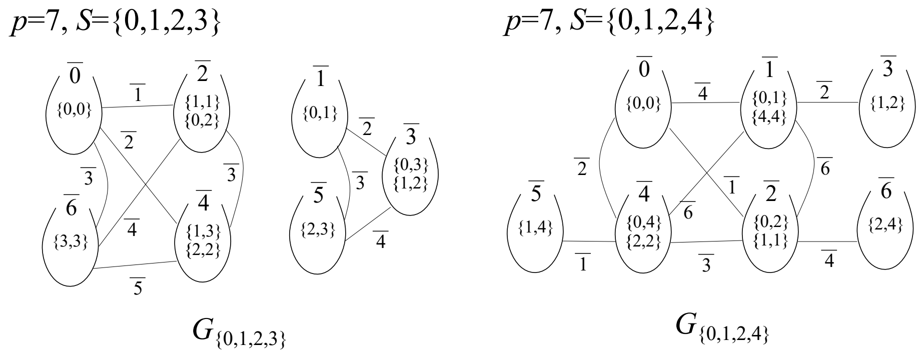


Figure 2: $\mathcal{R}$-palette graphs

Let G_S be the $\mathcal{R}$-palette graph of S. A graph $G = (V, E)$ is an $\mathcal{R}$-*subgraph* of G_S if G is a subgraph of G_S, and

$$e \in E \Longrightarrow \overline{b_e} \in V (i.e., e = \overline{b_1}\ \overline{b_2} \in E \Longrightarrow \overline{2^{-1}(b_1 + b_2)} \in V)$$

holds, where $\overline{b_e}$ is the label of e (see Figure 3 for example).

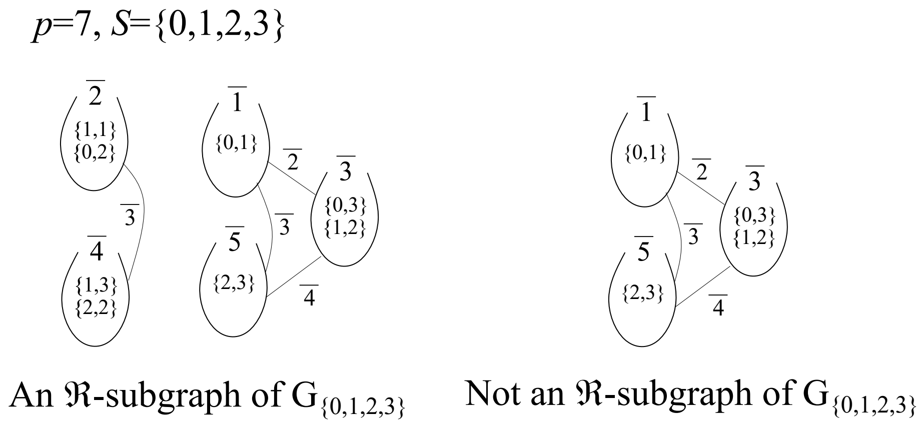
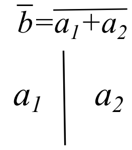
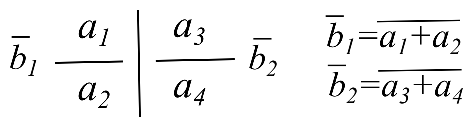


Figure 3: R-subgraph

Let (D, C) be a nontrivially Dehn p-colored diagram of a knot. The $\mathcal{R}$-*palette graph* of (D, C) is an $\mathcal{R}$-subgraph $G_{(D,C)} = (V_{(D,C)}, E_{(D,C)})$ of $G_{\mathcal{C}(D,C)}$ composed of the vertex set

$V_{(D,C)} = \{\bar{b} \mid$ there exists an arc on (D, C) with  $\}$

and the edge set $E_{(D,C)}$ satisfying that

$e = \overline{b_1}\ \overline{b_2} \in E_S \Longleftrightarrow$ there exists a crossing on (D, C) with 

As in the case of G_S, we attach the label $\overline{2^{-1}(b_1 + b_2)} \in \mathbb{Z}_p$ to the edge e between $\overline{b_1}$ and $\overline{b_2}$ (see Figure 4 for example).

Proposition 2.4. *Let $S \subset \mathbb{Z}_p$. If $S = \mathcal{C}(D, C)$ for some nontrivially* Dehn *p-colored diagram (D, C) of a knot, the $\mathcal{R}$-palette graph G_S includes a connected $\mathcal{R}$-subgraph with at least three vertices.*

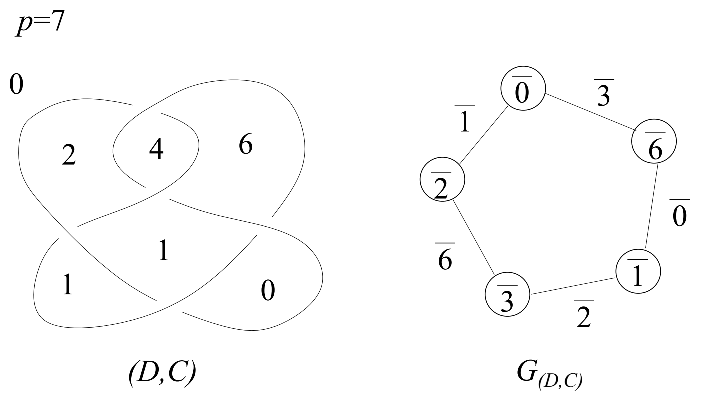


Figure 4: $\mathcal{R}$-palette graph

3 Main results

Theorem 3.1. *Let p be an odd prime number. For any* Dehn *p-colorable knot K, we have*

$$\mathrm{mincol}_p^{\mathrm{Dehn}}(K) \geq \lfloor \log_2 p \rfloor + 2.$$

Theorem 3.2. *Let p be an odd prime number with $p < 2^5$. If there exists a nontrivially* Dehn *p-colored diagram (D, C) of a knot such that $\sharp\mathcal{C}(D, C) = \lfloor \log_2 p \rfloor + 2$, then*

(i) $\mathcal{C}(D, C) \sim \{0, 1, 2\}$ *when* $p = 3$,

(ii) $\mathcal{C}(D, C) \sim \{0, 1, 2, 3\}$ *when* $p = 5$,

(iii) $\mathcal{C}(D, C) \sim \{0, 1, 2, 4\}$ *when* $p = 7$,

(iv) $\mathcal{C}(D, C) \sim \{0, 1, 2, 3, 6\}$ *or* $\{0, 1, 2, 4, 7\}$ *when* $p = 11$,

(v) $\mathcal{C}(D, C) \sim 0, 1, 2, 4, 7$ *when* $p = 13$,

(vi) $\mathcal{C}(D, C) \sim \{0, 1, 2, 3, 5, 9\}, \{0, 1, 2, 3, 5, 10\}, \{0, 1, 2, 3, 5, 12\}, \{0, 1, 2, 3, 6, 9\}, \{0, 1, 2, 3, 6, 10\}, \{0, 1,$
$\{0, 1, 2, 3, 6, 13\}, \{0, 1, 2, 3, 7, 11\}, \{0, 1, 2, 4, 5, 9\}, \{0, 1, 2, 4, 5, 10\}, \{0, 1, 2, 4, 5, 12\},$
or $\{0, 1, 2, 4, 10, 13\}$ *when* $p = 17$,

(vii) $\mathcal{C}(D, C) \sim \{0, 1, 2, 3, 5, 10\}, \{0, 1, 2, 3, 6, 10\}, \{0, 1, 2, 3, 6, 11\}, \{0, 1, 2, 3, 6, 12\}, \{0, 1, 2, 3, 6, 13\}, \{0,$
$\{0, 1, 2, 3, 7, 12\}, \{0, 1, 2, 4, 5, 10\}, \{0, 1, 2, 4, 5, 14\}, \{0, 1, 2, 4, 7, 12\},$ *or* $\{0, 1, 2, 4, 7, 15\}$
when $p = 19$,

(viii) $\mathcal{C}(D,C) \sim \{0,1,2,3,6,12\}, \{0,1,2,4,7,12\}, \{0,1,2,4,7,13\}$, $\{0,1,2,4,7,14\}, \{0,1,2,4,9,14\}$, *or* $\{0,1,2,4,10,19\}$ *when* $p = 23$,

(ix) $\mathcal{C}(D,C) \sim \{0,1,2,4,8,15\}$ *when* $p = 29$, *and*

(x) $\mathcal{C}(D,C) \sim \{0,1,2,4,8,16\}$ *when* $p = 31$.

Proposition 3.3. *For each odd prime p with $p < 2^5$ and $p \notin \{13, 29\}$, there exists a* Dehn *p-colorable knot K with* $\mathrm{mincol}_p^{\mathrm{Dehn}}(K) = \lfloor \log_2 p \rfloor + 2$.

References

[1] H. Abchir, M. Elhamdadi and S. Lamsifer, *On the minimum number of Fox colorings of knots*, Grad. J. Math. **5** (2020), no. 2, 122–137.

[2] F. Bento and P. Lopes, *The minimum number of Fox colors modulo 13 is 5*, Topology Appl. **216** (2017), 85–115.

[3] Y. Han and B. Zhou, *The minimum number of coloring of knots*, J. Knot Theory Ramifications **31** (2022), no. 2, Paper No. 2250013, 55 pp.

[4] T. Nakamura, Y. Nakanishi, and S. Satoh, *The pallet graph of a Fox coloring*, Yokohama Math. J. **59** (2013), 91–97.

[5] T. Nakamura, Y. Nakanishi, and S. Satoh, *11-colored knot diagram with five colors*, J. Knot Theory Ramifications **25** (2016), no. 4, Paper No. 1650017, 22 pp.

[6] K. Oshiro, *Any 7-colorable knot can be colored by four colors*, J. Math. Soc. Japan **62**(3) (2010) 963–973.

[7] S. Satoh, *5-colored knot diagram with four colors*, Osaka J. Math. **46** (2009), no. 4, 939–948.

(G. Yamagishi) Department of Information and Communication Sciences, Sophia University, Tokyo 102-8554, Japan
Email address: `g-yamagishi-3c9@eagle.sophia.ac.jp`

Minimum numbers of Dehn colors of knots

山岸 凱司 (上智大学)

松土恵理氏（日本大学），大城佳奈子氏(上智大学)との共同研究

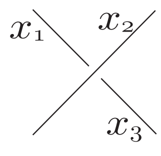
研究集会「カンドルと対称空間」　於 大阪公立大学

2022年12月8日

§ 1 Fox p-彩色と色の数

K: 結び目, D: Kの図式. p: 奇素数.

- DのFox p-彩色 $\cdots$ $C : \{D$の辺$\} \to \mathbb{Z}_p$ s.t.



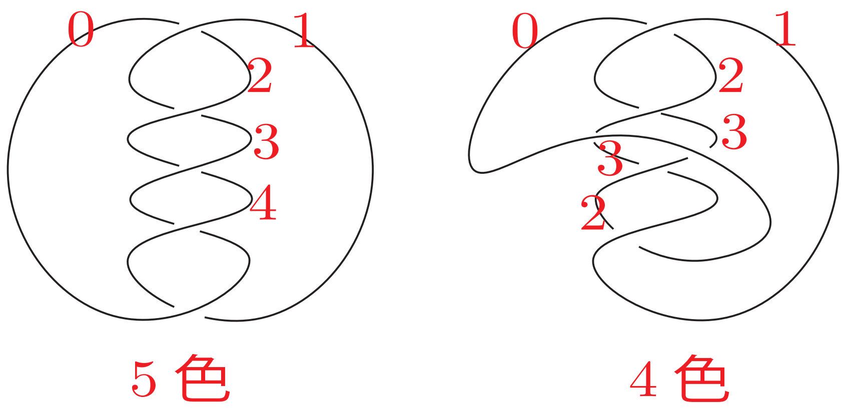
$2C(x_2) = C(x_1) + C(x_3)$

※ Fox p-彩色は二面体カンドル$(\mathbb{Z}_p, a * b = 2b - a)$-彩色

- Kのminimum number of Fox p-colors:

$$\mathrm{mincol}_p^{\mathrm{Fox}}(K) = \min\left\{\#\mathrm{Im}C \;\middle|\; (D, C) \in \left\{\begin{array}{l}\text{非自明にFox}\\ p\text{-彩色された}\\ K\text{の図式}\end{array}\right\}\right\}$$

$p = 5$



$\mathrm{mincol}_5^{\mathrm{Fox}}(K) \leq 4$

2

$\mathrm{mincol}_p^{\mathrm{Fox}}(K)$について，次の結果が知られている:

- K: Fox 3-彩色可能 $\Longrightarrow$ $\mathrm{mincol}_3^{\mathrm{Fox}}(K) = 3$
- K: Fox 5-彩色可能 $\Longrightarrow$ $\mathrm{mincol}_5^{\mathrm{Fox}}(K) = 4$ (S. Satoh)
- K: Fox 7-彩色可能 $\Longrightarrow$ $\mathrm{mincol}_7^{\mathrm{Fox}}(K) = 4$ (K. Oshiro)
- K: Fox 11-彩色可能 $\Longrightarrow$ $\mathrm{mincol}_{11}^{\mathrm{Fox}}(K) = 5$ (N-N-S)
- K: Fox 13-彩色可能 $\Longrightarrow$ $\mathrm{mincol}_{13}^{\mathrm{Fox}}(K) = 5$ (B-L)
- K: Fox 17-彩色可能 $\Longrightarrow$ $\mathrm{mincol}_{17}^{\mathrm{Fox}}(K) = 6$ (A-E-L)
- K: Fox 19-彩色可能 $\Longrightarrow$ $\mathrm{mincol}_{19}^{\mathrm{Fox}}(K) = 6$ (H-Z)
- $\mathrm{mincol}_p^{\mathrm{Fox}}(K) \geq \lfloor \log_2 p \rfloor + 2$ (Nakamura-Nakanishi-Satoh)

※ $\forall p$: 奇素数，$\forall K$: Fox p-彩色可能な結び目，

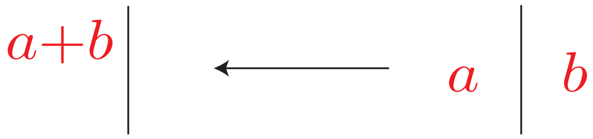
$$\mathrm{mincol}_p^{\mathrm{Fox}}(K) = \lfloor \log_2 p \rfloor + 2$$

となるかは分かっていない．

3

§ 2 Dehn p-彩色

$$K: \text{ Fox } p\text{-彩色可能} \iff K: \text{ Dehn } p\text{-彩色可能}$$



K: Dehn p-彩色可能な結び目.

- K の minimum number of Dehn p-colors:

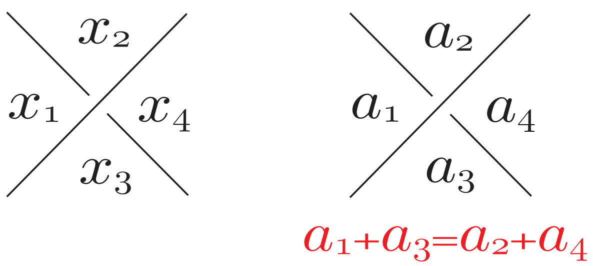
$$\mathrm{mincol}_p^{\mathrm{Dehn}}(K)$$
$$= \min\{\#\mathrm{Im}C \mid (D, C) \in \left\{\begin{array}{l}\text{非自明に Dehn } p\text{-彩色} \\ \text{された } K \text{ の図式}\end{array}\right\}\}$$

Q. $\mathrm{mincol}_p^{\mathrm{Dehn}}(K)$ をどのように評価できるか？
Q. $\mathrm{mincol}_p^{\mathrm{Dehn}}(K)$ も $\mathrm{mincol}_p^{\mathrm{Fox}}(K)$ のときのように K に依らない値を取るか？

D: 結び目図式, $\mathcal{R}(D) = \{D \text{の領域}\}$. p: 奇素数

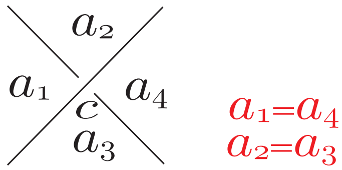
- D の Dehn p-彩色 $\cdots$ $C : \mathcal{R}(D) \to \mathbb{Z}_p$ s.t.

$$C(x_1) + C(x_3) = C(x_2) + C(x_4)$$

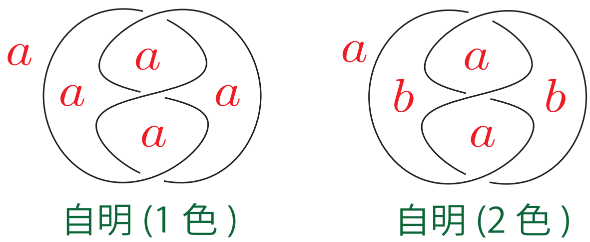


領域 x に対し, $C(x)$ を x の色という.

- (D,C) の交点 c が自明に彩色されている $:\overset{\text{def}}{\Leftrightarrow}$



- D の Dehn p-彩色 C が自明 $:\overset{\text{def}}{\Leftrightarrow}$ (D,C) の全ての交点が自明に彩色されている.



- 結び目 K が Dehn p-彩色可能 $:\overset{\text{def}}{\Leftrightarrow}$ $\exists(D,C)$: 非自明に Dehn p-彩色された K の図式.

6

§ 3 minimum number of Dehn p-colors と結果

K: Dehn p-彩色可能な結び目.

- K の minimum number of Dehn p-colors:

$$\mathrm{mincol}_p^{\mathrm{Dehn}}(K) = \min\{\#\mathrm{Im}C \mid (D,C) \in \left\{\begin{array}{l}\text{非自明に Dehn}p\text{-彩色} \\ \text{された }K\text{ の図式}\end{array}\right\}\}$$

- $\mathrm{mincol}_3^{\mathrm{Dehn}}(K) = 3$.
- $\mathrm{mincol}_5^{\mathrm{Dehn}}(K) = 4$. (S. Satoh)

<u>**定理 1**</u>　$\forall K$: Dehn p-彩色可能な結び目,

$$\mathrm{mincol}_p^{\mathrm{Dehn}}(K) \geq \lfloor \log_2 p \rfloor + 2.$$

7

<u>定理2</u>　p: 奇素数 s.t. $p < 2^5$.
$\exists$ (D, C): 非自明にDehn p-彩色された図式 s.t.
$\#\mathrm{Im}C = \lfloor \log_2 p \rfloor + 2 \Longrightarrow$

(i) $\mathrm{Im}C \sim \{0, 1, 2\}$ $(p = 3)$,

(ii) $\mathrm{Im}C \sim \{0, 1, 2, 3\}$ $(p = 5)$,

(iii) $\mathrm{Im}C \sim \{0, 1, 2, 4\}$ $(p = 7)$,

(iv) $\mathrm{Im}C \sim \{0, 1, 2, 3, 6\}$ または $\{0, 1, 2, 4, 7\}$ $(p = 11)$,

(v) $\cdots$

$$S \sim S' :\overset{\text{def}}{\Leftrightarrow} S' = sS + t \qquad (s \in \mathbb{Z}_p^\times, t \in \mathbb{Z}_p)$$

<u>命題3</u>　$\forall p \in \{3, 5, 7, 11, \cancel{13}, 17, 19, 23, \cancel{29}, 31\}$,
$\exists K$: Dehn p-彩色可能な結び目 s.t.

$$\mathrm{mincol}_p^{\mathrm{Dehn}}(K) = \lfloor \log_2 p \rfloor + 2.$$

<u>定理4</u>　$p \in \{13, 29\}$. $\forall K$: Dehn p-彩色可能な結び目,

$$\mathrm{mincol}_p^{\mathrm{Dehn}}(K) \geq \lfloor \log_2 p \rfloor + 3.$$

<u>定理5</u>　$\forall p \in \{7, 11, 13, \cancel{17}, 19, 23, 29, 31\}$,
$\exists K$: Dehn p-彩色可能な結び目 s.t.

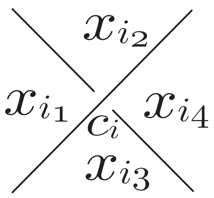
$$\mathrm{mincol}_p^{\mathrm{Dehn}}(K) \geq \lfloor \log_2 p \rfloor + 3.$$

- $\lfloor \log_2 p \rfloor + 2$は$p$によってはbest possibleでない.
- $\mathrm{mincol}_p^{\mathrm{Dehn}}(K) \neq \mathrm{mincol}_p^{\mathrm{Dehn}}(K')$となる場合がある.
- $\mathrm{mincol}_p^{\mathrm{Dehn}}(K)$はFoxのときとは全く違う振る舞いをする.

§ 4 定理 1($\mathrm{mincol}_p^{\mathrm{Dehn}}(K) \geq \lfloor \log_2 p \rfloor + 2$)について

D: Kの図式.

$c_1, \ldots, c_n$: Dの交点, $x_1, \ldots, x_{n+2}$: Dの領域.



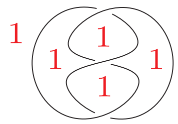
$$x_{i_1} - x_{i_2} + x_{i_3} - x_{i_4} = 0$$

$$(\star 1) \quad M_D^{\mathrm{col}} \begin{pmatrix} x_1 \\ \vdots \\ x_{n+2} \end{pmatrix} = 0.$$

{$\mathbb{Z}_p$上の($\star 1$)の解}

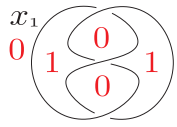
$\leftrightarrow$ {Dehn p-彩色}

※ 自明彩色に対応する解がある:



自明 (1 色)

$$\leftrightarrow a^{1T} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix};$$



自明 (2 色)

$$\leftrightarrow a^{2T} = \begin{pmatrix} 0 \\ * \\ \vdots \\ * \end{pmatrix}$$

- $\mathrm{rank}_{\mathbb{Z}} M_D^{\mathrm{col}} = n$
- {($\star 1$)の$\mathbb{Z}$上での整数解}$=\left\langle a^{1T}, a^{2T} \right\rangle_{\mathbb{Z}}$

10

$$A_D(-1) = \begin{pmatrix} & M_D^{\mathrm{col}} & & \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

$$(\star 2) \quad A_D(-1) \begin{pmatrix} x_1 \\ \vdots \\ x_{n+2} \end{pmatrix} = 0$$

- $A_D(t)$はKの結び目群のDehn表示から得られたアレクサンダー行列.

- $\mathrm{rank}_{\mathbb{Z}} A_D(-1) = n + 1$

- {($\star 2$)の$\mathbb{Z}$上での整数解}$=\left\langle a^{2T} = \begin{pmatrix} 0 \\ * \\ \vdots \\ * \end{pmatrix} \right\rangle_{\mathbb{Z}}$

11

定理 1($\mathrm{mincol}_p^{\mathrm{Dehn}}(K) \geq \lfloor \log_2 p \rfloor + 2$)の証明の概要

(D, C): 非自明にDehn p-彩色された図式 s.t.

$$\#\mathrm{Im}C = \mathrm{mincol}_p^{\mathrm{Dehn}}(K),\ C(x_1) = 0.$$

(ケース1) $\forall x_i, x_j \in \mathcal{R}(D)$,

$$C(x_i) = C(x_j) \Rightarrow C^{2\mathrm{T}}(x_i) = C^{2\mathrm{T}}(x_j).$$

$M_1 := A_D(-1)$,

$\mathrm{rank}_{\mathbb{Z}} M_1 = \#\{M_1\text{の列}\} - 1$ & $\mathrm{rank}_p M_1 \leq \#\{M_1\text{の列}\} - 2$

- $\{(\star 2)\text{の}\mathbb{Z}\text{上での整数解}\} = \left\langle a^{2\mathrm{T}} \right\rangle_{\mathbb{Z}}$
- $\{(\star 2)\text{の}\mathbb{Z}_p\text{上での解}\} = \left\langle a^{2\mathrm{T}}, a\ (\leftrightarrow C), \cdots \right\rangle_{\mathbb{Z}_p}$

$i < j$ s.t. $C(x_i) = C(x_j)$ において次の操作を行う.

$$\underbrace{(m_1, \ldots, m_{n+2})}_{M_1} \begin{pmatrix} x_1 \\ \vdots \\ x_{n+2} \end{pmatrix} = 0;\ \underbrace{\begin{pmatrix} C(x_1) \\ \vdots \\ C(x_{n+2}) \end{pmatrix}}_{a};\ \underbrace{\begin{pmatrix} C^{2\mathrm{T}}(x_1) \\ \vdots \\ C^{2\mathrm{T}}(x_{n+2}) \end{pmatrix}}_{a^{2\mathrm{T}}};$$

$$\rightsquigarrow (\ldots, m_i + m_j, \ldots, \widehat{m_j}, \ldots) \begin{pmatrix} \vdots \\ x_i \\ \vdots \\ \widehat{x_j} \\ \vdots \end{pmatrix} = 0;\ \begin{pmatrix} \vdots \\ C(x_i) \\ \vdots \\ \widehat{C(x_j)} \\ \vdots \end{pmatrix};\ \begin{pmatrix} \vdots \\ C^{2\mathrm{T}}(x_i) \\ \vdots \\ \widehat{C^{2\mathrm{T}}(x_j)} \\ \vdots \end{pmatrix}$$

この操作を繰り返すことにより,

$$M_2 \begin{pmatrix} y_1 \\ \vdots \\ y_\ell \end{pmatrix} = 0; \quad (\ell = \#\mathrm{Im}C)$$

とその$\mathbb{Z}$上での解$\tilde{a}^{2\mathrm{T}}$, $\mathbb{Z}_p$上での解$\tilde{a}$, $\tilde{a}^{2\mathrm{T}}$を得る.

$\mathrm{rank}_{\mathbb{Z}} M_2 = \#\{M_2\text{の列}\} - 1$ & $\mathrm{rank}_p M_2 \leq \#\{M_2\text{の列}\} - 2$

よって，$\exists M_3$: M_2の$(\ell-1)\times(\ell-1)$-部分行列 s.t.

$$\det M_3 \neq 0 \in \mathbb{Z} \ \&\ \det M_3 = 0 \in \mathbb{Z}_p.$$

故に，

$$p \leq |\det M_3|.$$

また，M_3の性質から次を示すことも可能:

$$|\det M_3| \leq 2^{\ell-1} (= 2^{\mathrm{mincol}_p^{\mathrm{Dehn}}(K)-1}).$$

以上より，

$$\mathrm{mincol}_p^{\mathrm{Dehn}}(K) \geq \log_2 p + 1,$$

$$\therefore \mathrm{mincol}_p^{\mathrm{Dehn}}(K) \geq \lfloor \log_2 p \rfloor + 2.$$

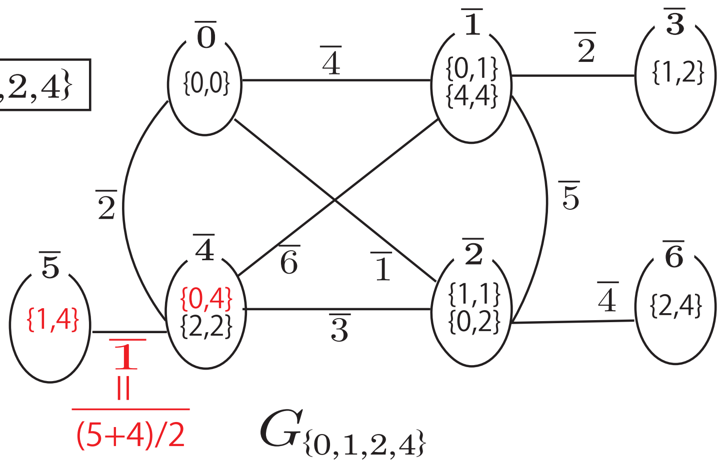
□

14

§ 5 $\mathcal{R}$-パレットグラフと定理 2

p: 奇素数，　$S \subset \mathbb{Z}_p$.

$\boxed{p=7,\ S=\{0,1,2,4\}}$



$G_{\{0,1,2,4\}}$

- $G_S = (V_S, E_S)$ がSの$\mathcal{R}$-パレットグラフ $:\overset{\mathrm{def}}{\Leftrightarrow}$

$$V_S = \{\overline{a_1 + a_2} \mid \{a_1, a_2\} \in S^2/_{(x,y)\sim(y,x)}\}$$

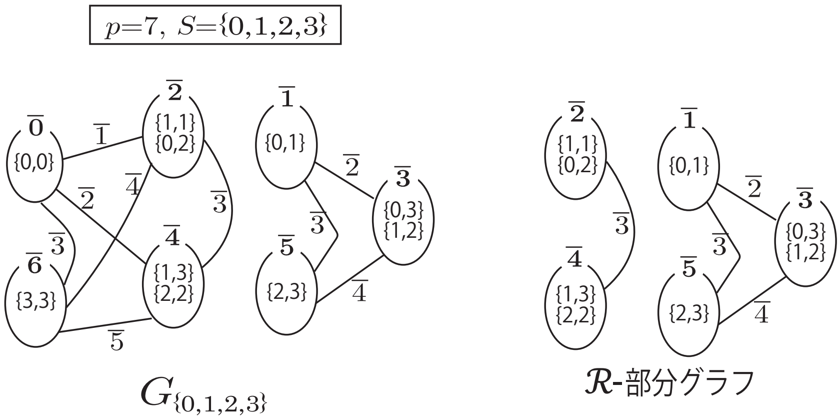
$$e = \overline{b_1}\,\overline{b_2} \in E_S \iff \begin{array}{l} \exists\{a_1,a_2\} \in \overline{b_1},\ \exists\{a_3,a_4\} \in \overline{b_2} \text{ s.t.} \\ a_1 + a_3 = a_2 + a_4 \in \mathbb{Z}_p \end{array}$$

15

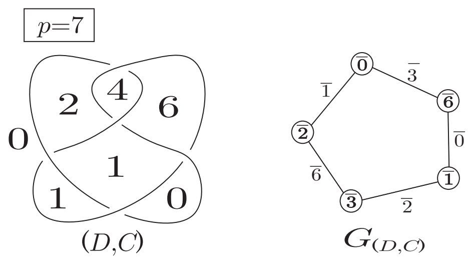
- $\boldsymbol{G = (V, E)}$が$\boldsymbol{G_S}$の$\mathcal{R}$-部分グラフ

  $:\overset{\text{def}}{\Longleftrightarrow}$ $\boldsymbol{G}$は$\boldsymbol{G_S}$の部分グラフ,

$$\boldsymbol{e \in E \Longrightarrow \overline{b_e} \in V} \quad (\overline{\boldsymbol{b_e}}\text{はの}\boldsymbol{e}\text{のラベル})$$



$\boldsymbol{(D, C)}$: 非自明にDehn $\boldsymbol{p}$-彩色された図式.



