The 3rd Japan-Taiwan Joint Conference on Differential Geometry

Osaka Central Advanced Mathematical Institute (OCAMI)
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The 3rd Japan-Taiwan Joint Conference on Differential Geometry

Organized by

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November 1–3, 2021

Abstract

"The 3rd Japan-Taiwan Joint Conference on Differential Geometry" was held on November 1–3, 2021. This volume records the abstracts and the slides of talks presented in this conference.

2020 *Mathematics Subject Classification.*
53B25, 53D37, 53D45, 53E10

Key words and Phrases.
CR manifold, hyperbolic geometry, minimal surface, submanifold theory, eigenvalues of the Laplacian, Lie group, quantum cohomology, Frobenius manifold, mirror symmetry, derived differential geometry, classical integrable system, Hodge theory, mean curvature flow, Yamabe flow

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Preface

The purpose of the Japan Taiwan Joint Conference on Differential Geometry is to foster discussions and interactions between the differential geometry communities of Japan and Taiwan. It is held approximately every two years.

The first and second conferences were:

- 1st Japan-Taiwan Conference on Differential Geometry & 8th OCAMI-TIMS Joint International Workshop on Differential Geometry and Geometric Analysis, 13-17 December 2016, Waseda University, Tokyo
- 2nd Taiwan-Japan Joint Conference on Differential Geometry, 1-5 November 2019, NCTS, National Taiwan University, Taipei

This report is a summary of the third conference:

- 3rd Japan-Taiwan Joint Conference on Differential Geometry, 1-3 November 2021 at OCAMI, Osaka City University, Osaka

It was held in hybrid format because of the COVID-19 pandemic. As the host institute, OCAMI (at Osaka City University) provided the lecture room for the conference and facilities for onsite participants as well as online participants. Taiwan participants gathered at a lecture room kindly provided by the NCTS (National Taiwan University). The two lecture rooms were connected by video link.

On the first day (1 November) the conference opened with short speeches by Prof. Yoshihiro Ohnita (Director of OCAMI) and Prof. Yng-Ing Lee (Director of NCTS). This was followed by 6 talks, 3 by Japan speakers and 3 by Taiwan speakers. On the second day (2 November) there were 6 talks, 2 by Japan speakers and 4 by Taiwan speakers. On the last day (3 November) there were 4 talks, 3 by Japan speakers and 1 by a Taiwan speaker.

A wide range of topics related to differential geometry were presented: geometry of CR manifolds, hyperbolic geometry, minimal surfaces and their moduli spaces, submanifold theory, eigenvalues of the Laplacian, Lie groups, quantum cohomology and Frobenius manifolds, mirror symmetry, derived differential geometry, classical integrable systems, Hodge theory, mean curvature flow and Yamabe flow.

Although personal interactions were restricted this time by the hybrid format, the conference was a valuable opportunity for geometry researchers on each side to see what kind of research is being carried out on the other side; it is hoped that this will lead to future contacts and collaborations.

The organisers are grateful to all speakers and participants, and to OCAMI and NCTS for providing facilities.

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JAPAN: Qing-Ming Cheng (Fukuoka University), Martin Guest (Waseda University), Miyuki Koiso (Kyushu University), Yoshihiro Ohnita (Osaka City University and OCAMI), Takashi Sakai (Tokyo Metropolitan University), Sumio Yamada (Gakushuin University)

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Quantum Flips and F-embeddings

Chin-Lung Wang (National Taiwan University)

We study analytic continuations of quantum cohomology under simple flips $f : X \to X'$ along the extremal ray variable q^l. Denote by $\Psi : H(X') \to H(X)$ the (inverse) graph correspondence. We show that there is a unique deformation $\widehat{\Psi}$ of Ψ which induces a non-linear imbedding $QH(X') \hookrightarrow QH(X)$ in the category of F (but not Frobenius) manifolds into the regular integrable loci of $QH(X)$ near $q^l = \infty$. This is a joint work with Yuan-Pin Lee and Hui-Wen Lin.

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Quantum Flips and F-embeddings

Chin-Lung Wang
National Taiwan University
(with Y.-P. Lee and H.-W. Lin)

The 3rd Japan-Taiwan Joint Conference on DG
November 1, 2021

Contents

1. What is Quantum Cohomology?

A: Deformation of $(H(X), \cup)$ by rational curves.

- Let $X/\mathbb{C}$ be smooth projective, $\overline{M}_n(X, \beta)$ the stable map moduli
$$f : (C, p_1, \ldots, p_n) \to X$$
from n-pointed nodal curves, $p_a(C) = 0, f_*(C) = \beta \in NE(X)$.
- For $i \in [1, n]$, let $e_i : \overline{M}_n(X, \beta) \to X$ be the evaluation map
$$e_i(f) := f(p_i) \in X.$$
- Let $\mathbf{t} \in H = H(X)$. The $g = 0$ Gromov–Witten potential
$$F(\mathbf{t}) = \langle\langle - \rangle\rangle(\mathbf{t}) := \sum_{n,\beta} \frac{q^\beta}{n!} \langle \mathbf{t}^{\otimes n} \rangle^X_{n,\beta}$$
$$= \sum_{n \geq 0, \beta \in NE(X)} \frac{q^\beta}{n!} \int_{[\overline{M}_n(X,\beta)]^{vir}} \prod_{i=1}^n e_i^* \mathbf{t}$$
is a formal function in $\mathbf{t}$ and q^β's (Novikov variables).

- We call $\mathscr{R} := \mathbb{C}[\![q^\bullet]\!]$ the (formal) Kähler moduli and denote

$$H_{\mathscr{R}} = H \otimes \mathscr{R}.$$

- Let $\{T_\mu\}$ be a basis of H and $\{T^\mu := \sum g^{\mu\nu} T_\nu\}$ the dual basis with respect to the Poincaré pairing

$$g_{\mu\nu} = (T_\mu . T_\nu), \qquad (g^{\mu\nu}) = (g_{\mu\nu})^{-1}.$$

- Let $\mathbf{t} = \sum t^\mu T_\mu$. The *big quantum ring* $(QH(X), *)$ is the $\mathbf{t}$-family of rings $Q_{\mathbf{t}}H(X) = (T_{\mathbf{t}}H_{\mathscr{R}}, *_{\mathbf{t}})$:

$$\begin{aligned} T_\mu *_{\mathbf{t}} T_\nu &:= \sum_{\epsilon,\kappa} \partial_\mu \partial_\nu \partial_\epsilon F(\mathbf{t})\, g^{\epsilon\kappa} T_\kappa \equiv \sum F_{\mu\nu\epsilon}\, g^{\epsilon\kappa} T_\kappa \\ &= \sum_{\epsilon,\kappa} \langle\langle T_\mu, T_\nu, T_\epsilon \rangle\rangle(\mathbf{t})\, g^{\epsilon\kappa} T_\kappa \\ &= \sum_{\kappa, n \geq 0, \beta \in NE(X)} \frac{q^\beta}{n!} \langle T_\mu, T_\nu, T^\kappa, \mathbf{t}^{\otimes n} \rangle^X_{n+3,\beta} T_\kappa. \end{aligned}$$

- The WDVV associativity equations equip $(H_{\mathscr{R}}, g_{\mu\nu}, F_{ijk}, T_0 = \mathbf{1})$ a structure of *formal Frobenius manifold* over $\mathscr{R}$.
- It is equivalent to the flatness of the Dubrovin connection

$$\nabla^z = d - \frac{1}{z} A := d - \frac{1}{z} \sum_\mu dt^\mu \otimes T_\mu *_{\mathbf{t}}$$

on the formal relative tangent bundle $TH_{\mathscr{R}}$ for all $z \in \mathbb{C}^\times$:

$$\partial_\mu A_\nu = \partial_\nu A_\mu, \qquad [A_\mu, A_\nu] = 0,$$

- where the (connection) matrix A_μ for $z\nabla^z_\mu$ is z-free:

$$A_\mu(\mathbf{t}) = T_\mu *_{\mathbf{t}}.$$

- This z-free property uniquely characterizes the constant frame $\{T_\mu\}$ among all frames $\{\tilde{T}_\mu\}$ with

$$\tilde{T}_\mu(q^\bullet, \mathbf{t}, z) \equiv T_\mu \pmod{\mathscr{R}}.$$

- Let $\psi = c_1(\mathbf{p}_1^* \omega_{\mathscr{C}/\overline{M}_n})$ be the class of cotangent line at the first marked section $\mathbf{p}_1 : \overline{M}_n \to \mathscr{C}$ of $\mathscr{C} \to \overline{M}_n$, then

$$J(\mathbf{t}, z^{-1}) := 1 + \frac{\mathbf{t}}{z} + \sum_{\beta,n,\mu} \frac{q^\beta}{n!} T_\mu \left\langle \frac{T^\mu}{z(z-\psi)}, \mathbf{t}^{\otimes n} \right\rangle_{n+1,\beta}^X$$

encodes all one-descendent invariants $\langle \psi^d T_i, \ldots \rangle$.
- The TRR:

$$\langle\langle \psi^{d+1} T_i, T_j, T_k \rangle\rangle = \sum_\mu \langle\langle \psi^d T_i, T_\mu \rangle\rangle \langle\langle T^\mu, T_j, T_k \rangle\rangle$$

implies the QDE:

$$z\partial_\mu \, z\partial_\nu J = \sum_\kappa A^\kappa_{\mu\nu} \, z\partial_\kappa J.$$

- Let $\mathscr{D}^z$ be the ring generated by $z\partial_i$ over $\mathscr{O} = \mathbb{C}[z][\![q^\bullet, \mathbf{t}]\!]$. The $\mathscr{D}^z$-module $\mathscr{O}^{\dim H}$ via $z\partial_i \mapsto z\nabla^z_i$ is isomorphic to $\mathscr{D}^z J$ (cyclic).

- In practice, one might be able to find element

$$I(\hat{\mathbf{t}}, z, z^{-1}) \in \mathscr{D}^z J(\mathbf{t}, z^{-1})$$

along some restricted variables $\hat{\mathbf{t}} \in H_1 \subset H$.
- For toric X, also hypersurfaces in it, ansatzs of I are found through $\mathbb{C}^\times$-localization data with $\hat{\mathbf{t}} \in H^{\leq 2}(X)$.
- [Lian–Liu–Yau 1996, Givental 1996] If $c_1(X) \geq 0$, $I = I(\hat{\mathbf{t}}, z^{-1})$ and $J(\hat{\mathbf{t}}, z^{-1})$ is obtained by a *mirror transform*.
- [Coates–Givental 2005, Iritani 2008, Brown 2010] An ansatz $I = I(\hat{\mathbf{t}}, z, z^{-1})$ is valid for any toric manifold.
- If $\langle H_1 \rangle = H$ (classical/quantum), there exists a recipe to get $J(\mathbf{t}, z^{-1})$ and ∇^z via $\mathscr{D}^z$-module techniques plus

Birkhoff Fatcorizations + Generalized Mirror Transform.

- Fix a presentation $T_\mu = \prod D_i$, define the naive quantization
$$\widehat{T}_\mu := \prod \widehat{D}_i \equiv \prod z\partial_i, \qquad \mu = 0, \ldots, R := \dim H - 1.$$
- Since $I \in \mathscr{D}^z J$, we have $\widehat{T}_\mu I \in \mathscr{D}^z J$ too. Hence
$$(\widehat{T}_\mu I)(\hat{\mathbf{t}}, z, z^{-1}) = z\nabla J(\sigma(\hat{\mathbf{t}}), z^{-1}) B(\hat{\mathbf{t}}, z).$$
- The unique $R \times R$ gauge transform $B(\hat{\mathbf{t}}, z)$ is called **BF**. Namely, $B^{-1}(z)$ removes the z-positive degree in I. In particular
$$J(\sigma(\hat{\mathbf{t}}), z^{-1}) = z\partial_0 J = \sum_\mu \widehat{T}_\mu I \cdot (B^{-1})^0_\mu =: P(\hat{\mathbf{t}}, z, z^{-1}) I(\hat{\mathbf{t}}, z, z^{-1}).$$
- The z^{-1} coefficient of PI gives the **GMT**:
$$\hat{\mathbf{t}} \mapsto \sigma(\hat{\mathbf{t}}) \in H_{\mathscr{R}}.$$

2. The Functoriality Problem:

Quantum Motives?

Which part of the structure on $QH(X)$ is functorial?

- $\mathcal{M}_k$: the category of Chow motives, k the ground field.
- Objects: $\hat{X}$, smooth k-variety. Morphisms are correspondences

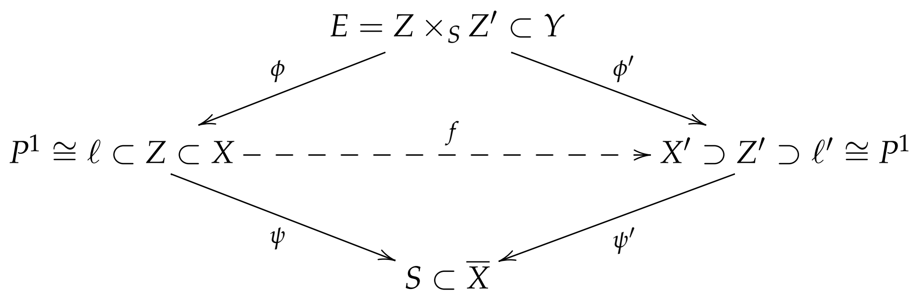
$$\Gamma \in \mathrm{Mor}(\hat{X}, \hat{X}') := A(X \times X').$$

- Induced map on Chow **groups**: $[\Gamma]_* : A(X) \to A(X')$:

$$\alpha \mapsto \pi'_*(\Gamma.\pi^*\alpha).$$

- Linear structures: if $\hat{X} \cong \hat{X}'$ then $A^i(X) \cong A^i(X')$ for all i. If k is a number field, X and X' have the same L functions for each i.
- However, the **ring structures are different**: $A(X) \not\cong A(X')$!
- Is there a *universal product structure* on Chow motives? Namely a universal family $(\mathscr{A}, *) \to T$ such that all geometric realizations $(A(X), \bullet)$ correspond to *special points*.
- Big quantum product provides partial solution to it.

- Typical examples come from ordinary (r, r')-flops/flips, $r \geq r'$:



- $\bar{\psi} : Z = P_S(F) \to S$, $\mathrm{rk}\, F = r + 1$, ψ-extremal ray $\ell = [C]$.
- $N_{Z/X}|_{\bar{\psi}^{-1}(s)} \cong \mathscr{O}_{P^r}(-1)^{\oplus(r'+1)}$ for all $s \in S$.
- $Y = \mathrm{Bl}_Z X = \mathrm{Bl}_{Z'} X'$,

$$\phi^* K_X = \phi'^* K_{X'} + (r - r')E.$$

- For flops $r = r'$, we have K-equivalence and $\hat{X} \cong \hat{X}'$ via

$$\Phi := [\bar{\Gamma}_f]_* = \phi'_* \circ \phi^* : H(X) \xrightarrow{\sim} H(X').$$

- It preserves the Poincaré pairing

$$(\Phi a.\Phi b)^{X'} = (\phi'^* \Phi a.\phi^* b)^Y = ((\phi^* a + \xi).\phi^* b)^Y = (a.b)^X,$$

but NOT the cup product!

- For the simple case ($S = \text{pt}$), let $\alpha_i \in H^{2l_i}(X)$, $\sum_{i=1}^3 l_i = \dim X$,

$$(\Phi\alpha_1.\Phi\alpha_2.\Phi\alpha_3)^{X'} = (\alpha_1.\alpha_2.\alpha_3)^X - \prod_{i=1}^3 (\alpha_i.h^{r-l_i})^Z,$$

where $h = c_1(\mathscr{O}_Z(1)) \in H^2(Z)$.

- Solution: use quantum product $(Q_{\mathbf{t}}H, *_{\mathbf{t}})$ instead.

- The effectivity of extremal curve is not preserved:

$$\Phi\ell = -\ell' \notin NE(X').$$

- It is necessary to consider analytic continuations $\overline{QH(X)}$ of $QH(X)$ along the Kähler moduli via the *partial compactification*

$$\Phi q^\beta = q^{\Phi\beta} \quad \text{toward} \quad \text{“}q^\ell = \infty\text{”}.$$

- For flops, the functoriality is simply the canonical isomorphism

$$\Phi : \overline{QH(X)} \xrightarrow{\sim} \overline{QH(X')}.$$

- In terms of Gromov–Witten invariants: for $\mathbf{t} \in H(X)$,

$$\Phi\langle\langle T_i, T_j, T_k \rangle\rangle^X(\mathbf{t}) = \langle\langle \Phi T_i, \Phi T_j, \Phi T_k \rangle\rangle^{X'}(\Phi\mathbf{t}).$$

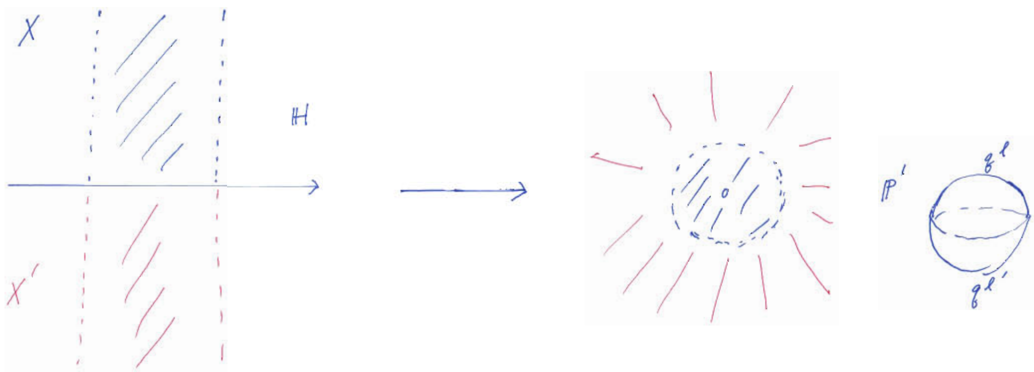
- [Li–Ruan] for 3-folds, [LLW, LLQW] for general ordinary flops.