- $\boldsymbol{G_{(D,C)} = (V_{(D,C)}, E_{(D,C)})}$ が$\boldsymbol{(D, C)}$の$\mathcal{R}$-パレットグラフ

  $:\overset{\text{def}}{\Longleftrightarrow}$

$$\boldsymbol{V_{(D,C)} = \{\overline{a_1 + a_2} \mid \exists} \quad \begin{array}{c|c} a_1 & a_2 \end{array} \quad \boldsymbol{\}}$$

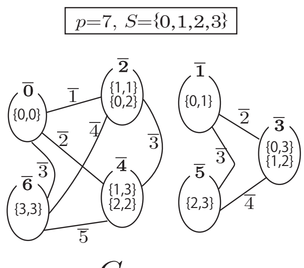
$$\boldsymbol{e = \overline{b_1}\,\overline{b_2} \in E_S} \iff \exists \begin{array}{c|c} a_1 & a_3 \\ \hline a_2 & a_4 \end{array} \quad \begin{array}{l} \overline{b_1} = \overline{a_1 + a_2} \\ \overline{b_2} = \overline{a_3 + a_4} \end{array}$$

$S = \mathrm{Im}C$.

- $G_{(D,C)}$ はG_Sの連結な$\mathcal{R}$-部分グラフ.
- $G_{(D,C)}$ は少なくとも3つの頂点を含む.

<u>定理6</u>　$S \subset \mathbb{Z}_p$.
$\exists(D,C)$: 非自明にDehn p-彩色された図式 s.t. $S = \mathrm{Im}C$
$\Longrightarrow$ G_S は連結な$\mathcal{R}$-部分グラフで$\#V_S \geq 3$となるものを含む.

定理6は図式に非自明彩色を与えるための色の候補を与える.



$G_{\{0,1,2,3\}}$

連結で頂点数3以上の
$\mathcal{R}$-部分グラフ
を持たない
↓
$\{0,1,2,3\}$ は非自明
な彩色を与える色
の候補にはならない

<u>定理2</u>　p: 奇素数 s.t. $p < 2^5$.
$\exists$ (D,C): 非自明にDehn p-彩色された図式 s.t.
$\#\mathrm{Im}C = \lfloor \log_2 p \rfloor + 2 \Longrightarrow$

(i) $\mathrm{Im}C \sim \{0,1,2\}$ $(p=3)$,

(ii) $\mathrm{Im}C \sim \{0,1,2,3\}$ $(p=5)$,

(iii) $\mathrm{Im}C \sim \{0,1,2,4\}$ $(p=7)$,

(iv) $\mathrm{Im}C \sim \{0,1,2,3,6\}$ または $\{0,1,2,4,7\}$ $(p=11)$,

(v) $\cdots$

$$S \sim S' :\overset{\mathrm{def}}{\Leftrightarrow} \quad S' = sS + t \qquad (s \in \mathbb{Z}_p^\times, t \in \mathbb{Z}_p)$$

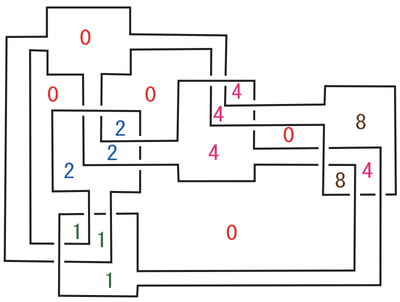
§ 6 命題 3について

<u>**命題3**</u> $\forall p \in \{3, 5, 7, 11, \cancel{13}, 17, 19, 23, \cancel{29}, 31\}$,
$\exists K$: Dehn p-彩色可能な結び目 s.t.

$$\mathrm{mincol}_p^{\mathrm{Dehn}}(K) = \lfloor \log_2 p \rfloor + 2.$$

$p = 11$. (D, C)の色の数$= \#\{0, 1, 2, 4, 8\} = 5 = \lfloor \log_2 11 \rfloor + 2$.
$\therefore$ 定理2より, $\mathrm{mincol}_{11}^{\mathrm{Dehn}}(K) = 5 = \lfloor \log_2 11 \rfloor + 2$.

(D, C)



ご清聴ありがとうございました。

The universality of groupoid racks on the colorings for spatial surface diagrams

Katsunori Arai

Abstract. A spatial surface is a compact oriented surface embedded in the 3-sphere $S^3 = \mathbb{R}^3 \sqcup \{\infty\}$ such that each connected component has a non-empty boundary. Spatial surfaces are represented by spatial trivalent graph diagrams ([1]). In this paper, we introduce a notion of a groupoid rack which defines the colorings for spatial surface diagrams. Furthermore, we show that a groupoid rack has the universality on the colorings.

1 Groupoid racks

Let $\mathcal{C}$ be a groupoid and $X = \mathrm{Hom}(\mathcal{C})$ be the set of all morphisms of $\mathcal{C}$. A pair of X and a binary operation $* : X \times X \to X$ is a *groupoid rack* if $* : X \times X \to X$ satisfies the following conditions:

- For any $x, f, g \in X$ satisfying $\mathrm{cod}(f) = \mathrm{dom}(g)$, $x*(fg) = (x*f)*g$ and $x*id_\lambda = x$, where id_λ is the identity morphism of the object λ.
- For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.
- For any For any $x, f, g \in X$ satisfying $\mathrm{cod}(f) = \mathrm{dom}(g)$, $\mathrm{cod}(f * x) = \mathrm{dom}(g * x)$ and $(fg) * x = (f * x)(g * x)$.

Let X be a groupoid rack, D be a Y-oriented spatial surface diagram and $\mathcal{A}(D)$ be the set of all arcs of D. A map $C : \mathcal{A}(D) \to X$ is an X-*coloring* if C satisfies the following conditions (Fig. 1).

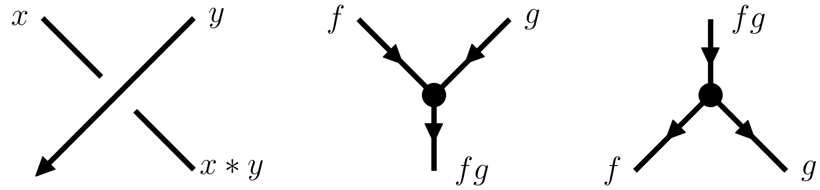


Figure 1: Groupoid rack coloring conditions $(x, y, f, g \in X,\ \mathrm{cod}(f) = \mathrm{dom}(g))$

We denote by $\mathrm{Col}_X(D)$ the set of all X-colorings of D.

Theorem 1.1. *Let X be a groupoid rack. If two Y-oriented diagrams D_1 and D_2 present equivalent spatial surfaces, then there is a bijection between $Col_X(D_1)$ and $Col_X(D_2)$. In particular, $|Col_X(D_1)|$ is a spatial surface invariant.*

2 The universality of groupoid racks

Theorem 2.1 says that a groupoid rack is a universal structure that defines the coloring for spatial surfaces, i.e. the algebraic structures defining the coloring for spatial surfaces must have a structure of a groupoid rack.

Theorem 2.1. *Let $R = (R, *, \rho)$ be a symmetric rack. We assume that a subset $P \subset R \times R$ and a map $\mu : P \to R$ satisfy the following conditions, where we denote $\mu(a, b)$ by ab.*

- *For any $a, b, c \in R$, the following are equivalent.*

$$[(a, b) \in P \wedge (ab, c) \in P], \quad [(b, c) \in P \wedge (a, bc) \in P].$$

- *For any $(a, b), (ab, c) \in P$, we have*

$$(ab)c = a(bc).$$

- *For any $a, b, x \in R$, the following are equivalent.*

$$(a, b) \in P, \quad (\rho(b), \rho(a)) \in P, \quad (a * x, b * x) \in P.$$

- *For any $(a, b) \in P$, we have*

$$(b, \rho(ab)) \in P \text{ and } (\rho(ab), a) \in P.$$

- *For any $(a, b) \in P$ and $x \in R$, we have*

$$\rho(b)\rho(a) = \rho(ab),\ (ab)\rho(b) = a,\ (ab) * x = (a * x)(b * x) \text{ and } x * (ab) = (x * a) * b.$$

Put $R' = \bigcup_{(a,b) \in P} \{a, b\}$. Then

(i) *R' is a subrack of $(R, *)$ with the good involution ρ.*

(ii) *$(R', *)$ is a groupoid rack.*

References

[1] S. Matsuzaki, *A diagrammatic presentation and its characterization of non-split compact surfaces in the 3-sphere*, J. Knot Theory Ramifications **30** (2021), no. 9, Paper No. 2150071, 32.

(K. Arai) Department of Mathematics, Graduate School of Science, Osaka University, 1-1, Machikaneyama, Toyonaka, Osaka, 560-0043, Japan
Email address: `u068111h@ecs.osaka-u.ac.jp`

The universality of groupoid racks on the colorings for spatial surface diagrams

カンドルと対称空間

新井 克典

大阪大学 M2

2022 年 12 月 8 日

Today's contents

1. Spatial surface
2. Groupoid rack
3. Universality

Definition

A *spatial surface* F is a compact surface embedded in S^3.

Remark

In this talk, we assume the following.

- F is an oriented surface.
- Each component of F has a non-empty boundary.
- F does not have disk components.

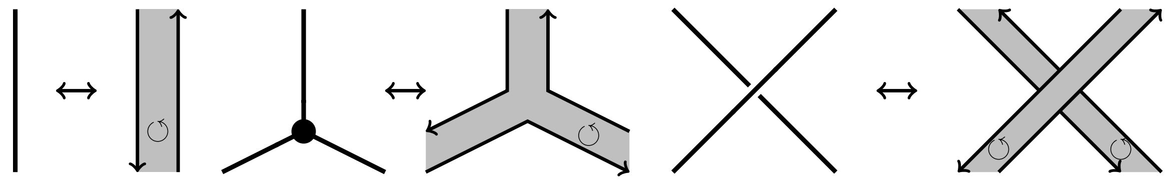
Two spatial surfaces are *equivalent* if they are ambient isotopic.

A spatial trivalent graph is a finite trivalent graph embedded in S^3.

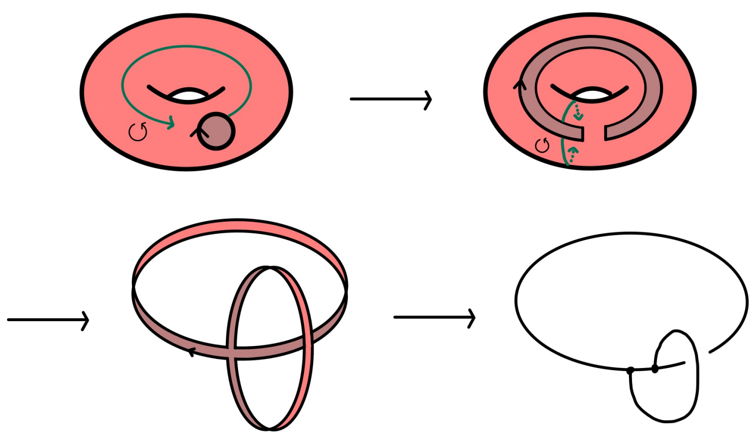
Remark

A spatial trivalent graph may have S^1-components.

Any spatial surface F is equivalent to the spatial surface obtained from some spatial trivalent graph diagram D by the following way.



We call D a *diagram* of F.



Theorem (Matsuzaki '21)

F_1, F_2: spatial surfaces, D_1, D_2: diagrams of F_1, F_2.
Then the following conditions are equivalent.

- F_1 and F_2 are equivalent.
- D_1 and D_2 are related by a finite sequence of Reidemeister moves and isotopies of S^2.

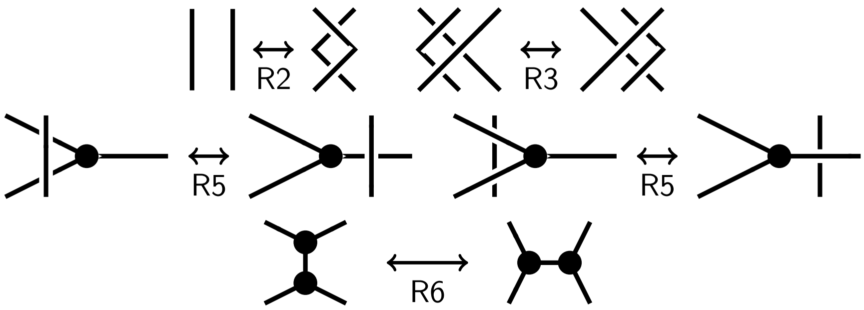
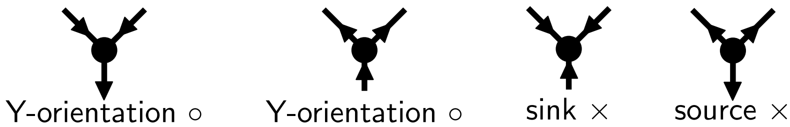


Figure: Reidemeister moves for spatial surface diagrams

A Y-oriented spatial trivalent graph is an oriented spatial trivalent graph without sinks and sources.



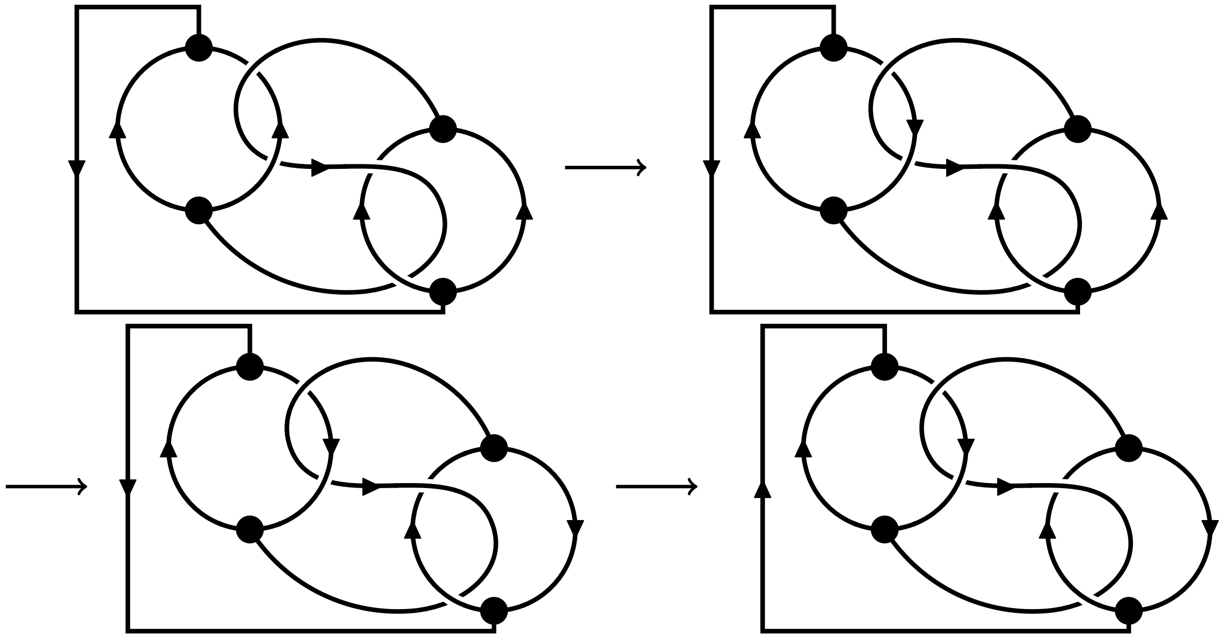
D: a Y-oriented spatial trivalent graph diagram.
D represents a spatial surface F by forgetting the Y-orientation.
We call D a Y-oriented diagram of F.

Remark

In general, spatial trivalent graphs have some Y-oriented diagrams.

Proposition (Ishii '15)

D: a spatial trivalent graph diagram.
Two Y-orientations of D are related by a finite sequence of reversing the orientation of one edge while preserving being Y-oriented.

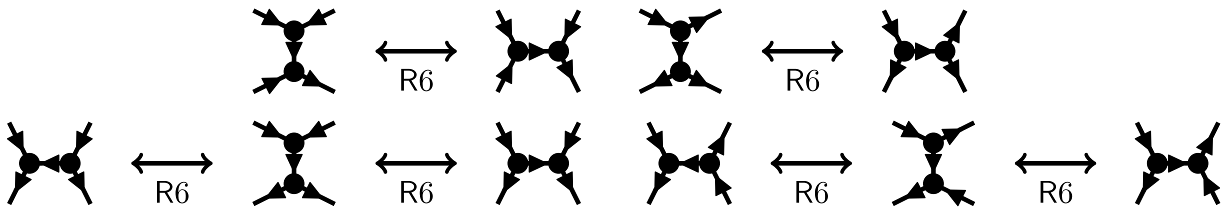


In this talk, we call the operation of reversing the orientation of one S^1-component *inverse move*.

Theorem (Matsuzaki)

F_1, F_2: spatial surfaces. D_1, D_2: Y-oriented diagrams of F_1, F_2.
Then the following conditions are equivalent.

- F_1 and F_2 are equivalent.
- D_1 and D_2 are related by a finite sequence of Y-oriented Reidemeister moves, inverse moves and isotopies of S^2.



Definition (Fenn-Rourke '92)

X: a set, $* : X \times X \to X$: a binary operation.
$X = (X, *)$: a *rack*
$\Leftrightarrow$ $*$ satisfies the following conditions.

- $\forall y \in X$, a map $S_y : X \ni x \mapsto x * y \in X$ is a bijection.
- $\forall x, y, z \in X, (x * y) * z = (x * z) * (y * z)$.

A *quandle* (Joyce, Matveev '82) is a rack $X = (X, *)$ satisfying

- $\forall x \in X, x * x = x$.

$\forall x, y \in X, \forall n \in \mathbb{Z}$, we denote $S_y^n(x)$ by $x *^n y$.
Aut(X): the automorphism group of X.

Definition (Kamada '07, Kamada-Oshiro '10)

$X = (X, *)$: a rack.
A map $\rho : X \to X$ is a *good involution*
$\Leftrightarrow$ ρ saitisfies the following conditions.

- $\rho \circ \rho = \mathrm{id}_X$,
- $\forall x, y \in X, \rho(x * y) = \rho(x) * y$,
- $\forall x, y \in X, x * \rho(y) = x *^{-1} y$.

$X = (X, \rho)$: a *symmetric rack*.

In particular, if X is a quandle, (X, ρ) is called a *symmetric quandle*.

Example

G: a group.
$\mathrm{Conj}(G) = (G, x * y = y^{-1}xy)$: the conjugation quandle.
$\rho : \mathrm{Conj}(G) \ni g \mapsto g^{-1} \in \mathrm{Conj}(G)$: a good involution of $\mathrm{Conj}(G)$.

A *groupoid* is a category in which all morphisms are invertible.

Definition

$\mathcal{C}$: a groupoid, $X = \mathrm{Hom}(\mathcal{C})$: the set of all morphisms of $\mathcal{C}$,
$* : X \times X \to X$: a binary operation on X.
$X = (X, *)$: a *groupoid rack*
$\Leftrightarrow$ $*$ satisfies the following conditions.

1. $\forall x \in X,\ \forall f \in \mathrm{Hom}(\lambda, \mu),\ \forall g \in \mathrm{Hom}(\mu, \nu)$,
   $x * (fg) = (x * f) * g$ and $x * \mathrm{id}_\xi = x$. (id_ξ: the identity morphism for $\xi \in \mathrm{Ob}(\mathcal{C})$)
2. $\forall x, y, z \in X,\ (x * y) * z = (x * z) * (y * z)$.
3. $\forall x \in X,\ \forall f \in \mathrm{Hom}(\lambda, \mu),\ \forall g \in \mathrm{Hom}(\mu, \nu)$,
   $\mathrm{cod}(f * x) = \mathrm{dom}(g * x)$ and $(fg) * x = (f * x)(g * x)$.

Remark

A groupoid rack X is a symmetric rack with a good involution $\rho : X \ni x \mapsto x^{-1} \in X$.

X: a groupoid rack,
D: a Y-oriented diagram of a spatial surface.
Arc(D): the set of all arcs of D.
An *X-coloring of D* is a map Arc$(D) \to X$ satisfying the following conditions.

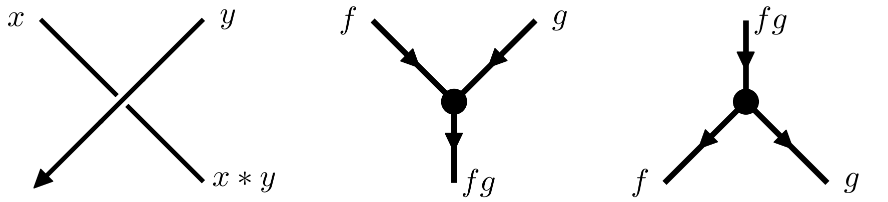


Figure: groupoid rack coloring conditions $(x, y, f, g \in X,\ \mathrm{cod}(f) = \mathrm{dom}(g))$

Col$_X(D)$: the set of all X-colorings of D.

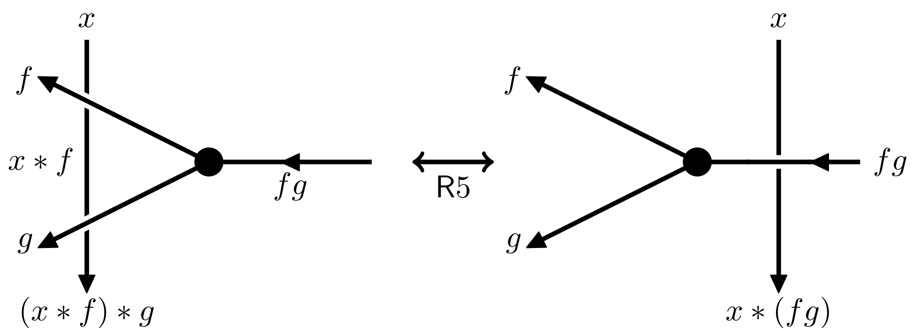


Figure: $x, f, g \in X,\ \mathrm{cod}(f) = \mathrm{dom}(g)$.

$$x * (fg) = (x * f) * g.$$

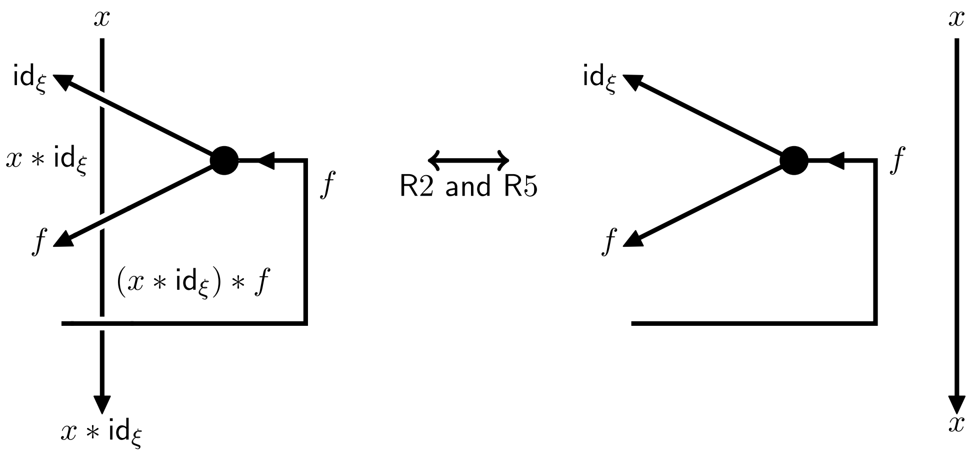


Figure: $x, f \in X,\ \xi = \mathrm{dom}(f)$.

$$x * (\mathrm{id}_\xi) = x.$$

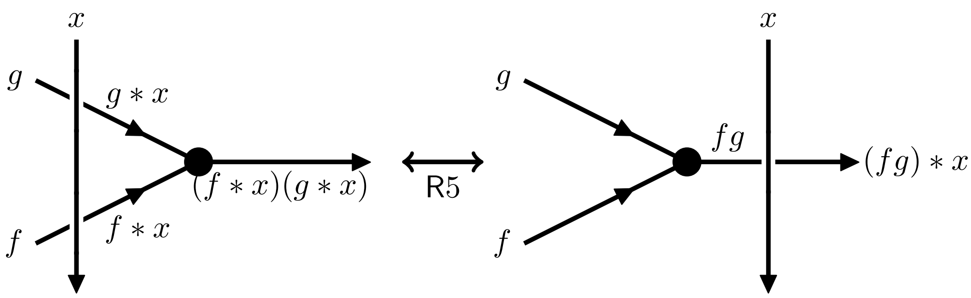


Figure: $x, f, g \in X,\ \mathrm{cod}(f) = \mathrm{dom}(g)$.

$$\mathrm{cod}(f * x) = \mathrm{dom}(g * x),\ (f * x)(g * x) = (fg) * x.$$

X: a groupoid rack.

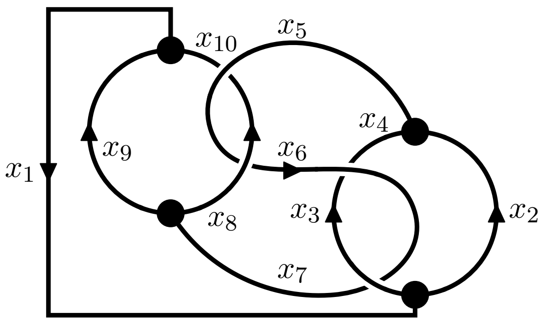


Figure: $x_i \in X$ $(i = 1, 2, \ldots, 10)$.

$$x_3 * x_6 = x_4, x_6 * x_3 = x_7, x_6 * x_8 = x_5, x_{10} * x_5 = x_8,$$

$$x_1 = x_2 x_3, x_5 = x_2 x_4, x_7 = x_8 x_9, x_1 = x_{10} x_9.$$

Theorem 1

X: a groupoid rack,
D_1, D_2: Y-oriented diagrams which represent equivalent spatial surfaces.
Then there is a bijection between $\mathrm{Col}_X(D_1)$ and $\mathrm{Col}_X(D_2)$.
In particular $|\mathrm{Col}_X(D_1)|$ is a spatial surface invariant.

The proof of this Theorem is similar to (Ishii-Matsuzaki-Murao '20).

Remark

Our invariants of spatial surfaces using groupoid racks include both of the invariants using MGR (Ishii-Matsuzaki-Murao '20) and the invariants using heap racks (Saito-Zappala).

Example 1 (Ishii-Matsuzaki-Murao '20)

$\{G_\lambda\}_{\lambda\in\Lambda}$: a family of groups, $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$, $* : X \times X \to X$: a binary operation on X.
$X = (X, *)$: a *multiple group rack* (MGR)
$\Leftrightarrow$ $*$ satisfies the following conditions.

1. $\forall x \in X,\ \forall y_1, y_2 \in G_\lambda,\ x * (y_1 y_2) = (x * y_1) * y_2$ and $x * e_\lambda = x$. (e_λ: the identity of G_λ)
2. $\forall x, y, z \in X,\ (x * y) * z = (x * z) * (y * z)$.
3. $\forall x_1, x_2 \in X,\ \forall y \in G_\lambda,\ \exists \mu \in \Lambda$ s.t. $x_1 * y,\ x_2 * y \in G_\mu$ and $(x_1 x_2) * y = (x_1 * y)(x_2 * y)$.

An MGR $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ is called a *multiple conjugation quandle* (MCQ) (Ishii '15)
$\Leftrightarrow$ $*$ satisfies the following condition.

- $\forall \lambda \in \Lambda, \forall x, y \in G_\lambda, x * y = y^{-1}xy$.

Example 2 (Saito-Zappala)

G: a group,
$* : G^2 \times G^2 \to G^2$; $(x, y) * (z, w) = (xz^{-1}w, yz^{-1}w)$: a binary operation on G^2.
$*$ is a rack operation.
In this talk, we call $G^2 = (G^2, *)$ *heap rack*.
$(x, y)(y, z) = (x, z)$: a partial product on G^2 $(x, y, z \in G)$.

Example 3

$R = (R, *)$: a rack,
$\triangleright : R^2 \times R^2 \to R^2$; $(x, y) \triangleright (z, w) = ((x *^{-1} z) * w, (y *^{-1} z) * w)$: a binary operation on R^2.
$\triangleright$ is a rack operation.
$(x, y)(y, z) = (x, z)$: a partial product on R^2 $(x, y, z \in R)$.

We can construct groupoid racks from augmented racks.

Definition (Fenn-Rourke '92)

G: a group, $R = (R, *)$: a rack with $R \curvearrowleft G$, $\partial : R \to G$.
(R, G, ∂): an *augmented rack*
$\Leftrightarrow$ ∂ satisfies the following conditions.

- $\forall x, y \in R, x * y = x \cdot \partial(y)$.
- $\forall x \in R, \forall g \in G, \partial(x \cdot g) = g^{-1}\partial(x)g$.

If R is a quandle, (R, G, ∂) is an *augmented quandle* (Joyce '82).