The quantum cohomology is parametrized by the complexified Kähler class $\omega = B + iH \in H^2(X,\mathbb{R})\otimes\mathbb{C}$, with $B \in H^{1,1}_{\mathbb{R}}(X)$, $H \in K_X$

with $q^\beta = e^{2\pi i(\omega.\beta)} = e^{-2\pi(H.\beta)} \cdot e^{2\pi i(B.\beta)}$

$\Phi : H^2(X) \cong H^2(X')$, $\quad \Phi\ell = -\ell'$, $\quad q^\ell = e^{-2\pi(H.\ell)} \cdot e^{2\pi i(B.\ell)}$, $\quad (H.\ell) > 0$

$q^{\ell'} = q^{-\ell}$, $\quad q^{\ell'} = e^{-2\pi(H.(-\ell))} \cdot e^{2\pi i(B.(-\ell))}$, $\quad (H.(-\ell)) < 0$



- The simplest non K-equivalent birational maps *preserving the dimension of Kähler moduli* are smooth ordinary flips.
- *Pseudo-abelian completion of Chow motives* $\widetilde{\mathcal{M}}$: objects $(\hat{X}, p)$, where $p \in \mathrm{End}(\hat{X}) = A(X \times X)$ is a projector: $p^2 = p$. Then
$$\hat{X} \equiv (\hat{X}, 1) = (\hat{X}, p) \oplus (\hat{X}, 1 - p).$$
- For flips with $r > r'$, $\Psi := [\bar{\Gamma}_{f^{-1}}]$ induces a sub-motive
$$\Psi : \hat{X}' \xrightarrow{\sim} (\hat{X}, p), \qquad p := \Psi \circ \Phi.$$
- On cohomology
$$\Psi : H(X') \hookrightarrow H(X),$$
the Poincaré pairing is still preserved $(\Psi a.\Psi b)^X = (a.b)^{X'}$, but not the cup product. **Not even the quantum product!**
- Solutions?

3. Statements of Results for Ordinary Flips

$$f : X \dashrightarrow X'$$

- **Claim:** $QH(X')$ is still a sub-theory of $QH(X)$ in a canonical, though *non-linear*, manner.
- The basic exact sequence is an orthogonal splitting

$$0 \longrightarrow K \longrightarrow H(X) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} H(X') \longrightarrow 0\,.$$

- The vanishing cycles K has dimension $d := (r - r') \dim H(S)$:

$$K = \bigoplus\nolimits_{j=r'+1}^{r} [P^j] \otimes H(S).$$

- The Dubrovin connection ∇ can be analytically continued *along the Kähler moduli* to a connection $\Phi\nabla$ by the rule

$$\Phi q^\beta = q^{\Phi\beta}, \qquad \beta \in NE(X).$$

- As before $\Phi\ell = -\ell'$ and analytic continuations are required.

- For divisor $D = \sum t^i D_i$, $(D_i.\beta_j) = \delta_{ij}$, we couple t^i with q^{β_i}:

$$q_i := q^{\beta_i} e^{t^i}, \qquad \partial_i = \frac{\partial}{\partial t^i} = q_i \frac{\partial}{\partial q_i}.$$

$$\nabla_\mu = \partial_\mu - \frac{1}{z} T_\mu *$$

has only (formal) *regular singularities* at $q_i = 0$.

- $\Phi\nabla$ turns out is analytic in q^ℓ and contains *irregular singularities* along K at $q^\ell = \infty$, that is $q^{\ell'} = 0$.
- Let $H' = H(X')$ and $\mathscr{R}' = \mathbb{C}[\![NE(X')]\!]$. The Dubrovin connection ∇' on $TH'_{\mathscr{R}'}$ is also (formally) regular.
- This suggests to extract ∇' from $\Phi\nabla$

by removing the K directions!

- We will show that there is a *bundle-decomposition*

$$TH \otimes \mathscr{R}'[1/q^{\ell'}] = \mathscr{T} \oplus^{\perp} \mathscr{K} \qquad (*)$$

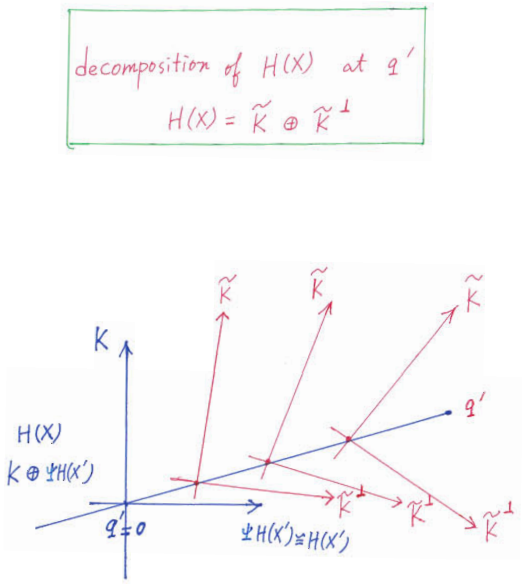
into irregular eigenbundle $\mathscr{K}$ which extends K over $\mathscr{R}'[1/q^{\ell'}]$ and the regular eigenbundle $\mathscr{T} = \mathscr{K}^{\perp}$.

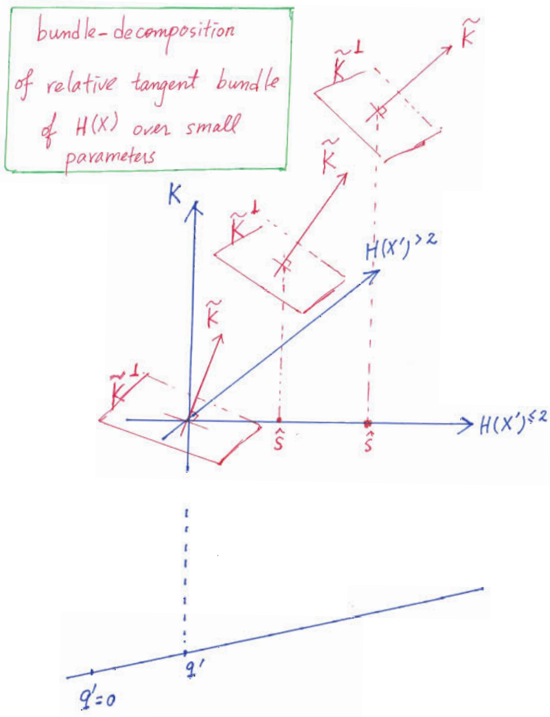
- From (coordinates free) WDVV equations, both $\mathscr{T}$ and $\mathscr{K}$ are shown to be integrable distributions. The integral submanifold

$$\mathcal{M}_{q'} \supset \{(q' \neq 0, \mathbf{t} = 0)\}$$

is the proposed manifold corresponding to $QH(X')$.

- To relate $\mathscr{T}$, and hence $\mathcal{M}_{q'}$, to $QH(X')$, we need to work on the connection (z-dependent) version of $(*)$.
- Hence there are BF/GMT involved, and it is unclear what kind of functoriality should exist.





Theorem (Lee–Lin–Wang, 2017, 2021)

For the projective local model $f : X \dashrightarrow X'$ of ordinary (r, r') flips, there is a unique $\mathscr{R}'$-point $\sigma_0(q') \in H'_{\mathscr{R}'}$ and a unique embedding $\widehat{\Psi}(q', \mathbf{s})$ over $\mathscr{R}'$:

$$\widehat{\Psi} : H(X')_{\mathscr{R}'} \longrightarrow \mathcal{M} \hookrightarrow H(X)_{\mathscr{R}'},$$
$$\sigma_0(q') + \mathbf{s} \longmapsto \widehat{\Psi}(q', \mathbf{s}).$$

where $\mathbf{s} \in H(X')$, such that

(1) *$(\widehat{\Psi}, \sigma_0)$ restricts to $(\Psi : H' \hookrightarrow H, 0)$ when modulo $q^{\ell'}$,*

(2) *$\widehat{\Psi}$ induces an F-embedding over $\mathscr{R}'[1/q^{\ell'}]$:*

$$(TH'_{\mathscr{R}'[1/q^{\ell'}]}, \nabla') \xhookrightarrow{d\widehat{\Psi}} (TH_{\mathscr{R}'[1/q^{\ell'}]}, \nabla)|_{\mathcal{M}} \longrightarrow \mathscr{K} \cong N_{\widehat{\Psi}} .$$

Remark: the simple flip case ($S = \mathrm{pt}$) was proved in 2017.

- In particular, outside the divisor $q^{\ell'} = 0$, the (big) quantum products on the corresponding tangent spaces are preserved.
- Denote $\widehat{\Psi}_i = \partial_i \widehat{\Psi}$, with induced metric

$$\mathbf{g}_{ij} = (\widehat{\Psi}_i, \widehat{\Psi}_j), \qquad \widehat{\Psi}^i := \sum \mathbf{g}^{ij} \widehat{\Psi}_j.$$

- Then $\widehat{\Psi}$ is an F-embedding means

$$\langle\langle \widehat{\Psi}_\mu, \widehat{\Psi}^i, \widehat{\Psi}_j \rangle\rangle^X (\widehat{\Psi}(q', \mathbf{s})) = \langle\langle T'_\mu, T'^i, T'_j \rangle\rangle^{X'} (\sigma_0(q') + \mathbf{s}).$$

- For simple flips, this leads to a family of ring decompositions:

$$Q_{\widehat{\Psi}(q',\mathbf{s})} H(X) \cong Q_{\sigma_0(q')+\mathbf{s}} H(X') \times \mathbb{C}^{r-r'},$$

  which depend on the points $(q', \mathbf{s})$.

4. STEP (i)

Irregular Singularity of $\overline{QH(X)}$ along Vanishing Cycles

(Local simple flip case)

$$f : X = P_{P^r}(\mathscr{O}(-1)^{r'+1} \oplus \mathscr{O}) \dashrightarrow X' = P_{P^{r'}}(\mathscr{O}(-1)^{r+1} \oplus \mathscr{O}).$$

$$H(X) = \mathbb{C}[h, \xi]/(h^{r+1}, \xi(\xi - h)^{r'+1}),$$
$$H(X') = \mathbb{C}[h', \xi']/(h'^{r'+1}, \xi'(\xi' - h')^{r+1}).$$

$$\Phi h = \xi' - h', \qquad \Phi \xi = \xi',$$

$$\Phi \ell = -\ell', \qquad \Phi \gamma = \gamma' + \ell'.$$

- Small parameters $\hat{\mathbf{t}} = t^0 T_0 + D \in H^{\leq 2}(X)$, $\hat{\mathbf{s}} = s^0 T_0' + D'$.

$$D = t^1 h + t^2 \xi = \Psi D' = \Psi(s^1 h' + s^2 \xi') = s^1(\xi - h) + s^2 \xi.$$

$$s^1 = -t^1, \qquad s^2 = t^2 + t^1.$$

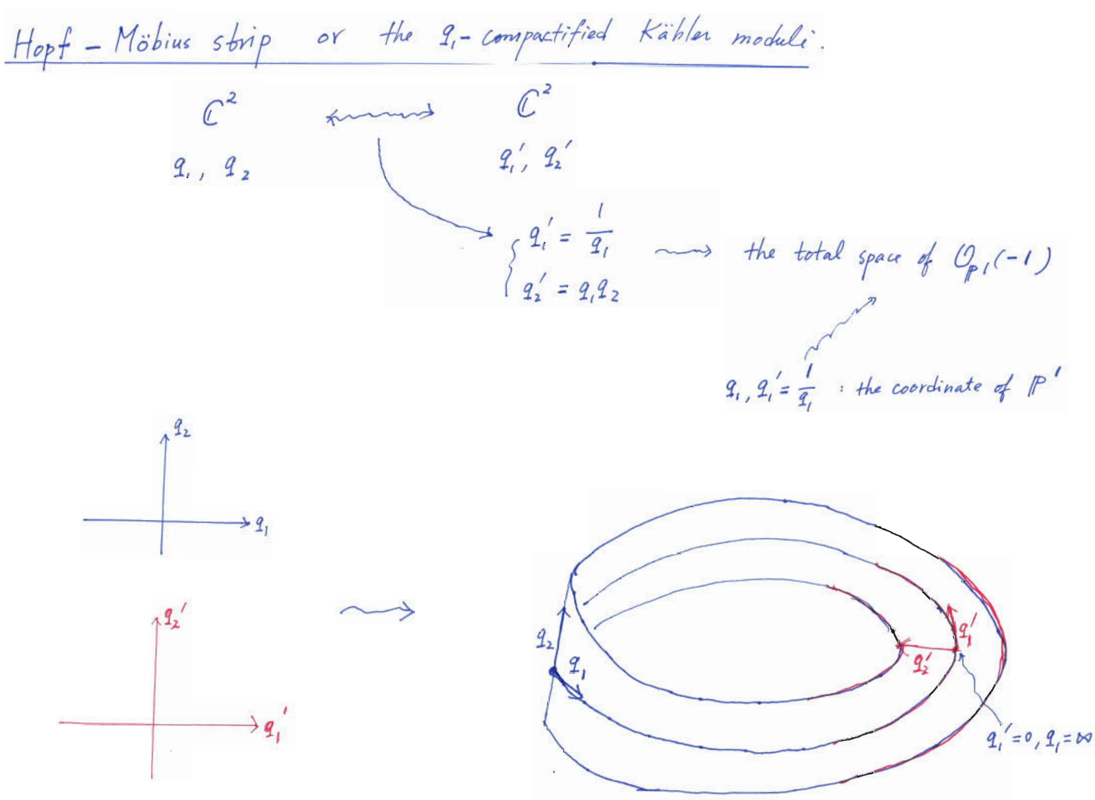
- Kähler moduli: $NE(X) = \mathbb{Z}\ell \oplus \mathbb{Z}\gamma$, $NE(X') = \mathbb{Z}\ell' \oplus \mathbb{Z}\gamma'$.

$$\begin{aligned} q_1 &= q^{\ell} e^{t^1}, \qquad & x = q_1' &= q^{\ell'} e^{s^1} = 1/q_1, \\ q_2 &= q^{\gamma} e^{t^2}, \qquad & y = q_2' &= q^{\gamma'} e^{s^2} = q_1 q_2. \end{aligned}$$

- Naive quantization, for $i \in [0, r]$, $j \in [0, r' + 1]$, $a = h^i \xi^j$,

$$\hat{a} \equiv \partial^{za} := \hat{h}^i \hat{\xi}^j = (z\partial_h)^i (z\partial_\xi)^j = (z\partial_1)^i (z\partial_2)^j.$$

- X is Fano: $c_1(X) = (r - r')h + (r' + 2)\xi$ is ample.
  X' is bad: $c_1(X') = (r' - r)h' + (r + 2)\xi'$ has no fixed sign.



- On X, for $\beta = d_1\ell + d_2\gamma \in NE(X)$,

$$I_\beta = \frac{1}{\prod_{m=1}^{d_1}(h+mz)^{r+1}\prod_{m=1}^{d_2-d_1}(\xi - h + mz)^{r'+1}\prod_{m=1}^{d_2}(\xi + mz)}$$

- $I = e^{\mathbf{t}/z}\sum_\beta e^{D.\beta}q^\beta I_\beta$ is annihilated by Picard–Fuchs equations:

$$\Box_\ell = (z\partial_h)^{r+1} - q_1(z\partial_{\xi-h})^{r'+1},$$
$$\Box_\gamma = z\partial_\xi(z\partial_{\xi-h})^{r'+1} - q_2.$$

- $I = I(z^{-1}) \Longrightarrow I = J_{small}$ and $QH(X)$ is "easy". Yet it is still non-trivial to write down ∇^X explicitly.
- The naive frame, for $\mathbf{e} = h^i\xi^j$ (or $h^i(\xi - h)^j$ w.r.t. $H(X')$),

$$\partial^{z\mathbf{e}} I \equiv \hat{h}^i\hat{\xi}^j I := (z\partial_h)^i(z\partial_\xi)^j I$$

does not lead to z-free connection matrices for $z\partial_1, z\partial_2$!

The Ψ-corrected quantum frame

- The quantized basis corresponding to $K = \ker\Phi$ is chosen to be

$$\hat{\kappa}_i I = \hat{h}^i(\hat{\xi} - \hat{h})^{r'+1} I, \qquad i \in [0, r - r' - 1].$$

- For $e_1 \in [0, r+1]$, $e_2 \in [0, r']$, we define

$$v_\mathbf{e} := \hat{h}^{e_1}(\hat{\xi} - \hat{h})^{e_2} I + \delta_{(e_1,e_2)}(-1)^{r'-e_2}\hat{\kappa}_{e_1+e_2-(r'+1)},$$

where

$$\begin{cases}\delta_{(e_1,e_2)} = 0 & \text{if } e_1 + e_2 \in [0, r'], \text{ and} \\ \delta_{(e_1,e_2)} = 1 & \text{otherwise.}\end{cases}$$

- The added term comes from $\ker\Phi \Longleftrightarrow e_1 + e_2 \in [r'+1, r]$. But $H^{2j}(X')$ with $j \geq r+1$ are also corrected accordingly.
- The frame reduces to a classical basis when modulo $NE(X)$.

The connection matrices for $z\partial_1$ and $z\partial_2$.

- **Proposition**. *For $i = 1, 2$, the connection matrix $C_i(q_1, q_2)$ in the Ψ corrected frame is independent of z. Moreover, $A_i(\hat{\mathbf{t}}) = C_i$.*
- Write $C_i = \begin{bmatrix} C_i^{11} & C_i^{12} \\ C_i^{21} & C_i^{22} \end{bmatrix}$ w.r.t. $H(X) = \Psi H(X') \oplus^\perp K$.
- Let $d = \dim K = r - r'$.
- For C_1, the $d \times d$ block corresponding to K is given by

$$C_1^{22} = \begin{bmatrix} & & (-1)^{r'+1} q_1 \\ 1 & & \\ & \ddots & \\ & 1 & \end{bmatrix}.$$

- Other entries in C_1 and C_2 have "good properties"!

- **Extract $QH(X')$ from $QH(X)$:** On X', let $\beta' = d_1' \ell' + d_2' \gamma'$, then

$$I_{\beta'}^{X'} = \frac{1}{\prod_1^{d_1'} (h' + mz)^{r'+1} \prod_1^{d_2' - d_1'} (\xi' - h' + mz)^{r+1} \prod_1^{d_2'} (\xi' + mz)}.$$

- It has Picard–Fuchs equations, irregular at $q_1' = 0$,

$$\Box_{\ell'} := (z\partial_2 - z\partial_1)^{r'+1} - q_1'(z\partial_1)^{r+1},$$
$$\Box_{\gamma'} := (z\partial_2)(z\partial_1)^{r+1} - q_2'.$$

- Since $\Box_{\ell'} = q_1^{-1} \Box_\ell$ and $\Box_{\gamma'} = z\partial_2 \Box_\ell - q_1 \Box_\gamma$, we get the
- **Key Lemma.** *Over $\mathbb{C}[q_1, q_1^{-1}, q_2] \cong \mathbb{C}[q_1', q_1'^{-1}, q_2']$, we have "the same" Picard–Fuchs ideal:*

$$\langle \Box_\ell, \Box_\gamma \rangle \cong \langle \Box_{\ell'}, \Box_{\gamma'} \rangle.$$

- **Corollary 1.** The Ψ-corrected frame corresponds to the constant frame for ∇^X. Hence C_i gives GW invariants on X directly.

- **Corollary 2.** Under the analytic continuation in the Kähler moduli over $NE(X')$, ∇^X is irregular in the divisor $(x = q'_1 = 0)$ precisely in the kernel block.

- **Corollary 3.** If C_1, C_2 can be simultaneously block-diagonalized to $\tilde{C}_1, \tilde{C}_2$, then the matrices $\tilde{C}_1^{11}, \tilde{C}_2^{11}$ can be used to compute $\nabla^{X'}$.

- Block-diagonalization is possible: Warow, Shibuya, Malgrange.

- **Issue:** $\tilde{C}_i$ must involve z, need BF/GMT.

5. STEP (ii)

Block Diagonalizations and BF/GMT over $NE(X')$

- We have $A_j(\hat{\mathbf{t}}) = C_j, j = 1, 2$:

$$C_1^{22} = \begin{bmatrix} 0 & 0 & \cdots & (-1)^{r'+1}q_1 \\ 1 & 0 & \cdots & 0 \\ & \ddots & & \\ 0 & \cdots & 1 & 0 \end{bmatrix} = \frac{1}{x}\begin{bmatrix} 0 & 0 & \cdots & (-1)^{r'+1} \\ x & 0 & \cdots & 0 \\ & \ddots & & \\ 0 & \cdots & x & 0 \end{bmatrix}.$$

- Irregular PDE system in (x, y) with parameter z.
- $R := \dim H(X)$, $R' := \dim H(X')$, $d = R - R' = r - r'$.
- To bring C_1^{22} into "semisimple" form, let $u = x^{1/d}$ and modify the constant frame to $\{T_i\}$:

$$\{T_i\}_{i=0}^{R'-1} = \{T_{\mathbf{e}}\}, \qquad \{T_{R'+i}\}_{i=0}^{d-1} = \{u^i \kappa_i\}_{i=0}^{d-1}.$$

- **Then we do shearing (= base change in $\mathscr{D}$-modules).**

- Let $Y(x) = \operatorname{diag}(1^{R'}, u^0, u^1, \cdots, u^{d-1})$. Let $S = YW$ and $x = u^d$,

$$zx\frac{\partial}{\partial x}S = C_1 S$$

becomes

$$zu\frac{\partial}{\partial u}W = D_1(u, z)W, \qquad (**)$$

$$D_1^{11} = d \cdot C_1^{11}, \qquad D_1^{12} = d \cdot C_1^{12} \cdot \operatorname{diag}(u^0, u^1, \cdots, u^{d-1}),$$

$$D_1^{21} = d \cdot \operatorname{diag}(u^0, u^{-1}, \ldots, u^{-d+1}) \cdot C_1^{21},$$

$$D_1^{22} = \frac{d}{u} \cdot \begin{bmatrix} 0 & 0 & \cdots & (-1)^{r'+1} \\ 1 & -z\frac{1}{d}u & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & 1 & -z\frac{d-1}{d}u \end{bmatrix}.$$

- D_1^{21} is polynomial in u. Thus, $(**)$ is irregular of Poincaré rank 1 in u, and the irregular part only appears in the $(2, 2)$ block D_1^{22}.

- Therefore, $D_1(z=0)$ has eigenvalues $0^{R'}$ and d *distinct nonzero* eigenvalues from $D_1^{22}(0)$ as solutions to

$$\omega^d = (-1)^{r'+1}.$$

- By the classical procedure (Wasow), and the flatness of ∇^X:

(i) C_1, C_2 are *simultaneously block diagonalized* to $\tilde{C}_1, \tilde{C}_2$, such that the $(2,2)$ blocks are *diagonalized*.

(ii) The new frame (gauge matrix) is z-dependent:

$$P = [\tilde{T}_0, \ldots, \tilde{T}_{R'-1}, \tilde{T}_{R'}, \ldots, \tilde{T}_{R-1}] = \begin{bmatrix} I_{R'} & * \\ * & I_d \end{bmatrix}.$$

It has the initial term $[T_0, \ldots, T_{R-1}]$ in u.

(iii) $\mathscr{T}$ spanned by $\tilde{T}_0, \ldots, \tilde{T}_{R'-1}$ and $\mathscr{K}$ spanned by $\tilde{T}_{R'}, \ldots, \tilde{T}_{R-1}$ lead to *orthogonal reduction of connection*.

- For $a, b \in H(X)$ we have

$$ab = a * b + \sum_{\beta \in NE(X)} q^\beta c_\beta$$

for some $c_\beta \in H(X)$. By induction we conclude that

$$T_\mu * = \sum_{\beta \in NE(X)} q^\beta P_\beta(h*, \xi*)$$

where P_β is a polynomial. Since X is Fano, the sum is finite.

- So the block diagonalization extends to all $T_\mu *$.
- In fact $\tilde{C}_1^{11}$ and $\tilde{C}_2^{11}$, hence all $\tilde{C}_\mu^{11}$, are expressible in x, y, z.
- Now we apply BF to remove the z-dependence in $\tilde{C}_\mu^{11}(x,y,z)$. Let $B = B(x,y,z)$ be the BF matrix and $B(0) := B(x,y,0)$.

$$[\mathbf{T}_0, \ldots, \mathbf{T}_{R'-1}] := \left([\tilde{T}_0, \ldots, \tilde{T}_{R'-1}]B^{-1}\right)(z=0).$$

- For $a = 0, 1, 2$, the "z-free" matrix

$$C'_a(\hat{\mathbf{s}}) = -(z\partial_a B)B^{-1} + B\tilde{C}_a^{11}B^{-1} = B(0)\tilde{C}_{a;0}^{11}B(0)^{-1}(x, y)$$

is related to $A'_\mu(\sigma)$ for $T'_\mu *'$ at the generalized mirror point

$$\sigma = \sigma(\hat{\mathbf{s}}) \in H(X')[\![x, y]\!].$$

- Under this GMT, we get relations of GW invariants:

$$C'_a(\hat{\mathbf{s}}) = \sum_\mu A'_\mu(\sigma(\hat{\mathbf{s}}))\frac{\partial \sigma^\mu}{\partial s^a}(\hat{\mathbf{s}}), \qquad a = 0, 1, 2,$$
$$\langle\!\langle T_a, \mathbf{T}_j, \mathbf{T}^i \rangle\!\rangle^X(\hat{\mathbf{s}}) = \sum_\mu \frac{\partial \sigma^\mu}{\partial s^a}(\hat{\mathbf{s}})\langle\!\langle T'_\mu, T'_j, T'^i \rangle\!\rangle^{X'}(\sigma(\hat{\mathbf{s}})).$$

- Since $(A'_\mu)^i_0 = \delta^i_\mu$, $\sigma(\hat{\mathbf{s}})$ is determined by the first column:

$$(C'_a)^\mu_0(\hat{\mathbf{s}}) = \langle\!\langle T_a, \mathbf{T}_0, \mathbf{T}^\mu \rangle\!\rangle^X(\hat{\mathbf{s}}) = \frac{\partial \sigma^\mu}{\partial s^a}(\hat{\mathbf{s}}).$$

- The next step is to transform $\mathbf{T}_0$ to the identity element (section) $e \in \mathscr{T}$ and normalized $\mathbf{T}_i$'s to $\tilde{\mathbf{T}}_i$'s accordingly.
- **Lemma.** There is a unique element $\mathbf{S}_0 \in \mathscr{T}$ such that

$$\mathbf{S}_0 * \mathbf{T}_0 = e,$$

and so e acts as zero on $\mathscr{K}$. (This requires delicate calculations!)
- Define the *normalized frame* on $\mathscr{T}$ by

$$\tilde{\mathbf{T}}_\mu := \mathbf{T}_\mu * \mathbf{S}_0.$$

- Theorem (Initial quantum invariance up to a shifting)

Let $\mathbb{T}_i(q') = \tilde{\mathbf{T}}_i(q', \hat{\mathbf{s}} = 0, z = 0)$ *and* $\sigma_0(q') = \sigma(q', \hat{\mathbf{s}} = 0)$*. Then we have*

$$\langle \mathbb{T}_\mu, \mathbb{T}^i, \mathbb{T}_j \rangle^X = \langle\!\langle T'_\mu, T'^i, T'_j \rangle\!\rangle^{X'}(\sigma_0(q')).$$

6. STEP (iii)

Non-Linear F-Embedding $QH(X') \hookrightarrow \overline{QH(X)}$

- An **F-manifold** M is a complex manifold with a commutative product structure on each T_pM, such that a WDVV-type integrability condition is forced when $p \in M$ varies.
- In $QH(X)$, this is the structure which remembers $*_p$ but forgets the metric g_{ij}. Hertling and Manin showed that the WDVV equations can be rewritten as

$$L_{X*Y}(*) = X * L_Y(*) + Y * L_X(*)$$

  for any local vector fields X and Y.
- I.e., for any local vector fields X, Y, Z, W:

$$\begin{aligned}&[X*Y, Z*W] - [X*Y, Z]*W - [X*Y, W]*Z \\ &\quad = X*[Y, Z*W] - X*[Y,Z]*W - X*[Y,W]*Z \\ &\qquad + Y*[X, Z*W] - Y*[X,Z]*W - Y*[X,W]*Z.\end{aligned}$$

- Denote by $\mathcal{K}$ the irregular eigenbundle and $\mathcal{T} := \mathcal{K}^\perp$ the regular eigenbundle, which extend $\mathscr{K}$ and $\mathscr{T}$ from $\mathbf{s} = 0$ to big $\mathbf{s}$.

- Lemma

  $\mathcal{T}$ is an integrable distribution of the relative tangent bundle $TH_{\mathscr{R}'}$.

  In particular, $\operatorname{Im} \widehat{\Psi}$ *is the integral submanifold* $\mathcal{M}$ *(over* $\mathscr{R}'$*) containing the slice* $(q^{\ell'} \neq 0, \mathbf{t} = 0)$ *which contains* $\operatorname{Im} \Psi$ *when modulo* $\mathscr{R}'$*.*

- Proof.

  Let X, Z be any local vector fields in $\mathcal{T} = \mathcal{K}^\perp$. Let $Y = e_i$ and $W = e_j$ be idempotents in $\mathcal{K}$. Since $a * b = 0$ for $a \in \mathcal{K}, b \in \mathcal{K}^\perp$,

$$0 = -X * Z * [e_i, e_j] - \delta_{ij} e_j * [X, Z].$$

  Let $i = j$ we get $e_j * [X, Z] = 0$ for all j. Hence $[X, Z] \in \mathcal{K}^\perp$. □

- The quantum product on the Frobenius manifold $H(X') \otimes \mathscr{R}'$ is semi-simple. Let $v'_0, \ldots, v'_{R'-1}$ be the idempotent vector fields.
- **Dubrovin 1996:** $[v'_i, v'_j] = 0$ for all $0 \leq i, j \leq R' - 1$. Hence the corresponding *canonical coordinates* $u'^0, \ldots, u'^{R'-1}$ satisfying

$$(u'^i(q', \mathbf{s} = 0)) = \sigma_0(q')$$

  and $v'_i = \partial/\partial u'^i$ exist.
- This was extended to F-manifolds by Hertling. The F-manifold $\mathcal{M}$ is semi-simple in the sense that $*_p$ on $T_p\mathcal{M}$ for $p \in \mathcal{M}$ is semi-simple. Denote the idempotent vector fields by $v_1 \ldots, v_{R'}$.
- **Hertling 2002:** $[v_i, v_j] = 0$ for all $0 \leq i, j \leq R' - 1$. Hence the canonical coordinates $u^0, \ldots, u^{R'-1}$ near each $p \in \mathcal{M}$ exist in the sense that $v_i = \partial/\partial u^i$.

- **Fixing the initial correspondence of frames:**
- We have constructed an analytic family of coordinate systems $(u^0(q', p), \ldots, u^{R'-1}(q', p))$ parametrized by $q' \in \mathscr{R}'$. Write

$$\mathbb{T}_i(q') = \sum_{j=0}^{R'-1} a_i^j(q')\, v_j(q', \mathbf{s} = 0)$$

for an invertible $R' \times R'$ matrix $(a_i^j(q'))$.

-

$$\langle \mathbb{T}_\mu, \mathbb{T}^i, \mathbb{T}_j \rangle^X = \langle\langle T'_\mu, T'^i, T'_j \rangle\rangle^{X'}(\sigma_0(q')). \tag{1}$$

From this relation, we see easily that:

- Lemma

After a possible reordering of $\{v'_j\}$, we have for all $i = 0, \ldots, R' - 1$:

$$T'_i = \sum_{j=0}^{R'-1} a_i^j(q')\, v'_j(\sigma_0(q')).$$

- Now we define the map $\hat{\Psi}$ by *matching the canonical coordinates*. Namely, $\hat{\Psi}(q', \mathbf{s}) \in \mathcal{M}$ is the unique point on $\mathcal{M}$ so that

$$u^i(\hat{\Psi}(q', \mathbf{s})) = u'^i(q', \mathbf{s}) = u'^i(\sigma_0(q') + \mathbf{s})$$

for $i = 0, \ldots, R' - 1$.

- Since the tangent map $\hat{\Psi}_*$ matches the idempotents

$$\hat{\Psi}_* \partial/\partial u'^i = \partial/\partial u^i,$$

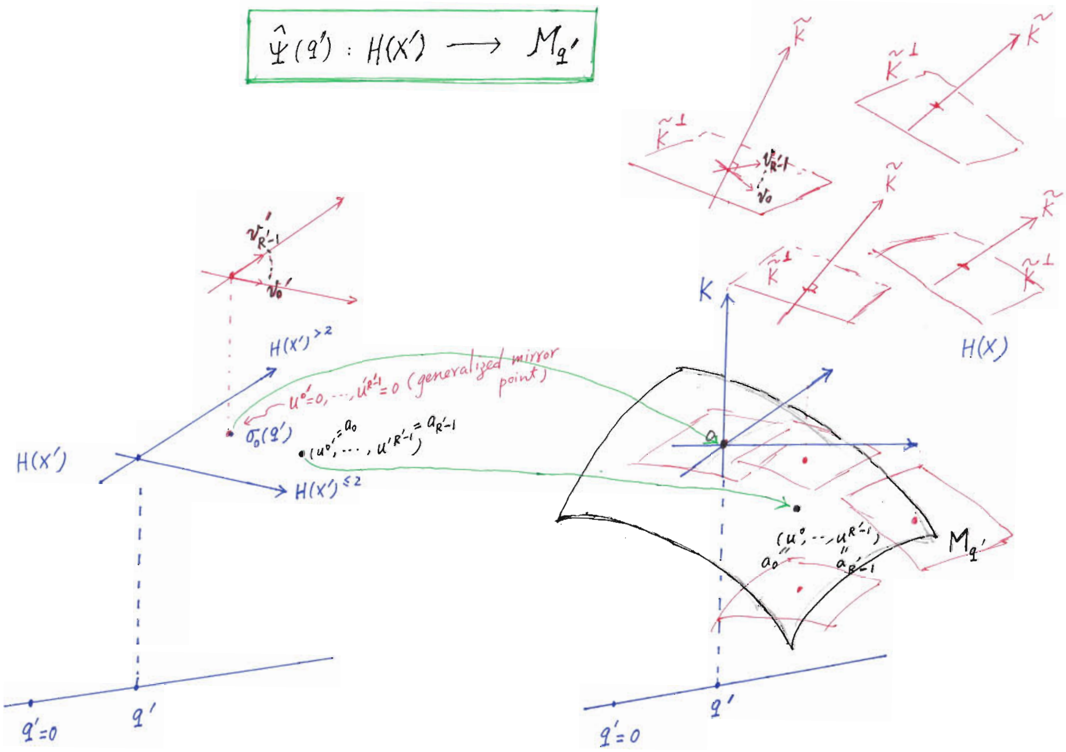
it induces a product structure isomorphism, and hence an F-structure isomorphism by "coordinates-free WDVV".

- Also along $\mathbf{s} = 0$, by Lemma we have

$$\hat{\Psi}_* T'_i = \mathbb{T}_i$$

which matches the initial condition along the $\mathscr{R}'$-axis.

- $H(X')$ is contractible $\Longrightarrow$ $\hat{\Psi}$ exists globally. **QED**



Ending Remarks

- Work in progress by LLW:

(1) Globalization to general (r, r') flips.

(2) Reconstruction of $QH(X)$ from $QH(X')$ and "the K-block".

- Other approaches:

(3) [Woodward et. al.] studying wall crossing of GW invariants in different GIT quotients.

(4) [Shoemaker et. al.] studying asymptotics of I functions in the toric setup.

(5) [Iritani] Equivariant GW for toric flips.

- Would be interesting to compare their approaches with ours.

日台友誼長存

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Mirror Symmetry and Rigid Structures of Generalized $K3$ Surfaces

Atsushi Kanazawa (Keio University)

Hitchin's invention of generalized Calabi-Yau structures is a key to unify the Calabi-Yau geometry (complex geometry of Calabi-Yau manifolds) and symplectic geometry. Such structures have been extensively studied in 2-dimensions by Huybrechts. Based upon his fundamental work, we introduce a formulation of mirror symmetry for generalized K3 surfaces, which generalizes mirror symmetry for lattice polarized K3 surfaces. Along the way, we investigate complex and Kahler rigid structures of generalized K3 surfaces.

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MS and Rigid Structures of Generalized K3 Surfaces

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Overview

Discuss mirror symmetry from the viewpoint of generalized CY geometry.