(R, G, ∂): an augmented rack.
Consider the groupoid $\mathcal{C}$ which consists of

- $\mathrm{Ob}(\mathcal{C}) = R$,
- $\mathrm{Hom}(x, y) = \{(x, y, g) \in R \times R \times G \mid y = x \cdot g\}$,
- composition: $\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \ni ((x, y, g),\ (y, z, h)) \mapsto (x, z, gh) \in \mathrm{Hom}(x, z)$,
- (x, x, e_G): the identity morphism for $x \in G$,
- $(y, x, g^{-1}) \in \mathrm{Hom}(y, x)$ is the inverse morphism of $(x, y, g) \in \mathrm{Hom}(x,\ y)$.

Example 4

$X = \mathrm{Hom}(\mathcal{C})$, $n \in \mathbb{Z}$, $\delta \in \{0, 1\}$,
$\triangleright : X \times X \to X$; $(x, y, g) \triangleright (z, w, h) = (x \cdot \partial(z)^n h^\delta \partial(w)^{-n}, y \cdot \partial(z)^n h^\delta \partial(w)^{-n}, g^{\partial(z)^n h^\delta \partial(w)^{-n}})$,
where $a^b = b^{-1}ab$ $(a, b \in G)$.
Then X is a groupoid rack.

$R_3 = (\mathbb{Z}_3, x * y = 2y - x)$: the dihedral quandle.
$\mathrm{Aut}(R_3) = \{f : R_3 \ni x \mapsto ax + b \in R_3 \mid a \in \mathbb{Z}_3^\times, b \in \mathbb{Z}_3\}$.

Example

$(R_3, \mathrm{Aut}(R_3), \partial : R_3 \ni x \mapsto S_x \in \mathrm{Aut}(R_3))$ is an augmented rack. The groupoid rack X, constituted from $(R_3, \mathrm{Aut}(R_3), \partial)$ by the method in Example 4, is not an MGR.

Remark

Example 3 and 4 show that it is possible to construct groupoid racks from racks. Furthermore, the way in Example 4 can also construct groupoid racks that are not MGRs.

Theorem 2

$R = (R, *, \rho)$: a symmetric rack. $P \subset R \times R$ and $\mu : P \to R$ satisfy the following conditions, where we denote $\mu(a, b)$ by ab.

- $\forall a, b, c \in R$, $[(a, b) \in P \wedge (ab, c) \in P] \Leftrightarrow [(b, c) \in P \wedge (a, bc) \in P]$.
- $\forall (a, b), (ab, c) \in P$, $(ab)c = a(bc)$.
- $\forall a, b, x \in R$, $(a, b) \in P \Leftrightarrow (\rho(b), \rho(a)) \in P \Leftrightarrow (a * x, b * x) \in P$.
- $\forall (a, b) \in P$, $(b, \rho(ab)) \in P, (\rho(ab), a) \in P$.
- $\forall (a, b) \in P, \forall x \in R$,
  $\rho(b)\rho(a) = \rho(ab), (ab)\rho(b) = a, (ab) * x = (a * x)(b * x), x * (ab) = (x * a) * b$.

Put $R' := \bigcup_{(a,b) \in P} \{a, b\}$. Then

(i) R' is a subrack of $(R, *)$ with the good involution ρ.

(ii) $(R', *)$ is a groupoid rack.

Before proving the theorem, we see how the conditions are obtained.

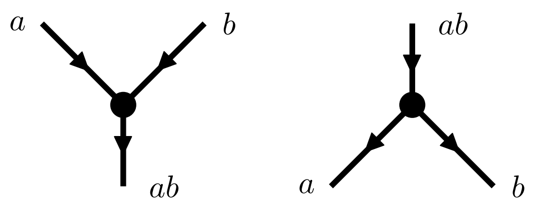


Figure: coloring conditions at trivalent vertices ($(a, b) \in P$)

Well-definedness of the coloring at a vertex:

- $\forall a, b, x \in R$, $(a, b) \in P \Leftrightarrow (\rho(b), \rho(a)) \in P$.
- $\forall (a, b) \in P$, $(b, \rho(ab)) \in P, (\rho(ab), a) \in P$.
- $\forall (a, b) \in P, \forall x \in R$, $\rho(b)\rho(a) = \rho(ab), (ab)\rho(b) = a$.

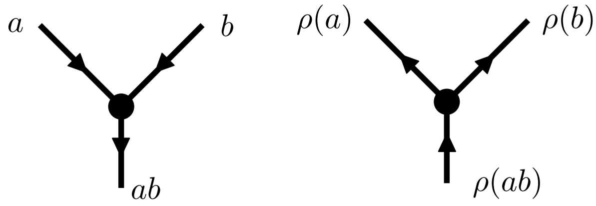


Figure: coloring at a vertex

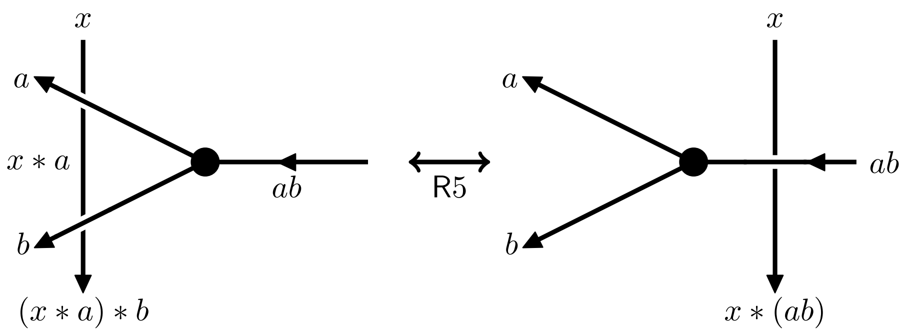
$$(a,b) \in P \Leftrightarrow (\rho(b), \rho(a)) \in P,$$
$$\rho(b)\rho(a) = \rho(ab).$$

R5-moves:

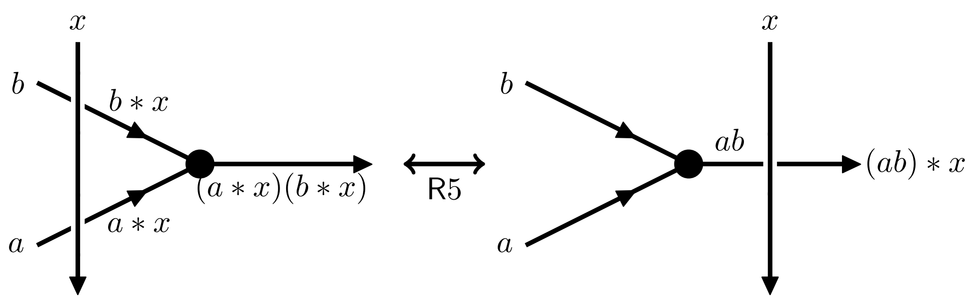
- $\forall a, b, x \in R,\ (a,b) \in P \Leftrightarrow (a * x, b * x) \in P.$
- $\forall (a,b) \in P, \forall x \in R,\ (ab) * x = (a * x)(b * x), x * (ab) = (x * a) * b.$

R6-moves

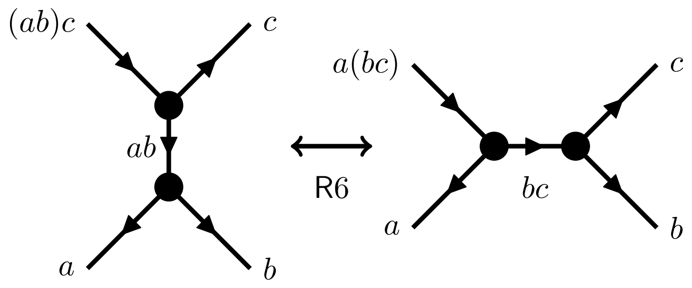
- $\forall a, b, c \in R,\ [(a,b) \in P \wedge (ab, c) \in P] \Leftrightarrow [(b,c) \in P \wedge (a, bc) \in P]\,.$
- $\forall (a,b), (ab,c) \in P,\ (ab)c = a(bc).$



$$(a, b) \in P \Rightarrow x * (ab) = (x * a) * b.$$



$$(a, b) \in P \Leftrightarrow (a * x, b * x) \in P,$$
$$(a * x)(b * x) = (ab) * x.$$



$$[(a.b) \in P \wedge (ab, c) \in P] \Leftrightarrow [(b, c) \in P \wedge (a, bc) \in P],$$

$$(a.b) \in P \wedge (ab, c) \in P \Rightarrow (ab)c = a(bc).$$

Sketch of proof:

1. $\forall a \in R', (\rho(a), a) \in P, (a, \rho(a)) \in P.$
2. $\forall x \in R, S_x(R') = R'.$
3. $\rho(R') = R'.$

$X = \bigcup_{(a,b) \in P} \{s_a, s_b, t_a, t_b\}$,
$\sim$: an equivalence relation generated by $\{(t_a, s_b) \mid (a, b) \in P\} \subset X \times X$.
Consider the groupoid $\mathcal{C}$ which consists of

- $\mathrm{Ob}(\mathcal{C}) = X/\sim$.
- $\forall x, y \in \mathrm{Ob}(\mathcal{C}), \mathrm{Hom}(x, y) = \{a \in R \mid s_a \in x, t_a \in y\}$.
- identity morphism for $x \in \mathrm{Ob}(\mathcal{C})$:$\mathrm{id}_x = a\rho(a)$, where $a : x \to y$ (well-defined).
- $\forall a \in \mathrm{Hom}(\mathcal{C})$, $\rho(a)$ is the inverse of a.

Then $R' = \mathrm{Hom}(\mathcal{C})$ and μ is equal to the composition of morphisms as a map. $*$ and μ satisfy the conditions of groupoid rack. Therefore R' is a groupoid rack.

Thank you for your attention!!

Good involutions of generalized Alexander quandles

Yuta Taniguchi

Abstract. Quandles with good involutions that satisfy certain conditions, which are called good involutions, can be used to construct invariants of unoriented link. In this note, we discuss good involutions of generalized Alexander quandles.

1 Introduction

A *quandle* [1, 4] is a set X with a binary operation $*: X^2 \to X$ satisfying the following conditions:

(Q1) For any $x \in X$, we have $x * x = x$.

(Q2) There exists a binary operation $\bar{*}: X^2 \to X$ such that $(x * y)\bar{*}y = (x\bar{*}y) * y = x$ for any $x, y \in X$.

(Q3) For any $x, y, x \in X$, we have $(x * y) * z = (x * z) * (y * z)$.

By (Q2), the map $S_y : X \to X; x \mapsto x * y$ is bijective for any y. A quandle X is *connected* if for any $x, y \in X$, there exist $z_1, z_2, \ldots, z_n \in X$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{\pm 1\}$ such that $S_{z_n}^{\varepsilon_n} \circ \cdots S_{z_2}^{\varepsilon_2} \circ S_{z_1}^{\varepsilon_1}(x) = y$.

Let X be a quandle. A map $\rho : X \to X$ is a *good involution* [2] if ρ is an involution such that $\rho(x * y) = \rho(x) * y$ and $x * \rho(y) = x\bar{*}y$ for any $x, y \in X$. Then, the pair (X, ρ) is called the *symmetric quandle* [3].

Example 1.1. A quandle X is a *kei* if the operation $*$ coincides with $\bar{*}$. Then, the identity map on X is a good involution.

Example 1.2. Let G be a group. Let us define the operation $*$ on G by $g * h := h^{-1}gh$. Then, $\mathrm{Conj}(G) = (G, *)$ is a quandle, which is called the *conjugation quandle* of G. The inversion $\mathrm{inv}(G) : G \to G; g \mapsto g^{-1}$ is a good involution of $\mathrm{Conj}(G)$.

The following problems naturally arise.

Problem 1.3. *Determine the necessary and sufficient condition for good involutions of a quandle X to exist.*

Problem 1.4. *Determine the set of all good involutions of a quandle X.*

In this note, we consider this problem if a quandle is a *generalized Alexander quandle*.

The author was supported by JSPS KAKENHI Grant Number 21J21482.

2 Main results

Let G be a group and $\varphi : G \to G$ be a group automorphism of G. Let us define the operation $*$ on G by $g * h := \varphi(gh^{-1})h$. Then, $\mathrm{GAlex}(G, \varphi) = (G, *)$ is a quandle, which is called the *generalized Alexander quandle* of (G, φ). If G is an abelian group, we call $\mathrm{GAlex}(G, \varphi)$ the *Alexander quandle.*

At first, we give an answer of Problem 1.3 if a quandle X is the generalized Alexander quandle of (G, φ) for some G and φ.

Theorem 2.1. *Let G be a group and $\varphi : G \to G$ a group automorphism of G. There exists a good involution of* $\mathrm{GAlex}(G, \varphi)$ *if and only if the quandle* $\mathrm{GAlex}(G, \varphi)$ *is a kei.*

Next, let us consider Problem 1.4 when the generalized Alexander quandle $\mathrm{GAlex}(G, \varphi)$ is connected. Let G be a group and $\varphi : G \to G$ be a group automorphism. Suppose that the generalized Alexander quandle $\mathrm{GAlex}(G, \varphi)$ is a kei and a connected quandle. By Theorem 2.1, there exists a good involution $\rho : \mathrm{GAlex}(G, \varphi) \to \mathrm{GAlex}(G, \varphi)$. We put $r := \rho(e)$, where e is the identity element of G. Then, r satisfies the following conditions:

- We have $\varphi(r) = r$.
- For any $x, y \in G$, we have $r(x * y) = (rx) * y$.

Since $\mathrm{GAlex}(G, \varphi)$ is connected, we see that $\rho(x) = rx$ for any $x \in \mathrm{GAlex}(G, \varphi)$. This implies that $r^2 = e$.

Conversely, if there exists an element $r \in G$ such that $r^2 = e$ and $\varphi(r) = r$, we see the map $\rho_r : \mathrm{GAlex}(G, \varphi) \to \mathrm{GAlex}(G, \varphi); x \mapsto rx$ is a good involution. Thus, we have the following proposition:

Proposition 2.2. *Let G be a group and $\varphi : G \to G$ a group automorphism. If the quandle* $\mathrm{GAlex}(G, \varphi)$ *is a kei and a conncted quandle, there is a bijection*

$$\{\rho : \text{good involutions of } \mathrm{GAlex}(G, \varphi)\} \overset{1:1}{\longleftrightarrow} \{r \in G \mid \varphi(r) = r, r^2 = e\}.$$

Remark 2.3. If a generalized Alexander quandle is not connected, Proposition 2.2 does not hold. Let us consider the quandle $R_4 = \mathrm{GAlex}(\mathbb{Z}/4\mathbb{Z}, \mathrm{inv}(\mathbb{Z}/4\mathbb{Z}))$, which is called the *dihedral quandle of order 4*. By [3], R_4 has four good involutions. However, we see that $\{r \in \mathbb{Z}/4\mathbb{Z} \mid \mathrm{inv}(\mathbb{Z}/4\mathbb{Z})(r) = r, 2r = 0\}$ consists of two elements $0, 2 \in \mathbb{Z}/4\mathbb{Z}$.

References

[1] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra, **23**, no. 1 (1982), 37–65.

[2] S. Kamada, *Quandles with good involutions, their homologies and knot invariants*, in Intelligence of Low Dimensional Topology 2006, 2007, pp. 101-108.

[3] S. Kamada, K. Oshiro, *Homology groups of symmetric quandles and cocycle invariants of links and surface-links*, Trans. Am. Math. Soc. **362**, (10) (2010), 5501-5527.

[4] S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb., **161**, no. 1 (1982) 78–88.

(Y. Taniguchi) DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, 1-1, MACHIKANEYAMA, TOYONAKA, OSAKA, 560-0043, JAPAN
Email address: `u660451k@ecs.osaka-u.ac.jp`

Good involutions of generalized Alexander quandles

谷口 雄大

大阪大学大学院理学研究科

カンドルと対称空間
December 8, 2022

定義 (Joyce '82, Matveev '82)

X: (空でない) 集合, $* : X^2 \to X$: 2 項演算.
$X = (X, *)$: **カンドル (quandle)** $\Leftrightarrow$ $*$ は以下を満たす:

1. ${}^\forall x \in X$, $x * x = x$.
2. ${}^\forall y \in X, S_y : X \to X; x \mapsto x * y$ は全単射.
3. ${}^\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

(2) $\Leftrightarrow$ 「${}^\exists \bar{*} : X^2 \to X$: 2 項演算 s.t.
${}^\forall x, y \in X, (x * y)\bar{*}y = (x\bar{*}y) * y = x$.」
$\bar{*}$ を $*$ の**双対演算**と呼ぶ.

注意.
$(X, *)$: カンドル $\Rightarrow (X, \bar{*})$: カンドル.
一般に $(X, *)$ と $(X, \bar{*})$ はカンドルとして同型でない.

定義 (Kamada '07, Kamada-Oshiro '10)

$X = (X, *)$: カンドル.
$\rho : X \to X$: **良い対合 (good involution)** $\Leftrightarrow \rho$ は以下を満たす:

1. ρ: 対合写像 i.e. $\rho^2 = \mathrm{id}_X$.
2. ${}^\forall x, y \in X, \rho(x * y) = \rho(x) * y$.
3. ${}^\forall x, y \in X, x * \rho(y) = x \bar{*} y$.

ρ が良い対合のとき,
組 (X, ρ) を**対称カンドル (symmetric quandle)** と呼ぶ.

$(X, \rho), (X', \rho')$: 対称カンドル.
(X, ρ) と (X', ρ') が **(対称カンドルとして) 同型**
$\Leftrightarrow$ あるカンドル同型 $f : X \to X'$ が存在して $f \circ \rho = \rho' \circ f$.
$\mathrm{Aut}(X, \rho) := \{f : X \to X \mid f : \text{カンドル同型}, f \circ \rho = \rho \circ f\}$.

例.
X: **自明なカンドル** i.e. ${}^\forall x, y \in X, x * y = x$.
このとき任意の対合写像は良い対合.

例.
X: **圭** i.e. X: カンドル s.t. $* = \bar{*}$.
このとき恒等写像 Id_X は良い対合.

例.
G: 群. $\mathrm{Conj}(G) = (G, x * y := y^{-1}xy)$: G の**共役カンドル**.
$\mathrm{Inv}(G) : G \to G; g \mapsto g^{-1}$: $\mathrm{Conj}(G)$ の良い対合.
例.
$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.
$x * y :=$ (x を y と原点を通る軸でθ 回転させた点) ($\theta \in [0, 2\pi)$)
$\rightsquigarrow (S^2, *)$: カンドル.
$\rho : S^2 \to S^2; x \mapsto -x$: $(S^2, *)$ の良い対合.

問題

カンドル X が性質 $(\star)$ を持つとする. ($(\star)$ には連結, アレキサンダー等が入る)

- X が良い対合を持つ必要 (十分) 条件を求めよ.
- X の良い対合をリストアップし, 対称カンドルとしての同型類を決定せよ.

事実.
カンドル $(X,*)$ が良い対合を持つ $\Rightarrow$ $(X,*)$ と $(X,\bar{*})$ は同型.
$(\because)$ $\rho(x\bar{*}y) = \rho(x*\rho(y)) = \rho(x)*\rho(y)$.
逆は一般に不成立 (4 面体カンドルなど).

注意.
2 面体カンドル R_n については [Kamada-Oshiro '10] でリストアップが完了している:

- n が奇数 $\Rightarrow$ 良い対合は 1 つ.
- $n=2m$ かつ m が奇数 $\Rightarrow$ 良い対合は 2 つ.
- $n=2m$ かつ m が偶数 $\Rightarrow$ 良い対合は 4 つ.

モチベーション

結び目理論の観点から
カンドル $\rightsquigarrow$ 向きのついた結び目の不変量.
対称カンドル $\rightsquigarrow$ 向きのついていない結び目 (特に向きつけ不可能な曲面結び目) の不変量.

- 対称カンドルの具体例を大量に構成して結び目の不変量を増やしたい!
- 対称カンドルの構造を理解して結び目の不変量の性質を理解したい!

カンドルと対称空間の観点から

定義 (Kubo-Nagashiki-Okuda-Tamaru '22)

X: カンドル, $A \subset X$: **対蹠的** $\Leftrightarrow \forall x, y \in A, x * y = x$.

任意の $x \in X$ に対して集合 $\{\rho(x) \mid \rho : \text{良い対合}\} \subset X$ は対蹠的
$\Rightarrow$ 良い対合が豊富にあると (極大) 対蹠的な部分集合は大きくなる.

問題

カンドル X が性質 $(\star)$ を持つとする.

- X が良い対合を持つ必要 (十分) 条件を求めよ.
- X の良い対合をリストアップし, 対称カンドルとしての同型類を決定せよ.

今日は $(\star)$ = **一般化されたアレキサンダーカンドル** を考える.

定義

G: 群, $\varphi : G \to G$: 群同型, $x * y := \varphi(xy^{-1})y$ $(x, y \in G)$.
$\Rightarrow$ $\mathrm{GAlex}(G, \varphi)$: **一般化されたアレキサンダーカンドル**.
特に G が可換群のときには**アレキサンダーカンドル**と呼ぶ.

主結果

定理

${}^{\exists}\rho : \mathrm{GAlex}(G, \varphi) \to \mathrm{GAlex}(G, \varphi)$: 良い対合 $\Leftrightarrow$ $\mathrm{GAlex}(G, \varphi)$: 圭.

(証明)
($\Rightarrow$) を示せばよい. ${}^{\forall}x \in \mathrm{GAlex}(G, \varphi)$ に対して
$\rho(x) = \rho(x * x) = \rho(x) * x = \varphi(\rho(x)x^{-1})x = \varphi(\rho(x))\varphi(x^{-1})x$.
よって ${}^{\forall}x \in \mathrm{GAlex}(G, \varphi)$ に対して $\varphi(\rho(x)^{-1})\rho(x) = \varphi(x^{-1})x$.
したがって
$x \bar{*} y = x * \rho(y) = \varphi(x\rho(y)^{-1})\rho(y) = \varphi(xy^{-1})y = x * y$ なので,
$\mathrm{GAlex}(G, \varphi)$ は圭. □

カンドル X が**連結**
$\Leftrightarrow \langle \{S_y \mid y \in X\} \rangle$ が X に推移的に作用する.

定理

G: **有限**可換群, $\varphi : G \to G$: 群同型.
$\mathrm{GAlex}(G, \varphi)$: 連結かつ圭
$\Rightarrow$ $\mathrm{GAlex}(G, \varphi)$ の良い対合は恒等写像しか存在しない.

注意
命題の仮定から有限や連結を無くすと, 非自明な良い対合を持つようになる.

一般化されたアレキサンダーカンドルの良い対合

ここから G: 群, $\varphi : G \to G$: 群同型は固定し,
$X := \mathrm{GAlex}(G, \varphi)$ は連結な圭であると仮定する.
X の良い対合 ρ に対し, $r := \rho(e)$ と置くと次が成り立つ:

- $\varphi(r) = \varphi(re^{-1})e = r * e = \rho(e) * e = \rho(e * e) = \rho(e) = r.$
- ${}^\forall x, y \in X, (rx) * y = \varphi(rxy^{-1})y = r\varphi(xy^{-1})y = r(x * y).$

よって X は連結なので $\rho(x) = rx$ が成り立つことがわかる.
また ρ は対合なので $r^2 = e$ であることもわかる.

一般化されたアレキサンダーカンドルの良い対合

逆に $\varphi(r) = r$ かつ $r^2 = e$ を満たす元に対して
$\rho_r : X \to X; x \mapsto rx$ と定めると ρ_r は良い対合になる.
以上をまとめると次を得る:

命題

$\{\mathrm{GAlex}(G, \varphi)$ の良い対合$\} \overset{1:1}{\longleftrightarrow} \{r \in G \mid \varphi(r) = r, r^2 = e\}.$

一般化されたアレキサンダーカンドルの良い対合の分類

$X = \mathrm{GAlex}(G, \varphi)$ は連結なので,
${}^\forall f : X \to X$: カンドル同型, ${}^\exists f_\# : G \to G$:群同型, ${}^\exists b \in G$ s.t.
$f(x) = f_\#(x)b$ (cf. [Higashitani-Kurihara]).
よって X の良い対合 $\begin{cases} \rho_1 : X \to X; x \mapsto r_1 x \\ \rho_2 : X \to X; x \mapsto r_2 x \end{cases}$ に対して
$f \circ \rho_1 = \rho_2 \circ f$ ならば $f_\#(r_1) = r_2$ が成り立つ.
また群同型 $f : G \to G$ が $f \circ \varphi = \varphi \circ f$ かつ $f(r_1) = r_2$ を満たすならば f は (X, ρ_1) と (X, ρ_2) の間の同型になる.

命題

$\{\mathrm{GAlex}(G, \varphi)$ の良い対合 $\}_{/\text{同型}}$
$\xleftrightarrow{1:1} \{r \in G \mid r^2 = e, \varphi(r) = r\}_{/\{f \in \mathrm{Aut}_{\mathrm{grp}}(G) \mid f \circ \varphi = \varphi \circ f\}}$.

例.
$G := S^1 = \mathbb{R}/\mathbb{Z}$ に演算 $x * y = 2y - x$ でカンドル構造を入れる.
このとき, 右辺の集合は $\left\{0, \dfrac{1}{2}\right\}$ となる.
よってこのカンドルの良い対合は Id_{S^1} と $\rho(x) = x + \dfrac{1}{2}$ の 2 つ.
また, 命題より (X, Id_{S^1}) と (X, ρ) は対称カンドルとして同型でない.

連結の仮定を外すと今までのことは成り立たない.
<u>例.</u>
$R_{4n} = (\mathbb{Z}/4n\mathbb{Z}, x * y = 2y - x)$: 位数 $4n$ の 2 面体カンドル
$(n \in \mathbb{Z}_{>0})$.
このとき, 良い対合は以下の 4 つ [Kamada-Oshiro '10]:

(1) $\rho_1(x) = x$.
(2) $\rho_2(x) = x + 2n$.
(3) $\rho_3(x) = (2n+1)x$.
(4) $\rho_4(x) = (2n+1)x + 2n$.

(R_{4n}, ρ_3) と (R_{4n}, ρ_4) は対称カンドルとしては同型になる.
$\Rightarrow$ R_{4n} の対称カンドルとしての同型類は 3 種類.

カンドル X が**等質** $\Leftrightarrow$ $\mathrm{Aut}(X)$ が X に推移的に作用する.
対称カンドル (X, ρ) が**等質**
$\Leftrightarrow$ $\mathrm{Aut}(X, \rho)$ が X に推移的に作用する.
<u>事実.</u> 一般化されたアレキサンダーカンドルは等質.

カンドル X が等質でも (X, ρ) が等質だとは限らない.
実際, $(R_{4n}, \rho_1), (R_{4n}, \rho_2)$ は等質だが (R_{4n}, ρ_3) は等質でない.

$$\left(\mathrm{Aut}(R_{4n}, \rho_3) = \left\{ f(x) = ax + b \;\middle|\; \begin{array}{c} a \in (\mathbb{Z}/4n\mathbb{Z})^\times \\ b \in \{2, 4, \ldots, 2(2n-1)\} \end{array} \right\}\right).$$

一般化されたアレキサンダーカンドルは等質なカンドルの「親玉」である.

定理 (Joyce '82)

等質なカンドルは一般化されたアレキサンダーカンドルの商として実現出来る.

最初の定理より, この定理の「対称カンドル版」は成り立たない.
(商をとる前のカンドルが圭ならば商をとったカンドルも圭)

問題

等質な対称カンドルの「親玉」と呼べるような対称カンドルは何か?

カンドル X のホモロジー群 $H_n^Q(X)$ と同様に**対称カンドル** (X, ρ) **のホモロジー群** $H_n^{Q,\rho}(X)$ も定義される. しかし $H_n^{Q,\rho}(X)$ は $H_n^Q(X)$ とふるまいが異なる.

Ex.

X: 有限かつ連結
$\Rightarrow H_3^Q(X)$ のベッチ数は 0 ([Litherland-Nelson '03]).
一方 X が有限かつ連結でも $H_3^{Q,\rho}(X) = \mathbb{Z}$ となる対称カンドル (X, ρ) が存在する. ([Carter-Oshiro-Saito '10])

問題

対称カンドルの (コ) ホモロジー群の性質を調べよ.

ご清聴ありがとうございました.

Homogeneous quandles with a commutative inner automorphism group

Takuya Saito and Sakumi Sugawara
(Presenter: Takuya Saito)

Abstract. There is a subclass of the flat quandles, which consists of quandles having a commutative inner automorphism group. In this work, we determined the structure of homogeneous quandles with commutative inner automorphism groups. We also give the number of isomorphism classes for small orders.

1 Preliminary

The quandles are usually defined as magmas that satisfy several conditions, but we define them as sets with symmetric transformations. We write $\mathrm{Map}(X, X)$ as the set of maps on X. Let Q be a set and let $s : Q \to \mathrm{Map}(Q, Q); x \to s_x$ be a map. The pair (Q, s) is a *quandle* if following three conditions:

- for any $x \in Q$, $s_x(x) = x$;
- for any $x \in Q$, the map s_x is a bijection;
- for any $x, y \in Q$, $s_x s_y = s_{s_x(y)} s_x$.

For example, if s_x is an identity map at any $x \in X$, then (X, s) satisfies the above axioms and is a quandle. This is called a *trivial* quandle.

A map on a quandle $f : (Q, s) \to (Q, s)$ is an *automorphism* if f satisfies $f \circ s_x = s_{f(x)} \circ f$ for any $x \in Q$. We write $\mathrm{Aut}(Q, s)$ as the group of all automorphisms on (Q, s). The quandle (Q, s) is *homogeneous* if $\mathrm{Aut}(Q, s)$ acts Q transitive. Also, We write $\mathrm{Inn}(Q, s)$ the subgroup of $\mathrm{Aut}(Q, s)$ generated by the set $\{s_x \in \mathrm{Aut}(Q, s) \in\mid x \in Q\}$ and call it the *inner automorphism group* of (Q, s).

2 Main results

Let X be a set and let A be an abelian group. Take a map $d : X \times X \to A$ with $d(x, x) = 0$ for any $x \in X$. We define the quandle $Q(X, A, d)$ on $X \times A$ by $s_{(x,a)}(y, b) := (y, b + d(x, y))$. Sometimes this is called abelian extension of a trivial quandle X by A.

Proposition 1. Let $\mathrm{Aut}(X, d) := \{f \in \mathrm{Map}(X, X) \mid d(x, y) = d(f(x), f(y))(\forall x, y \in X), f \text{ is bijective}\}$. If $\mathrm{Aut}(X, d)$ acts transitively on X, then $Q(X, A, d)$ is a homogeneous quandle with a commutative inner automorphism group.

This work was partly supported by JSPS KAKENHI22J20470.

Example 2. Let $X = \{a, b, c\}$ be a set acted by $\mathbb{Z}/3\mathbb{Z}$ transitively. Then the orbits are $O_0 = \{(a, a), (b, b), (c, c)\}, O_1 = \{(a, b), (b, c), (c, a)\}$ and $O_2 = \{(a, c), (b, a), (c, b)\}$. Take $A = \mathbb{Z}O_1 + \mathbb{Z}O_2 = \mathbb{Z}^2$ the free module generated by orbits O_1 and O_2. And take $d(x, y) = O_i((x, y) \in O_i))$ where we define $O_0 := 0$. Then $Q(X, A, d)$ is a homogeneous quandle with a commutative inner automorphism group.

If the "isometry group" $\mathrm{Aut}(X, d)$ of the pair (X, d) is transitive, then $Q(X, A, d)$ satisfies the desired condition. Conversely, when a homogeneous quandle (Q, s) has a commutative inner automorphism group, there exists a quandle $Q(X, A, d)$ which is isomorphic to (Q, s) First, given the commutativity of the inner automorphism group, the following lemma can be found.

Lemma 3. Let (Q, s) be a quandle. If $\mathrm{Inn}(Q, s)$ is commutative, then the map $s : Q \to \mathrm{Map}(X, X)$ is a constant on each $\mathrm{Inn}(Q, s)$-orbit, that is, $s_x \equiv s_y$ if there exists $f \in \mathrm{Inn}(Q, s)$ such that $y = f(x)$.

A quandle with only one $\mathrm{Inn}(Q, s)$-orbit is called a *connected* quandle, we immediately get that connected quandle with commutative inner automorphism group is the trivial quandle of order 1.

The following theorem is obtained by adding the assumption that (Q, s) is homogeneous to the above lemma.

Theorem 4. Let (Q, s) be a homogeneous quandle with a commutative inner automorphism group. Then there is a there is a set X, an abel group A and a map $d : X \times X \to A$ satisfying $d(x, x) = 0$ for any $x \in X$ such that $Q(X, A, d)$ is isomorphic to (Q, s). In particular, we can take X as $Q/\mathrm{Inn}(Q)$ and A as some quotient group of $\mathrm{Inn}(Q, s)$.

Corollary 5. Let (Q, s) be a homogeneous quandle with a commutative inner automorphism group of prime order. Then the quandle (Q, s) is trivial.

Finally, we give the number of isomorphism classes of such quandle for small orders.

order	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\cdots$
#{isomorphism classes}	1	1	1	2	1	4	1	7	4	7	1	36	1	15	19	$\cdots$

References

[1] J. Scott Carter, Mohamed Elhamdadi, Marina Appiou Nikiforou, and Masahico Saito. Extensions of quandles and cocycle knot invariants. *J. Knot Theory Ramifications*, 12(6):725–738, 2003.

[2] J. Scott Carter, Seiichi Kamada, and Masahico Saito. Diagrammatic computations for quandles and cocycle knot invariants. In *Diagrammatic morphisms and applications (San Francisco, CA, 2000)*, volume 318 of *Contemp. Math.*, pages 51–74. Amer. Math. Soc., Providence, RI, 2003.

[3] K Furuki and H Tamaru. Flat homogeneous quandles and vertex-transitive graphs. *preprint*.

[4] Akihiro Higashitani and Hirotake Kurihara. Generalized alexander quandles of finite groups and their characterizations. *arXiv preprint arXiv:2210.16763*, 2022.

[5] Akihiro Higashitani and Hirotake Kurihara. Homogeneous quandles arising from automorphisms of symmetric groups. *Communications in Algebra*, pages 1–18, 2022.

[6] Yoshitaka Ishihara and Hiroshi Tamaru. Flat connected finite quandles. *Proc. Amer. Math. Soc.*, 144(11):4959–4971, 2016.

[7] Přemysl Jedlička, Agata Pilitowska, David Stanovský, and Anna Zamojska-Dzienio. The structure of medial quandles. *J. Algebra*, 443:300–334, 2015.

[8] David Joyce. A classifying invariant of knots, the knot quandle. *J. Pure Appl. Algebra*, 23(1):37–65, 1982.

[9] Akira Kubo, Mika Nagashiki, Takayuki Okuda, and Hiroshi Tamaru. A commutativity condition for subsets in quandles—a generalization of antipodal subsets. In *Differential geometry and global analysis—in honor of Tadashi Nagano*, volume 777 of *Contemp. Math.*, pages 103–125. Amer. Math. Soc., [Providence], RI, [2022] ©2022.

[10] Sam Nelson and Chau-Yim Wong. On the orbit decomposition of finite quandles. *J. Knot Theory Ramifications*, 15(6):761–772, 2006.

(T.Saito) Department of Mathematics, Faculty of Science, Hokkaido University Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan
Email address: saito.takuya.p6@elms.hokudai.ac.jp

(S.Sugawara) Department of Mathematics, Faculty of Science, Hokkaido University Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan
Email address: sugawara.sakumi.f5@elms.hokudai.ac.jp

内部自己同型群が可換な等質カンドルについて

カンドルと対称空間 2022

齋藤琢弥

北海道大学大学院理学院数学専攻博士後期課程 1 年

2022/12/9

本発表は北海道大学 菅原朔見氏との共同研究に基づく.

内部自己同型群が可換な等質カンドルについて

北大-理-数
齋藤琢弥

準備
$Q(X, A, d)$
定義
$\mathrm{Inn}(Q(X, A, d))$
例
等質性
等長変換
等質な集合からの構成
主結果
主結果
系
位数が小さい場合
参考文献

内部自己同型群が可換な等質カンドルについて

北大-理-数
齋藤琢弥

準備
$Q(X, A, d)$
定義
$\mathrm{Inn}(Q(X, A, d))$
例
等質性
等長変換
等質な集合からの構成
主結果
主結果
系
位数が小さい場合
参考文献

定義

集合 Q，写像 $s : Q \to \mathrm{Map}(Q, Q)(= Q$ 上の写像全体$); x \to s_x$ の組 (Q, s) は，以下の公理を満たすときカンドルという．

Q1. 任意の $x \in Q$ に対して $s_x(x) = x$,

Q2. 任意の $x \in Q$ に対して s_x は全単射,

Q3. 任意の $x, y \in Q$ に対して $s_x s_y = s_{s_x(y)} s_x$.

また，s_x を x での対称変換という．

Example (自明カンドル)

任意の $x \in Q$ で $s_x := \mathrm{id}_Q$ とするとカンドルの公理を満たす．これを自明カンドルという．

自己同型群

定義

- カンドルの間の全単射 $f : (Q, s) \to (Q', s')$ がカンドル同型 $:\overset{\text{def}}{\Longleftrightarrow}$ 任意の $x \in Q$ で $f \circ s_x = s'_{f(x)} \circ f$.
- 自己同型群 $\mathrm{Aut}(Q, s) := \{$カンドル (Q, s) の自己同型$\}$.
- 内部自己同型群 $\mathrm{Inn}(Q, s) := \langle s_x \in \mathrm{Aut}(Q, s) \mid x \in Q \rangle$.

群 $G^0(Q, s) := \langle s_x \circ s_y \mid x, y \in Q \rangle$ ($\subset \mathrm{Inn}(Q, s)$) が可換なカンドルは平坦と呼ばれる．平坦なリーマン対称空間のカンドル版（らしい）.

内部自己同型群が可換な等質カンドルについて

北大-理-数
齋藤琢弥

準備
$Q(X, A, d)$
定義
$\mathrm{Inn}(Q(X, A, d))$
例
等質性
等長変換
等質な集合からの構成
主結果
主結果
系
位数が小さい場合
参考文献

$Q(X, A, d)$

定義

集合 X，アーベル群 A，任意の $x \in X$ で $d(x,x) = 0$ なる写像 $d : X \times X \to A$ に対してカンドル $Q(X, A, d) = (X \times A, s)$ を以下で定義する．

$$s_{(x,a)}(y, b) := (y, b + d(x, y)).$$

※1 $A = \{0\}$ とすると $Q(X, A, d)$ は自明カンドルになる．
※2 自明カンドル X の 2-cocycle によるアーベル拡大．

内部自己同型群が可換な等質カンドルについて

北大-理-数
齋藤琢弥

準備
$Q(X, A, d)$
定義
$\mathrm{Inn}(Q(X, A, d))$
例
等質性
等長変換
等質な集合からの構成
主結果
主結果
系
位数が小さい場合
参考文献

$Q(X, A, d)$

定義

$Q(X, A, d) := (X \times A, s)$，集合 X，アーベル群 A，$[\forall x \in X, d(x,x) = 0]$ なる写像 $d : X \times X \to A$，$s_{(x,a)}(y,b) := (y, b + d(x,y))$.

Q1. $s_{(x,a)}(x, a) = (x, a + 0)$,
Q2. $s^{-1}_{(x,a)}(y, b) = (y, b - d(x, y))$,
Q3.

$$\begin{tikzcd}
(z,c) \arrow[r, mapsto, "s_{(x,a)}"] \arrow[d, mapsto, "s_{(y,b)}"'] & (z, c+d(x,z)) \arrow[d, mapsto, "s_{s_{(x,a)}(y,b)} = s_{(y,b+d(x,y))}"] \\
 & (z, c+d(x,z)+d(y,z)) \arrow[d, equal] \\
(z, c+d(y,z)) \arrow[r, mapsto, "s_{(x,a)}"'] & (z, c+d(y,z)+d(x,z))
\end{tikzcd}$$

$\mathrm{Inn}(Q(X, A, d))$

定義

$Q(X, A, d) := (X \times A, s)$，集合 X，アーベル群 A，$[\forall x \in X, d(x, x) = 0]$ なる写像 $d : X \times X \to A$，$s_{(x,a)}(y, b) := (y, b + d(x, y))$.

命題

$\mathrm{Inn}(Q(X, A, d))$ はアーベル群である.

Proof.

$$\begin{tikzcd}
(z,c) \arrow[r, mapsto, "s_{(x,a)}"] \arrow[dd, mapsto, "s_{(y,b)}"'] & (z, c+d(x,z)) \arrow[d, mapsto, "s_{(y,b)}"] \\
& (z, c+d(x,z)+d(y,z)) \arrow[d, equal] \\
(z, c+d(y,z)) \arrow[r, mapsto, "s_{(x,a)}"'] & (z, c+d(y,z)+d(x,z))
\end{tikzcd}$$

内部自己同型群が可換な等質カンドルについて

北大-理-数
齋藤琢弥

準備
$Q(X, A, d)$
定義
$\mathrm{Inn}(Q(X, A, d))$
例
等質性
等長変換
等質な集合からの構成
主結果
主結果
系
位数が小さい場合
参考文献

定義

$Q(X, A, d) := (X \times A, s)$，集合 X，アーベル群 A，$[\forall x \in X, d(x, x) = 0]$ なる写像 $d : X \times X \to A$，$s_{(x,a)}(y, b) := (y, b + d(x, y))$.

Example (古木-田丸)

- 有向実グラスマン多様体 $G_k(\mathbb{R}^n)^{\sim} = \{\mathbb{R}^n$の k 次元有向部分空間 $\}$ は $x \in G_k(\mathbb{R}^n)^{\sim}$ に関する折り返しでカンドルになる.
- $\mathbb{R}^n$ の標準的な基底 $e_1, \ldots, e_n$ に対してそれらの張る有向部分空間のなす集合 $A(k, n) := \{\pm \langle e_{i_1}, \ldots, e_{i_k} \rangle \mid 1 \leq i_1 < \ldots < i_k \leq n\}$ はその部分カンドルになる.
- このとき $A(k, n)$ は次で定まる $Q(X, \mathbb{Z}/2\mathbb{Z}, d)$ と同型.
  $X = \{x \in 2^{\{1,\ldots,n\}} \mid |x| = k\}, d(x, y) = \begin{cases} 1 & |y \setminus x| \text{ is odd} \\ 0 & \text{others} \end{cases}$.

※対蹠集合の一般化になっている.

定義

$Q(X, A, d) := (X \times A, s)$，集合 X，アーベル群 A，$[\forall x \in X, d(x, x) = 0]$ なる写像 $d : X \times X \to A$，$s_{(x,a)}(y, b) := (y, b + d(x, y))$.

Example

1. $G = (V, E)$ 単純グラフ，$X = V$，$d : E \to \mathbb{Z}/2\mathbb{Z}$, $d(x, y) = 1$ iff $\overline{xy} \in E$.
2. 距離空間 (X, d)，$A = \mathbb{R}$.
3. X = 群，体の乗法群 $A = \mathbb{K}^\times$，指標 $\chi : X \to \mathbb{K}^\times$, $d(x, y) = \chi(xy^{-1})$.
4. Lie 代数 X，$A = (X, +)$，$d(x, y) = [x, y]$.

等長変換と自己同型

定義

- 三つ組 (X, A, d) に対して，全単射 $f : X \to X$ が d に関する等長変換 $:\overset{\text{def}}{\Longleftrightarrow} d(x, y) = d(f(x), f(y))$ を満たす．
- $\mathrm{Aut}(X, d) := \{d$ に関する X 上の等長変換$\} =$ $\{f \in \mathrm{Map}(X, X) \mid d(x, y) = d(f(x), f(y)), f$ は全単射$\}$

命題

f が d に関する等長変換であるとき，
$f' : X \times A \to X \times A; (x, a) \mapsto (f(x), a)$ は $Q(X, A, d)$ の自己同型．
特に，$\mathrm{Aut}(X, d) \subset \mathrm{Aut}(Q(X, A, d))$．

内部自己同型群が可換な等質カンドルについて

北大-理-数
齋藤琢弥

準備
$Q(X, A, d)$
定義
$\mathrm{Inn}(Q(X, A, d))$
例
等質性
等長変換
等質な集合からの構成
主結果
主結果
系
位数が小さい場合
参考文献

等長変換と等質性

定義

カンドル (Q, s) が

- 等質 $:\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Aut}(Q, s)$ が (Q, s) に推移的に作用する．
- 連結 $:\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Inn}(Q, s)$ が (Q, s) に推移的に作用する．

※内部自己同型群が可換かつ連結 $\Rightarrow$ 位数 1 の自明カンドル（後述）．
$\to$ 内部自己同型群が可換な等質カンドルはどうなっているか．

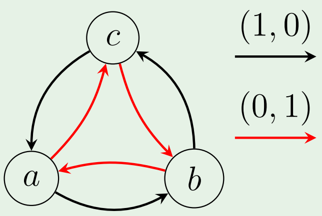
定理

$\mathrm{Aut}(X, d)$ が X に対して推移的である $\Rightarrow$ $Q(X, A, d)$ は等質．

$\to$ 簡単に内部自己同型群が可換な等質カンドルを構成できる．

等質な集合 X からの構成

Example



$d(x,y)$

$x\backslash y$	a	b	c
a	$(0,0)$	$(1,0)$	$(0,1)$
b	$(0,1)$	$(0,0)$	$(1,0)$
c	$(1,0)$	$(0,1)$	$(0,0)$

- $X=\{a,b,c\}$.
- $\mathbb{Z}/3\mathbb{Z}$ の推移的な作用 ρ の軌道は
  - $O_0=\{(a,a),(b,b),(c,c)\}$
  - $O_1=\{(a,b),(b,c),(c,a)\}$
  - $O_2=\{(a,c),(b,a),(c,b)\}$
- $A=\mathbb{Z}O_1+\mathbb{Z}O_2=\mathbb{Z}^2$ として，d を各軌道上で値が一定になるように左のように取る．
- $\mathrm{Aut}(X,d)$ は X に対して推移的に作用．
- このとき $Q(X,A,d)$ は内部自己同型群が可換で等質．

主結果

定理

内部自己同型群が可換な等質カンドル Q に対して，ある集合 X とアーベル群 A，$d : X \times X \to A$ が存在して $Q(X, A, d)$ と同型になる．

特に

- X：Q の $\mathrm{Inn}(Q)$ による軌道分解 $Q = \bigsqcup_{x \in X} Q_x$ の添字集合
- A：$\mathrm{Inn}(Q)$ の適当な剰余群

で取れる．

内部自己同型群が可換な等質カンドルについて
北大-理-数
齋藤琢弥
準備
$Q(X, A, d)$
定義
$\mathrm{Inn}(Q(X, A, d))$
例
等質性
等長変換
等質な集合からの構成
主結果
主結果
系
位数が小さい場合
参考文献

証明の概略

- $\mathrm{Inn}(Q)$ が可換
  $\Rightarrow \forall p, q \in Q, s_p s_q = s_q s_p = s_{s_q(p)} s_q$
  $\Leftrightarrow \forall p, q \in Q, s_p = s_{s_q(p)}$
  $\Leftrightarrow \forall p \in Q, \forall f \in \mathrm{Inn}(Q), s_p = s_{f(p)}$
  $\Leftrightarrow$ Q の $\mathrm{Inn}(Q)$-軌道上で対称変換は一定
- Q の $\mathrm{Inn}(Q)$ による軌道分解を $\bigsqcup_{x \in X} Q_x$ でとると，等質性から任意の x, y で
$$Q_x \cong Q_y$$
- したがって任意に x を固定して $Q = X \times Q_x$ のように書ける．
- Q_x は自然な作用で等質な $\mathrm{Inn}(Q)$-集合だから Q_x は $\mathrm{Inn}(Q)$ の剰余 A として取れる． □

系

各 $\mathrm{Inn}(Q)$-軌道 $Q_x \subset Q$ が自明な部分カンドルになることに注意すると以下がすぐにわかる.

系

連結（つまり内部自己同型群による軌道が一つ）で内部自己同型群が可換なカンドルは位数 1 の自明カンドルである.

系

内部自己同型群が可換な素数位数の等質カンドルは自明カンドルである.

内部自己同型群が可換な等質カンドルについて

北大-理-数
齋藤琢弥

準備
$Q(X, A, d)$
定義
$\mathrm{Inn}(Q(X, A, d))$
例
等質性
等長変換
等質な集合からの構成
主結果
主結果
系
位数が小さい場合
参考文献

位数が小さい場合

位数が小さい内部自己同型群が可換な等質カンドルの同型類の個数

位数	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\cdots$
個数	1	1	1	2	1	4	1	7	4	7	1	36	1	15	19	$\cdots$

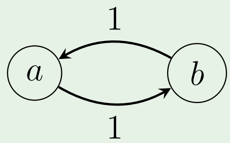
例：$n = 4$

Example

$|X \times A| = 4$ より，$(|X|, |A|) = (4, 1), (2, 2)$.

$(4, 1)$: 自明カンドル．

$(2, 2)$: $\mathbb{Z}/2\mathbb{Z} \curvearrowright X$: 推移的，$A = \mathbb{Z}/2\mathbb{Z}$，$d \not\equiv 0$．$\therefore$ 次の場合に限る．



$d(x, y)$

$x \backslash y$	a	b
a	0	1
b	1	0

内部自己同型群が可換な等質カンドルについて

北大-理-数
齋藤琢弥

準備
$Q(X, A, d)$
定義
$\mathrm{Inn}(Q(X, A, d))$
例
等質性
等長変換
等質な集合からの構成
主結果
主結果
系
位数が小さい場合
参考文献

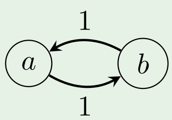
例：$n = 6$

Example

$|X \times A| = 6$ より，$(|X|, |A|) = (6, 1), (2, 3), (3, 2)$.

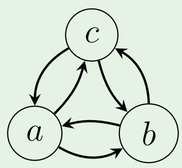
$(6, 1)$: 自明カンドル．

$(2, 3)$: $\mathbb{Z}/2\mathbb{Z} \curvearrowright X$: 推移的，
$A = \mathbb{Z}/3\mathbb{Z}$，$d \not\equiv 0$．
$\therefore$ 次の場合に限る．$d(x, y)$;

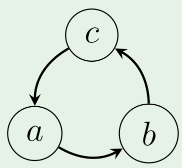


$x \backslash y$	a	b
a	0	1
b	1	0

$(3, 2)$: $\mathbb{Z}/3\mathbb{Z} \curvearrowright X$: 推移的，
$A = \mathbb{Z}/2\mathbb{Z}$，$d \not\equiv 0$．
$\therefore$ 右の二通り．$d(x, y)$;



$x \backslash y$	a	b	c
a	0	1	1
b	1	0	1
c	1	1	0



$x \backslash y$	a	b	c
a	0	1	0
b	0	0	1
c	1	0	0

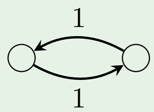
例：$n = 10$

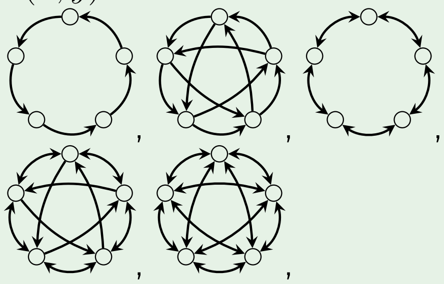
Example

$|X \times A| = 10$ より，$(|X|, |A|) = (10, 1), (2, 5), (5, 2)$.

$(10, 1)$: 自明カンドル． $(5, 2)$: $\mathbb{Z}/5\mathbb{Z} \curvearrowright X$: 推移的，$A = \mathbb{Z}/2\mathbb{Z}$

$(2, 5)$: $A = \mathbb{Z}/5\mathbb{Z}$ $d(x, y) = 1$ iff $x \longrightarrow y$;





内部自己同型群が可換な等質カンドルについて

北大-理-数
齋藤琢弥

準備
$Q(X, A, d)$
定義
$\mathrm{Inn}(Q(X, A, d))$
例
等質性
等長変換
等質な集合からの構成
主結果
主結果
系
位数が小さい場合
参考文献

参考文献 I

[1] J. Scott Carter, Mohamed Elhamdadi, Marina Appiou Nikiforou, and Masahico Saito.
Extensions of quandles and cocycle knot invariants.
J. Knot Theory Ramifications, 12(6):725–738, 2003.

[2] J. Scott Carter, Seiichi Kamada, and Masahico Saito.
Diagrammatic computations for quandles and cocycle knot invariants.
In *Diagrammatic morphisms and applications (San Francisco, CA, 2000)*, volume 318 of *Contemp. Math.*, pages 51–74. Amer. Math. Soc., Providence, RI, 2003.

[3] W. Edwin Clark, Masahico Saito, and Leandro Vendramin.
Quandle coloring and cocycle invariants of composite knots and abelian extensions.
J. Knot Theory Ramifications, 25(5):1650024, 34, 2016.

[4] K Furuki and H Tamaru.
Flat homogeneous quandles and vertex-transitive graphs.
preprint.

[5] Akihiro Higashitani and Hirotake Kurihara.
Generalized alexander quandles of finite groups and their characterizations.
arXiv preprint arXiv:2210.16763, 2022.

参考文献 II

[6] Akihiro Higashitani and Hirotake Kurihara.
Homogeneous quandles arising from automorphisms of symmetric groups.
Communications in Algebra, pages 1–18, 2022.

[7] Yoshitaka Ishihara and Hiroshi Tamaru.
Flat connected finite quandles.
Proc. Amer. Math. Soc., 144(11):4959–4971, 2016.

[8] Přemysl Jedlička, Agata Pilitowska, David Stanovský, and Anna Zamojska-Dzienio.
The structure of medial quandles.
J. Algebra, 443:300–334, 2015.

[9] David Joyce.
A classifying invariant of knots, the knot quandle.
J. Pure Appl. Algebra, 23(1):37–65, 1982.

[10] Akira Kubo, Mika Nagashiki, Takayuki Okuda, and Hiroshi Tamaru.
A commutativity condition for subsets in quandles—a generalization of antipodal subsets.
In *Differential geometry and global analysis—in honor of Tadashi Nagano*, volume 777 of *Contemp. Math.*, pages 103–125. Amer. Math. Soc., [Providence], RI, [2022] ©2022.

内部自己同型群が可換な等質カンドルについて

北大-理-数
齋藤琢弥

準備
$Q(X, A, d)$
定義
$\mathrm{Inn}(Q(X, A, d))$
例
等質性
等長変換
等質な集合からの構成
主結果
主結果
系
位数が小さい場合
参考文献

参考文献 III

[11] Sam Nelson and Chau-Yim Wong.
On the orbit decomposition of finite quandles.
J. Knot Theory Ramifications, 15(6):761–772, 2006.

[12] Mituhisa Takasaki.
Abstraction of symmetric transformations.
Tôhoku Math. J., 49:145–207, 1943.

[13] Hiroshi Tamaru.
Two-point homogeneous quandles with prime cardinality.
J. Math. Soc. Japan, 65(4):1117–1134, 2013.

[14] Leandro Vendramin.
Doubly transitive groups and cyclic quandles.
J. Math. Soc. Japan, 69(3):1051–1057, 2017.

[15] Koshiro Wada.
Two-point homogeneous quandles with cardinality of prime power.
Hiroshima Math. J., 45(2):165–174, 2015.

参考文献 IV

[16] 田丸 博士.
対称空間の離散化とカンドル代数，Part I.
Geometry and Analysis 2014（福岡大学微分幾何研究会）記録集, pages 99–107, 2015.

[17] 田丸 博士.
対称空間の離散化とカンドル代数，Part II.
部分多様体論・湯沢 *2014* 記録集, pages 55–60, 2015.

[18] 田丸 博士.
対称空間の離散化とカンドル代数，Part III.
第 *35* 回代数的組合せ論シンポジウム記録集, pages 67–73, 2018.

[19] 田丸 博士.
対称空間の離散化とカンドル代数，Part IV.
部分多様体論・湯沢 *2018* 記録集, 2019.

[20] 田丸 博士.
対称空間の離散化とカンドル代数，Part V.
数理解析研究所講究録, 2210:57–65, 2022+.

Maximal antipodal sets of F_4 and FI

Sasaki Yuuki

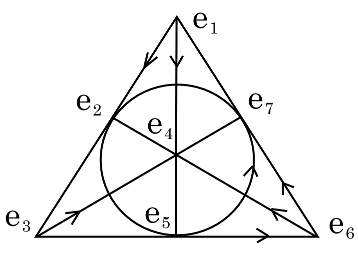
1 Introduction

Let M be a compact Riemannian symmetric space and denote the geodesic symmetry at $x \in M$ by s_x. In this paper, we assume that M is connected. If $s_x(y) = y$ for two points $x, y \in M$, we say that x, y are antipodal. A subset S of M is an antipodal set, if any two points of S are antipodal. The 2-number $\#_2 M$ of M is the maximum of the cardinalities of antipodal sets of M. We call an antipodal set S in M great if $\#S = \#_2 M$. An antipodal set S is called maximal if there are no antipodal sets including S properly. These notions were introduced by Chen-Nagano [1]. In general, any antipodal set of any Riemannian symmetric space of noncompact type is a one-point set, so we consider only compact symmetric spaces in this paper. It is known that antipodal sets are finite sets and 2-number is finite [1] [3]. In the present paper, we observe maximal antipodal sets of the exceptional compact Lie group F_4 and the compact symmetric space of FI type. Remark that FII is a symmetric R-space, so maximal antipodal sets of FII is already classfied by Tanaka- Tasaki [3].

2 Maximal antipodal sets of F_4

Let $\mathbb{O} = \bigoplus_{i=0}^{7} \mathbb{R}e_i$ be the octonions. The multiplicity of $\mathbb{O}$ is defined satisfying following: (1) e_0 is the unit element of this multiplicity, (2) $e_i^2 = -e_0$ and $e_i e_j = -e_j e_i$ for any $1 \leq i \neq j \leq 7$, (3) the multiplicity satisfies the distributive law, (4) the multiplicity among $e_1, \cdots, e_7$ is defined by Figure 1 (for example, $e_1 e_2 = e_3, e_2 e_3 = e_1$ and $e_3 e_1 = e_2$). Remark that the associative law does not follow in the octonions. For each

Figure 1:



$x = \sum_{i=0}^{7} x_i e_i \in \mathbb{O}$ ($x_i \in \mathbb{R}$), we set the conjugation $\overline{x} = x_0 e_0 - \sum_{i=1}^{7} x_i e_i$ of x. Let $y = \sum_{i=0}^{7} y_i e_i \in \mathbb{O}$ ($y_i \in \mathbb{R}$). Then, the standard inner product $(\ ,\ )_{\mathbb{O}}$ of $\mathbb{O}$ is defined by

$(x, y)_{\mathbb{O}} = \sum_{i=0}^{7} x_i y_i$. Let $SO(\mathbb{O})$ be the set of all isometric linear automorphisms of $\mathbb{O}$ whose determinant is 1. Then, the exceptional compact Lie group G_2 is given by

$$G_2 = \{g \in SO(\mathbb{O})\ ;\ g(xy) = g(x)g(y)\ (x, y \in \mathbb{O})\}.$$

The triality principle of $SO(\mathbb{O})$ is well known.