- Mirror Symmetry:
  duality between complex geometry and symplectic geometry
- Generalized Calabi-Yau Geometry:
  unification of CY geometry and symplectic geometry

The philosophies are different but there are some similarities.

Show that generalized CY geometry brings a new insight into rigid structure of K3 surfaces and settle a problem for singular K3 surfaces (complex rigid K3 surfaces).

Mirror Symmetry

A Calabi-Yau (CY) manifold X is a compact Kähller manifold such that $c_1(T_X) = 0$ and $\pi_1(X) = 0$. Mirror symmetry (MS) conjectures that CY manifolds show up in pairs, say X and Y, in such a way that

$$\text{Complex Geometry of } X \quad \cong \quad \text{Symplectic Geometry of } Y$$

There are various formulations;

- Hodge theoretic, homological, SYZ, ...
- They are decategorified to the level of cohomologies:

Hodge diamond 3-dim

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$h^{2,1} = \dim H^{2,1}(X) = \dim H^1(T_X)$ complex moduli

$h^{1,1} = \dim H^{1,1}(X)$ symplectic moduli

K3 surfaces (2-dim CY)

MS for a K3 surface S is very subtle as the complex and Kähler structures are somewhat mixed.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$H^2(S,\mathbb{Z}) \cong U^{\oplus 3} \oplus E_8^{\oplus 2}$

σ : holomprhic 2-form

ω : Kähler form

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

There are sublattices of $H^2(S,\mathbb{Z})$ reflecting the complex structure: the Néron-Severi, transcendental lattices

$$NS(S) = \{\delta \in H^2(S,\mathbb{Z}) \mid \langle \delta, [\sigma] \rangle = 0\} \quad \text{"algebraic 2-cycles"},$$

$$T(S) = NS(S)^\perp \subset H^2(S,\mathbb{Z}) \quad \text{"transcendental 2-cycles"},$$

It is useful to consider the algebraic lattice ("algebraic cycles")

$$NS'(S) = H^0(S,\mathbb{Z}) \oplus NS(S) \oplus H^4(S,\mathbb{Z}) = NS(S) \oplus U.$$

Mirror symmetry for K3 surfaces

Formulation: Two families of K3 surfaces $\{S\}$ and $\{S^\vee\}$ are mirror symmetric if for generic members S and $S^\vee$

$$NS'(S) \cong T(S^\vee), \quad T(S) \cong NS'(S^\vee).$$

This can be realized by lattice polarizations (Dolgachev). Given a lattice M of sgn $(1,*)$ and a primitive embedding $M \hookrightarrow H^2(S,\mathbb{Z})$ such that

$$M^\perp \cong N \oplus U, \quad \exists N \quad \text{(Asm.)}$$

S is called M-polarized if $M \subset NS(S)$. Then

- a family of M-polarized K3 surfaces $\{S\}$,
- a family of N-polarized K3 surfaces $\{S^\vee\}$ $\quad (N^\perp \cong M \oplus U)$

are mirror symmetric.

$$NS'(S) \cong M \oplus U \cong T(S^\vee), \quad T(S) \cong N \oplus U \cong NS'(S^\vee).$$

Drawbacks

This formulation has some drawbacks (although it works beautifully in many cases).

- $NS'(S) = NS(S) \oplus U$ and $T(S)$ are not really symmetric:

$$\min\{\mathrm{rank} NS'(S)\} = 3, \quad \min\{\mathrm{rank} T(S)\} = 2.$$

- (Asm.) does not hold in general.
- MS for singular K3 surfaces ($\mathrm{rank} NS(S) = 20$) fails.

	singular K3 surface	??
Kähler	20-dim	0-dim
complex	0-dim	20-dim

$\rightsquigarrow$ These are all solved by generalized CY geometry.

Generalized CY structures (2-dim)

M: C^∞-manifold underlying a K3 surface,
$A^{2*}_{\mathbb{C}}(M) = \oplus_{i=0}^{2} A^{2i}_{\mathbb{C}}(M)$: even diff forms with $\mathbb{C}$-coeff with Mukai pairing

$$\langle \varphi, \psi \rangle = \varphi_2 \wedge \psi_2 - \varphi_0 \wedge \psi_4 - \varphi_4 \wedge \psi_0 \in A^4_{\mathbb{C}}(M)$$

Definiton 4.1 (generalized CY structure (2-dim), Hitchin)

A generalized CY structure on M is a closed form $\varphi \in A^{2*}_{\mathbb{C}}(M)$ such that

$$\langle \varphi, \varphi \rangle = 0, \quad \langle \varphi, \overline{\varphi} \rangle > 0$$

Example 4.2

- *symplectic form* ω, $\varphi = e^{\sqrt{-1}\omega} = 1 + \sqrt{-1}\omega - \frac{1}{2}\omega^2$.
- *holomorphic* 2*-form (w.r.t. a complex structure),* $\varphi = \sigma$.

B-field transform

For $B \in A^2_{\mathbb{C}}(M)$, e^B acts on $A^{2*}_{\mathbb{C}}(M)$ by exterior product:

$$e^B \varphi = (1 + B + \frac{1}{2} B \wedge B) \wedge \varphi.$$

This action is orthogonal w.r.t. the Mukai pairing

$$\langle e^B \varphi, e^B \psi \rangle = \langle \varphi, \psi \rangle.$$

A real closed 2-form is called a B-field.

Theorem 4.3

For a B-field B and a gCY structure φ, the B-field transform $e^B \varphi$ is a gCY structure.

Classification of gCY structures

Theorem 4.4 (Hitchin)

Let φ be a gCY structure.

- *(type A) $\varphi_0 \neq 0$: $\exists$ a symplectic form ω, a B-field B.*

$$\varphi = \varphi_0 e^{B+\sqrt{-1}\omega}$$

- *(type B) $\varphi_0 = 0$: $\exists$ a hol 2-form σ (w.r.t. a complex str) and a B-field B.*

$$\varphi = e^B\sigma = \sigma + \sigma \wedge B^{0,2}$$

Definiton 4.5

gCY structures φ, φ' are isomorphic if $\exists$ an exact B-field B and $f \in \mathrm{Diff}_*(M)$ such that $\varphi = e^B f^*\varphi'$.

$\mathrm{Diff}_*(M) = \mathrm{Ker}(\mathrm{Diff}(M) \to O(H^2(M,\mathbb{Z})))$.

Unification of A- and B-structures

The most fascinating aspects of gCY structures is the occurrence of the classical CY structure σ and symplectic gCY structure $e^{\sqrt{-1}\omega}$ in the same moduli.

Example 4.6 (Hitchin)

For a hol 2-form σ, the real and imaginary parts $\mathrm{Re}(\sigma), \mathrm{Im}(\sigma)$ are symplectic forms. A family of gCY structures of type A

$$\varphi_t = te^{\frac{1}{t}(\mathrm{Re}(\sigma)+\sqrt{-1}\mathrm{Im}(\sigma))}$$

converges, as $t \to 0$, to the gCY structure σ of type B. The B-fields interpolate between gCY structures of type A and B.

Kähler structure

For a gCY structure φ, we define a distribution of real 2-planes:

$$P_\varphi = \mathbb{R}\mathrm{Re}\varphi \oplus \mathbb{R}\mathrm{Im}\varphi \subset A^*(M)$$

gCY structures φ and φ' are called orthogonal if P_φ and $P_{\varphi'}$ are pointwise orthogonal. This is a stronger condition than $\langle \varphi, \varphi' \rangle = 0$.

Definiton 4.7 (Kähler)

A gCY structure φ is called Kähler if $\exists$ another gCY structure φ' orthogonal to φ. Such φ' is called a Kähler structure for φ.

A Kähler structure for $\varphi = \sigma$ is of the form $\varphi' = \varphi'_0 e^{B+\sqrt{-1}\omega}$. The orthogonality is equivalent to

$$\sigma \wedge B = \sigma \wedge \omega = 0.$$

Therefore B is a closed real $(1,1)$-form and $\pm\omega$ is a Kähler form w.r.t. σ.

HyperKähler structure

A Kähler form ω on a K3 surface is a hyperKähler form if for some $C \in \mathbb{R}$

$$2\omega^2 = C\sigma \wedge \overline{\sigma}.$$

Definiton 4.8 (hyperKähler)

A gCY structure φ is hyperKähler if $\exists$ a Kähler structure φ' such that

$$\langle \varphi, \overline{\varphi} \rangle = \langle \varphi', \overline{\varphi'} \rangle.$$

Such φ' is called a hyperKähler structure for φ.

Remark 4.9

If φ' a (hyper)Kähler for φ, then $e^B\varphi'$ is a (hyper)Kähler structure for $e^B\varphi$.

Generalized CY structures

Classification of hyperKähler structures

(details are not important)

- $\varphi = \sigma$: a hyperKähler structure is $\varphi' = \lambda e^{B+\sqrt{-1}\omega}$, where B is a closed $(1,1)$-form and $\pm\omega$ is a hyperKähler form such that

$$2|\lambda|^2\omega^2 = \sigma \wedge \overline{\sigma}.$$

- $\varphi = \lambda e^{\sqrt{-1}\omega}$: a hyperKähler structure is either
  - $\varphi' = \sigma$, where $\pm\omega$ is a hyperKähler form,
  - $\varphi' = \lambda' e^{B'+\sqrt{-1}\omega'}$ such that
    - $\omega \wedge \omega' = \omega \wedge B' = \omega' \wedge B = 0,\ B'^2 = \omega^2 + \omega'^2$,
    - $|\lambda|^2\omega^2 = |\lambda'|^2\omega'^2$.

By Remark 4.9, any hyperKähler structure is a B-field transform of one of the above cases. There are 3 cases:

(type A, type B), (type B, type A), (type A, type A)

Generalized CY structures

Generalized K3 surfaces

Definiton 4.10

A generalized K3 surface is a pair (φ, φ') of gCY structures such that φ is a hyperKähler structure for φ'.

- A K3 surface $S = M_\sigma$ with a chosen hyperKähler structure ω is considered as a gK3 surface $(e^{\sqrt{-1}\omega}, \sigma)$.
- gK3 surfaces (φ, φ') and (ψ, ψ') are called isomorphic if $\exists$ $f \in \mathrm{Diff}_*(M)$ and exact $B \in A^2(M)$ such that

$$(\varphi, \varphi') = e^B f^*(\psi, \psi') = (e^B f^*\psi, e^B f^*\psi').$$

$\Rightarrow$ isom classes are classified by cohomology classes

Generalized CY structures

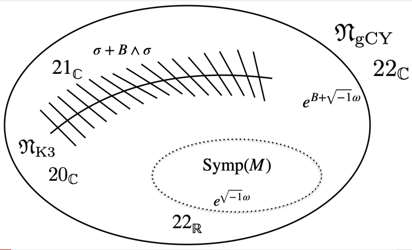
Period domains and period maps

$\mathfrak{N}_{\mathrm{gCY}} = \{\mathbb{C}\varphi\}/\cong$: moduli space of gCY structures of hyperKähler type
$\mathfrak{N}_{\mathrm{K3}} = \{\mathbb{C}\sigma\}/\mathrm{Diff}_*(M)$: moduli space of complex structures

Theorem 4.11 (Huybrechts)

$$\begin{array}{ccc} \mathfrak{N}_{\mathrm{gCY}} & \xrightarrow[\mathbb{C}\varphi\to[\phi]]{\mathfrak{per}_{\mathrm{gCY}}} & \widetilde{\mathfrak{D}} = \{[\varphi] \in \mathbb{P}(H^*(M,\mathbb{C})) \mid \langle\varphi,\varphi\rangle = 0, \langle\varphi,\overline{\varphi}\rangle > 0\} \\ \cup & & \cup \\ \mathfrak{N}_{\mathrm{K3}} & \xrightarrow[\mathbb{C}\sigma\to[\sigma]]{\mathfrak{per}_{\mathrm{K3}}} & \mathfrak{D} = \{[\sigma] \in \mathbb{P}(H^2(M,\mathbb{C})) \mid \langle\sigma,\sigma\rangle = 0, \langle\sigma,\overline{\sigma}\rangle > 0\} \end{array}$$

$\mathfrak{per}_{\mathrm{gCY}}$: *étale surjective*



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Generalized CY structures

Néron-Severi and transcendental lattices (new!)

We define sublattices of the Mukai lattice $H^*(M,\mathbb{Z}) \cong U^{\oplus 4} \oplus E_8^{\oplus 2}$ reflecting a gCY structure.

Definiton 4.12

The Néron–Severi and transcendental lattices of a gK3 surface $X = (\varphi, \varphi')$ *are defined respectively by*

$$\widetilde{NS}(X) = \{\delta \in H^*(M,\mathbb{Z}) \mid \langle\delta,[\varphi']\rangle = 0\},$$
$$\widetilde{T}(X) = \{\delta \in H^*(M,\mathbb{Z}) \mid \langle\delta,[\varphi]\rangle = 0\}.$$

- $\widetilde{NS}(X)$ and $\widetilde{T}(X)$ are defined on an equal footing.

$$2 \le \mathrm{rank}(\widetilde{NS}(X)),\ \mathrm{rank}(\widetilde{T}(X)) \le 22.$$

- In general, pt and $[M]$ are no longer ”algebraic”.

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RIgidity

Complex and Kähler rigidity

Definiton 5.1

A gK3 surface $X = (\varphi, \varphi')$ is called

- *complex rigid if φ' is of type B and $\mathrm{rank}(\widetilde{NS}(X)) = 22$.*
- *Kähler rigid if φ is of type A and $\mathrm{rank}(\widetilde{T}(X)) = 22$.*

Theorem 5.2

A complex rigid gK3 surface is of the form $e^{B'}(\lambda e^{B+\sqrt{-1}\omega}, \sigma)$:

- *M_σ: singular K3 surface*
- *$B \in H^{1,1}(M_\sigma, \mathbb{R})$,*
- *$B' \in H^2(M, \mathbb{Q})$,*
- *$\pm\omega$ is a Kähler form w.r.t. σ.*

RIgidity

Mukai lattice polarization and mirror symmetry

$\kappa, \lambda \geq 2$, $\kappa + \lambda = 24$,
K, L: even lattices of signature $(2, \kappa - 2)$ & $(2, \lambda - 2)$

Definiton 5.3 (Mukai lattice polarization)

Given a primitive embedding $K \oplus L \subset H^*(M, \mathbb{Z})$, a (K, L)-polarization of $X = (\varphi, \varphi')$ is defined by the conditions:

- *$K \subset \widetilde{NS}(X)$ and $K_{\mathbb{C}}$ contains gCY structure of type A,*
- *$L \subset \widetilde{T}(X)$ and $L_{\mathbb{C}}$ contains gCY structure of type B.*

polarization ⊂ lattice polarization ⊂ Mukai lattice polarization

Definiton 5.4

A family of (K, L)-polarized gK3 surfaces and a family of (L, K)-polarized gK3 surfaces are mirror symmetric.

MS for complex and Kähler rgid gK3 surfaces

$K = \langle -2n \rangle^{\oplus 2} \oplus U \oplus E_8^{\oplus 2}$, $L = \langle 2n \rangle^{\oplus 2}$, $(n > 0)$

- (K, L)-polarized gK3 surfaces
  = singular K3 surfaces $\{X = (e^{B+\sqrt{-1}\omega}, \sigma)\}$,
  $(T(M_\sigma) = L,\ B, \omega \in NS(M_\sigma)_{\mathbb{R}})$
- (L, K)-polarized gK3 surfaces
  $\supset \{X^\vee = (e^{\sqrt{-1}H}, \sigma^\vee)\}$, $(NS(M_{\sigma^\vee}) = \mathbb{Z}H, H^2 = 2n)$
  (19-dimensional subfamily of classical K3 surfaces)

	(K, L)-pol. gK3	(L, K)-pol. gK3
A-deform	20-dim	0-dim
B-deform	0-dim	20-dim

Punchline classical geometry: $H^2(M, \mathbb{Z})$, generalized geometry: $H^*(M, \mathbb{Z})$.

多謝你! Thank you!

The initial motivation of this work comes from the attractor mechanisms of moduli space of CY3s, applied to “K3 surface × elliptic curve”.

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On exact triangles consisting of projectively flat bundles on higher dimensional complex tori

Kazushi Kobayashi

In general, for a given n-dimensional complex torus X^n, a simple projectively flat bundle E on X^n is constructed from each affine Lagrangian submanifold in a mirror partner of X^n with a unitary local system along it (see [5, 7, 6, 1, 2, 4] etc.). In this talk, we focus on a certain class of exact triangles consisting of three simple projectively flat bundles E on a higher dimensional complex torus X^n ($n \geq 2$), and explain that such an exact triangle on X^n is obtained as the pullback of an exact triangle on an elliptic curve X^1 by a suitable holomorphic projection $X^n \to X^1$ (cf. [3]).

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Derived Differential Geometry and Virtual Fundamental Classes

Adeel A. Khan (Academia Sinica)

Virtual counts of pseudoholomorphic curves on a symplectic manifold play an important role in Gromov-Witten theory and Lagrangian Floer theory. These counts are defined using the virtual fundamental class of the moduli space of pseudoholomorphic curves. I will explain a simple new construction of the virtual fundamental class based on a theory of derived differential geometry.

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Derived differential geometry and virtual fundamental classes

Adeel Khan (Academia Sinica)

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Introduction

Pseudoholomorphic curves

The study of pseudoholomorphic curves on symplectic manifolds, following Gromov (1985), has led to interesting new developments in symplectic topology through the introduction of invariants such as:

- Gromov–Witten theory,
- Floer theory,
- Fukaya categories and homological mirror symmetry.

1

Example: Arnold conjecture

- Let M be a closed symplectic manifold and $\phi : M \to M$ a nondegenerate Hamiltonian symplectomorphism.
- Arnold conjectured that the number of fixed points of ϕ is at least equal to the Morse number of M (the number of critical points of a smooth function on M).
- By Morse theory, the Arnold conjecture implies in particular that

$$\#\{\text{fixed points of } \phi\} \geqslant \dim H_*(M, \mathbf{Q}).$$

2

Example: Arnold conjecture

- Floer used Hamiltonian Floer theory $HF_*(M)$, defined in terms of a Hamiltonian $H : M \times S^1 \to \mathbf{R}$, to give an approach to such bounds.
- The main input is an isomorphism $HF_*(M) \simeq H_*(M, \mathbf{Q})$.
- By construction of Hamiltonian Floer theory, we get: for any $H : M \times S^1 \to \mathbf{R}$ whose time $t = 1$ Hamiltonian flow $\phi_H : M \to M$ has nondegenerate fixed points, we have

$$\#\{\text{fixed points of } \phi_H\} \geqslant \dim H_*(M, \mathbf{Q}).$$

- The definition of $HF_*(M)$ involves moduli spaces of "Floer trajectories" or pseudoholomorphic cylinders.