Proposition 2.1 (The triality principle of $SO(\mathbb{O})$). *For any $g_1 \in SO(\mathbb{O})$ there are $g_2, g_3 \in SO(\mathbb{O})$ such that*

$$\overline{(g_1x)(g_2y)} = g_3(\overline{xy}), \quad \overline{(g_2x)(g_3y)} = g_1(\overline{xy}), \quad \overline{(g_3x)(g_1y)} = g_2(\overline{xy}).$$

Moreover, such (g_2, g_3) are (g_2, g_3) or $(-g_2, -g_3)$.

Set D as follows:

$$D := \{(g_1, g_2, g_3) \in SO(\mathbb{O})^3\ ;\ g_1, g_2, g_3 \text{ satisfy the triality principle of } SO(\mathbb{O})\}.$$

Then, D is isomorphic to $Spin(8)$, so we denote D to $Spin(8)$. There are four totally geodesic embeddings $f_0, \cdots, f_3$ from G_2 to $Spin(8)$:

$$\begin{aligned} &f_0 : G_2 \to Spin(8)\ ;\ g \mapsto (\ \ g,\ \ g,\ \ g), && f_1 : G_2 \to Spin(8)\ ;\ g \mapsto (\ \ g, -, g, -g), \\ &f_2 : G_2 \to Spin(8)\ ;\ g \mapsto (-g,\ \ g, -g), && f_3 : G_2 \to Spin(8)\ ;\ g \mapsto (-g-, g,\ \ g). \end{aligned}$$

Let $M(3, \mathbb{O})$ be the set of all 3×3 matrices whose components are octonions. Set $\mathfrak{J} := \{X \in M(3, \mathbb{O})\ ;\ {}^t\overline{X} = X\}$ and we call $\mathfrak{J}$ the exceptional Jordan algebra. The Jordan product $\circ$ of $\mathfrak{J}$ is defined by $X \circ Y = \frac{1}{2}(XY + YX)$ $(X, Y \in \mathfrak{J})$. Then, the exceptional compact Lie group F_4 is defined by

$$F_4 = \{g \in \mathrm{Isom}(\mathfrak{J})\ ;\ g(X \circ Y) = g(X) \circ g(Y)\},$$

where $\mathrm{Isom}(\mathfrak{J})$ is the set of all automorphisms of $\mathfrak{J}$. Then, the following ϕ is an one-to-one homomorphism from $Spin(8)$ to F_4:

$$\phi(g_1, g_2, g_3) \begin{pmatrix} \xi_1 & x_3 & \bar{x_2} \\ \bar{x_3} & \xi_2 & x_1 \\ x_2 & \bar{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & g_3 x_3 & g_2 \bar{x}_2 \\ g_3 \bar{x}_3 & \xi_2 & g_1 x_1 \\ g_2 x_2 & g_1 \bar{x}_1 & \xi_3 \end{pmatrix} \quad (x_i \in \mathbb{O}, \xi_i \in \mathbb{R}, i = 1, 2, 3).$$

Theorem 2.2. *[2] Let Δ_{G_2} be any maximal antipodal sets of G_2 and $\Delta_{F_4} := \phi\big(f_0(\Delta_{G_2}) \cup \cdots \cup f_3(\Delta_{G_2})\big)$. Then, Δ_{F_4} is a maximal antipodal set of F_4 and any maximal antipodal set of F_4 is congruent to Δ_{F_4}. Moreover, $\#_2 F_4 = 32$ because $\#_2 G_2 = 8$.*

3 Maximal antipodal sets of FI

A subspace V of $\mathbb{O}$ satisfying $(vu)w = v(uw)$ $(v, u, w \in V)$ is called an associative submanifold and we call the set of all associative subspaces the associative Grassmann manifold. The associative Grassmann is denoted to $G_{\mathbb{H}}(\mathbb{O})$. For example, the standard quaternions $\mathbb{H}$ is an associative subspace of $\mathbb{O}$. It is known that $G_{\mathbb{H}}(\mathbb{O})$ is a compact symmetric space of G type. Let $G_4(\mathbb{O})$ be the set of all 4-dimensional subspaces of $\mathbb{O}$. Then, $G_{\mathbb{H}}(\mathbb{O}) \subset G_4(\mathbb{O})$ and this inclusion is totally geodesic. The following triality principle of $G_4(\mathbb{O})$ is true.

Lemma 3.1. *[2] For any $V_1 \in G_4(\mathbb{O})$, there are $V_2, V_3 \in G_4(\mathbb{O})$ such that*

$$\overline{v_1 v_2} \in V_3, \quad \overline{v_2 v_3} \in V_1, \quad \overline{v_3 v_1} \in V_2 \quad (v_i \in V_i,\ i = 1, 2, 3).$$

Moreover, (V_2, V_3) is (V_2, V_3) or $(V_2^{\perp}, V_3^{\perp})$.

Set $G_4^T(\mathbb{O})$ as follows.

$$G_4^T(\mathbb{O}) := \{(V_1, V_2, V_3) \in (G_4(\mathbb{O}))^3 \ ; \ \overline{v_i v_{i+1}} \in V_{i+2} \ (v_i \in V_i,\ i \text{ is mod3})\}.$$

We easily check that $Spin(8)$ acts on $G_4^T(\mathbb{O})$ transitively and $G_4^T(\mathbb{O}) \cong SO(8)/SO(4) \times SO(4)$. There are four totally geodesic embeddings $g_0, \cdots, g_3$ from $G_{\mathbb{H}}(\mathbb{O})$ to $G_4^T(\mathbb{O})$:

$$g_0 : G_{\mathbb{H}}(\mathbb{O}) \to G_4^T(\mathbb{O}) \ ; \ V \mapsto (\ \ V,\ \ V,\ \ V),\ g_1 : G_{\mathbb{H}}(\mathbb{O}) \to G_4^T(\mathbb{O}) \ ; \ V \mapsto (\ \ V, -, V, -V),$$
$$g_2 : G_{\mathbb{H}}(\mathbb{O}) \to G_4^T(\mathbb{O}) \ ; \ V \mapsto (-V,\ \ V, -V),\ g_3 : G_{\mathbb{H}}(\mathbb{O}) \to G_4^T(\mathbb{O}) \ ; \ V \mapsto (-V-, V,\ \ V).$$

Set $\mathfrak{J}_{\mathbb{H}} = \{X \in M(3, \mathbb{H}) \ ; \ {}^t\bar{X} = X\}$. Then, $\mathfrak{J}_{\mathbb{H}}$ is a subalgebra of $\mathfrak{J}$ with respect to $\circ$. Denote the set of all subalgebras of $\mathfrak{J}$ which are isomorphisc to $\mathfrak{J}_{\mathbb{H}}$ to $G_{\mathbb{H}}(\mathfrak{J})$. We see that F_4 acts on $G_{\mathbb{H}}(\mathfrak{J})$ transitively and $G_{\mathbb{H}}(\mathfrak{J})$ is a compact symmetric space of FI type. Then, the following ψ is a totally geodesic embedding from $G_4^T(\mathbb{O})$ to $G_{\mathbb{H}}(\mathfrak{J})$.

$$\psi : G_4^T(\mathbb{O}) \to G_{\mathbb{H}}(\mathfrak{J}) \ ; \ (V_1, V_2, V_3) \mapsto E + V_1 + V_2 + V_3,$$

where E is the subspace of all diagonal matrices of $\mathfrak{J}$.

Theorem 3.2. *[2] Let Δ_G be any maximal antipodal set of $G_{\mathbb{H}}(\mathbb{O})$ and $\Delta_{FI} := \psi\big(g_0(\Delta_G) \cup \cdots \cup g_3(\Delta_G)\big)$. Then, Δ_{FI} is a maximal antipodal set of $G_{\mathbb{H}}(\mathfrak{J})$ and any maximal antipodal set of $G_{\mathbb{H}}(\mathfrak{J})$ is isomorphic to Δ_{FI}. Moreover, $\#_2 FI = 28$ since $\# G_{\mathbb{H}}(\mathbb{O}) = 7$.*

References

[1] B.Y.Chen, T.Nagano, *A Riemannian geometric invariant and its applications to a problem of Borel and Serre*, Trans.Amer.Math.Soc, **308**(1988), 273-297

[2] Y.Sasaki, *Maximal Antipodal Sets of F_4 and FI*, Journal of Lie Theory, **32-1**(2022), 281-300

[3] M.S.Tanaka, H.Tasaki, *Antipodal sets of symmetric R-spaces*, Osaka J. Math., **50**(2013), 161-169

[4] I.Yokota, *Exceptional Lie groups*, arXiv:0902.043lvl

(Y.Sasaki) Department pf Liberal Arts, National Institute of Technology, Tokyo College, 1220-2, Kunugida-machi, Hachioji-shi, Tokyo, 193-0997, Japan

Email address: `y_sasaki@tokyo.kosen-ac.jp`

例外型コンパクトリー群 F_4 および FI 型コンパクト対称空間の対蹠集合

佐々木　優

東京工業高等専門学校

2022/12/09 研究集会「カンドルと対称空間」

例外型コンパクトリー群 F_4 および FI 型コンパクト対称空間の対蹠集合

佐々木　優

東京工業高等専門学校

2022/12/09 研究集会「カンドルと対称空間」

佐々木　優（東京高専）　F_4 および FI 型コンパクト対称空間の対蹠集合　1 / 46

今日の講演内容

本講演では,

「例外型コンパクトリー群 F_4」

「FI 型コンパクト対称空間」

の極大対蹠集合の分類・構成を紹介する

1. コンパクト対称空間の対蹠集合

対称空間

定義 2.1

リーマン多様体 M について，各点 $x \in M$ に対して次を満たす等長変換 s_x が存在するとき，M を対称空間という．

(1) x は s_x の孤立固定点である．

(2) s_x は対合的である ($s_x^2 = \mathrm{id}_M$).

s_x を x における点対称と呼ぶ．

カンドルについて

定義 2.2

X を集合とし，$* : X \times X \to X$ とする．このとき，$(X, *)$ がカンドルであるとは，以下を満たすこと．

(Q1) 任意の $x \in X$ について $x * x = x$.

(Q2) 任意の $x, y \in X$ について $z * x = y$ を満たす $z \in X$ がただ一つ存在する．

(Q3) 任意の $x, y, z \in X$ について $(x * y) * z = (x * z) * (y * z)$.

対称空間について

補題 2.3

M を対称空間とし g を M の等長変換とする．任意の $y, z \in M$ について

$$s_{g(y)}g = g\,s_y \qquad \text{とくに} \qquad s_{s_z(y)}s_z = s_z s_y$$

- M 上の 2 項演算 $*$ を次で定める．

$$x * y = s_y(x)$$

(Q1) $\underline{x * x = x}$
$x * x = s_x(x) = x$ より従う．

(Q2) $\underline{(\cdot) * x \text{ が全単射}}$
点対称 s_x が等長変換であることから従う．

(Q3) $\underline{(x * y) * z = (x * z) * (y * z)}$
$(x * y) * z = s_z s_y(x) = s_{s_z(y)} s_z(x) = (x * z) * s_z(y) = (x * z) * (y * z)$ より．

- したがって，対称空間における点対称は，カンドルの構造を与えている．

対蹠集合

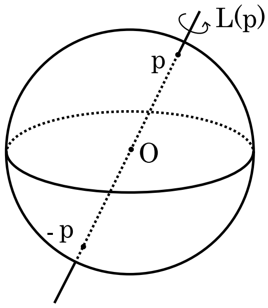
定義 2.4 (Chen-Nagano, 1988)

M を連結な対称空間とする．

- $p, q \in M$ が対蹠的 $\overset{\text{def}}{\Longleftrightarrow}$ $s_p(q) = q (\Longleftrightarrow s_q(p) = p)$.
- M の部分集合 S が対蹠集合 $\overset{\text{def}}{\Longleftrightarrow}$ S の任意の 2 点が対蹠的.
- 濃度が最大の対蹠集合を大対蹠集合と呼ぶ．
  大対蹠集合の濃度を M の 2-number といい，$\#_2 M$ とかく．
  対蹠集合間の包含関係に関して極大なものを，極大対蹠集合という．

- 以下，対称空間はコンパクトであると仮定する．
- 対蹠集合は常に有限集合になり，2-number は有限である．

例：球面 S^2



- $p \in S^2$ とし，$L(p)$ を中心 o と p を通る直線とする．
- p における点対称 s_p は $L(p)$ を回転軸とした 180 度回転となる．
- $\{x \in S^2\ ;\ s_p(x) = x\} = \{p, -p\}$ であり，$s_p = s_{-p}$ なので $s_{-p}(p) = p$．
  よって，$\{p, -p\}$ は S^2 の大対蹠集合となり，$\#_2 S^2 = 2$．

例：コンパクトリー群

G をコンパクトリー群とする．

- コンパクトリー群 G は，両側不変計量によりコンパクト対称空間になる．
- このとき，$g \in G$ における点対称は

$$s_g : G \to G\ ;\ h \mapsto gh^{-1}g.$$

- 単位元を含む極大対蹠集合は，各元の位数が 2 であるアーベル群で極大なもの (maximal elementary abelian 2-subgroup) になる．

  逆に，maximal elementary abelian 2-subgroup は極大対蹠集合になる．
- 例えば，ユニタリ群 $U(n)$ においては

$$\Delta_n = \left\{ \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \in U(n) \right\}$$

が大対蹠集合となっている．とくに，$\#_2 U(n) = 2^n$ となる．

対蹠集合の性質 1

定理 2.5 (Chen-Nagano, 1988)
$\#_2 M$ は M の不変量である．

- 2-number が異なる対称空間は，互いに同型にならない．
- $\#_2 S^1 = \#_2 S^2 = 2$ だが，S^1 と S^2 は同型にならない．

定理 2.6 (Chen-Nagano, 1988)
M をコンパクト対称空間とし，$\chi(M)$ を M のオイラー数とする．このとき，

$$\chi(M) \leq \#_2 M$$

対蹠集合の性質 2

定理 2.7 (Takeuchi, 1989)
M を対称 R 空間とする．このとき，大対蹠集合を臨界点集合とする $\mathbb{Z}_2$-perfect Morse 関数が存在し，

$$\#_2 M = \dim H_*(M\,;\,\mathbb{Z}_2).$$

定理 2.8 (Amman, 2021)
任意のコンパクト対称空間について

$$\#_2 M \leq \dim H_*(M\,;\,\mathbb{Z}_2)$$

$<$ と $=$ の違いはまだよくわかっていない…

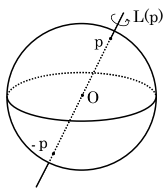
2. 対蹠集合の分類

極大対蹠集合の分類

全てのコンパクト対称空間で,

極大対蹠集合の分類・構成が完成しているわけではない！

- とくに，等長変換群の単位連結成分で移り合うものを同じとみなして (合同類)，極大対蹠集合を分類する.



- 既約コンパクト型対称空間は次のように分類されている．
- 各型について，極大対蹠集合の合同類の分類および構成がなされている．
- 対称 R 空間においては，極大対蹠集合の分類・構成は完成している．
（任意の極大対蹠集合は互いに合同になる）

古典型

I型	単連結	局所等長類	II 型	単連結	局所等長類
AI 型	$SU(n)/SO(n)$	いくつか	A型	$SU(n)$	いくつか
AII 型	$SU(2n)/Sp(n)$	いくつか			
$AIII$ 型	複素グラスマン多様体	1 or 2			
BDI 型	有向実グラスマン多様体	2 or 4	BD型	$Spin(n)$	2 or 4
$DIII$ 型	$SO(2n)/U(n)$	1 or 2			
CI 型	$Sp(n)/U(n)$	2	C型	$Sp(n)$	2
CII 型	四元数グラスマン多様体	1 or 2			

例外型

I 型	単連結	局所等長類	II 型	単連結	局所等長類
G 型	$G_2/SO(4)$	1	G_2型	G_2	1
FI 型	$\boxed{F_4/(Sp(1)\cdot Sp(3))}$	$\boxed{1}$	F_4型	$\boxed{F_4}$	$\boxed{1}$
FII 型	$F_4/Spin(9)$	1			
EI 型	$\boxed{E_6/(Sp(4)/\mathbb{Z}_2)}$	$\boxed{2}$	E_6型	$\boxed{E_6}$	$\boxed{2}$
EII 型	$\boxed{E_6/(U(1)\times Spin(10))/\mathbb{Z}_4}$	$\boxed{1}$			
$EIII$ 型	$\boxed{E_6/(Sp(1)\cdot SU(6))}$	$\boxed{1}$			
EIV 型	$\boxed{E_6/F_4}$	$\boxed{2}$			
EV 型	$E_7/(SU(8)/\mathbb{Z}_2)$	2	E_7型	E_7	2
EVI 型	$E_7/(SU(2)\cdot Spin(12))$	1			
$EVII$ 型	$E_7/((U(1)\times E_6)/\mathbb{Z}_3)$	2			
$EVIII$ 型	$E_8/Ss(16)$	1	E_8型	E_8	1
EIX 型	$E_8/SU(2)\cdot E_7$	1			

極大対蹠集合の構成

- 今回は，それぞれのコンパクト対称空間で分類結果を見るのではなく，全測地的部分多様体の観点から極大対蹠集合を考える．
- 対称空間の全測地的部分多様体は，「部分対称空間」になっている．

全測地的部分多様体の対蹠集合は，元の対称空間の対蹠集合になる．

	全測地的部分多様体		コンパクト対称空間
$f:$	N	$\to$	M
	N の極大対蹠集合 Δ		M の極大対蹠集合 $f(\Delta)$

3. F_4 の極大対蹠集合

次の包含列を考えていく．

$$G_2 \subset Spin(8) \subset F_4$$

例外型コンパクトリー群 G_2

定義 4.1

以下のようにして積を定めた 8 次元ベクトル空間 $\sum_{1=0}^{7}\mathbb{R}e_i$ を八元数 $\mathbb{O}$ という．

(1) e_0 は積の単位元とする．単に 1 と書く．

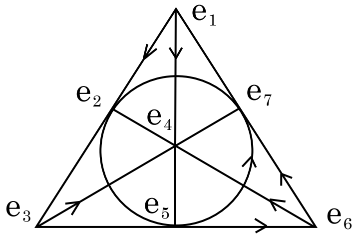
(2) 各 $1 \leq i \neq j \leq 7$ について，

$$e_i^2 = -1, e_ie_j = -e_je_i$$

(3) 積は分配法則を満たしている．

(4) 右の図により積を定める．

（例：$e_1e_2 = e_3, e_1e_4 = e_5$）



- $\mathrm{Im}\mathbb{O} = \sum_{i=1}^{7}\mathbb{R}e_i$ とおく．
- $x = \sum_{i=0}^{7} x_ie_i$ について，x の共役 $\bar{x}$ を，$\bar{x} = x_0 - \sum_{i=1}^{7} x_ie_i$ により定める．
- $\mathbb{O}$ の標準内積 $(\ ,\ )$ を以下で定める．

$$(x, y) = \frac{1}{2}(x\bar{y} + y\bar{x}) = \sum_{i=0}^{7} x_iy_i, \quad (x = \sum_{i=0}^{7} x_ie_i, y = \sum_{i=0}^{7} y_ie_i)$$

定義 4.2

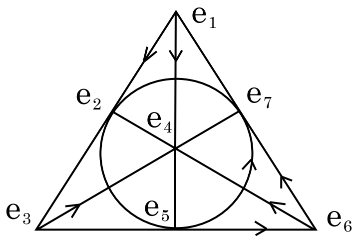
$\mathbb{O}$ の線形変換 f で $f(xy) = f(x)f(y)$ を満たすものを，$\mathbb{O}$ の自己同型という．
$\mathbb{O}$ の自己同型全体による群を例外型コンパクトリー群 G_2 という．

- 各 $g \in G_2$ について，$(g(x), g(y)) = (x, y) \ \ (x, y \in \mathbb{O})$
- 各 $g \in G_2$ について，$g(e_0) = e_0$．とくに，$g(\mathrm{Im}\mathbb{O}) \subset \mathrm{Im}\mathbb{O}$ となり，

$$G_2 \subset SO(\mathrm{Im}\mathbb{O}) = SO(7).$$

- $g_1, \cdots, g_7 \in G_2$ を次で定める．

$$\begin{aligned}
g_1 &= \mathrm{I}_{\mathbb{R}e_1+\mathbb{R}e_2+\mathbb{R}e_3} - \mathrm{I}_{\mathbb{R}e_4+\mathbb{R}e_5+\mathbb{R}e_6+\mathbb{R}e_7}\\
g_2 &= \mathrm{I}_{\mathbb{R}e_1+\mathbb{R}e_4+\mathbb{R}e_5} - \mathrm{I}_{\mathbb{R}e_2+\mathbb{R}e_3+\mathbb{R}e_6+\mathbb{R}e_7}\\
g_3 &= \mathrm{I}_{\mathbb{R}e_1+\mathbb{R}e_6+\mathbb{R}e_7} - \mathrm{I}_{\mathbb{R}e_2+\mathbb{R}e_3+\mathbb{R}e_4+\mathbb{R}e_5}\\
g_4 &= \mathrm{I}_{\mathbb{R}e_2+\mathbb{R}e_4+\mathbb{R}e_6} - \mathrm{I}_{\mathbb{R}e_1+\mathbb{R}e_3+\mathbb{R}e_5+\mathbb{R}e_7}\\
g_5 &= \mathrm{I}_{\mathbb{R}e_2+\mathbb{R}e_5+\mathbb{R}e_7} - \mathrm{I}_{\mathbb{R}e_1+\mathbb{R}e_3+\mathbb{R}e_4+\mathbb{R}e_6}\\
g_6 &= \mathrm{I}_{\mathbb{R}e_3+\mathbb{R}e_4+\mathbb{R}e_7} - \mathrm{I}_{\mathbb{R}e_1+\mathbb{R}e_2+\mathbb{R}e_5+\mathbb{R}e_6}\\
g_7 &= \mathrm{I}_{\mathbb{R}e_3+\mathbb{R}e_5+\mathbb{R}e_6} - \mathrm{I}_{\mathbb{R}e_1+\mathbb{R}e_2+\mathbb{R}e_3+\mathbb{R}e_4}
\end{aligned}$$



- g_0 を単位元とし，

$$\Delta_{G_2} = \{g_i \ ; \ 0 \le i \le 7\}.$$

定理 4.3 (Tanaka-Tasaki-Yasukura,2022)

Δ_{G_2} は G_2 の極大対蹠集合である．
また，G_2 の任意の極大対蹠集合は Δ_{G_2} と合同で，$\#_2 G_2 = 8$．

$SO(8)$- 3 対原理

以下，$SO(\mathbb{O})$ を $SO(8)$ と記す．

定理 4.4 ($SO(8)$-三対原理)

任意の $g_1 \in SO(8)$ に対して，$g_2, g_3 \in SO(8)$ で次を満たすものが符号を除いて一意に存在する．

$$g_i(\overline{xy}) = \overline{g_{i+1}(x)g_{i+2}(y)} \quad (x, y \in \mathbb{O},\ 添え字は\ \mathrm{mod}3)$$

- D を次で定める．

$$D := \left\{(g_1, g_2, g_3) \in SO(8)^3 \ ; \ \begin{matrix} \overline{g_i(x)g_{i+1}(y)} = g_{i+2}(\overline{xy}) \\ x, y \in \mathbb{O},\ 添え字は\ \mathrm{mod}3 \end{matrix} \right\}.$$

D はコンパクトリー群になり，次は 2 重被覆．

$$\pi : D \to SO(8) \ ; \ (g_1, g_2, g_3) \mapsto g_1$$
$$\pi^{-1}(g_1) = \{(g_1, g_2, g_3), (g_1, -g_2, -g_3)\}$$

- とくに，$D \cong Spin(8)$ となる．以下，D を $Spin(8)$ と記す．

$G_2 \subset Spin(8)$ について

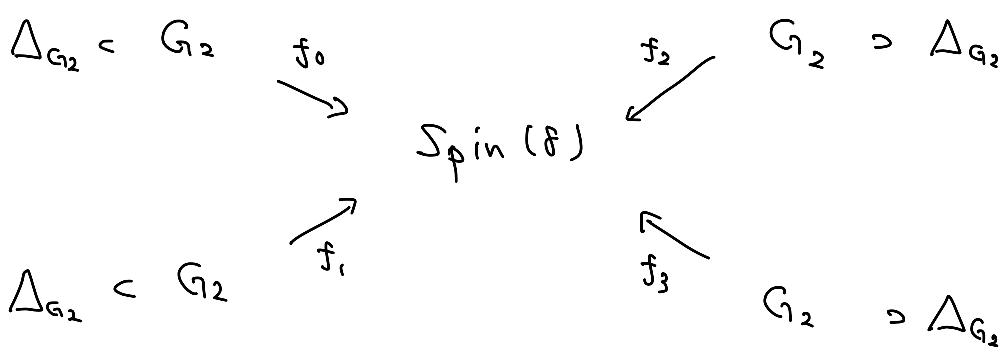
- G_2 から $Spin(8)$ へ，4 種類の全測地的埋め込み f_0, f_1, f_2, f_3 が存在する．

$$
\begin{aligned}
f_0 &: G_2 \to Spin(8)\ ;\ g \mapsto (\ \ g,\ \ g,\ \ g),\\
f_1 &: G_2 \to Spin(8)\ ;\ g \mapsto (\ \ g, -g, -g),\\
f_2 &: G_2 \to Spin(8)\ ;\ g \mapsto (-g,\ \ g, -g),\\
f_3 &: G_2 \to Spin(8)\ ;\ g \mapsto (-g, -g,\ \ g).
\end{aligned}
$$

- $\bigcup_{i=0}^{3} f_i(G_2)$ は $Spin(8)$ の部分群で，$G_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ と同型．

$Spin(8)$ の極大対蹠集合

- $\Delta_{Spin(8)} = \bigcup_{i=0}^{3} f_i(\Delta_{G_2})$ と定める．



補題 4.5 (Wood, 1989)

$\Delta_{Spin(8)} = \bigcup_{i=0}^{3} f(\Delta_{G_2})$ は $Spin(8)$ の極大対蹠集合．
$Spin(8)$ の任意の極大対蹠集合は $\Delta_{Spin(8)}$ と合同．とくに，$\#_2 Spin(8) = 32$．

F_4 について

- $\mathfrak{J}$ を次で定める

$$\mathfrak{J} = \{X \in M(3, \mathbb{O})\ ;\ {}^*X = X\} = \left\{ \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix}\ ;\ \xi_i \in \mathbb{R}, x_i \in \mathbb{O} \right\}$$

$X, Y \in \mathfrak{J}$ について，$X \circ Y = \frac{1}{2}(XY + YX)$ によりジョルダン積を定める．

- $(\mathfrak{J}, \circ)$ を例外ジョルダン代数という．

定義 4.6

$\mathfrak{J}$ の線形自己同型写像 f で，$f(X \circ Y) = f(X) \circ f(Y)$ を満たすもの全体による群を，例外型コンパクトリー群 F_4 として定める．

F_4 の極大対蹠集合

- $Spin(8)$ は次の ϕ により，F_4 の部分群とみなせる．

$$\phi : Spin(8) \to F_4\ ,\ \phi(g_1, g_2, g_3) \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & g_3(x_3) & \overline{g_2(x_2)} \\ \overline{g_3(x_3)} & \xi_2 & g_1(x_1) \\ g_2(x_2) & \overline{g_1(x_1)} & \xi_3 \end{pmatrix}.$$

定理 4.7 (S)

$\Delta_{F_4} = \phi(\Delta_{Spin(8)})$ とすれば，Δ_{F_4} は F_4 の極大対蹠集合となる．
また，任意の極大対蹠集合は Δ_{F_4} と合同である．とくに，$\#_2 F_4 = 32$.

- F_4 では，全測地部分多様体の列

$$G_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \subset Spin(8) \subset F_4$$

において，極大対蹠集合が極大対蹠集合として含まれている．

4. FI 型の極大対蹠集合

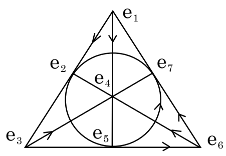
次の包含列を考えていく．

$$G_{\mathbb{H}}(\mathbb{O}) \subset \tilde{G}_4(\mathbb{R}^8) \subset FI$$

(結合的グラスマン多様体)　　(有向実グラスマン多様体)

結合的グラスマン多様体 $G_{\mathbb{H}}(\mathbb{O})$

- $\mathbb{O}$ では結合則が成り立たない．
  - $(e_1e_2)e_4 = e_3e_4 = e_7$
  - $e_1(e_2e_4) = e_1(-e_6) = -e_7$



- しかし，$\mathbb{H} = \mathbb{R}e_0 \oplus \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$ などでは結合則が成り立つ．
  - $(e_1e_2)e_3 = e_3e_3 = -1$
  - $e_1(e_2e_3) = e_1e_1 = -1$
- $\mathbb{O}$ の 4 次元部分空間 V で．任意の $u, v, w \in V$ について

$$(uv)w = u(vw)$$

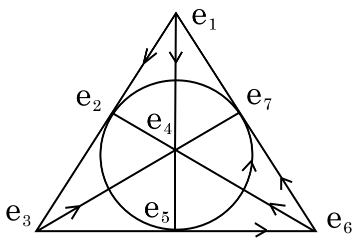
が成り立つものを，結合的部分空間という．

定義 5.1
結合的部分空間全体を $G_{\mathbb{H}}(\mathbb{O})$ と記し，結合的グラスマン多様体という．

$G_{\mathbb{H}}(\mathbb{O})$ の極大対蹠集合

- とくに，$G_{\mathbb{H}}(\mathbb{O})$ は G 型コンパクト対称空間.
- $\mathbb{H}_1, \cdots, \mathbb{H}_7 \in G_{\mathbb{H}}(\mathbb{O})$ を次で定める.

$$\begin{aligned}
\mathbb{H}_1 &= \mathbb{R}e_0 \oplus \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3\\
\mathbb{H}_2 &= \mathbb{R}e_0 \oplus \mathbb{R}e_1 \oplus \mathbb{R}e_4 \oplus \mathbb{R}e_5\\
\mathbb{H}_3 &= \mathbb{R}e_0 \oplus \mathbb{R}e_1 \oplus \mathbb{R}e_6 \oplus \mathbb{R}e_7\\
\mathbb{H}_4 &= \mathbb{R}e_0 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_4 \oplus \mathbb{R}e_6\\
\mathbb{H}_5 &= \mathbb{R}e_0 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_5 \oplus \mathbb{R}e_7\\
\mathbb{H}_6 &= \mathbb{R}e_0 \oplus \mathbb{R}e_3 \oplus \mathbb{R}e_4 \oplus \mathbb{R}e_7\\
\mathbb{H}_7 &= \mathbb{R}e_0 \oplus \mathbb{R}e_3 \oplus \mathbb{R}e_5 \oplus \mathbb{R}e_6
\end{aligned}$$



補題 5.2 (Tanaka-Tasaki-Yasukura, 2022)

$\Delta_{G_{\mathbb{H}}(\mathbb{O})} = \{\mathbb{H}_1, \cdots, \mathbb{H}_7\}$ は，$G_{\mathbb{H}}(\mathbb{O})$ の極大対蹠集合.
また，任意の極大対蹠集合は $\Delta_{G_{\mathbb{H}}(\mathbb{O})}$ と合同である．とくに，$\#_2 G_{\mathbb{H}}(\mathbb{O}) = 7$.

$G_4(\mathbb{O})$ の 3 対原理

- $\mathbb{O}$ の 4 次元部分空間全体を $G_4(\mathbb{O})$ と記す.

系 5.3

任意の $V_1 \in G_4(\mathbb{O})$ に対して，$V_2, V_3 \in G_4(\mathbb{O})$ で次を満たすものが存在する.

$$\overline{v_i v_{i+1}} \in V_{i+2} \quad (v_i \in V_i,\ \text{添え字は } \mathrm{mod}3).$$

さらに，そのような (V_2, V_3) は $(V_2, V_3), (V_2^{\perp}, V_3^{\perp})$ に限る.

- $G_4^T(\mathbb{O})$ を次で定める.

$$G_4^T(\mathbb{O}) := \left\{ (V_1, V_2, V_3) \in \big(G_4(\mathbb{O})\big)^3 \ ; \ \begin{array}{l} \overline{v_i v_{i+1}} \in V_{i+2}, \\ v_i \in V_i,\ \text{添え字は } \mathrm{mod}3 \end{array} \right\}$$

有向実グラスマン多様体 $G_4^T(\mathbb{O})$

- 次は 2 重被覆である．

$$\pi : G_4^T(\mathbb{O}) \to G_4(\mathbb{O}) ; (V_1, V_2, V_3) \mapsto V_1$$
$$\pi^{-1}(V_1) = \{(V_1, V_2, V_3), (V_1, V_2^\perp, V_3^\perp)\}$$

- $G_4^T(\mathbb{O})$ は $\mathbb{R}^8$ の向き付き 4 次元部分空間全体による有向実グラスマン多様体とみなせる．

$G_4^T(\mathbb{O})$ の極大対蹠集合

- $G_{\mathbb{H}}(\mathbb{O})$ から $G_4^T(\mathbb{O})$ へ，4 種類の全測地的埋め込み $g_0, \cdots, g_3$ が存在する．

$$g_0 : G_{\mathbb{H}}(\mathbb{O}) \to G_4^T(\mathbb{O}) ; V \mapsto (V, V, V),$$
$$g_1 : G_{\mathbb{H}}(\mathbb{O}) \to G_4^T(\mathbb{O}) ; V \mapsto (V, V^\perp, V^\perp),$$
$$g_2 : G_{\mathbb{H}}(\mathbb{O}) \to G_4^T(\mathbb{O}) ; V \mapsto (V^\perp, V, V^\perp),$$
$$g_3 : G_{\mathbb{H}}(\mathbb{O}) \to G_4^T(\mathbb{O}) ; V \mapsto (V^\perp, V^\perp, V),$$

補題 5.4 (Tasaki,2014)

$\Delta_{G_4^T(\mathbb{O})} = \bigcup_{i=0}^3 g_i(\Delta_{G_{\mathbb{H}}(\mathbb{O})})$ は，$G_4^T(\mathbb{O})$ の極大対蹠集合．

任意の極大対蹠集合は $\Delta_{G_4^T(\mathbb{O})}$ と合同である．とくに，$\#_2 G_4^T(\mathbb{O}) = 28$．

FI 型コンパクト対称空間 $G_{\mathbb{H}}(\mathfrak{J})$

- $\mathfrak{J}$ のジョルダン積 $\circ$ に関する部分代数 $\mathfrak{J}_{\mathbb{H}}$ を次で定める.

$$\mathfrak{J}_{\mathbb{H}} = \left\{ \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} ; \xi_i \in \mathbb{R}, x_i \in \mathbb{H} \right\}$$

定義 5.5

$\mathfrak{J}$ の $\circ$-部分代数 V で，$\mathfrak{J}_{\mathbb{H}}$ と同型であるもの全体を $G_{\mathbb{H}}(\mathfrak{J})$ と記す.

- F_4 はジョルダン積 $\circ$ に関する同型なので，$G_{\mathbb{H}}(\mathfrak{J})$ に作用している.

補題 5.6 (S)

F_4 が $G_{\mathbb{H}}(\mathfrak{J})$ に推移的に作用し，$G_{\mathbb{H}}(\mathfrak{J})$ は FI 型コンパクト対称空間.

$G_4^T(\mathbb{O}) \subset G_{\mathbb{H}}(\mathfrak{J})$ について

- $G_4^T(\mathbb{O})$ から $G_{\mathbb{H}}(\mathfrak{J})$ への全測地的埋め込み ψ が存在する.

$$\psi : G_4^T(\mathbb{O}) \to G_{\mathbb{H}}(\mathfrak{J}) \ ; \ (V_1, V_2, V_3)$$
$$\mapsto \left\{ \begin{pmatrix} x_1 & v_3 & \bar{v_2} \\ \bar{v_3} & x_2 & v_1 \\ v_2 & \bar{v_1} & x_1 \end{pmatrix} ; \begin{array}{l} x_i \in \mathbb{R} \\ v_i \in V_i \ (i = 1, 2, 3) \end{array} \right\}$$

- $\Delta_{FI} = \psi(\Delta_{G_4^T(\mathbb{O})})$ とおく.

$G_{\mathbb{H}}(\mathfrak{J})$ の極大対蹠集合

定理 5.7 (S)

Δ_{FI} は，$G_{\mathbb{H}}(\mathfrak{J})$ の極大対蹠集合．
また，任意の極大対蹠集合は Δ_{FI} と合同である．とくに，$\#_2 FI = 28$．

- FI 型では，全測地的部分多様体の列

$$G_{\mathbb{H}}(\mathbb{O}) \sqcup \cdots \sqcup G_{\mathbb{H}}(\mathbb{O}) \quad \subset \quad G_4^T(\mathbb{O}) \quad \subset \quad G_{\mathbb{H}}(\mathfrak{J})$$

  において，極大対蹠集合が極大対蹠集合として含まれている．
- この包含列は，四元数ケーラー多様体の包含列にもなっている．

5. F_4 における極地と対蹠集合

極地について

定義 6.1

対称空間 M および $x \in M$ に対して，$F(s_x, M) = \{y \in M \,;\, s_x(y) = y\}$ と定める．
$F(s_x, M)$ の各連結成分を，x の極地という．
1 点集合になる極地を，極と呼ぶ．

- $\{x\}$ も，x の極地になっている．
  $\{x\}$ を x の自明な極などという
- 極地は全測地的部分多様体になることが知られている．
- x の (x 以外の) 極地を $M_1^+, \cdots, M_k^+$ とすれば

$$F(s_x, M) = \{x\} \sqcup M_1^+ \sqcup \cdots \sqcup M_k^+.$$

- x を含む対蹠集合の各点は，x の極地たちに振り分けられる．

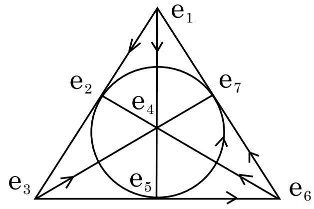
G_2 の極地たち

G_2 および G_2 の極地たちの極地は次のよう．

- G_2　:　自明な極，G 型
- G 型　:　自明な極，$S^2 \cdot S^2$
- $S^2 \cdot S^2$　:　自明な極，極が 1 つ，$S^1 \cdot S^1$
- $S^1 \cdot S^1$　:　自明な極，極が 3 つ

G_2 の極大対蹠集合 Δ_{G_2} の各点が，どの極地に含まれていくのかを観察する．

$\mathbb{O}$ における，$\mathbb{R}e_0 \oplus \mathbb{R}e_i \oplus \mathbb{R}e_j \oplus \mathbb{R}e_k$ に関する鏡映を r_{ijk} と記す．



G_2 の極地 ：自明な極，G 型

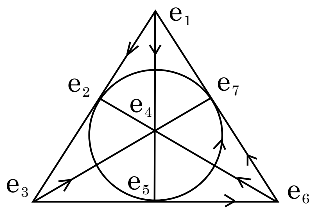
G型

e $\quad r_{123} \quad r_{145} \quad r_{167} \quad r_{246} \quad r_{257} \quad r_{346} \quad r_{356}$

G 型の極地 ：自明な極，$S^2 \cdot S^2$

$S^2 \cdot S^2$

$r_{123} \quad r_{145} \quad r_{167} \quad r_{246} \quad r_{257} \quad r_{346} \quad r_{356}$



$S^2 \cdot S^2$ の極地 ：自明な極，極が 1 つ，$S^1 \cdot S^1$

$S^1 \cdot S^1$

極

$r_{123} \quad r_{145} \quad r_{167} \quad r_{246} \quad r_{257} \quad r_{346} \quad r_{356}$

1 点を指定すると，これにより定まる複素構造 (e_i たちのいずれか) に対応した点が極になる

$S^1 \cdot S^1$ の極地 ：自明な極，極が 3 つ

極

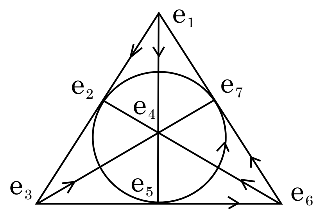
$r_{246} \quad r_{257} \quad r_{346} \quad r_{356}$

F_4 の極地たち

F_4 および F_4 の極地たちの極地は次のよう．

- F_4 : 自明な極, FII, FI
- FI : 自明な極, $\mathbb{H}P^2$, $S^2 \cdot CI(3)$
- $S^2 \cdot CI(3)$: 自明な極, 極が 1 つ, $\mathbb{C}P^2$ が 2 つ, $S^1 \cdot UI(3)$
- $S^1 \cdot UI(3)$: 自明な極, 極が 3 つ, $\mathbb{R}P^2$ が 3 つ

以下，F_4 の極大対蹠集合 Δ_{F_4} の各点が，どの極地に含まれていくのかを観察する．



F_4 の極地 : 自明な極, FII, FI

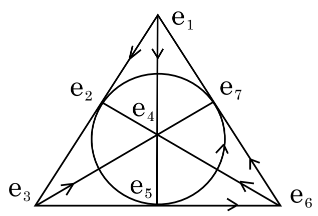
FII　FI

$f_0(e)$	$f_2(e)$	$f_0(r_{123})$	$f_2(r_{123})$	$f_0(r_{145})$	$f_2(r_{145})$	$f_0(r_{167})$	$f_2(r_{167})$
$f_1(e)$	$f_3(e)$	$f_1(r_{123})$	$f_3(r_{123})$	$f_1(r_{145})$	$f_3(r_{145})$	$f_1(r_{167})$	$f_3(r_{167})$
$f_0(r_{246})$	$f_2(r_{246})$	$f_0(r_{257})$	$f_2(r_{257})$	$f_0(r_{347})$	$f_2(r_{347})$	$f_0(r_{356})$	$f_2(r_{356})$
$f_1(r_{246})$	$f_3(r_{246})$	$f_1(r_{257})$	$f_3(r_{257})$	$f_1(r_{347})$	$f_3(r_{347})$	$f_1(r_{356})$	$f_3(r_{356})$

- 1 点を指定すれば，同じ組のその他の点たち → FII 型の極地へ
  その他の点たち → FI 型の極地へ

FI の極地：

自明な極, $\mathbb{H}P^2$, $S^2 \cdot CI(3)$



$\mathbb{H}P^2$ $S^2 \cdot CI(3)$

		$f_0(r_{123})$	$f_2(r_{123})$	$f_0(r_{145})$	$f_2(r_{145})$	$f_0(r_{167})$	$f_2(r_{167})$
		$f_1(r_{123})$	$f_3(r_{123})$	$f_1(r_{145})$	$f_3(r_{145})$	$f_1(r_{167})$	$f_3(r_{167})$
$f_0(r_{246})$	$f_2(r_{246})$	$f_0(r_{257})$	$f_2(r_{257})$	$f_0(r_{347})$	$f_2(r_{347})$	$f_0(r_{356})$	$f_2(r_{356})$
$f_1(r_{246})$	$f_3(r_{246})$	$f_1(r_{257})$	$f_3(r_{257})$	$f_1(r_{347})$	$f_3(r_{347})$	$f_1(r_{356})$	$f_3(r_{356})$

- 1 点を指定すれば，同じ組のその他の点たち $\to \mathbb{H}P^2$ 型の極地へ
  その他の点たち $\to S^2 \cdot CI(3)$ 型の極地へ

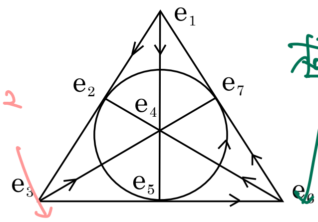
$S^2 \cdot CI(3)$ の極地：

自明な極，　極が 1 つ，

$\mathbb{C}P^2$ が 2 つ，　$S^1 \cdot UI(3)$

極

$\mathbb{C}P^2$ $\mathbb{C}P^2$



$S^1 \cdot UI(3)$

		$f_0(r_{123})$	$f_2(r_{123})$	$f_0(r_{145})$	$f_2(r_{145})$	$f_0(r_{167})$	$f_2(r_{167})$
		$f_1(r_{123})$	$f_3(r_{123})$	$f_1(r_{145})$	$f_3(r_{145})$	$f_1(r_{167})$	$f_3(r_{167})$
$f_0(r_{246})$	$f_2(r_{246})$	$f_0(r_{257})$	$f_2(r_{257})$	$f_0(r_{347})$	$f_2(r_{347})$	$f_0(r_{356})$	$f_2(r_{356})$
$f_1(r_{246})$	$f_3(r_{246})$	$f_1(r_{257})$	$f_3(r_{257})$	$f_1(r_{347})$	$f_3(r_{347})$	$f_1(r_{356})$	$f_3(r_{356})$

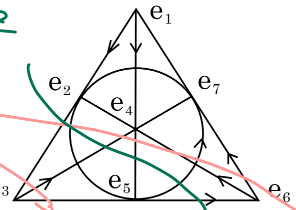
- 1 点を指定すれば，同じ組のその他の点たち $\to \mathbb{C}P^2$ 型の極地へ
  複素構造で対応する 1 点 $\to$ 極へ
  上記の 1 点と同じ組の点たち $\to \mathbb{C}P^2$ 型の極地へ
  その他の点たち $\to S^1 \cdot UI(3)$ 型の極地へ

$S^1 \cdot U l(3)$ の極地 :

自明な極，極が 3 つ，$\mathbb{R}P^2$ が 4 つ

$\mathbb{R}P^2$

極



$f_0(r_{246})$	$f_2(r_{246})$	$f_0(r_{257})$	$f_2(r_{257})$	$f_0(r_{347})$	$f_2(r_{347})$	$f_0(r_{356})$	$f_2(r_{356})$
$f_1(r_{246})$	$f_3(r_{246})$	$f_1(r_{257})$	$f_3(r_{257})$	$f_1(r_{347})$	$f_3(r_{347})$	$f_1(r_{356})$	$f_3(r_{356})$

F_4 の極大対蹠集合は，G_2 の極大対蹠集合から構成することができたが，
極地による分解に関しても G_2 と類似する性質を有している．

本日のまとめ

- 対称空間はカンドルの一種である．
- 対称空間では，対蹠集合と呼ばれる有限離散集合が定義される．
- F_4, FI 型の極大対蹠集合の構成を紹介した．
  - F_4 の極大対蹠集合の合同類の数は 1 つ
    $G_2 \subset Spin(8) \subset F_4$ の包含列から構成ができる．
  - FI 型の極大対蹠集合の合同類の数は 1 つ
    $G_2/SO(4) \subset SO(8)/SO(4) \times SO(4) \subset FI$ の包含列から構成できる．
- F_4 の極大対蹠集合の，極地による分解の様子を紹介した．
  - G_2 の場合と類似した性質を有していた．

ご清聴ありがとうございました！

Generalized s-manifolds and compact symmetric triads

Shinji Ohno

Abstract. In this paper, we define generalized s-manifolds as a generalization of Riemann symmetric spaces and construct examples using compact symmetric triads. The content of this paper is based on joint research with Professor Takashi Sakai of Tokyo Metropolitan University.

1 Introduction

Riemann symmetric spaces are a class of Riemann manifolds introduced by Cartan, including Euclidean spaces, unit hyperspheres, real hyperbolic spaces, Grassmann manifolds, and compact Lie groups. This is a well-known class that plays a fundamental role in differential geometry.

For Riemannian symmetric spaces, curvature, fundamental groups, classification theory, etc. are described in terms of Lie groups and Lie algebras. For compact Riemann symmetric spaces, special subsets defined by point symmetries such as polar, meridian and antipodal sets define invariants. The relation between these invariants and topology is known ([1, 2]).

It is natural to consider generalizations of Riemann symmetric spaces for the purpose of extending these results. There are two main directions of generalization. One is the generalization of geodesic symmetry, and the other is the generalization of Riemann symmetric pairs. By generalizing point symmetries, symmetric spaces ([9, 7]), Riemannian s-maifolds, regular s-manifolds, etc. ([3, 4, 5, 6]) are obtained. From the generalization of symmetric pairs, symmetric Γ-symmetric spaces ([8]) are obtained. In order to deal with these generalized notions of symmetric spaces in a unified way, we define a generalized s manifold as follows.

Definition 1.1. Let M be a C^∞-manifold and Γbe a group. Then M is called a generalized s-manifold if for each point $x \in M$, there exists a group homomorphism $\varphi_x : \Gamma \to \mathrm{Diff}(M)$ such that

(1) For $\gamma \in \Gamma$, the map $\mu^\gamma : M \times M \to M;\ (x, y) \mapsto \varphi_x(\gamma)(y)$ is C^∞. If $\Gamma$is a Lie group, then $\mu : \Gamma \times M \times M \to M;\ (\gamma, x, y) \mapsto \varphi_x(\gamma)(y)$ is C^∞.

(2) For each $x \in M$, $x$is an isolated fixed point of the action of $\varphi_x(\Gamma)$ on M.

(3) For each $x, y \in M,\ \gamma, \delta \in \Gamma$, $\varphi_x(\gamma) \circ \varphi_y(\delta) \circ \varphi_x(\gamma)^{-1} = \varphi_{\varphi_x(\gamma)(y)}(\gamma\delta\gamma^{-1})$ holds.

symmetric spaces ares a generalized s-manifolds when $\Gamma = \mathbb{Z}_2$. Therefore, the generalized s-manifold is an extension of the symmetric space. More generally, we also find that k-symmetric spaces have the generalized s-structure when $\Gamma = \mathbb{Z}_k$. Using compact symmetric triads, we can construct examples such that Γ is not a commutative group.

2 Compact symmetric triads

Using compact symmetric triads, we can construct examples such that Γ is not a commutative group. Let G be a compact semisimple Lie group and θ_1, θ_2 be involutive automorphisms of G. Fix $(G^{\theta_i})_0 \subset K_i \subset G^{\theta_i} = \{k \in G \mid \theta_i(k) = k\}$ for each $i = 1, 2$. (That is, (G, K_1) and (G, K_2) are compact symmetric pairs.) Then (G, K_1, K_2) is called a compact symmetric triad.

Proposition 2.1. *Let (G, K_1, K_2) be a compact symmetric triad and set $H = K_1 \cap K_2$. Denotes Γ be the group generated by θ_1 and θ_2.*

For $\gamma \in \Gamma, g, g' \in G$, define

$$\varphi_{gH}(\gamma)(g'H) = g\gamma(g^{-1}g')H.$$

Then $(G/H, \Gamma, \{\varphi_x\}_{x \in G/H})$ is a generalized s-manifolds.

Remark 2.2.

- If $\theta_1 = \theta_2$, then G/H is a symmetric space.
- If $\theta_1\theta_2 = \theta_2\theta_1$, then G/H is a Γ-symmetric space.
- If $\theta_1\theta_2 \neq \theta_2\theta_1$, then G/H is a generalized s-manifold which is not a Γ-symmetric space.

Corollary 2.3. *Let (G, K) be a compact symmetric pair. For each $g \in G$, $G/(K \cap \mathrm{I}_g(K))$ is a generalized s-manifold. Here I_g is the inner automorphism with respect to g.*

If g is the identity element, then G/H is a symmetric space. By replacing g, we can construct nontrivial examples of generalized s manifolds. By using compact symmetric triads, we can also construct k-symmetric spaces and $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces.

References

[1] B.-Y. Chen and T. Nagano, *Totally geodesic submanifolds of symmetric spaces. II*, Duke Math. J. **45** (1978), no. 2, 405–425.

[2] B.-Y. Chen and T. Nagano, *A Riemannian geometric invariant and its applications to a problem of Borel and Serre*, Trans. Amer. Math. Soc. **308** (1988), 273–297.

[3] O. Kowalski, *Generalized symmetric Riemannian spaces*, Period. Math. Hungar. **8** (1977), no. 2, 181–184.

[4] O. Kowalski, *Smooth and affine s-manifolds*, Period. Math. Hungar. **8** (1977), no. 3–4, 181–184.

[5] A. J. Ledger, *Espaces de Riemann symétriques généralisés*, C. R. Acad. Sci. Paris Sér. A-B **264**, (1967), A947–A948.

[6] A. J. Ledger and M. Obata, *Affine and Riemannian s-manifolds*, J. Differential Geometry, **2** (1968), 451–459.

[7] O. Loos, *Symmetric spaces. I: General theory*, W. A. Benjamin, Inc., New York-Amsterdam 1969 viii+198 pp.

[8] R. Lutz, *Sur la géométrie des espaces Γ-symétriques*, C. R. Acad. Sci. Paris Sér. I Math. **293**, (1981), no. 1, 55–58.

[9] Nagano, Tadashi *Geometric theory of symmetric spaces. (Japanese)*, Geometry of submanifolds (Japanese) (Kyoto, 2001). Sūrikaisekikenkyūsho Kōkyūroku No. 1206 (2001), 55–82.

(Shinji Ohno) Department of Mathematics, College of Humanities and Sciences, Nihon University, 3-25-40, Sakurajosui, Setagaya-ku, Tokyo, 156-8550, Japan

Email address: `ohno.shinji@nihon-u.ac.jp`

一般化された s 多様体とコンパクト対称三対

大野 晋司 (日本大学)

カンドルと対称空間 2022

酒井 高司 (東京都立大学), 寺内 泰紀との共同研究

2022 年 12 月 9 日

目次

1. 対称空間とその一般化概念
2. 一般化された s 多様体の例
3. 対蹠集合

Riemann 対称空間

Def (Riemann 対称空間)

(M, g): (連結な)Riemann 多様体,
各 $x \in M$ に対して等長変換 $s_x : M \to M$ が定まっていて,

1. $s_x \circ s_x = \mathrm{id}_M$
2. $\forall x \in M$ に対して, x は s_x の孤立固定点.

を満たすとき, (M, g) を Riemann 対称空間と呼ぶ.

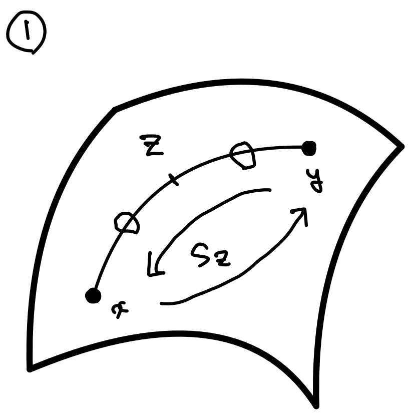
例

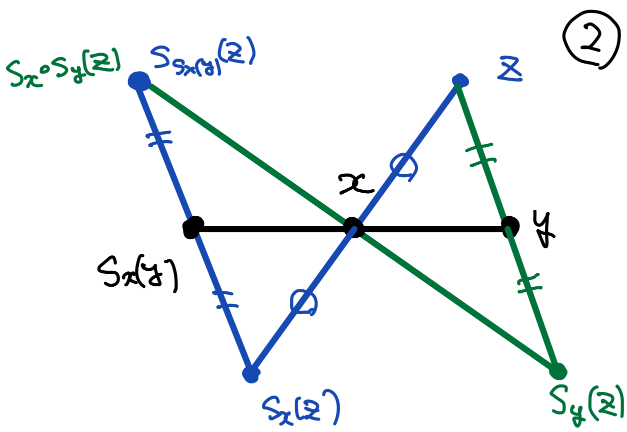
$(\mathbb{R}^n, \{s_x\}_{x\in\mathbb{R}^n})$ ただし $s_x(y) = -y + 2x$
$(S^n, \{s_x\}_{x\in S^n})$ ただし $s_x(y) = -y + 2\langle x, y\rangle x$
$(\mathbb{R}P^n, \{s_x\}_{x\in\mathbb{R}P^n})$

Fact

(M, g): Riemann 対称空間

1. $\mathrm{Isom}(M, g)$ の M への作用は推移的.
2. $\forall x, y \in M$ に対して, $s_x \circ s_y = s_{s_x(y)} \circ s_x$





Def (対称空間 (Loos, Nagano))

M: C^∞ 級多様体,
$\mu : M \times M \to M$: C^∞ 級. $x \in M$ に対して,
$s_x(y) := \mu(x, y) \quad (y \in M)$ で定める.

1. $s_x \circ s_x = \mathrm{id}_M$
2. $\forall x \in M$ に対して, x は s_x の孤立固定点.
3. $\forall x, y \in M$ に対して, $s_x \circ s_y = s_{s_x(y)} \circ s_x$

を満たすとき, $(M, \{s_x\}_{x \in M})$ を対称空間と呼ぶ.

Def (対称対)

G:連結 Lie 群. K: G の閉部分群. θ: G の対合的自己同型.
$(G^\theta)_0 \subset K \subset G^\theta = \{k \in G \mid \theta(k) = k\}$
このとき, (G, K) を対称対と呼ぶ.

Def (regular な s 多様体 (Kowalski))

M: C^∞ 級多様体, $\mu : M \times M \to M$: C^∞ 級. $x \in M$ に対して, $s_x(y) := \mu(x, y) \quad (y \in M)$ で定める. 次を満たす時 M を regular な s 多様体という.

1. $\forall x \in M$ について, s_x は微分同相写像.
2. $\forall x \in M$ に対して, x は s_x の孤立固定点.
3. $\forall x, y \in M$ に対して, $s_x \circ s_y = s_{s_x(y)} \circ s_x$

Def (Γ 対称空間 (Lutz))

Γ: 可換な有限群. G:連結 Lie 群. K: G の閉部分群.
$\rho : \Gamma \to \mathrm{Aut}(G)$: 単射な群準同型 s.t.
$(G^\Gamma)_0 \subset K \subset G^\Gamma = \{g \in G \mid \rho(\gamma)(g) = g \ (\gamma \in \Gamma)\}$
このとき, (G, K) を Γ 対称対と呼び, G/K を Γ 対称空間と呼ぶ.

Riemann 対称空間の一般化概念

1. 局所 Riemann 対称空間
2. affine 対称空間，局所 affine 対称空間
3. 対称空間 (Loos, Nagano)
4. 擬 Riemann 対称空間
5. Riemanniann s-manifold, affine s-manifold, regular s-manifold, tangentially regular s-manifold, k-symmetric space, (Ledger, Obata, Kowalski, ...)
6. Γ 対称空間 (Lutz)
7. weakly symmetric space (Serberg)

一般化された s 多様体

Def (一般化された s 多様体 (O-Sakai))

M: C^∞ 級多様体，Γ: 群.
各 $x \in M$ に対して群準同型 $\varphi_x : \Gamma \to \mathrm{Diff}(M)$ が定まっていて，

1. $\forall x, \forall y \in M$, $\forall \gamma, \forall \delta \in \Gamma$,
$$\varphi_x(\gamma) \circ \varphi_y(\delta) \circ \varphi_x(\gamma)^{-1} = \varphi_{\varphi_x(\gamma)(y)}(\gamma\delta\gamma^{-1}).$$
2. $\forall x \in M$ に対して，x は $\varphi_x(\Gamma)$ の M への作用の孤立固定点.
3. 各 $\gamma \in \Gamma$ について $\mu_\gamma : M \times M \to M; (x, y) \mapsto \varphi_x(\gamma)(y)$ は C^∞ 級.

を満たすとき，$(M, \Gamma, \{\varphi_x\}_{x \in M})$ を一般化された s 多様体と呼び，$(\Gamma, \{\varphi_x\}_{x \in M})$ を M の一般化された s 構造と呼ぶ.

Remark

- 各 φ_x は単射でなくてもよい.
- 各 φ_x が単射のとき.
  - Γ が巡回群または $\mathbb{Z}$ の時, $(M, \Gamma, \{\varphi_x\}_{x \in M})$ は regular な s 多様体と呼ばれる (O. Kowalski, A, J. Ledger, M. Obata ....).
  - $\Gamma = \mathbb{Z}_k$ のとき, $(M, \Gamma, \{\varphi_x\}_{x \in M})$ は k 対称空間.
  - $\Gamma = \mathbb{Z}_2$ のとき, $(M, \Gamma, \{\varphi_x\}_{x \in M})$ は対称空間.