3

Counting pseudoholomorphic curves

Like Floer theory, the invariants we have in mind are all defined by "counting pseudoholomorphic maps".

- Construct a moduli space $\mathcal{M}$ of (stable) pseudoholomorphic maps from Riemann surfaces.
- If $\mathcal{M}$ is "cut out transversally", then there is a fundamental class $[\mathcal{M}]$.
- We get the number $\int_{[M]} 1$.

However, in practice we typically do not have transversality for $\mathcal{M}$. We only get a virtual fundamental class $[\mathcal{M}]^{\mathrm{vir}}$ (Kontsevich).

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The transversality problem

Augmenting foundations

- In order to construct the virtual fundamental class $[\mathcal{M}]^{\mathrm{vir}}$, some augmentation of the traditional foundations of symplectic geometry is required.
- That is, the moduli space must be considered with some kind of additional structure.

Approaches

- In 1999, Fukaya and Ono introduced a so-called *Kuranishi structure* on $\mathcal{M}$ in order to define its virtual fundamental class.
- A complete account of the technical details of Kuranishi structures appeared in a book of Fukaya–Oh–Ohta–Ono (2020). McDuff and Wehrheim have also written some further details about Kuranishi structures (2016).
- Pardon has introduced a simpler variant called "implicit atlases", which is sufficient for many applications (2016).
- Another approach, involving infinite-dimensional manifolds, is the Polyfolds project (Hofer–Wysocki–Zehnder, 2007–).

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Kuranishi structures

Definition (Fukaya–Ono)

A **Kuranishi chart** *of a space X consists of:*

- *a smooth orbifold M,*
- *an orbibundle $E \to M$ (the obstruction bundle),*
- *a smooth section $s : M \to E$,*
- *and a homeomorphism between X and the zero locus $s^{-1}(0)$.*

A **Kuranishi structure** *on a space X is, roughly, a compatible system of local Kuranishi charts.*

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Strategy for constructing virtual fundamental classes

1. Show that the moduli space $\mathcal{M}$ admits a Kuranishi structure.
2. Construct a virtual fundamental class $[X]^{\mathrm{vir}}$ in the presence of a Kuranishi structure on a space X.

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Step 1

Theorem (Fukaya–Oh–Ohta–Ono)
Moduli spaces of pseudoholomorphic maps admit Kuranishi structures.

- *Exponential decay estimates and smoothness of the moduli space of pseudo-holomorphic curves* (arXiv:1603.07026)
- *Construction of Kuranishi structures on the moduli spaces of pseudo-holomorphic disks: I* (arXiv:1710.01459)
- *Construction of Kuranishi structures on the moduli spaces of pseudo-holomorphic disks: II* (arXiv:1808.06106)

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Step 2

- When X admits a global Kuranishi chart $(M, E \to M, s, G)$, $[X]^{\mathrm{vir}}$ should be the localized Euler class $e(E, s)$.
- Given a local Kuranishi structure on X, we have to glue the local virtual classes to a global one. This is done in:

  Fukaya–Oh–Ohta–Ono, *Kuranishi structures and virtual fundamental chains* (2020 book).

Derived differential geometry

Grothendieck's viewpoint on spaces: affine schemes

In algebraic geometry, an affine k-**scheme** is a formal symbol X which admits an arbitrary commutative k-algebra A as its ring of functions. We write $X = \mathrm{Spec}(A)$.

- Example: $X = \mathbf{A}^n_k = \mathrm{Spec}(k[t_1, \ldots, t_n])$ is the scheme-theoretic incarnation of affine space k^n.
- Example: $X = \mathrm{Spec}(k[t_1, \ldots, t_n]/(f_1, \ldots, f_m))$, for polynomials $f_1, \ldots, f_m \in k[t_1, \ldots, t_n]$, is the scheme-theoretic incarnation of the zero locus inside $\mathbf{A}^n_k$ of the system of polynomial equations $\{f_i = 0\}_i$.
- Roughly speaking, X is a "singular algebraic manifold".

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Grothendieck's viewpoint on spaces: moduli problems

Often we are not given a space not via a presentation as a zero locus, but as a moduli problem.

- A **moduli problem** is a functor $\mathcal{M} : \mathrm{AffSch}_k^{\mathrm{op}} \to \mathrm{Grpd}$ which assigns to every affine scheme S a groupoid of "objects over S".
- Example: $X = \mathrm{Spec}(k[t_1, \ldots, t_n]/(f_1, \ldots, f_m))$ solves the moduli problem

  $$S = \mathrm{Spec}(A) \mapsto \{(a_1, \ldots, a_n) \in A^n \text{ satisfying } f_i(a_1, \ldots, a_n) = 0 \; \forall i\}.$$

  In other words we can think of X as the scheme of solutions to the system $\{f_i = 0\}_i$ in all k-algebras of coefficients A.
- Example: curves, and isomorphisms between them, define a moduli problem $S = \mathrm{Spec}(A) \mapsto \{\text{relative curves over } S\}$.

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Schemes and stacks

If a moduli problem can be covered by affine schemes (an "atlas"), then it is called a scheme or a stack.

- Schemes: Zariski covers ($\Rightarrow$ all automorphism groups trivial).
- Deligne–Mumford stacks: étale covers ($\Rightarrow$ all automorphism groups finite).
- Artin stacks: smooth covers ($\Rightarrow$ infinite automorphism groups).

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Derived algebraic geometry

- Around 2004, Toën–Vezzosi and Lurie introduced a theory of derived algebraic geometry. This is obtained by deriving the notion of commutative k-algebra.
- Roughly speaking, a derived commutative algebra ($\sim$ commutative dg-algebra) can be thought of as a derived k-vector space ($\sim$ object of the derived category of k-vector spaces) with a multiplicative structure (commutative).
- We then get *derived affine k-schemes*, *derived schemes*, and *derived stacks*.
- Note that the target category of groupoids also has to be enlarged to ∞-groupoids.

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Example: derived intersections

- Let M be a smooth scheme ("algebraic manifold"), $E \to M$ an algebraic vector bundle, and $s : M \to E$ a section. Form the zero locus Z:

$$\begin{array}{ccc} Z & \longrightarrow & M \\ \downarrow & & \downarrow{\scriptstyle s} \\ M & \xrightarrow{0} & E. \end{array}$$

  When s does not meet the zero section transversely, Z will not be of the expected dimension $\dim(M) - \mathrm{rk}(E)$.
- We can replace Z by the derived fibre product Z^{der}. Algebraically, this means we replace $\mathcal{O}_Z = \mathcal{O}_M/s$ by the Koszul complex

$$\mathcal{O}_{Z^{\mathrm{der}}} = [\Lambda^{\mathrm{rk}(E)}(E) \to \cdots \to \Lambda^1(E) \to \mathcal{O}_M].$$

  Note that Z^{der} remembers the expected dimension.

Quasi-smooth derived stacks

- A derived scheme is called **quasi-smooth** if it is locally of the form Z^{der} for some (M, E, s) as above. (Similarly for quasi-smooth Deligne–Mumford stacks, where M is allowed to be a smooth Deligne–Mumford stack instead of a smooth scheme.)
- In 2019, I gave a construction of virtual fundamental classes for derived Artin stacks which is completely global and intrinsic to the stack: no choice of local "Kuranishi" presentation is involved in the construction.

Derived differential geometry

- Differential-geometric variants of DAG have been considered by Lurie (2009), Spivak (2010), Joyce (2012), Behrend–Liao–Xu (2020), ...
- I will work in the formalism of "derived C^∞-algebraic geometry". Roughly speaking, this is constructed by replacing (derived) commutative rings by (derived) C^∞-rings.
- This is similar to Joyce's formalism, except that he doesn't consider Artin stacks (which will be important for us).
- I will explain how to adapt my construction of virtual fundamental classes to the setting of derived Artin C^∞-stacks.

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C^∞-rings

- C^∞-rings should be thought of as rings of functions on "singular manifolds" (C^∞-schemes). For example, $C^\infty(\mathbf{R}^n)$ should be a C^∞-ring, but so should all quotients.
- Roughly speaking, the precise definition is: a C^∞-**ring structure** on a set A is a collection of operations $A^n \to A$ indexed by C^∞-maps $\mathbf{R}^n \to \mathbf{R}$, $n \geqslant 0$, satisfying certain relations.
- Note that commutative algebra structures can be defined similarly, where the $\mathbf{R}^n$ are replaced by the affine spaces $\mathbf{A}^n_k$, and C^∞-maps by polynomial maps (Lawvere).
- There is also a corresponding derived theory.

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Derived C^∞-stacks

- We consider moduli problems $C^\infty\text{-Ring} \to \text{Grpd}$, and derived moduli problems where the source is replaced by derived C^∞-rings and the target by ∞-groupoids.
- We define derived schemes and stacks (Deligne–Mumford and Artin) similarly as in the algebraic case.
- There is a notion of smoothness (in the algebro-geometric sense). The category of C^∞-manifolds (resp. orbifolds) embeds as a full subcategory of smooth C^∞-schemes (resp. Deligne–Mumford stacks).
- We define **quasi-smooth** derived stacks similarly as in the algebraic case. This is an intrinsic way to speak of spaces with Kuranishi structures.

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Virtual fundamental classes

Deformation to the normal cone

- Let X be a quasi-smooth derived scheme (or Artin stack). The construction of $[X]^{\mathrm{vir}}$ involves an intermediate geometric construction called deformation to the normal stack.
- Deformation to the normal stack is a generalization of Verdier's deformation to the normal cone.
- Recall that deformation to the normal cone associates, to any closed immersion $i : Z \to X$, a family of closed immersions over the affine line $\mathbf{A}^1$ which deforms i to the zero section of the normal cone.
- A differential-geometric analogue is considered e.g. by Kashiwara–Schapira (for the embedding of a submanifold in a manifold).

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The normal stack

- The *cotangent complex* was introduced in the algebraic setting by Illusie, building on work of Quillen.
- In derived geometry, we can form "derived vector bundles" as total spaces of complexes.
- Given a morphism $f : X \to Y$ of derived stacks, the **normal stack** $N_{X/Y}$ is the total space of the (-1)-shifted cotangent complex $L_{X/Y}[-1]$:

$$N_{X/Y} := \mathbf{V}_X(L_{X/Y}[-1]).$$

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The normal stack

- This definition makes sense in the C^∞-category.
- For the inclusion of a smooth submanifold N inside a smooth manifold M, $N_{N/M}$ is just the normal bundle.
- If $f : X \to \mathrm{pt}$ is the projection of a smooth manifold, then $N_{X/\mathrm{pt}} = [X/T_X]$ is an Artin stack, the classifying stack of the tangent bundle.

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Deformation to the normal stack

Deformation to the normal stack is an $\mathbf{A}^1$-family of algebraic stacks which deforms $f : X \to Y$ to the zero section $0 : X \to N_{X/Y}$.

Theorem

There exists a commutative diagram of derived Artin stacks

$$\begin{array}{ccccc} X & \xrightarrow{0} & X \times \mathbf{A}^1 & \xleftarrow{1} & X \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \widehat{f}} & & \downarrow{\scriptstyle f} \\ N_{X/Y} & \longrightarrow & D_{X/Y} & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{pt} & \xrightarrow{0} & \mathbf{A}^1 & \xleftarrow{1} & \mathrm{pt} \end{array}$$

where each square is cartesian.

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The construction

Let X be a quasi-smooth derived stack and $f : X \to \mathrm{pt}$.

- The long exact sequence for the closed-open decomposition

$$N_X \xrightarrow{\hat{i}} D_X \leftarrow Y \times (\mathbf{A}^1 \setminus \{0\})$$

  gives rise to a *specialization map*

$$\mathrm{sp}_X : \mathrm{H}^*(\mathrm{pt}) \to \mathrm{H}^{\mathrm{BM}}_*(N_X)$$

  where the target is Borel–Moore homology (extended to derived Artin stacks).
- By homotopy invariance for derived vector bundles, we have

$$\mathrm{H}^{\mathrm{BM}}_*(N_X) \simeq \mathrm{H}^{\mathrm{BM}}_{*+d}(X)$$

  where d is the virtual dimension of X.

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The virtual fundamental class

- The image of 1 by

$$\mathrm{H}^*(\mathrm{pt}) \xrightarrow{\mathrm{sp}_X} \mathrm{H}^{\mathrm{BM}}_*(N_X) \simeq \mathrm{H}^{\mathrm{BM}}_{*+d}(X)$$

  is a canonical element we call

$$[X]^{\mathrm{vir}} \in \mathrm{H}^{\mathrm{BM}}_d(X).$$

- If X is smooth, this is the usual fundamental class in Borel–Moore homology.
- If X is the derived zero locus of a Kuranishi chart (M, E, s), then $[X]^{\mathrm{vir}}$ is the localized Euler class $e(E, s)$.

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Pseudoholomorphic curves

- Let (X, ω, J) be a closed symplectic manifold with almost complex structure. Pseudoholomorphic maps $C \to X$ from a Riemann surface C can be organized into a derived moduli problem, i.e., a derived C^∞-stack $\mathcal{M}$.
- To apply this construction to pseudoholomorphic curves, we need a representability result for $\mathcal{M}$, i.e., that it is Deligne–Mumford. This part is still highly nontrivial: the only proof I currently know goes the work of Hofer–Wysocki–Zehnder on polyfolds.

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Lagrangian Mean Curvature Flows with Perpendicular Symmetries

Akifumi Ochiai (Tokyo Metropolitan University)

We show a method of constructing an invariant Lagrangian mean curvature flow in a Calabi–Yau manifold with the use of generalized perpendicular symmetries. We use moment maps of the action of Lie groups, which are not necessarily abelian. By our method, we construct non-trivial examples in $\mathbb{C}^n$ including self-similar solutions and translating solitons of mean curvature flows.

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Lagrangian mean curvature flows with generalized perpendicular symmetries

Tokyo Metropolitan University,　Akifumi Ochiai

November 1, 2021
The 3rd Japan-Taiwan Joint Conference on Differential Geometry

§1.　Goals

•Goal（general cases）:
To construct mean curvature flows by symmetries of Lie groups in Riemannian mfds.

•Goal（special cases）:
To construct Lagrangian mean curvature flows by generalized perpendicular symmetries of Lie groups in Calabi-Yau mfds.

§2. **Previous Researches**

•Previous Researches:

Yamamoto(2016)	
construct	generalized Lag MCF
in	toric almost Calabi-Yau mfds
using	moment map & toric symm.
Konno(2018)	
construct	Lag MCF
in	Calabi-Yau mfds
using	moment map & perp. symm. of abelian actions

•Our Researches:

Ours (general cases)	
construct	MCF
in	Riem. mfds
using	symm. of general actions
Ours (special cases)	
construct	Lag MCF
in	Calabi-Yau mfds
using	moment map & generalized perp. symm. of general actions

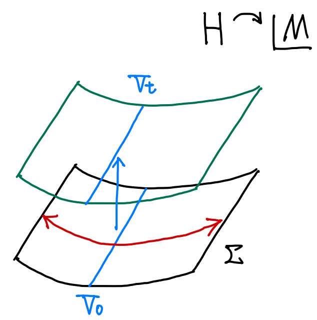
§3. **Overview**

How to construct MCF by symm. of Lie groups

M : Riem. mfd, H: Lie grp s.t. $H \curvearrowright M$,

Σ: H-invariant submfd of M.

Step.1 Find a nice sumfd $V_0 \subset \Sigma$ s.t. $H \cdot V_0 = \Sigma$.



Step.2 Study how V_0 is deformed by the MCF of Σ.

Step.3 Have the MCF of Σ by $\Sigma_t := H \cdot V_t$.

§4. Preliminaries

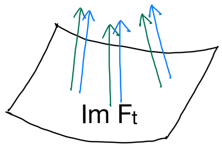
Def. 1 Let $\phi : \overset{\text{mfd}}{\Sigma} \to \overset{\text{mfd}}{M}$ be an immersion. For a smooth map

$$\begin{cases} F : \Sigma \times [0, T) \to M; & (p, t) \mapsto F_t(p) \\ F_0 = \phi \end{cases},$$

if $F_t(\cdot) : \Sigma \to M$ is an immersion for $\forall t \in [0, T)$, then we call F the **deformation** of ϕ (or Σ).

Let $\phi : \overset{\text{mfd}}{\Sigma} \to \overset{\text{Riem.mfd}}{(M, g)}$ be an immersion. The **mean curvature flow** $F = (F_t)_{t\in[0,T)}$ of ϕ is the deformation of ϕ s.t. it is a smooth solution of the following PDE:

$$\frac{\partial}{\partial t} F(p, t) = \mathcal{H}^t(p) \quad \text{w/ } \mathcal{H}^t : \text{mcv of } F_t.$$



Fact. 2 *MCF preserves the "Lagrangeness" in Kähler-Einstein mfds.*

§5. Constructions of MCFs

•Setting $(*1)$:

- (M, g): Riem. mfd,
- H: Lie grp s.t. $H \curvearrowright M$,
- K: closed subgrp of H,
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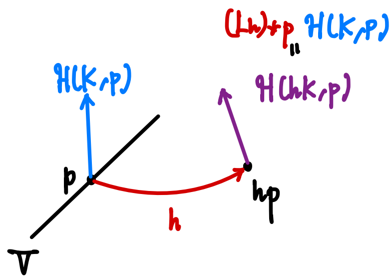


Def. 3 Under $(*1)$,

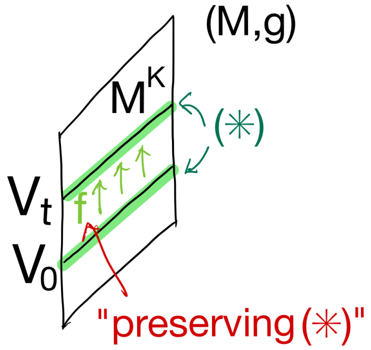
$$Z(\mathfrak{h}^*) : \text{the center of the Lie coalgebra } \mathfrak{h}^*,$$
$$L^K := \{p \in L \mid H_p = K\}, \quad \text{w/}L : \text{any submfd of } M,$$
$$\phi_V : (H/K) \times V \to M; \quad (hK, p) \mapsto hp.$$

Def. 4 (property (∗)) Under $(*1)$, if ϕ_V is an immersion & its mean curvature vectors are H-invariant, i.e., it holds that

$$\mathcal{H}(hK, p) = (L_h)_{*p}\mathcal{H}(K, p), \qquad (*)$$

then we say that V has the **property (∗)** wrt the H-actions.