例

$M = \mathbb{R}^2, \Gamma = O(2)$ とし,

$$\varphi_x(g)(y) = g(y - x) + x \quad (g \in O2, y \in \mathbb{R}^2)$$

とすれば. $(\mathbb{R}^2, O(2), \{\varphi_x\}_{x \in M})$ は一般化された s 多様体.
($O(2)$ は線対称で生成される群)

Γ 対称空間について

G/K: Γ 対称空間
各 $g \in G, \gamma \in \Gamma$ について,

$$\varphi_{gK}(\gamma)(g'K) = g\gamma(g^{-1}g')K \quad (g' \in G)$$

と定義すると, $(G/K, \Gamma, \{\varphi_x\}_{x \in G/K})$ は一般化された s 多様体.

Remark

- Γ 対称空間は一般化された s 多様体.
- Lutz の定義では Γ は可換群.
- Γ を可換とは限らない有限群としても一般化された s 構造を定める (寺内).

一般化された s 多様体の例

- 対称空間
- k 対称空間
- regular な s 多様体
- Γ 対称空間
- いくつかの R 空間 (cf. P. Quast-T. Sakai) $(\Gamma = (\mathbb{Z}_2)^r)$
- Kähler C-space $(\Gamma = \mathbb{Z}_k, (\mathbb{Z}_2)^r, Z(K) \cong T^r)$
- $(\mathbb{R}^n, \mathrm{GL}(n;\mathbb{R}), \{\varphi_x\}_{x\in\mathbb{R}^n})$
  ただし，$x \in \mathbb{R}^n$，$g \in \mathrm{GL}(n;\mathbb{R})$ に対して，

$$\varphi_x(g)(y) = x + g(y - x) \qquad (y \in \mathbb{R}^n)$$

旗多様体

$\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ とする．
$n_1 + n_2 + \cdots + n_r < n$ を満たす，$n, n_1, \ldots, n_r \in \mathbb{N}$ に対して，旗多様体 $F_{n_1,\ldots,n_r}(\mathbb{K}^n)$ を

$$F_{n_1,\ldots,n_r}(\mathbb{K}^n) = \left\{ f = (V_1, \ldots, V_r) \;\middle|\; \begin{array}{l} \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{K}^n \text{ 部分空間} \\ \dim V_i = n_1 + \cdots + n_i \ (i = 1, \ldots, r) \end{array} \right\}$$

で定める．

旗多様体

各 $f = (V_1, \ldots, V_r)$ に対して，$s_{V_i} := 2P_{V_i} - \mathrm{Id}_{\mathbb{K}^n}$
$(i = 1, \ldots r)$ とおく．ただし P_{V_i} は $\mathbb{K}^n$ から V_i への直交射影．
$s_{V_1}, \ldots, s_{V_r}$ で生成される群を Γ とおくと，$\Gamma \cong (\mathbb{Z}_2)^r$.
この Γ に関して，旗多様体 $F_{n_1,\ldots,n_r}(\mathbb{K}^n)$ は一般化された s 多様体となる．

2 つの対称対からできる一般化された s 多様体

Def (コンパクト対称三対)

G: コンパクト半単純 Lie 群，θ_1, θ_2:G の対合的自己同型．
$(G^{\theta_i})_0 \subset K_i \subset G^{\theta_i} = \{k \in G \mid \theta_i(k) = k\}$ を一つずつ固定する．(つまり，(G, K_1) と (G, K_2) はコンパクト対称対)
このとき，(G, K_1, K_2) をコンパクト対称三対と呼ぶ

G/K_i は，$g, g' \in G$ について，

$$s_{gK_i}(g'K_i) = g\theta_i(g^{-1}g)K_i$$

で $s_{gK_i} : G/K_i \to G/K_i$ を定めると，G/K_i は対称空間．
K_2 の G/K_1 への自然な作用と，K_1 の G/K_2 を Hermann 作用と呼ぶ．
コンパクト型既約対称空間への余等質性 2 以上の超極作用は Hermann 作用と軌道同値．

$$\begin{tikzcd}
 & G \arrow[ddl] \arrow[d] \arrow[ddr] & \\
 & G/(K_1 \cap K_2) \arrow[dl] \arrow[dr] & \\
G/K_1 & & G/K_2
\end{tikzcd}$$

Prop

コンパクト対称三対 (G, K_1, K_2) について，$\{\theta_1, \theta_2\}$ で生成される群を Γ とおき，$H = K_1 \cap K_2$ とおく．

$\gamma \in \Gamma, g, g' \in G$ について，

$$\varphi_{gH}(\gamma)(g'H) = g\gamma(g^{-1}g')H$$

とすれば $(G/H, \Gamma, \{\varphi_x\}_{x \in G/H})$ は一般化された s 多様体．

Proof. 次の 3 条件をチェックすれば良い．

1. $\forall x, \forall y \in M$, $\forall \gamma, \forall \delta \in \Gamma$,
$$\varphi_x(\gamma) \circ \varphi_y(\delta) \circ \varphi_x(\gamma)^{-1} = \varphi_{\varphi_x(\gamma)(y)}(\gamma\delta\gamma^{-1}).$$
2. $\forall x \in M$ に対して，x は $\varphi_x(\Gamma)$ の M への作用の孤立固定点．
3. 各 $\gamma \in \Gamma$ について $\mu_\gamma : M \times M \to M; (x, y) \mapsto \varphi_x(\gamma)(y)$ は C^∞ 級．

$o = eH \in G/H$ とおく.
$\gamma, \delta \in \Gamma, g, g' \in G$ について,

$$\begin{aligned}
&\varphi_o(\gamma) \circ \varphi_{gH}(\delta) \circ \varphi_o(\gamma)^{-1}(g'H)\\
=&\varphi_o(\gamma) \circ \varphi_{gH}(\delta)(\gamma^{-1}(g')H)\\
=&\varphi_o(\gamma)(g\delta(g^{-1}\gamma^{-1}(g'))H)\\
=&\gamma(g\delta(g^{-1}\gamma^{-1}(g')))H\\
=&\gamma(g)\gamma \circ \delta(g^{-1}\gamma^{-1}(g')))H\\
=&\gamma(g)(\gamma \circ \delta \circ \gamma^{-1}(\gamma(g)^{-1}g'))))H\\
=&\varphi_{\gamma(g)H}(\gamma\delta\gamma^{-1})(g'H)\\
=&\varphi_{\varphi_o(\gamma)(gH)}(\gamma\delta\gamma^{-1})(g'H)
\end{aligned}$$

$T_o(G/H) \cong \mathfrak{m}_1 + \mathfrak{m}_2$ とみなせば,
o の十分小さい近傍 U において,

$$\exists X \in \mathfrak{m}_1 + \mathfrak{m}_2 \text{ s.t. } x = \exp(X)H \quad (x \in U).$$

$i = 1, 2$ について, $\varphi_o(\theta_i)(x) = x$ であれば

$$\varphi_o(\theta_i)(\exp(X)H) = \exp(\theta_i(X))H$$

よって,

$$X = \theta_1(X) = \theta_2(X) \in \mathfrak{k}_1 \cap \mathfrak{k}_2 \cap (\mathfrak{m}_1 + \mathfrak{m}_2) = \{0\}$$

したがって o は $F(\varphi_x(\Gamma), M)$ の孤立点.

Remark

- $\theta_1 = \theta_2$ の場合，G/H は対称空間．
- $\theta_1\theta_2 = \theta_2\theta_1$(あるいは $\theta_1\theta_2$ の位数が有限) の場合，G/H は Γ 対称空間．
- $\theta_1\theta_2$ の位数が無限の場合は G/H は Γ 対称空間でない一般化された s 多様体．

Cor

(G, K):コンパクト対称対．
$g \in G$ について，$G/(K \cap \mathrm{I}_g(K))$ は一般化された s 多様体

$\theta_1\theta_2 \neq \theta_2\theta_1$ の場合

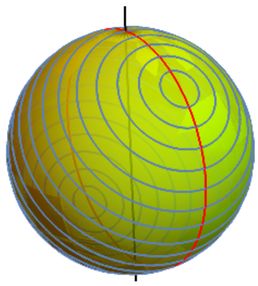
$G = \mathrm{SO}(n)$ $(n \geq 3)$，$t \in \mathbb{R}$ について，
$\theta_1 = \mathrm{I}_{I_{1,n-1}}, \theta_2 = \mathrm{I}_{g_t}\theta_1\mathrm{I}_{g_t}^{-1}$ とおく．
ただし，

$$I_{m,n-m} = \begin{bmatrix} -I_m & 0 \\ 0 & I_{n-m} \end{bmatrix}, g_t = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix}.$$

$t \notin \pi\mathbb{Z}$ について，コンパクト対称三対
$(G, K_1, K_2) = (\mathrm{SO}(n), \mathrm{SO}(n-1), \mathrm{I}_{g_t}(\mathrm{SO}(n-1)))$
を考える．$(t \in \pi\mathbb{Z} \Rightarrow \theta_1 = \theta_2)$

$t \notin \pi\mathbb{Z} \Rightarrow K_1 \cap K_2 \cong \mathrm{SO}(n-2)$
$G/K_1 = S^{n-1}, G/K_2 = S^{n-1}$,
$G/K_1 \cap K_2 = \mathrm{SO}(n)/\mathrm{SO}(n-2)$

$$
\begin{tikzcd}
& K_1/H \arrow[d] \arrow[dr] & \\
K_2/H \arrow[r] \arrow[dr] & G/H \arrow[r] \arrow[d] & G/K_2 \\
& G/K_1 &
\end{tikzcd}
$$



対蹠集合

Chen-Nagano はコンパクト Riemann 対称空間に対して，極地と対蹠集合の概念を導入し，対蹠集合の濃度の上限として，2-number と呼ばれる不変量を定義した．
対蹠集合はコンパクト Hermite 対称空間の実形の交差としてもあらわれる．2-number はコンパクト連結対称空間のオイラー数とも関係している．
これらの諸概念を一般化された s 多様体に拡張する．

対蹠集合

Def（対蹠集合，対蹠数）

$A \subset M$ とする．

- A が M の対蹠集合 $\iff \forall x, \forall y \in A, \forall \gamma \in \Gamma$,
$$\varphi_x(\gamma)(y) = y, \quad \varphi_y(\gamma)(x) = x.$$
- A が M の極大対蹠集合 $\iff \forall A'$: M の対蹠集合，$A \subset A' \Rightarrow A = A'$.
- 対蹠集合の濃度の上限を $\#_\Gamma(M)$ で表し，M の対蹠数と呼ぶ．
- A が M の大対蹠集合 $\iff$ A は対蹠集合で，$\#_\Gamma(M) = \#(A)$ を満たす．

$\Gamma = \mathbb{Z}_2$，つまり M が対称空間であるとき，$\#_\Gamma(M)$ を $\#_2(M)$ と表す．

部分空間

Def（部分空間）

$(M, \Gamma, \{\varphi_x\}_{x\in M})$:一般化された s 多様体，
$N \subset M$:部分多様体とする．
N:M の部分空間
$\iff \forall \gamma \in \Gamma, x \in N,\ \varphi_x(\gamma)(N) = N$.

Proposition

- $(M, \Gamma, \{\varphi_x\}_{x\in M})$ の部分空間 N はまた一般化された s 多様体である．
- N が M の部分空間であれば，$\#_\Gamma(M) \geq \#_\Gamma(N)$

Proof.
$A \subset N$: 対蹠集合とすると，A は M の対蹠集合．

例

$\{x\} \subset \mathbb{R}^n$ は $(\mathbb{R}^n, \{s_x\}_{x\in\mathbb{R}^n})$ の大対蹠集合.
$\{x, -x\} \subset S^n$ は $(S^n, \{s_x\}_{x\in S^n})$ の大対蹠集合.
$\{[e_1], [e_2], \dots, [e_{n+1}]\}$ は $(\mathbb{R}P^n, \{s_x\}_{x\in\mathbb{R}P^n})$ の大対蹠集合.

Corollary

$n \geq 2$ のとき,
$(\mathbb{R}P^n, \{s_x\}_{x\in\mathbb{R}P^n})$ は $(S^m, \{s_x\}_{x\in S^m})$ $(m > 1)$ の部分空間にならない.

極地

Def (極地)

$x \in M$ に対して,

$$F(\varphi_x(\Gamma), M) := \{y \in M \mid \varphi_x(\gamma)(y) = y \quad (\gamma \in \Gamma)\}$$

の連結成分を M の x における極地と呼ぶ. 特に一点からなる極地を極と呼ぶ.
$\{x\}$ は定義から M の x における極である. $\{x\}$ を自明な極と呼ぶ.

Proposition

Γ が可換であるとき, 一般化された s 多様体の極地は部分空間

Theorem

Γ を有限可換群，または可換なコンパクト Lie 群とする．
$A \subset M$ を M の大対蹠集合とする．
$x \in A$ について，M の x における極地が高々加算個であれば，
$F(\varphi_x(\Gamma), M)$ は加算個の極地の非交和

$$F(\varphi_x(\Gamma), M) = \bigcup_{i=0}^{\infty} M_i^+$$

と表され，

$$\#_\Gamma(M) \leq \sum_{i=0}^{\infty} \#_\Gamma(M_i^+)$$

が成り立つ．

コンパクト連結対称空間の場合はさらに次の定理が知られている．

Theorem(Chen-Nagano)

$(M, \{s_x\}_{x \in M})$:コンパクト連結対称空間とし，$\chi(M)$ で M のオイラー標数を表す．このとき，

$$\chi(M) \leq \#_2(M)$$

Theorem(Takeuchi)

$(M, \{s_x\}_{x \in M})$:対称 R 空間とし，$H(M, \mathbb{Z}_2)$ で M の $\mathbb{Z}_2$ 係数のホモロジー群を表す．このとき，

$$\dim H_*(M, \mathbb{Z}_2) = \#_2(M)$$

$F_{1,2}(\mathbb{R}^5)$ について

$M = F_{1,2}(\mathbb{R}^5) = \{(l_1, V_2) \mid l_1 \subset V_2\}$ について，
$f_0 := (\mathrm{Span}(e_1), \mathrm{Span}(e_1, e_2, e_3)) \in F_{1,2}(\mathbb{R}^5)$ とおく．このとき，$F(\varphi_{f_0}(\Gamma), M)$ は

$$F(\varphi_{f_0}(\Gamma), M) = M_0^+ \cup \cdots M_{10}^+$$

と 11 個の極地の非交和に分解される．

以下，簡単のために，$\mathrm{Span}(e_{n_1}, \ldots, e_{n_k}) = (n_1, \ldots, n_k)$ と表す．(たとえば，$f_0 = ((1), (1, 2, 3))$)

$$M_0^+ = \{((1), (1, 2, 3))\}, M_1^+ = \{((1), (1, 4, 5))\}$$
$$M_2^+ = \{(1), (1) \oplus l_1 \oplus l_2) \mid l_1 \in G_1((1, 2)), l_2 \in G_1((4, 5))\}$$
$$M_3^+ = \{(V_1, (1, 2, 3) \mid V_1 \in G_1((2, 3))\}$$
$$M_4^+ = \{(V_1, V_1 \oplus (4, 5)) \in M \mid V_1 \in G_1((2, 3))\}$$
$$M_5^+ = \{(V_1, (1) \oplus V_1 \oplus l) \mid V_1 \in G_1((2, 3)), l \in G_1((4, 5))\}$$
$$M_6^+ = \{(V_1, (2, 3) \oplus l) \mid V_1 \in G_1((2, 3)), l \in G_1((4, 5))\}$$
$$M_7^+ = \{(V_1, (1, 4, 5)) \mid V_1 \in G_1((4, 5))\}$$
$$M_8^+ = \{(V_1, (2, 3) \oplus V_1) \mid V_1 \in G_1((4, 5))\}$$
$$M_9^+ = \{(V_1, l \oplus (4, 5)) \mid l \in G_1((2, 3)), V_1 \in G_1((4, 5))\}$$
$$M_{10}^+ = \{(V_1, (1) \oplus l \oplus V_1) \mid l \in G_1((2, 3)), V_1 \in G_1((4, 5))\}$$

Theorem

(1) 旗多様体 $F_{n_1,\dots,n_r}(\mathbb{K}^n)$ の極大対蹠集合は

$$A = \{(\langle e_{i_1}, \dots, e_{i_{n_1}}\rangle_{\mathbb{K}}, \langle e_{i_1}, \dots, e_{i_{n_1+n_2}}\rangle_{\mathbb{K}}, \dots, \langle e_{i_1}, \dots, e_{i_{n_1+\cdots+n_r}}\rangle_{\mathbb{K}}) \\ \mid 1 \le i_1 < \cdots < i_{n_1} \le n,\ 1 \le i_{n_1+1} < \cdots < i_{n_1+n_2} \le n, \dots, \\ 1 \le i_{n_1+\cdots+n_{r-1}+1} < \cdots < i_{n_1+\cdots+n_r} \le n, \\ \#\{i_1, \dots, i_{n_1+\cdots+n_r}\} = n_1 + \cdots + n_r\}$$

と合同になる．ここで，$e_1, \dots, e_n$ は $\mathbb{K}^n$ の標準基底である．

(2)

$$\#_\Gamma(F_{n_1,\dots,n_r}(\mathbb{K}^n)) = \frac{n!}{n_1!n_2!\cdots n_r!n_{r+1}!}.$$

ただし，$n_{r+1} = n - (n_1 + \cdots + n_r)$.

Remark

$\mathbb{K} = \mathbb{C}$ のとき，$F_{n_1,\dots,n_r}(\mathbb{K}^n)$ には先に定めた，$\Gamma = (\mathbb{Z}_2)^r$ の他にも，$\Gamma = \mathbb{Z}_k$, $Z(G_X)$ などの一般化された s 構造が入る．これらの一般化された s 構造は一般には異なるが，極大対蹠集合は一致する．極大対蹠集合は $\mathrm{SU}(n+1)$ の Weyl 群の軌道となる．

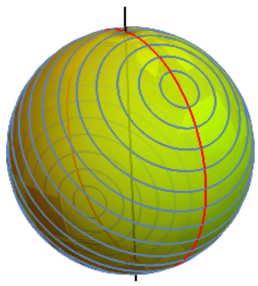
さらに，Sánchez の結果と合わせると次の系を得る．

Corollary

$$\#_\Gamma(F_{n_1,\dots,n_r}(\mathbb{K}^n)) = \frac{n!}{n_1!n_2!\cdots n_r!n_{r+1}!} \\ = \dim H_*(F_{n_1,\dots,n_r}(\mathbb{K}^n); \mathbb{Z}_2).$$

$\mathrm{SO}(n)/\mathrm{SO}(n-2)$ の場合

$$\begin{tikzcd}
& K_1/H \arrow[d] \arrow[dr] & \\
K_2/H \arrow[r] \arrow[dr] & G/H \arrow[r] \arrow[d] & G/K_2 \\
& G/K_1 &
\end{tikzcd}$$



$\mathrm{SO}(n)/\mathrm{SO}(n-2)$ の $o := e(K_1 \cap K_2)$ 対蹠集合を考える．
射影 $\pi_1 : \mathrm{SO}(n)/\mathrm{SO}(n-2) \to \mathrm{SO}(n)/\mathrm{SO}(n-1)$ について，$\pi_1(o)$ を含む $\mathrm{SO}(n)/\mathrm{SO}(n-1) \cong S^{n-1}$ の大対蹠集合は，$\{\pi_1(o), \pi_1(o')\}$．
ただし，$o' = I_{2,n-2} \cdot K_1 \cap K_2$
従って，$\mathrm{SO}(n)/\mathrm{SO}(n-2)$ の o を含む対蹠集合は，

$$\pi_1^{-1}(\{\pi_1(o), \pi_1(o')\}) = K_1 \cdot o \cup \mathrm{I}_{I_{2,n-2}}(K_1) \cdot o'$$

の部分集合．
$F(\varphi_o(\theta_2), G/K_1 \cap K_2) \cap \pi_1^{-1}(\{\pi_1(o), \pi_1(o')\}) = \{o, o'\}$
であるから，$\mathrm{SO}(n)/\mathrm{SO}(n-2)$ の o を含む対蹠集合は,$\{o, o'\}$
に含まれる．

$\theta_1\theta_2 = \theta_2\theta_1$

$$
\begin{tikzcd}
G/K_3 & K_1/H \arrow[l] \arrow[d] \arrow[dr] & \\
K_2/H \arrow[u] \arrow[r] \arrow[dr] & G/H \arrow[ul] \arrow[r] \arrow[d] & G/K_2 \\
 & G/K_1 & K_3/H \arrow[l] \arrow[u] \arrow[ul]
\end{tikzcd}
$$

$F_{1,2}(\mathbb{R}^5)$

$$
\begin{tikzcd}
G_2(\mathbb{R}^5) & G_2(\mathbb{R}^4) \arrow[l] \arrow[d] \arrow[dr] & \\
\mathbb{R}P^2 \arrow[u] \arrow[r] \arrow[dr] & F_{1,2}(\mathbb{R}^5) \arrow[ul] \arrow[r] \arrow[d] & G_3(\mathbb{R}^5) \\
 & \mathbb{R}P^4 & \mathbb{R}P^2 \arrow[l] \arrow[u] \arrow[ul]
\end{tikzcd}
$$

今後の課題

- 一般化された s 多様体の分類
- 等質性
- Γ を取り変えたときの構造の変化
- 対蹠集合とトポロジーの関係

Symmetric Clifford systems and quandle structures on real Grassmannian manifolds

Shinobu FUJII

Abstract. In this article, we introduce subspace arrangements associated with symmetric Clifford systems. We show some properties of these arrangements and quandles generated by the arrangements.

1 Symmetric Clifford systems

We denote by $\mathrm{Sym}_{2n}(\mathbb{R})$ a set of real symmetric matrices of degree $2n$.

Definition 1.1. Let m and n be positive integers. A *symmetric Clifford system on* $\mathbb{R}^{2n}$ is a finite subset $\{P_0, P_1, \ldots, P_m\} \subsetneq \mathrm{Sym}_{2n}(\mathbb{R})$ satisfying

$$P_iP_j + P_jP_i = 2\delta_{ij}I_{2n} \quad \text{for any} \quad i \quad \text{and} \quad j \in \{0, 1, \ldots, m\}. \tag{1.1}$$

It is known that there exists a one-to-one correspondence between symmetric Clifford systems and representations of real Clifford algebras (cf. Cecil [1]).

Definition 1.2. Let $\mathcal{P} = \{P_0, P_1, \ldots, P_m\}$ and $\mathcal{Q} = \{Q_0, Q_1, \ldots, Q_m\} \subsetneq \mathrm{Sym}_{2n}(\mathbb{R})$ be symmetric Clifford systems on $\mathbb{R}^{2n}$.

1. $\mathcal{P}$ and $\mathcal{Q}$ are said to be *algebraically equivalent* if there exists an orthogonal transformation $A \in \mathrm{O}(2n)$ such that $Q_i = AP_i\,{}^tA$ for every $i \in \{0, 1, \ldots, m\}$,
2. $\mathcal{P}$ and $\mathcal{Q}$ are said to be *geometrically equivalent* if there exists an orthogonal transformation $B \in \mathrm{O}(\mathrm{span}\{P_0, P_1, \ldots, P_m\})$ such that $B\mathcal{P}$ is algebraically equivalent to $\mathcal{Q}$. Here, $B\mathcal{P} := \{BP_0, BP_1, \ldots, BP_m\}$.

2 Quandle structures on real Grassmannian manifolds

For a set X, $\mathrm{Map}(X, X)$ denotes a set of all maps from X to X.

Definition 2.1. For a set X, a map $s : X \to \mathrm{Map}(X, X)$, $x \mapsto s_x$ is called a *quandle structure* on X if it satisfies that

1. $s_x(x) = x$ for any $x \in X$,
2. s_x is bijective for any $x \in X$,
3. $s_x \circ s_y = s_{s_x(y)} \circ s_x$ for any x and $y \in X$.

Then, a pair (X, s) is called a *quandle* simply.

Let $(\mathbb{R}^{2n}, \langle -,\ - \rangle_{2n})$ be the standard Euclidean space of dimension $2n$, and $\mathrm{Gr}_n^{2n}(\mathbb{R})$ be a real Grassmannian manifold of all linear subspaces of dimension n in $\mathbb{R}^{2n}$. For any $V \in \mathrm{Gr}_n^{2n}(\mathbb{R})$, we denote its orthogonal complement by $V^\perp$, and an orthogonal projections to V by $\pi_V : \mathbb{R}^{2n} \to V$. Then, a *reflection at V* $\sigma_V : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is defined by a linear transformation which acts on V as the identity and acts on $V^\perp$ as multiplication by -1, that is,

$$\sigma_V = \mathrm{id}_V - \mathrm{id}_{V^\perp}\,. \tag{2.1}$$

For any $V \in \mathrm{Gr}_n^{2n}(\mathbb{R})$, the reflection σ_V at V induces a *symmetry* s_V on $\mathrm{Gr}_n^{2n}(\mathbb{R})$ at V in natural way. Thus, $s_V : \mathrm{Gr}_n^{2n}(\mathbb{R}) \to \mathrm{Gr}_n^{2n}(\mathbb{R})$ can be written as

$$s_V : \mathrm{Gr}_n^{2n}(\mathbb{R}) \to \mathrm{Gr}_n^{2n}(\mathbb{R}); \quad W \mapsto s_V(W), \tag{2.2}$$

where $s_V(W)$ is a linear subspace in $\mathbb{R}^{2n}$ defined by

$$s_V(W) := \{\sigma_V(w) \mid w \in W\}\,. \tag{2.3}$$

Then, a map $s : \mathrm{Gr}_n^{2n}(\mathbb{R}) \to \mathrm{Map}(\mathrm{Gr}_n^{2n}(\mathbb{R}), \mathrm{Gr}_n^{2n}(\mathbb{R}))$; $V \mapsto s_V$ gives a quandle structure on $\mathrm{Gr}_n^{2n}(\mathbb{R})$.

3 Clifford arrangements

In this section, we consider the case of a standard inner product for $\mathbb{R}^{2n}$.

Definition 3.1 (F. [2])**.** Let n and m be positive integers.

1. A finite subset $\{V_0, V_1, \ldots, V_m\} \subseteq \mathrm{Gr}_n^{2n}(\mathbb{R})$ is called a *Clifford arrangement* in $\mathbb{R}^{2n}$ if we have $s_{V_i}(V_j) = V_j^\perp$ for $i \neq j$.
2. Two Clifford arrangements $\{V_0, V_1, \ldots, V_m\}$ and $\{W_0, W_1, \ldots, W_m\}$ in $\mathbb{R}^{2n}$ are said to be *equivalent* if there exists an orthogonal transformation $A \in \mathrm{O}(2n)$ such that $W_i = AV_i$ for every $i \in \{0, 1, \ldots, m\}$.

Example 3.2. We consider a Clifford arrangement in $\mathbb{R}^2$ with the standard inner product $\langle \cdot, \cdot \rangle_2$. Let $\{e_1, e_2\}$ be the orthonormal basis of $\mathbb{R}^2$. We define two lines ℓ_0 and ℓ_1 as follows:

$$\ell_0 = \mathrm{span}_{\mathbb{R}}\{e_1\}, \quad \ell_1 = \mathrm{span}_{\mathbb{R}}\{e_1 + e_2\}. \tag{3.1}$$

Then, $\mathcal{L} := \{\ell_0, \ell_1\}$ is a Clifford arrangement in $\mathbb{R}^2$.

Example 3.3. We consider a Clifford arrangement in $\mathbb{R}^4$ with the standard inner product $\langle \cdot, \cdot \rangle_4$. Let $\{e_1, e_2, e_3, e_4\}$ be the orthonormal basis of $\mathbb{R}^4$. We define three subspaces V_0, V_1 and V_2 in $\mathbb{R}^4$ of dimension two as follows:

$$V_0 = \mathrm{span}_{\mathbb{R}}\{e_1, e_2\}, \quad V_1 = \mathrm{span}_{\mathbb{R}}\{e_1 + e_3, e_2 + e_4\}, \quad V_2 = \mathrm{span}_{\mathbb{R}}\{e_1 - e_4, e_2 + e_3\}. \tag{3.2}$$

Then, $\mathcal{V} := \{V_0, V_1, V_2\}$ is a Clifford arrangement in $\mathbb{R}^4$.

Theorem 3.4 (F. [2])**.** *Let $\mathcal{V} = \{V_0, V_1, \ldots, V_m\}$ be a Clifford arrangement in $\mathbb{R}^{2n}$. For each $i \in \{0, 1, \ldots, m\}$, we define P_i as a $2n \times 2n$-matrix corresponding to reflection σ_{V_i} with respect to the fixed orthonormal basis of $\mathbb{R}^{2n}$. Then, we have*

1. *$\mathcal{P} := \{P_0, P_1, \ldots, P_m\}$ is a symmetric Clifford system on $\mathbb{R}^{2n}$.*

2. *Another Clifford arrangement $\mathcal{W} = \{W_0, W_1, \ldots, W_m\}$ in $\mathbb{R}^{2n}$ is equivalent to $\mathcal{V}$ if and only if the corresponding Clifford systems are algebraically equivalent.*

For a Clifford arrangement $\mathcal{V}$ in $\mathbb{R}^{2n}$, we consider as $\mathcal{V} \subseteq \mathrm{Gr}_n^{2n}(\mathbb{R})$. And we define $\mathcal{V}^\perp$ by

$$\mathcal{V}^\perp := \left\{V^\perp \mid V \in \mathcal{V}\right\} \subsetneq \mathrm{Gr}_n^{2n}(\mathbb{R}). \tag{3.3}$$

Moreover, we write $Q(\mathcal{V}) := \mathcal{V} \sqcup \mathcal{V}^\perp$. Note that, for all V and $W \in Q(\mathcal{V})$, we have

$$s_V(W) = \begin{cases} W & \text{if } W \in \{V, V^\perp\}, \\ W^\perp & \text{otherwise.} \end{cases} \tag{3.4}$$

Hence, for the quandle structure s on $\mathrm{Gr}_n^{2n}(\mathbb{R})$, a map $s_V|_{Q(\mathcal{V})} : Q(\mathcal{V}) \to Q(\mathcal{V})$ is well-defined for each $V \in Q(\mathcal{V})$. Then, the following is obtained:

Theorem 3.5 (F. [2])**.** *For a Clifford arrangement $\mathcal{V}$ in $\mathbb{R}^{2n}$, $Q(\mathcal{V})$ is a subquandle in $\mathrm{Gr}_n^{2n}(\mathbb{R})$.*

4 Our expectation for s-commutative subsets in oriented real Grassmannian manifolds

Definition of s-commutative subsets in quandles are introduced by Nagashiki [3].