Def. 5 (preserve the property (∗)) Let V_0 is a submfd of M s.t. $V_0 \subset M^K$ & has the property (∗). Under (∗1), if ∃ a deformation of V_0 in M^K & $V_t := f_t(V)$ also has the property (∗), we say that f **preserves** the property (∗) of V_0.



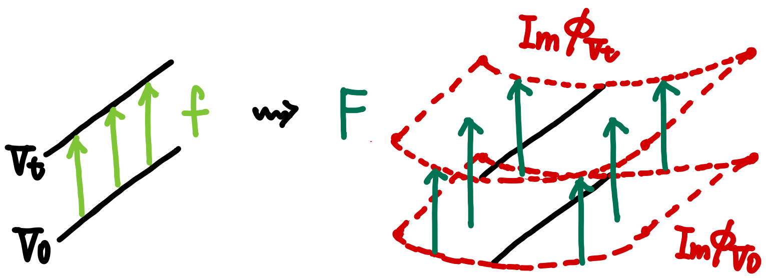
Under (∗1), suppose that ∃ a deformation $f : V_0 \times [0, T) \to M^K$.

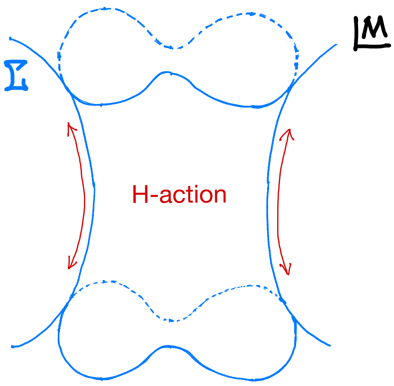
Def. 6 (expansion of deformation) If ϕ_{V_t} is an immersion for $\forall t \in [0, T)$, we can define a deformation F of ϕ_{V_0} by

$$F : (H/K) \times V_0 \times [0, T) \to M; \quad (hK, p, t) \mapsto hf_t(p) =: F_t(hK, p).$$

We call F the **expansion** of f.

We denote the mean curvature vector of F_t by $\mathcal{H}^t$.





•Setting (∗2):

- (M, g): Riem. mfd,
- H: Lie grp s.t. $H \curvearrowright M$,
- K: closed subgrp of H,
- V_0: submfd with (∗) of M (s.t. $V_0 \subset M^K$).

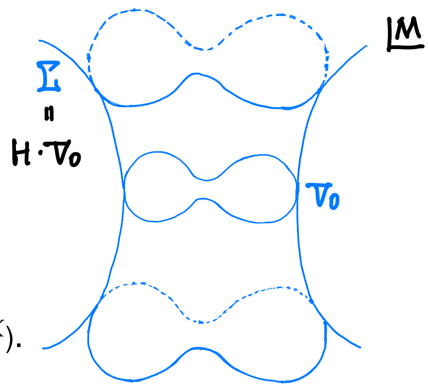
Thm. 7 *Under* (∗2), *suppose that* ∃ *a deformation* f *of* V_0 *with its expansion* F *satisfying (i) & (ii):*

(i) *For* $\forall t \in [0, T), \forall p \in V_0$,

$$\frac{\partial}{\partial t} F_t(K, p) = \mathcal{H}^t(K, p) \quad (\text{"restricted MCF condition"}),$$

(ii) f *preserves the property* (∗).

Then, $(F_t)_{t\in[0,T)}$ *is the MCF of* ϕ_{V_0}.



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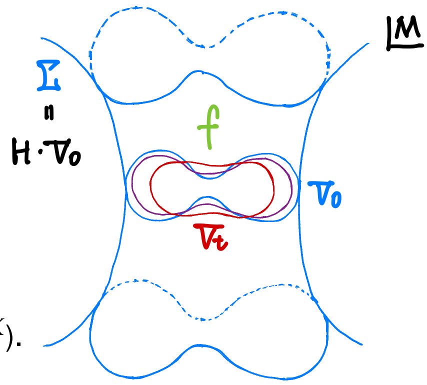

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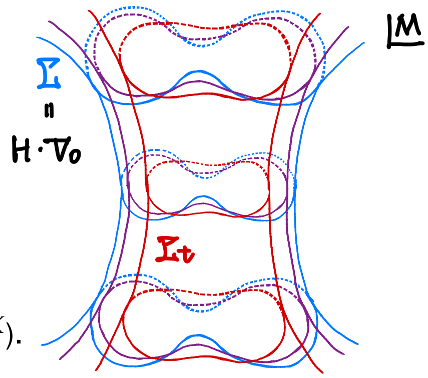

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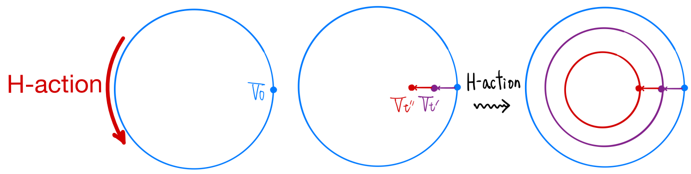
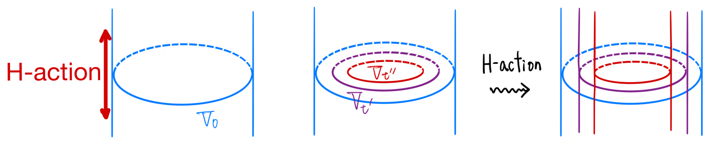
e.g. 8 (circle, sphere)

$\phi : S^n \to \mathbb{R}^{n+1},\ V_0 := \text{single point},\ H := SO(n+1).$



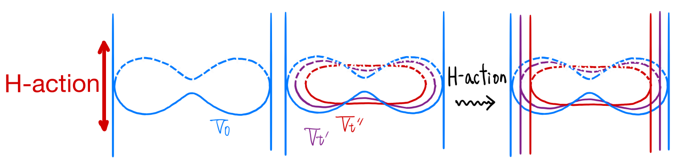
e.g. 9 (cylinder)

$\phi : S^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n+1}, V_0 := S^m,\ H := \mathbb{R}^{n-m}.$



e.g. 10 (generalized cylinder)

$\phi : M^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n+1},\ V_0 := M,\ H := \mathbb{R}^{n-m}.$



Question: How to reduce the restricted MCF eq to an ODE ?

► Additional assumption:

The evolution of the restricted MCF forms a vector field of the mean curvature vectors.

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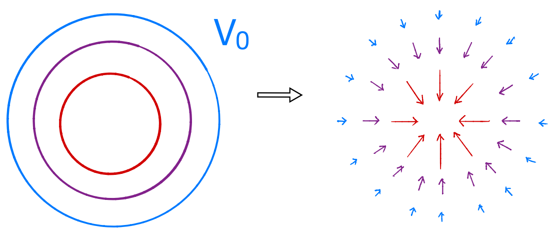

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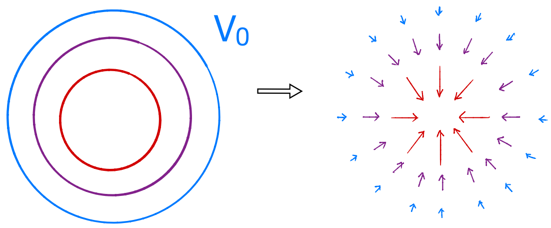

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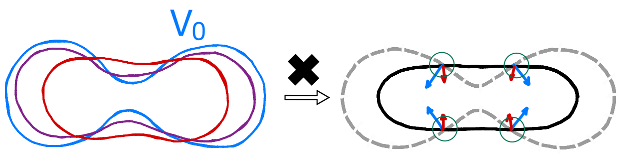

Cor. 12 *Under* $(*2)$, *suppose that the restricted MCF of* (V_0, ϕ_{V_0}) *forms a vector filed* A, *i.e.,* $\exists$ *a vector field* A *satisfying* (i.a) & (i.b)*:*

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$$\frac{d}{dt}F_t(K, p) = A_{f_t(p)} \quad (\forall p \in V_0, \forall t \in [0, T)) \quad \leftarrow \text{ODE}$$

(i.b) For $\forall t \in [0, T)$ & $p \in V_0$,

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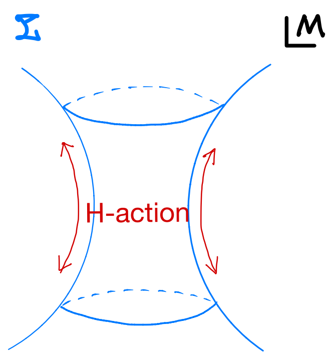

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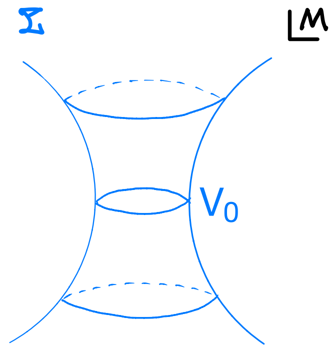

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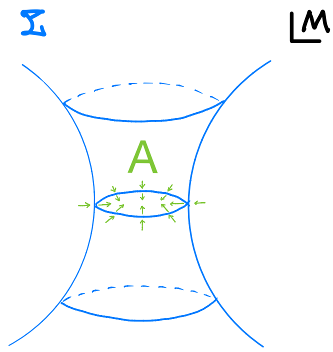

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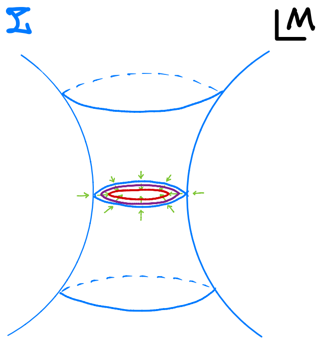

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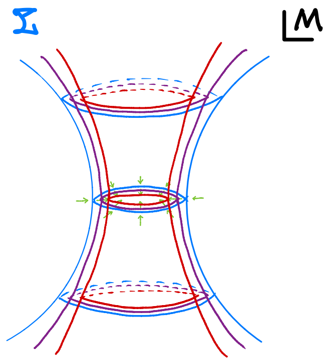

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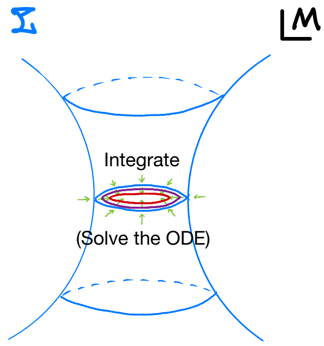

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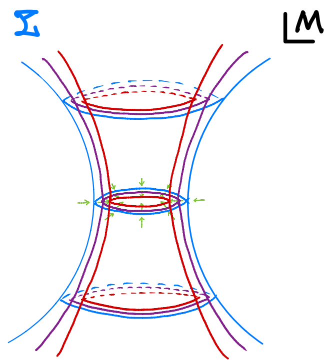

$\rightsquigarrow$ How to find V_0 with A satisfying (i.b) for constructing Lag MCFs in CY mfds ?

§6. Constructions of Lag MCFs

•Setting $(*3)$:

- (M,ω): $2n$-$\dim_{\mathbb{R}}$ symp. mfd,
- H: Lie grp s.t. $H \curvearrowright (M,\omega)$ with moment map $\mu : M \to \mathfrak{h}^*$,
- K: closed subgrp of H,
- V_c: submfd of M s.t. $V_c \subset M^K$,
- ϕ_{V_c}: immersion.

Prop. 13 *Under* $(*3)$, *suppose*

(i) V_c *is isotropic,*
(ii) *("moment map condition")* $V_c \subset \mu^{-1}(c)$ *for* $c \in Z(\mathfrak{h}^*)$.
(iii) $\dim H/K + \dim V_c = n$

Then ϕ_{V_c} *is Lagrangian. Conversely, if* ϕ_{V_c} *is connected & Lagrangian, then* (i), (ii) *and* (iii) *hold.*

Def. 14 (Lagrangian angle) (M, I, g, Ω): Calabi-Yau mfd, L: oriented Lag submafd of M,

$$\theta : L \to \mathbb{R}/2\pi\mathbb{Z} : \textbf{Lagrangian angle} :\Leftrightarrow \iota^*\Omega = e^{\sqrt{-1}\theta}\mathrm{vol}_{\iota^* g}$$

w/ $\iota : L \to M$: inclusion map.

$$L : \textbf{special Lagrangian submfd} :\Leftrightarrow \theta \equiv \mathrm{const}.$$

Prop. 15 $\mathcal{H}(p)$*: mean curvature vector of* L *at* $p \in L$. *Then,*

$$\mathcal{H}(p) = I_{\iota(p)}\left\{\iota_{*p}(\mathrm{grad}_{\iota^* g}\theta)_p\right\}.$$

•Setting $(*4)$:

- (M, I, ω, Ω): connected Calabi-Yau mfd,
- H: connected Lie grp s.t. $H \curvearrowright (M, I, \omega)$ with moment map $\mu : M \to \mathfrak{h}^*$,
- K: closed subgrp of H s.t. H/K: orientable & $K \curvearrowright \Omega$,
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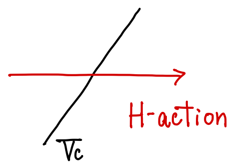

Prop. 16 *Under* $(*4)$,

(1) $\theta_c(hK, p) = \exists\, \theta_H(hK) + \exists\, \theta_{V_c}(p)$, *w/* θ_c*: Lag angle of* ϕ_{V_c},

(2) $\mathcal{H}^c(hK, p) = \exists\, (A_H)_{hp} + (L_h)_{*p} I_p \left\{ (\mathrm{grad}_{\phi^*_{V_c} g} \theta_{V_c})_p \right\}$,

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If $\theta_{V_c} \equiv \mathrm{const.} \rightsquigarrow \mathcal{H}^c = A_H$ holds and V_c accomodates to Cor.12

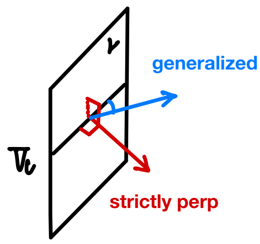
•Setting $(*5)$:

- (M, I, ω, Ω): connected Calabi-Yau mfd,
- H: connected Lie grp s.t. $H \curvearrowright (M, I, \omega)$
  with moment map $\mu : M \to \mathfrak{h}^*$,
- K: closed subgrp of H s.t. H/K: orientable & $K \curvearrowright \Omega$,
- A_H: vector field along M^K as in Prop.16
- L: special Lag submfd with Lag angle $\theta(p) \equiv \theta$,
- $c \in Z(\mathfrak{h}^*)$,
- V_c: $\big(n - \dim(H/K)\big)$-dim submfd of M s.t. $V_c \subset \mu^{-1}(c) \cap L^K$.

Prop. 17 *Under* $(*5)$, *suppose*

$$\forall p \in V_c, \forall \xi \in \mathfrak{h}, \quad \xi_p^\# \in \mathrm{T}_p^\perp L \oplus \mathrm{T}_p V_c \ \&\ \xi_p^\# \notin \mathrm{T}_p V_c \setminus \{0\}.$$

("generalized perp. condition")

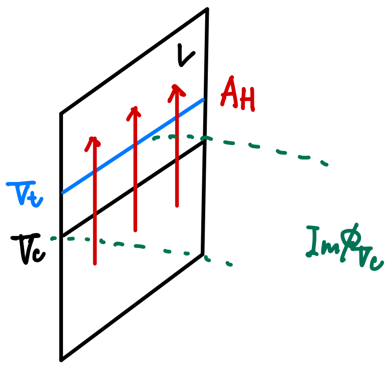


Then,

(1) $\theta_{V_c}(p) = \theta - \frac{\pi}{2} \dim(H/K), \quad \leftarrow \mathrm{const.}$

(2) $\mathcal{H}^c(hK, p) = (A_H)_{hp}$.

Thm. 18 *Under* $(*5)$, *suppose that* A_H *generates a deformation* $f : V_c \times [0, T) \to L^K$ *with its expansion* F, *and for* $\forall t \in [0, T)$ *and* $V_t := f_t(V_c)$, *the generalized perpendicular condition holds.*
Then, A_H *and* V_c *satisfies the condition of Cor.12 and* $(F_t)_{t \in [0,T)}$ *is a Lag MCF of* ϕ_{V_c}.



§7. **Examples**

e.g. 19

construct	*Lag self-similar solution*
in	$\mathbb{C}^4$
using	*strictly perp. symm. of* $U(1) \times SO(3)$

e.g. 20

construct	*Lag MCF*
in	$\mathbb{C}^5$
using	*gen. perp. symm. of* $\mathbb{R} \times SO(2)$

e.g. 21

construct	*Lag translating soliton*
in	$\mathbb{C}^5$
using	*strictly perp. symm. of* $U(1) \times SO(3)$

e.g. 22

construct	*Lag translating soliton*
in	$\mathbb{C}^6$
using	*gen. perp. symm. of* $\mathbb{R} \times SO(2)$

Thank you very much for your attention.

Lagrangian Mean Curvature Flow with Boundary

Albert Wood (National Taiwan University)

The Lagrangian mean curvature flow is the name given to the remarkable fact that mean curvature flow preserves the class of Lagrangian submanifolds in Kahler-Einstein manifolds. A natural follow-up question that springs to mind is whether there exists a suitable boundary condition for this flow, such that the resulting flow with boundary still preserves the Lagrangian condition. Remarkably, standard Neumann and Dirichlet boundary conditions do not work, but there is a symplectically natural mixed Dirichlet-Neumann boundary condition involving a boundary Lagrangian flow which does. In this talk I will describe the condition and give an overview of the proof, as well as describe some examples of the flow's behaviour.

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Third Japan-Taiwan Joint Conference on Differential Geometry

Lagrangian Mean Curvature Flow with Boundary

Albert Wood

National Taiwan University

November 1, 2021

Preliminaries

Mean Curvature Flow

Mean curvature flow is the gradient descent for the volume functional of submanifolds of Riemannian manifolds.
Let N^n be a smooth manifold, and M^m a smooth Riemannian manifold. A family of immersions $F_t : N^n \to (M^m, \overline{g})$ is a **mean curvature flow** if

$$\frac{dF}{dt} = \vec{H},$$

where $\vec{H}$ is the vector-valued second fundamental form of the embedding,

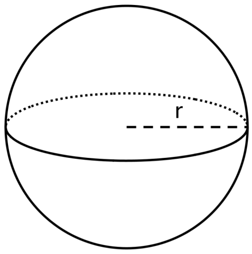
$$\vec{H} := \text{trace}(g^{-1}\vec{A}).$$

Preliminaries

Examples of Mean Curvature Flow

Shrinking Sphere in $\mathbb{R}^n$:

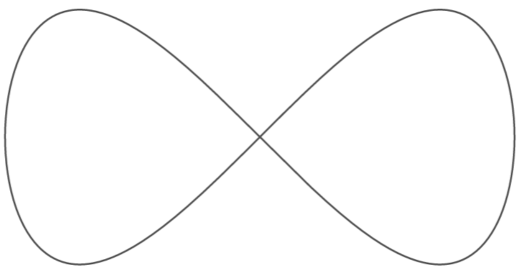
$$\frac{dr}{dt} = -\frac{n}{r}$$
$$\implies r = \sqrt{R - 2nt}.$$



Preliminaries

Examples of Mean Curvature Flow

Curve Shortening Flows:

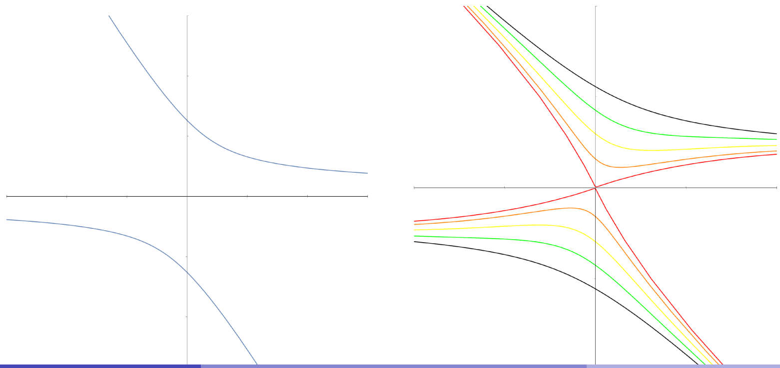


Preliminaries

Examples of Mean Curvature Flow

$O(n)$-Equivariant Flows in $\mathbb{C}^n$: Flows of the form $L(s,\alpha) = (a(s)\alpha, b(s)\alpha) \in \mathbb{C}^n$ for $\alpha \in S^{n-1}$, a, b real functions. Quotienting by the spherical symmetry, we obtain the **profile curve** $\gamma(s) = a(s) + ib(s) \in \mathbb{C}$. The mean curvature flow reduces to the following flow of the profile curve:

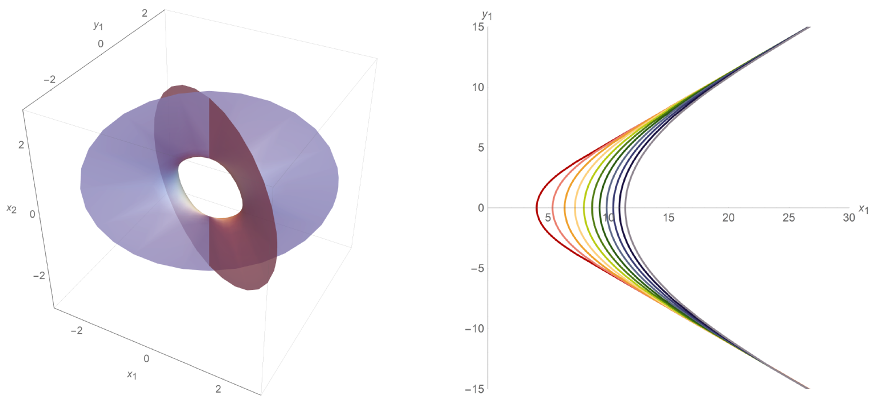
$$\frac{d\gamma}{dt} = \vec{k} - (n-1)\frac{\gamma^{\perp}}{|\gamma|^2}.$$



Preliminaries

Examples of Mean Curvature Flow: $O(n)$-Equivariant Flows in $\mathbb{C}^n$

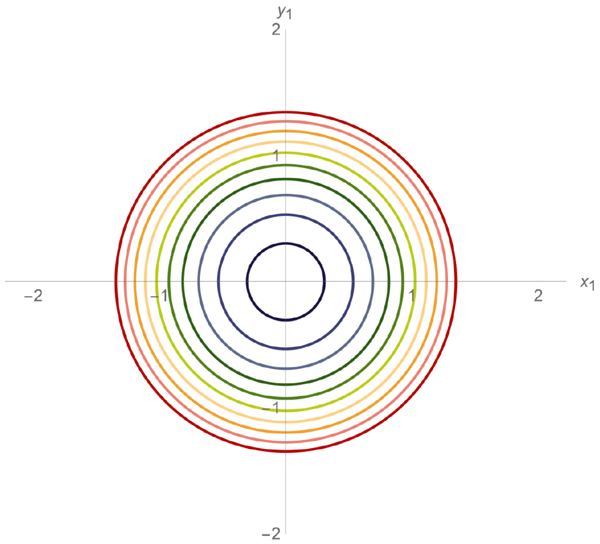
Here is a self-expanding equivariant flow known as the **Anciaux expander**:



Preliminaries

Examples of Mean Curvature Flow: $O(n)$-Equivariant Flows in $\mathbb{C}^n$

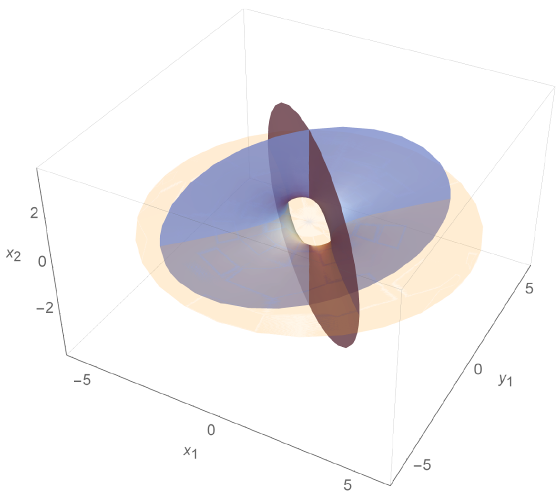
Here is a self-shrinking equivariant flow - the **Clifford torus**.

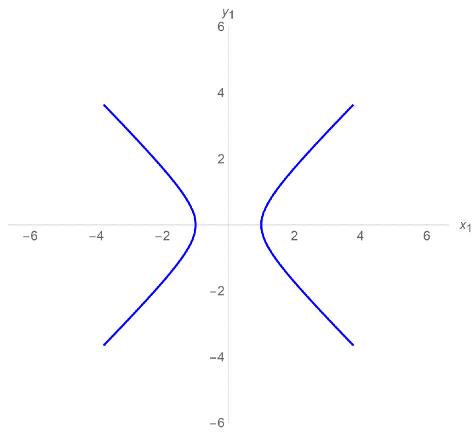


Preliminaries

Examples of Mean Curvature Flow: $O(n)$-Equivariant Flows in $\mathbb{C}^n$

Finally, here is a static flow (i.e. a minimal submanifold): the **Lawlor Neck**.





Preliminaries

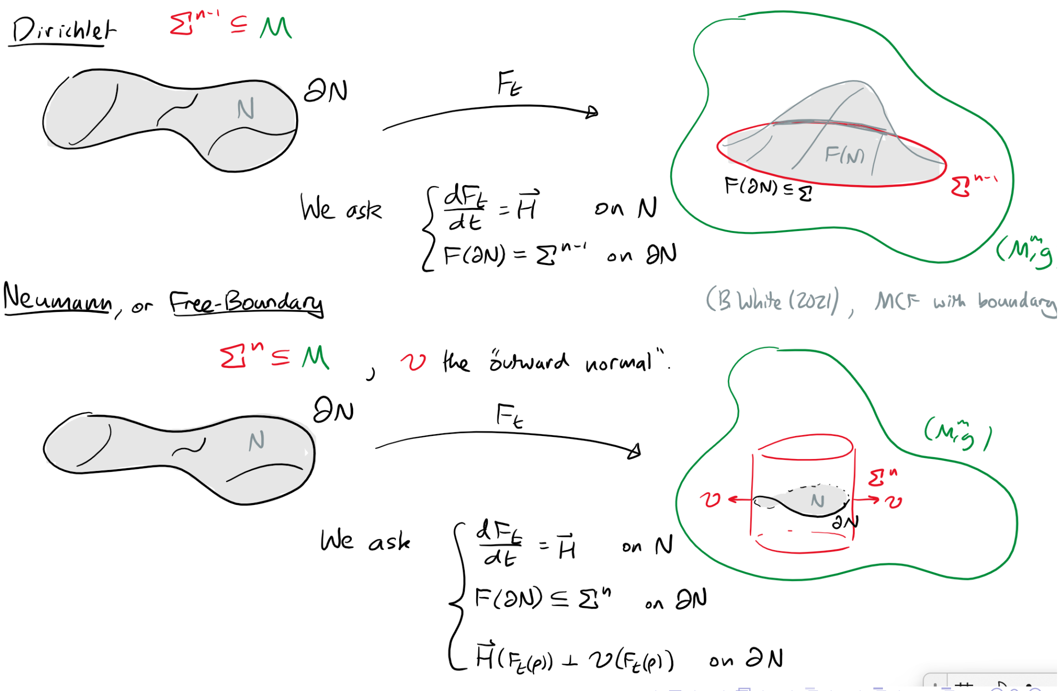
Mean Curvature Flow with Boundary

How may one extend the concept of mean curvature flow $F : N \to (M^m, \overline{g})$ for manifolds N with a boundary ∂N?

One natural option is to ask for the boundary $F(\partial N) = \Sigma^{n-1}$ to remain fixed during the flow - a **Dirichlet boundary condition**.

Another is to require that $F(\partial N) \subset \Sigma^n$ for a submanifold $\Sigma^n \subset M$. We then must demand one extra condition, for example a perpendicularity condition at the boundary - this is known as a **free boundary condition.**

Preliminaries



Lagrangian Mean Curvature Flow

Kahler Manifolds

A smooth manifold $(M, \overline{g}, J, \omega)$ with compatible smooth, complex and symplectic structures is known as a Kähler manifold. A submanifold $L \subset M$ is **Lagrangian** if $\omega|_L = 0$.

- $J : TL \to TL^{\perp}$ is an isomorphism.
- H can be considered a 1-form, h a fully symmetric $(0, 3)$-tensor.
- In a Kähler-Einstein manifold, the mean curvature 1-form H is closed.

If M is a Calabi-Yau manifold, there is a holomorphic volume form Ω which may be used to define a primitive called the **Lagrangian angle**:

$$\Omega|_L = e^{i\theta} vol_L,$$

$$d\theta = H.$$

L is **special Lagrangian** if it is minimal. If M is Calabi-Yau, then this is equivalent to θ being constant.

Lagrangian Mean Curvature Flow

Lagrangian Graphs in $\mathbb{C}^n$

Let $f : \mathbb{R}^n \to \mathbb{R}^n$. When is f Lagrangian?

$$f(x) = (x^1, \ldots, x^n, f^1(x), \ldots, f^n(x))$$

$$\frac{\partial f}{\partial x^i} = \left(0, \ldots, 1, \ldots, 0, \frac{\partial f^1}{\partial x^i}, \ldots, \frac{\partial f^n}{\partial x^i}\right)$$

$$\omega = \sum dx^i \wedge dy^i$$

$$\implies \omega\left(\frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x^j}\right) = \frac{\partial f}{\partial x^j} - \frac{\partial f}{\partial x^i}.$$

So, defining $\alpha = f^i dx^i$, the graph of f is Lagrangian if and only if α is a closed 1-form.

Lagrangian Mean Curvature Flow

Theorem (K. Smoczyk)

In a Kähler-Einstein manifold, the class of closed Lagrangian submanifolds is preserved under MCF.

Proof.

A calculation shows that under a normal variation $\vec{N}$, $\frac{d\omega_t}{dt} = dN$, where N is the associated 1-form to $\vec{N}$.
Under MCF, since H is closed, initially $\frac{d\omega|_L}{dt} = 0$. This isn't enough, as this calculation only holds while L is Lagrangian, and this may immediately cease to be true! Instead, work with **totally real** submanifolds, and show that

$$\frac{d}{dt}|\omega_t|^2 \leq \Delta|\omega_t|^2 + c|\omega_t|^2.$$

□

Lagrangian Mean Curvature Flow

Lagrangian MCF of Graphs in $\mathbb{C}^n$

Let α be a closed form in $\mathbb{R}^n$, so for some $u : \mathbb{R}^n \to \mathbb{R}$, $\alpha = du$.

$$F(x) = \begin{pmatrix} x \\ (u_k)_{k=1}^n \end{pmatrix}$$

$$\frac{\partial F}{\partial x^i} = \begin{pmatrix} e_i \\ (u_{ik})_{k=1}^n \end{pmatrix}$$

$$g_{ij} = \delta_{ij} + u_{ik}u_{jk}$$

$$h_{ijk} = \left\langle \frac{\partial^2 F}{\partial x^i \partial x^j}, J\frac{\partial F}{\partial x^k} \right\rangle = u_{ijk}.$$

$$H_k = g^{ij}h_{ijk}.$$

Lagrangian Mean Curvature Flow

Lagrangian MCF of Graphs in $\mathbb{C}^n$

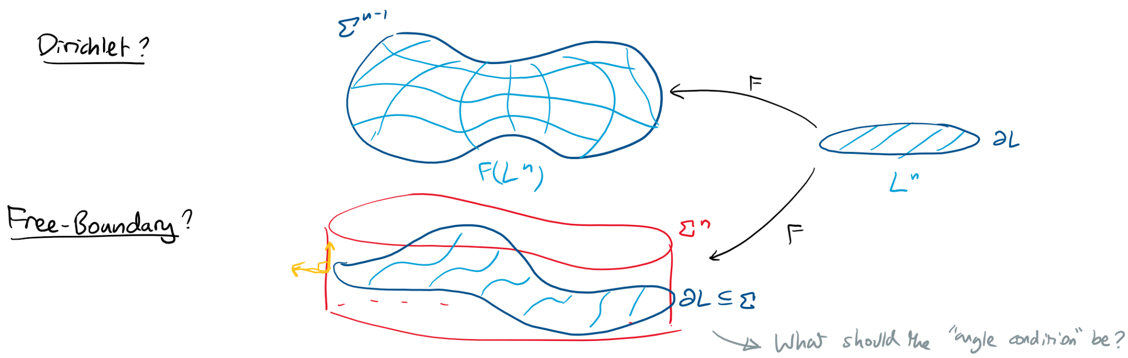
Remembering that $d\theta = H$ for the associated 1-form H of the mean curvature vector $\vec{H}$,

$$
\begin{aligned}
& & H_k &= g^{ij} h_{ijk} \\
\left(\frac{\partial F}{\partial t}\right)^{\perp} = \vec{H} & \iff & du &= H \\
& \iff & \frac{du}{dt} &= \theta + C(t).
\end{aligned}
$$

Note that θ constant (special Lagrangian condition) implies that u changes only by a global constant, and therefore the immersion remains static.

Lagrangian MCF with Boundary

We now ask the question - May the Lagrangian mean curvature flow be extended to submanifolds with boundary L? In other words, is there a boundary condition we can put on $F(\partial L)$ which preserves the boundary condition?



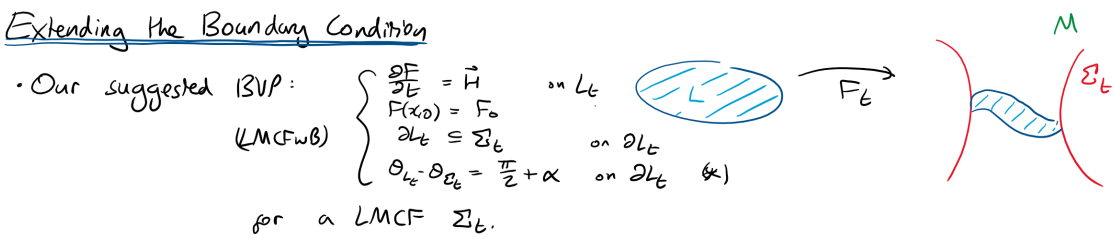
Lagrangian MCF with Boundary

Unfortunately, Dirichlet conditions don't work. If Neumann is to work, what should the boundary condition be? Remember that

$$d\theta = H.$$

So, constant θ means that the submanifold is static under mean curvature flow (a minimal submanifold). Such Lagrangians are known as **special Lagrangians**. So a simple example of a Lagrangian MCF with boundary is a special Lagrangian immersion with boundary on another special Lagrangian.
Perhaps demanding a constant Lagrangian angle difference at the boundary is a way to extend this example?

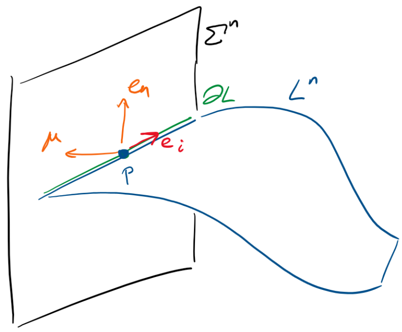
Lagrangian MCF with Boundary



Unfortunately this doesn't work, as we need to prove that the flow remains Lagrangian for the concept of the Lagrangian angle θ_L to make sense. We need a generalised Lagrangian angle.