Definition 4.1 (Nagashiki [3])**.** Let (X, s) be a quandle.

1. A subset $A \subseteq X$ is said to be *s-commutative* if it holds that $s_x \circ s_y = s_y \circ s_x$ for every x, $y \in A$.

2. An s-commutative subset $A \subseteq X$ is said to be *maximal* if A is maximal for inclusion.

Nagashiki [3] studied on maximal s-commutative subsets in oriented real Grassmannian manifolds $\mathrm{Gr}_k^n(\mathbb{R})^\sim := \{V \subseteq \mathbb{R}^n \mid V \text{ is an oriented subspace of dimension } k\}$. Moreover she determined maximal s-commutative subsets in $\mathrm{Gr}_k^n(\mathbb{R})^\sim$ for the following cases:

1. $n \neq 2k$,

2. k is odd and $n = 2k$,

3. $k = 2$ and $n = 4$ (i.e. $n = 2k$).

The following proposition gives us expectation that Clifford arrangements are related to maximal s-commutative subsets in $\mathrm{Gr}_n^{2n}(\mathbb{R})$ and $\mathrm{Gr}_n^{2n}(\mathbb{R})^\sim$ for the cases that k is even and $n = 2k$:

Proposition 4.2 (F.)**.** *A maximal s-commutative subset in $\mathrm{Gr}_2^4(\mathbb{R})^\sim$ can be obtained from quandles generated by Clifford arrangements explained in Example 3.3.*

References

[1] T. E. Cecil, *Lie Sphere Geometry.* With applications to submanifolds. Second edition. Universitext. Springer, New York, 2008.

[2] S. Fujii, *Subspace arrangements associated with symmetric Clifford systems*, in preparation.

[3] M. Nagashiki, *Maximal s-commutative subsets in oriented real Grassmannian manifolds*, Master's thesis, Hiroshima University.

(S. Fujii) Chitose Institute of Science and Technology, 758-65 Bibi, Chitose, Hokkaido, Japan 066-8655, Japan

Email address: s-fujii@photon.chitose.ac.jp

対称 Clifford 系と実 Grassmann 多様体のカンドル構造

藤井　忍 (公立千歳科技大・理工)

研究集会「カンドルと対称空間」
於 大阪公立大学数学研究所

2022 年 12 月 9 日

研究テーマ

我々の期待

以下の二つは関係があるだろう:

- 4 つの主曲率をもつ, 球面内の等径超曲面,
- 運動量写像.

注意

我々の期待通りなら...

- 4 つの主曲率をもつ, 球面内の等径超曲面すべてを統一的に扱うことが出来る... かも,
- 4 つの主曲率をもつ, 球面内の等径超曲面の分類に応用できる... かも.

研究テーマ

研究のテーマ

4 つの主曲率をもつ, 球面内の OT–FKM 型等径超曲面を運動量写像で記述したい.

OT–FKM 型等径超曲面とは, 対称 Clifford 系 $\{P_0, P_1, \ldots, P_m\}$ によって構成される Cartan–Münzner 多項式

$$F(x) = \|x\|^4 - 2\sum_{i=0}^{m} \langle P_i x, x\rangle^2$$

のレベル集合と単位球面の共通部分として得られる球面内の超曲面.

研究テーマ

問題

対称 Clifford 系 $\{P_0, P_1, \ldots, P_m\}$ を群作用の言葉で特徴づけできないか?

- 対称 Clifford 系は実対称行列の有限族,
- 実対称行列は直交行列で対角化可能 (群作用),
- { 実対称行列 }/対称行列 $\simeq$ 実 Grassmann 多様体

今日の話の大まかな流れ

- Clifford 代数の表現に付随する部分空間配置 (Clifford 配置) を定義する,
- Clifford 配置から Clifford 代数の表現が復元されることを示す,
- Clifford 配置を Grassmann 多様体のカンドル構造の観点から眺める,
- Clifford 配置と有向 Grassmann 多様体の極大 s–可換集合の関係について触れる.

Contents of this talk

対称 Clifford 系について

Clifford 代数の定義

一般に,

- V: 有限次元実線型空間,
- $q : V \times V \to \mathbb{R}$: V 上の 2 次形式,

定義

(V, q) に付随する Clifford 代数

$$\mathrm{Cl}(V, q) := T(V)/I_q$$

- $T(V) := \bigoplus_k T^k(V)$: テンソル代数,
- $I_q := \left\langle v \otimes v + q(v) \;\middle|\; v \in V \right\rangle_{\text{bi–sided}}$: 両側イデアル.

Clifford 代数の定義

定義

(V, q) に付随する Clifford 代数

$\mathrm{Cl}(V, q) := T(V)/I_q$

- $T(V) := \bigoplus_k T^k(V)$: テンソル代数,
- $I_q := \langle v \otimes v + q(v, v) \mid v \in V \rangle_{\text{bi–sided}}$: 両側イデアル.

以下, $V = \mathbb{R}^{m-1}$, $q = \langle -, - \rangle$: Euclid 内積, とする.
$\mathrm{Cl}(V, q) = \mathrm{Cl}_{m-1}$ と表す.

Clifford 代数の表現

定義

Clifford 代数 Cl_{m-1} の n 次表現
$\overset{\text{def}}{\Longleftrightarrow}$ $\mathbb{R}$-代数準同型 $\rho : \mathrm{Cl}_{m-1} \to M_n(\mathbb{R})$ のこと.
ただし,

- $M_n(\mathbb{R}) := \{n$ 次実係数正方行列$\}$,
- $\mathbb{R}$–代数準同型とは, 以下が成り立つこと:
  1. $\rho(\xi + \eta) = \rho(\xi) + \rho(\eta)$,
  2. $\rho(\xi \otimes \eta) = \rho(\xi)\rho(\eta)$,
  3. $\rho(a\xi) = a\rho(\xi)$,

代数準同型を決めるには, 生成系の行き先を決めるだけで十分.

Clifford 代数の表現

代数準同型を決めるには，生成系の行き先を決めるだけで十分．

事実

Cl_{m-1} は V の基底 $e_1, e_2, \dots, e_{m-1}$ で $\mathbb{R}$–代数として生成される．

$\rho(e_i) =: E_i \in M_n(\mathbb{R})$ とすると ρ は決まるが，特に $E_i \in \mathrm{Alt}_n(\mathbb{R}) := \{n$ 次実交代行列$\}$ とできる．

Clifford 代数の表現

$\rho(e_i) =: E_i \in M_n(\mathbb{R})$ とすると ρ は決まるが，特に $E_i \in \mathrm{Alt}_n(\mathbb{R}) := \{n$ 次実交代行列$\}$ とできる．

定義

Clifford 代数の表現 $\rho : \mathrm{Cl}_{m-1} \to M_n(\mathbb{R})$ は以下を満たす $\{E_1, E_2, \dots, E_{m-1}\} \subset \mathrm{Alt}_n(\mathbb{R})$ で決まる:

$$E_i E_j + E_j E_i = -2\delta_{ij} I_n.$$

このような $\{E_1, E_2, \dots, E_{m-1}\} \subset \mathrm{Alt}_n(\mathbb{R})$ を **Clifford 系**という．

対称 Clifford 系

- $\rho : \mathrm{Cl}_{m-1} \to M_n(\mathbb{R})$: 表現,
- $\{E_1, E_2, \dots, E_{m-1}\}$: ρ に付随する Clifford 系,

このとき, P_0, P_1, $P_2, \dots, P_m \subset \mathrm{Sym}_{2n}(\mathbb{R})$ を以下のように定める:

$$P_0 := \left(\begin{array}{c|c} I_n & 0_n \\ \hline 0_n & -I_n \end{array}\right), \quad P_1 := \left(\begin{array}{c|c} 0_n & I_n \\ \hline I_n & 0_n \end{array}\right),$$

$$P_i := \left(\begin{array}{c|c} 0_n & E_{i-1} \\ \hline -E_{i-1} & 0_n \end{array}\right) \quad (i \in \{2, 3, \dots, m\})$$

命題

$\{P_0, P_1, \dots, P_m\}$ は以下の関係式を満たす:

$$P_i P_j + P_j P_i = 2\delta_{ij} I_{2n}.$$

対称 Clifford 系

定義

$\{P_0, P_1, \dots, P_m\}$: $2n$ 次の対称 Clifford 系

$\overset{\text{def}}{\Longleftrightarrow}$

- $P_i \in \mathrm{Sym}_{2n}(\mathbb{R})$ $({}^\forall i)$,
- $P_i P_j + P_j P_i = 2\delta_{ij} I_{2n}$.

事実

$2n$ 次の対称 Clifford 系から Clifford 系を構成することができる.

対称 Clifford 系から Clifford 系を構成する

事実

$2n$ 次の対称 Clifford 系 $\{P_0, P_1, \ldots, P_m\}$ から Clifford 系を構成することができる.

∵ (概略だけ)

- 各 P_i の固有値は ± 1 のみ,
- 各 P_i の各固有空間は n 次元,
- $V_0 := \left\{ v \in \mathbb{R}^{2n} \mid P_0 v = v \right\} \simeq \mathbb{R}^n$,
- ${}^\forall i,\ E_i := P_1 P_{1+i}|_{V_0}$,
- $\{E_1, \ldots, E_{m-1}\}$ は Clifford 系.

対称 Clifford 系と Clifford 球面

対称 Clifford 系と Clifford 球面

- $\mathrm{Sym}_{2n}(\mathbb{R})$ 上の内積 $\langle -, - \rangle$ を以下で定義:

$$\langle P, Q \rangle := \frac{1}{2n} \mathrm{Tr}(PQ) \quad \text{for} \quad P, Q \in \mathrm{Sym}_{2n}(\mathbb{R}),$$

- $\mathcal{P} = \{P_0, P_1, \ldots, P_m\} \subsetneq \mathrm{Sym}_{2n}(\mathbb{R})$: 対称 Clifford 系,
- $L(\mathcal{P}) := \mathrm{span}_{\mathbb{R}}\{P_0, P_1, \ldots, P_m\} \subsetneq \mathrm{Sym}_{2n}(\mathbb{R})$,

定義

Clifford 球面 $\Sigma(\mathcal{P}) := \left\{ P \in L(\mathcal{P}) \;\middle|\; \|P\| = 1 \right\}$.

対称 Clifford 系と Clifford 球面

命題

Clifford 球面 $\Sigma(\mathcal{P})$ に対して, ${}^{\forall}P \in \Sigma(\mathcal{P})$, $P^2 = I_{2n}$.

この逆も成り立つ.

命題

$\Sigma \subseteq \mathrm{Sym}_{2n}(\mathbb{R})$: 部分集合 s.t.

- $P^2 = I_{2n}$ for ${}^{\forall}P \in \Sigma$,
- $\Sigma \subsetneq \mathrm{span}_{\mathbb{R}}(\Sigma)$: 単位球面,

$\Longrightarrow$ $\mathrm{span}_{\mathbb{R}}(\Sigma)$ の任意の正規直交基底は対称 Clifford 系.

対称 Clifford 系は Clifford 球面で決まる!

Clifford 球面と Grassmann 多様体

Clifford 球面 $\Sigma \subsetneq \mathrm{Sym}_{2n}(\mathbb{R})$ について

- ${}^\forall P \in \Sigma$, P の固有値は ± 1 のみ,
- ${}^\forall P \in \Sigma$, P の $(+1)$–固有空間も (-1)–固有空間も n 次元,

$\Longrightarrow$

問題

Clifford 球面と Grassmann 多様体
$\mathrm{Gr}_n^{2n}(\mathbb{R}) = \left\{ V \underset{\text{linear}}{\subsetneq} \mathbb{R}^{2n} \,\middle|\, \dim_{\mathbb{R}} V = n \right\}$ はどのような関係があるか?

Clifford 球面と Grassmann 多様体

Clifford 球面 $\Sigma \subsetneq \mathrm{Sym}_{2n}(\mathbb{R})$ は Grassmann 多様体 $\mathrm{Gr}_n^{2n}(\mathbb{R})$ の幾何と密接な関係がある.

命題 (Wang (1990), Wolf (1963))

$f_\Sigma : \Sigma \to \mathrm{Gr}_n^{2n}(\mathbb{R})$ を以下で定義:

$$\Sigma \ni P \xmapsto{f_\Sigma} P \text{ の } (+1)\text{–固有空間} \in \mathrm{Gr}_n^{2n}(\mathbb{R}).$$

このとき, f_Σ は全測地的埋め込みで, その像は全測地的球面.
逆に, $\mathrm{Gr}_n^{2n}(\mathbb{R})$ 内の全測地的球面は, Clifford 球面の f_Σ の像として得られる.

$f_\Sigma(\Sigma)$ は isoclinic sphere と呼ばれる.

Clifford 配置の定義

Clifford 配置

- $\mathcal{P} = \{P_0, P_1, \ldots, P_m\} \subsetneq \mathrm{Sym}_{2n}(\mathbb{R})$: 対称 Clifford 系,
- Σ: $\mathcal{P}$ に付随する Clifford 球面,
- $f_\Sigma : \Sigma \to \mathrm{Gr}_n^{2n}(\mathbb{R})$: Wang の定理の写像.

$\Longrightarrow$

問題

$V_i := f_\Sigma(P_i)$ と定めると,
$f_\Sigma(\mathcal{P}) = \{V_0, V_1, \ldots, V_m\} \subsetneq \mathrm{Gr}_n^{2n}(\mathbb{R})$ はどのような性質をもった有限部分集合か?

対称空間としての $\mathrm{Gr}_n^{2n}(\mathbb{R})$

定義

Riemann 多様体 (M, g) が **(Riemann) 対称空間**

$\overset{\text{def}}{\iff}$ $^\exists s : M \to \mathrm{Aut}(M)$, $x \mapsto s_x$ (点対称) s.t.

1. $^\forall x \in M$, x は s_x の孤立固定点,
2. $^\forall x \in M$, $s_x \circ s_x = \mathrm{id}_M$,
3. $^\forall x \in M$, s_x は等長変換.

$\mathrm{Gr}_n^{2n}(\mathbb{R})$ の場合, $^\forall V \in \mathrm{Gr}_n^{2n}(\mathbb{R})$ に対して,
$s_V : \mathrm{Gr}_n^{2n}(\mathbb{R}) \to \mathrm{Gr}_n^{2n}(\mathbb{R})$ を以下で定義:

$$W \mapsto s_V(W) := \left\{ \pi_V(w) - \pi_{V^\perp}(w) \;\middle|\; w \in W \right\}.$$

ただし, $V^\perp$: V の直交補空間, π_V: V への直交射影.

Clifford 配置と Grassmann 多様体

- $\mathbb{R}^{2n}$ の正規直交基底を任意に一組選び**固定**,
- $\mathcal{P} = \{P_0, \ldots, P_m\} \subsetneq \mathrm{Sym}_{2n}(\mathbb{R})$:
  Clifford 代数の表現から構成される対称 Clifford 系,
- $\Sigma \subsetneq \mathrm{Sym}_{2n}(\mathbb{R})$: Clifford 球面,
- $f_\Sigma : \Sigma \to \mathrm{Gr}_n^{2n}(\mathbb{R})$: Wang の定理の写像,
- $V_0 := \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R}^n \right\}$, $V_1 := \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \;\middle|\; x \in \mathbb{R}^n \right\}$,
  $V_i := \left\{ \begin{pmatrix} x \\ E_{i-1}x \end{pmatrix} \;\middle|\; x \in \mathbb{R}^n \right\}$ $(i = 2, \ldots, m)$.

命題 (F., in preparation)

$i \neq j \Longrightarrow s_{V_i}(V_j) = V_j^\perp$.

Clifford 配置から対称 Clifford 系を構成する

定理 (F., in preparation)

以下の条件を満たす $\mathcal{V} = \{V_0, \dots, V_\ell\} \subsetneq \mathrm{Gr}_n^{2n}(\mathbb{R})$ を任意にとる:

$$\begin{cases} V_i \not\perp V_j & (i \neq j), \\ s_{V_i}(V_j) = V_j^\perp & (i \neq j). \end{cases}$$

${}^\forall i \in \{0, 1, \dots, \ell\}$ に対して, $P_i \in \mathrm{Sym}_{2n}(\mathbb{R})$ を

$$P_i := s_{V_i} \text{ の表現行列}$$

と定義するとき, $\{P_0, P_1, \dots, P_\ell\}$ は対称 Clifford 系.

Clifford 配置から対称 Clifford 系を構成する

$\because$ (概略だけ)

- ${}^\forall i$, P_i は明らかに対称行列,
- ${}^\forall i$, $P_i^2 = I_{2n}$ は O.K.,
  $\because$ s_{V_i} は対合.
- $i \neq j \implies P_iP_j + P_jP_i = 0$ が面倒くさい,
  $\because$ 次を示せばよい:

$${}^\forall x \in \mathbb{R}^{2n}, \quad s_{V_i} \circ s_{V_j}(x) + s_{V_j} \circ s_{V_i}(x) = 0.$$

- 計算の途中で $s_{V_i}(V_j) = V_j^\perp$ を使う.

Clifford 配置

定義 (F.)

$\mathcal{V} = \{V_0, \ldots, V_\ell\}$: $\mathbb{R}^{2n}$ 内の Clifford 配置
$\overset{\text{def}}{\Longleftrightarrow}$

- $V_i \subsetneq \mathbb{R}^{2n}$: n 次元線型部分空間,
- $\sigma_{V_i}(V_j) = V_j{}^\perp$ if $i \neq j$.

以下では, $\mathbb{R}^{2n}$ 内の Clifford 配置 $\{V_0, \ldots, V_\ell\}$ を $\mathcal{C}\ell_\ell^n$ と表す.

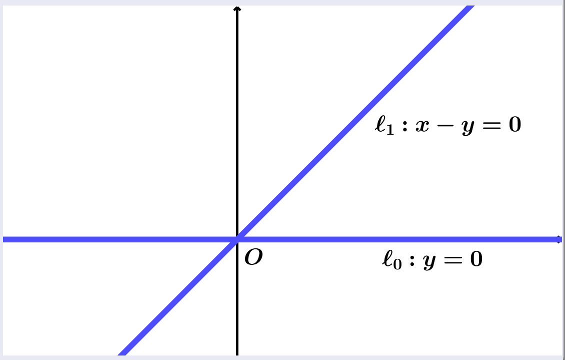
Clifford 配置の例: その 1

例 $(n = 1)$

$\mathbb{R}^2$ 内の直線 ℓ_0, ℓ_1 を

$$\ell_0 : y = 0,$$
$$\ell_1 : x - y = 0$$

とするとき, $\{\ell_0, \ell_1\}$ は Clifford 配置.



$\mathbb{R}^2$ の標準基底で表すと

$$\ell_0 = \mathbb{R}e_1, \quad \ell_1 = \mathbb{R}(e_1 + e_2).$$

Clifford 配置の例: その 2

例 ($n = 2$)

$\mathbb{R}^4$ 内の 2 次元線型部分空間 V_0, V_1, V_2 を

$$\begin{aligned} V_0 &:= \left\{ {}^t(x, y, 0, 0) \,\middle|\, x, y \in \mathbb{R} \right\}, \\ V_1 &:= \left\{ {}^t(x, y, x, y) \,\middle|\, x, y \in \mathbb{R} \right\}, \\ V_2 &:= \left\{ {}^t(x, y, y, -x) \,\middle|\, x, y \in \mathbb{R} \right\} \end{aligned}$$

とするとき, $\{V_0, V_1, V_2\}$ は Clifford 配置.

$\mathbb{R}^4$ の標準基底で表すと

$$V_0 = \mathrm{span}_{\mathbb{R}} \{e_1, e_2\}, \ V_1 = \mathrm{span}_{\mathbb{R}} \{e_1 + e_3, e_2 + e_4\},$$
$$V_2 = \mathrm{span}_{\mathbb{R}} \{e_1 - e_4, e_2 + e_3\}.$$

Clifford 配置の性質

Clifford 配置の正体

問題

主定理に現れる $\mathcal{V} = \mathcal{C}\ell_\ell^n$ は何を表しているのか?

対蹠集合ではないことは簡単に分かる.
注) S: 対蹠集合 $\overset{\text{def}}{\Longleftrightarrow}$ ${}^\forall x, {}^\forall y \in S,\ s_x(y) = y$.

命題 (F., in preparation)

$\mathcal{C}\ell_\ell^n$ に対して, $(\mathcal{C}\ell_\ell^n)^\perp := \{V^\perp \mid V \in \mathcal{C}\ell_\ell^n\}$ とするとき,
$s_{s_V(W)} \circ s_V = s_V \circ s_W$ for ${}^\forall V, {}^\forall W \in \mathcal{C}\ell_\ell^n \sqcup (\mathcal{C}\ell_\ell^n)^\perp$.

上の命題は s が $Q(\mathcal{C}\ell_\ell^n) := \mathcal{C}\ell_\ell^n \sqcup (\mathcal{C}\ell_\ell^n)^\perp$ 上のカンドル構造を定めることを意味する.

カンドル構造の復習

定義 (by Tamaru (2013))

- X: 集合,
- $s : X \to \mathrm{Map}(X, X),\ x \mapsto s_x$: 写像,

s が X 上のカンドル構造 (or (X, s): カンドル) $\overset{\text{def}}{\Longleftrightarrow}$

1. ${}^\forall x \in X,\ s_x(x) = x$,
2. ${}^\forall x \in X,\ s_x$ は全単射,
3. ${}^\forall x, {}^\forall y \in X,\ s_{s_x(y)} \circ s_x = s_x \circ s_y$.

一般に, 連結 Riemann 対称空間はカンドル構造を持つ (点対称がカンドル構造).

Clifford 配置に付随するカンドル

定義

(X, s), (X', s'): カンドル,
写像 $f : X \to X'$ がカンドル準同型 $\overset{\text{def}}{\Longleftrightarrow}$
${}^\forall x \in X$, $f \circ s_x = s'_{f(x)} \circ f$.

定理 (F., in preparation)

- $\mathcal{V} = \left\{ V_i \in \mathrm{Gr}_n^{2n}(\mathbb{R}) \;\middle|\; V_i \not\perp V_j,\ \ s_{V_i}(V_j) = V_j{}^\perp \ \ (i \neq j) \right\}$,
- $\mathcal{V}^\perp := \left\{ V^\perp \;\middle|\; V \in \mathcal{V} \right\}$,

このとき,

1. 点対称は $Q(\mathcal{V}) := \mathcal{V} \sqcup \mathcal{V}^\perp$ のカンドル構造,
2. 包含写像 $\iota : Q(\mathcal{V}) \to \mathrm{Gr}_n^{2n}(\mathbb{R})$ はカンドル準同型.

Clifford 配置に付随するカンドルの性質

事実 (cf. Furuki–Tamaru)

- $S^{N-1} \subsetneq \mathbb{R}^N$: 単位球面,
- S^{N-1} のカンドル構造: $s_x(y) := 2\langle x, y\rangle x - y$,
- $A^N := \{\pm e_1, \ldots, \pm e_N\} \subsetneq S^{N-1}$,

このとき, A^N は S^{N-1} の平坦, 等質, 非連結な有限部分カンドル.

命題 (F., in preparation)

カンドルとして $Q(\mathcal{C}\ell_\ell^n) \simeq A^{\ell+1}$.

$\because$ $f : Q(\mathcal{C}\ell_\ell^n) \to A^{\ell+1}$ を
$f(V_i) := e_{i+1}$, $f(V_i^\perp) := -e_{i+1}$ と定めれば良い.

カンドルの s-可換性

s–可換性

(X, s): カンドル.

定義

$x,\ y \in X$: s–可換 $\overset{\text{def}}{\Longleftrightarrow}$ $s_x \circ s_y = s_y \circ s_x$.

定義

- $S \subset X$: 部分集合,

S: s–可換 $\overset{\text{def}}{\Longleftrightarrow}$ ${}^\forall x,\ {}^\forall y \in S$, x と y が s–可換.

- 極大 s–可換部分集合 $\overset{\text{def}}{\Longleftrightarrow}$ 包含関係において極大な s–可換部分集合,
- 対蹠集合 $\Longrightarrow$ s–可換部分集合,
- 極大 s–可換部分集合 $\Longrightarrow$ 部分カンドル.

有向 Grassmann 多様体の極大 s–可換部分集合

事実 (cf. Nagashiki–Tamaru)

k, n が以下の場合，有向 Grassmann 多様体 $\mathrm{Gr}_k^n(\mathbb{R})^\sim$ の任意の極大 s–可換部分集合は

$$\left\{\pm\,\mathrm{span}_{\mathbb{R}}\{e_{i_1},\ldots,e_{i_k}\}\;\middle|\; 1\le i_1<\ldots<i_k\le n\right\}$$

に $\mathrm{O}(n)$–作用で合同:

1. $n\neq 2k$,
2. k: 奇数，$n=2k$.

有向 Grassmann 多様体の極大 s–可換部分集合

事実 (cf. Nagashiki–Tamaru)

有向 Grassmann 多様体 $\mathrm{Gr}_2^4(\mathbb{R})^\sim$ の極大 s–可換部分集合は

- $\pm\,\mathrm{span}_{\mathbb{R}}\{e_i,e_j\}\quad(1\le i<j\le 4)$,
- $\pm\,\mathrm{span}_{\mathbb{R}}\{e_i\pm e_j, e_k\pm e_\ell\}$
  $(1\le i<j\le 4,\ 1\le k<\ell\le 4)$

からなる集合に $\mathrm{O}(4)$–作用で合同なもののみ.

$k\ge 4$: 偶数，$n=2k$ のとき，有向 Grassmann 多様体 $\mathrm{Gr}_k^{2k}(\mathbb{R})^\sim$ の極大 s–可換部分集合はよくわかっていない.

有向 Grassmann 多様体の極大 s–可換部分集合

事実 (cf. Nagashiki–Tamaru)

有向 Grassmann 多様体 $\mathrm{Gr}_2^4(\mathbb{R})^\sim$ の極大 s–可換部分集合は

- $\pm\,\mathrm{span}_{\mathbb{R}}\{e_i, e_j\} \quad (1 \leq i < j \leq 4)$,
- $\pm\,\mathrm{span}_{\mathbb{R}}\{e_i \pm e_j, e_k \pm e_\ell\}$
  $(1 \leq i < j \leq 4,\ 1 \leq k < \ell \leq 4)$

からなる集合に $\mathrm{O}(4)$–作用で合同なもののみ.

全部で 36 枚の 2 次元部分空間から成る有限部分集合.

有向 Grassmann 多様体の極大 s–可換部分集合

命題 (F.)

有向 Grassmann 多様体 $\mathrm{Gr}_2^4(\mathbb{R})^\sim$ の極大 s–可換部分集合は $\mathbb{R}^4$ 内の full な Clifford 配置から構成可能である.

$\mathbb{R}^4$ 内の full な Clifford 配置は以下の部分空間からなる:

$$V_0 = \mathrm{span}_{\mathbb{R}}\{e_1, e_2\},$$
$$V_1 = \mathrm{span}_{\mathbb{R}}\{e_1 + e_3, e_2 + e_4\},$$
$$V_2 = \mathrm{span}_{\mathbb{R}}\{e_1 + e_4, e_2 - e_3\}.$$

有向 Grassmann 多様体の極大 s–可換部分集合

命題 (F.)

有向 Grassmann 多様体 $\mathrm{Gr}_2^4(\mathbb{R})^\sim$ の極大 s–可換部分集合は $\mathbb{R}^4$ 内の full な Clifford 配置から構成可能である.

1 組目	e_1, e_2	$e_1 + e_3, e_2 + e_4$	$e_1 + e_4, e_2 - e_3$
2 組目	e_1, e_3	$e_1 + e_4, e_2 + e_3$	$e_1 + e_2, e_3 - e_4$
3 組目	e_1, e_4	$e_1 + e_2, e_3 + e_4$	$e_1 + e_3, e_2 - e_4$

- 上に挙げた基底で張られる 2 次元空間: 9 枚,
- それらの直交補空間: 9 枚,
- 向きを考慮: 2 通り.

まとめ & 今後の課題

まとめ

- Clifford 代数の表現に付随する部分空間配置を定義した,
- Clifford 配置から対称 Clifford 系が復元されることを示した,
- Clifford 配置は Grassmann 多様体の部分カンドルを生成することを示した,
- Clifford 配置は有向 Grassmann 多様体の極大 s–可換集合と関係がありそうなことを確認した.

今後の課題

部分空間配置の観点から

- n を固定することの同値類の数: 計算中.
- Hilbert 多項式: Derksen の定理から容易に計算可能.
- 自由性: 不明.

カンドル, 対蹠集合の観点から

- $n \in 4\mathbb{Z}_+$ の場合の極大 s-可換集合との関係.

その他の観点から

- OT–FKM 型等径超曲面との関係.

Thank you for your attention!