Lagrangian MCF with Boundary

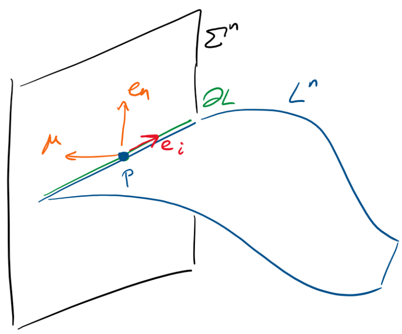
Extending the Lagrangian Angle



At $p \in F(\partial L)$, choose an orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ of $T_p\partial L$, which we complete to bases of T_pL and $T_p\Sigma$.
Then, $\Omega(e_1, \ldots, e_{n-1}, \mu) = re^{i\theta_L}$ defines a Lagrangian angle θ_L, as long as $r \neq 1$ (totally real condition).

Lagrangian MCF with Boundary

Extending the Lagrangian Angle



Moreover, since Σ is Lagrangian, we may define a 'relative holomorphic volume form',

$$\Omega' = \bigwedge_{i=1}^{n} (e_i)^* + i(Je_i)^*.$$

Then $\Omega'(e_1, \ldots, e_{n-1}, \mu) = re^{i(\theta_L - \theta_\Sigma)}$ defines the **relative Lagrangian angle** $\theta_L - \theta_\Sigma$ without need for a global holomorphic volume form!

Lagrangian MCF with Boundary

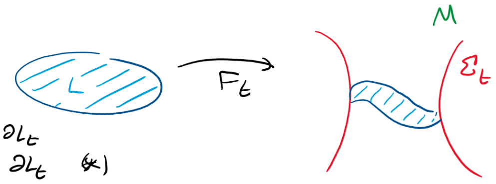
Main Theorem

Extending the Boundary Condition

• Our suggested BVP: (LMCFwB)

$$\begin{cases} \frac{\partial F}{\partial t} = \vec{H} & \text{on } L_t \\ F(x,0) = F_0 & \\ \partial L_t \subseteq \Sigma_t & \text{on } \partial L_t \\ \theta_{L_t} - \theta_{\Sigma_t} = \frac{\pi}{2} + \alpha & \text{on } \partial L_t \quad (*) \end{cases}$$

for a LMCF Σ_t.



Theorem (Evans, Lambert, W.)

Given Σ_t a Lagrangian MCF in a Kähler-Einstein manifold M, defined on $(0, T)$, $F_0 : L \to M$ a Lagrangian immersion of a manifold L with boundary ∂L, then the solution to (LMCFwB) exists for short time, is unique, and remains Lagrangian.

Lagrangian MCF with Boundary

Proof Sketch

Proof.

From Smoczyk, on the interior $\frac{\partial}{\partial t}|\omega|^2 \leq \Delta|\omega|^2 + C|\omega|^2$.
We must complement this with a boundary estimate of the form $\nabla_\mu|\omega|^2 \leq C|\omega|^2$.
We are able to achieve this by proving symmetries of the second fundamental form of L inherited from Σ by the boundary condition.
Then, choosing a distance function ρ from the boundary, and considering $f = |\omega|^2 e^{A\rho - Bt}$, it follows from the above estimates that at the boundary, $\nabla_\mu f \leq |\omega|^2 e^{A\rho - Bt}(C - A)$, which is negative if A is large.
At an interior increasing maximum, $0 \leq \left(\frac{\partial}{\partial t} - \Delta\right) f = |\omega|^2 e^{A\rho - Bt}(C - B)$, which is a contradiction if we pick B sufficiently large. So there is no increasing maximum. □

Equivariant Examples

- Choose $\gamma : \mathbb{R} \to \mathbb{C}$ "profile curve"

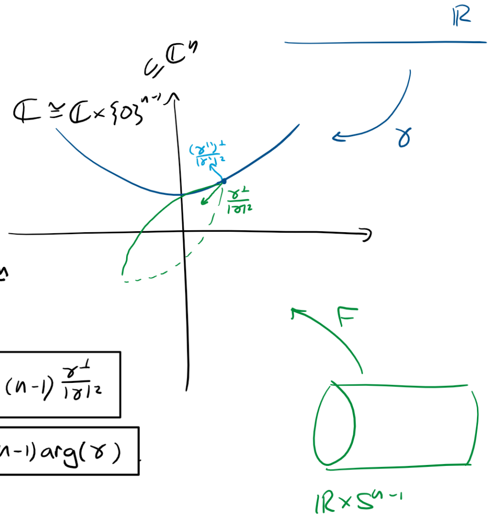
Then $F : \mathbb{R} \times S^{n-1} \to \mathbb{C}^n$

$F(s,\sigma) = \gamma(s) \cdot \sigma$

is an $SO(n)$-equivariant Lagrangian for the diagonal $SO(n)$ action on $\mathbb{C}^n$.

The mean curvature is given by $\vec{H}(s,\sigma) = \frac{(\gamma'')^{\perp}}{|\gamma'|^2} - (n-1)\frac{\gamma^{\perp}}{|\gamma|^2}$

Lagrangian angle: $\theta(s,\sigma) = \arg(\gamma') + (n-1)\arg(\gamma)$

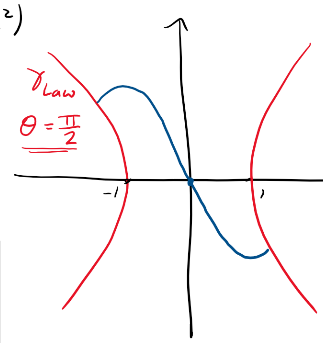


Equivariant Examples

Equivariant LMCF with boundary on the Lawlor Neck (in $\mathbb{C}^2$)

(LMCFwB) becomes

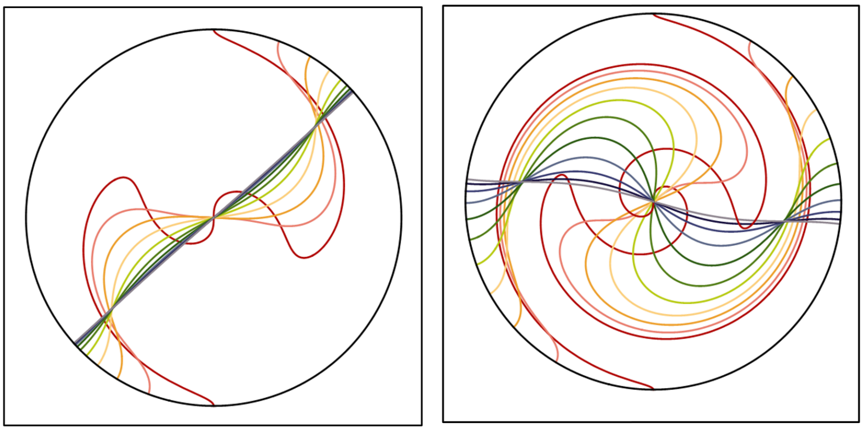
$$\begin{cases} \left(\frac{dF}{dt}\right)^{\perp} = \vec{H} & \text{on } L_t \\ F_0 = L_0 & \\ \partial L_t \subseteq \Sigma_{Law} & \\ \theta_t|_{\partial L_t} = -\alpha & \text{on } \partial L_t \end{cases}$$



Theorem Let L_0 be an S^1-equivariant Lagrangian disc in $\mathbb{C}^2$, with Lagrangian angle θ_0, $\cos(\theta_0) > \varepsilon$.

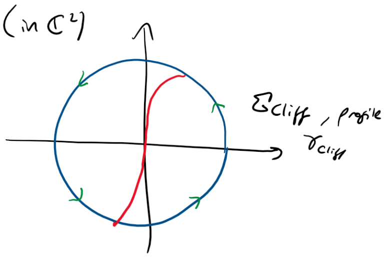
Then there is a unique immortal solution to (LMCFwB) which converges in infinite time to the special Lagrangian disc with $\theta = -\alpha$

Equivariant Examples



Equivariant Examples

Equivariant LMCF with boundary on the Clifford Torus (in $\mathbb{C}^2$)



- More interesting situation.
- If F_t solves $\left(\frac{\partial F}{\partial t}\right)^\perp = \vec{H}$,
  then
  $$\overline{F}_\tau := \frac{1}{\sqrt{-t}} F_t \Big|_{t=-e^{-\tau}}$$
  solves $\left(\frac{\partial \overline{F}}{\partial \tau}\right)^\perp = \vec{H} + \frac{\overline{F}^\perp}{2}$.

Theorem Let $\overline{L}_0$ be an equivariant Lagrangian disc in $\mathbb{C}^2$, with boundary on the Clifford torus

$$\Sigma_{cliff} = \{2e^{i\phi}(\cos(\psi), \sin(\psi)) \in \mathbb{C}^2 : \phi, \psi \in [0, 2\pi)\}$$

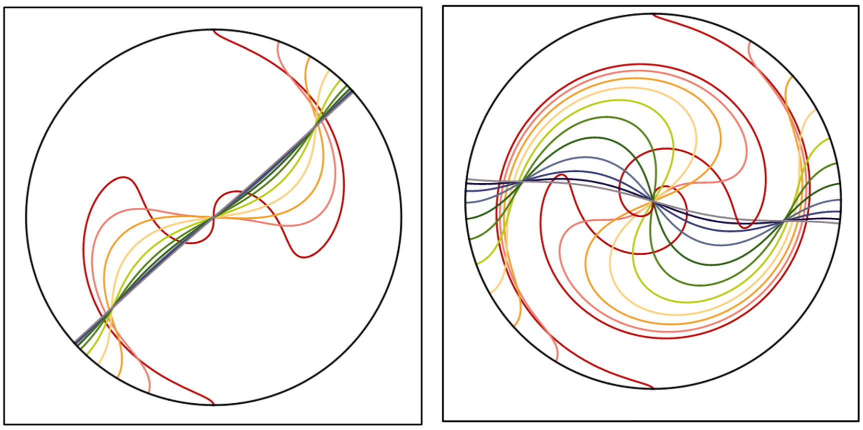
Assume its Lagrangian angle satisfies

$$\cos(\theta_0(s) - 2\arg(\overline{\gamma}_0(s))) > \varepsilon.$$

Then there is a unique, eternal solution to the rescaled LMCF problem:

(Rescaled LMCFwB)
$$\begin{cases} \left(\frac{\partial \overline{F}}{\partial \tau}\right)^\perp = \vec{H} + \frac{\overline{F}^\perp}{2} \\ F_{\tau_0} = \overline{L}_0 \\ \partial \overline{L}_\tau \subseteq \Sigma_{cliff} \\ \theta_\tau|_{\partial \overline{L}_\tau} - 2\arg(\overline{\gamma}_0) = 0 \end{cases}$$

Equivariant Examples



Homotopy Fiber Product of Manifolds

Hsuan-Yi Liao (National Tsing Hua University)

A main motivation of developing derived differential geometry is to deal with singularities arising from zero loci or intersections of submanifolds. Both zero loci and intersections can be considered as fiber products of manifolds. Thus, we extend the category of differentiable manifolds to a larger category in which one has "homotopy fiber products". In this talk, I would like to show a construction, using vector bundles and sections, of homotopy fiber products of manifolds and explain the structures behind the construction. The talk is mainly based on a joint work with Kai Behrend and Ping Xu.

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Homotopy fiber product of manifolds

Hsuan-Yi Liao
joint work with Kai Behrend and Ping Xu

NATIONAL TSING HUA UNIVERSITY

Third Japan-Taiwan Joint Conference on Differential Geometry
Nov 1 - 3, 2021

NOTE:
In this talk, all the manifolds are C^∞ manifolds over $\mathbb{R}$, and all the maps are C^∞ maps.

Recall (fiber products):
Given smooth maps $X \xrightarrow{f} Z \xleftarrow{g} Y$ between manifolds, one can form the fiber product (as topological spaces)

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

If f or g is a submersion (i.e. the tangent map is surjective at each point), then $X \times_Z Y$ is a manifold.

In general, $X \times_Z Y$ is NOT a manifold.

Example

Let X, Y be submanifolds of M. The intersection $X \cap Y$ of X and Y in M can be identified with

$$X \times_M Y = \{(x, y) \in X \times Y \mid \iota_1(x) = \iota_2(y)\},$$

where $\iota_1 : X \hookrightarrow M$, $\iota_2 : Y \hookrightarrow M$ are embeddings of submanifolds.

Example

Let $f \in C^\infty(M, \mathbb{R})$. The zero set $Z(f)$ of f in M can be identified with $M \times_{M\times\mathbb{R}} M$, where the maps are

$$M \to M \times \mathbb{R} : x \mapsto (x, f(x)),$$
$$M \to M \times \mathbb{R} : y \mapsto (y, 0).$$

Purpose of the talk

- Want to study: $X \times_Z Y$.
- Approach: resolve the singular spaces $X \times_Z Y$ by vector bundles and sections.
- Goal of today:
  - introduce quasi-smooth derived manifolds (= vector bundle + a global section)
  - construct homotopy fiber products of manifolds
  - explain the categorical structure behind the construction, i.e., the following theorem:

Theorem (Behrend, L, Xu)

The category of derived manifolds is a category of fibrant objects.

Here, derived manifold = finite-dimensional bundle of curved $L_\infty[1]$ algebras of positive amplitudes.

Quasi-smooth derived manifolds

Idea: Resolve $Z(f)$ by $(M \times \mathbb{R}, f)$.

- A quasi-smooth derived manifold $\mathcal{M} = (M, L, \lambda)$ is a vector bundle $L \to M$ together with a global section $\lambda \in \Gamma(M, L)$.
- A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ is a vector bundle map such that the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\phi} & L' \\ \lambda \uparrow\downarrow & & \downarrow\uparrow \lambda' \\ M & \xrightarrow[f]{} & M' \end{array}$$

- $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ is called a **fibration** / **submersion** if f is a submersion and $\phi|_p : L|_p \to L'|_{f(p)}$ is surjective $\forall p \in M$.

Problem: There are too many (M, L, λ) with same zero locus $Z(\lambda)$, so we need a certain notion of equivalence.

Tangent complex and weak equivalence

Assume $p \in M$ is a classical/Maurer-Cartan locus of $\mathcal{M} = (M, L, \lambda)$, i.e., $\lambda(p) = 0_p \in L|_p$. Define $D_p\lambda$ by

$$D_p\lambda : TM|_p \xrightarrow{T\lambda|_p} TL|_{0_p} \cong TM|_p \oplus L|_p \xrightarrow{\mathrm{pr}} L|_p$$

The tangent complex of $\mathcal{M}$ at $p \in Z(\lambda)$ is the two-term complex

$$T\mathcal{M}|_p \quad := \quad TM|_p \xrightarrow{D_p\lambda_0} L|_p$$

The derived dimension $\dim^h(\mathcal{M})$ of $\mathcal{M}$ = the Euler characteristic of $T\mathcal{M}|_p = \dim(M) - \mathrm{rk}(L)$.
A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ of derived manifolds induces a cochain map

$$T(f, \phi)|_p : T\mathcal{M}|_p \to T\mathcal{M}'|_{f(p)}$$

Definition

A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ of derived manifolds is called a **weak equivalence** if

- f induces a bijection on classical loci, and
- $T(f, \phi)|_p : T\mathcal{M}|_p \to T\mathcal{M}'|_{f(p)}$ is a quasi-isomorphism at each classical locus $p \in Z(\lambda)$.

In particular, if there is a weak equivalence $\mathcal{M} \to \mathcal{M}'$, then $\mathcal{M}$ and $\mathcal{M}'$ have the same derived dimension.

Example

If $\lambda \in \Gamma(M, L)$ is a regular section (i.e. $D_p\lambda : TM|_p \to L|_p$ is surjective $\forall p \in Z(\lambda)$), then $Z(\lambda)$ is a manifold of dimension $\dim^h \mathcal{M} = \dim M - \mathrm{rk}\, L$. And the inclusion map $Z(\lambda) = (Z(\lambda), Z(\lambda) \times 0, 0) \to (M, L, \lambda)$ is a weak equivalence.

Algebraic model

The **function algebra** $C^\infty(\mathcal{M})$ of $\mathcal{M} = (M, L, \lambda)$ is the commutative differential graded algebra $(\Gamma(\Lambda^{-\bullet} L^\vee), \iota_\lambda)$:

$$\cdots \xrightarrow{\iota_\lambda} \Gamma(\Lambda^2 L^\vee) \xrightarrow{\iota_\lambda} \Gamma(\Lambda^1 L^\vee) \xrightarrow{\iota_\lambda} C^\infty(M) \to 0$$

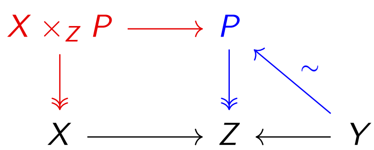
A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ induces a morphism of cdga's by pullback: $\phi^* : C^\infty(\mathcal{M}') \to C^\infty(\mathcal{M})$.

Proposition

$(f, \phi) : \mathcal{M} \to \mathcal{M}'$ is a weak equivalence iff $\phi^* : C^\infty(\mathcal{M}') \to C^\infty(\mathcal{M})$ is a quasi-isomorphism of cdga's.

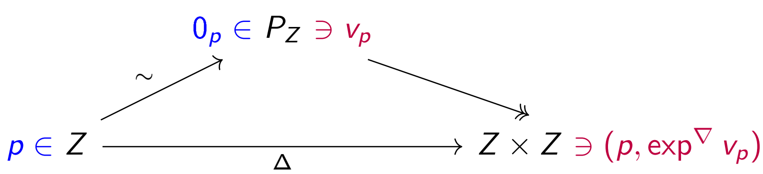
Homotopy fiber product

Idea of homotopy fiber products:



Construction of P:

First construct an important case: diagonal map $\Delta : Z \to Z \times Z$



$P_Z = (TZ, TZ \times_Z TZ, \delta)$, $\delta(v_p) = (v_p, v_p)$

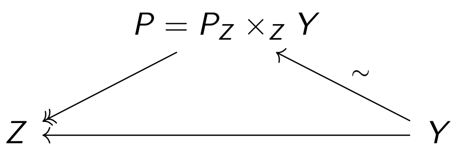
Note: base space TZ is actually a neighborhood of the image of zero section in TZ where $\exp^\nabla$ is defined.

- $Z = (Z, Z, 0) \xrightarrow{\sim} P_Z = (TZ, TZ \times_Z TZ, \delta) : (p, 0_p) \mapsto (0_p, (0_p, 0_p))$ is a weak equivalence because it induces
  - bijection between zero loci: $Z(0) = Z \to \{0_p \in T_Z\} = Z(\delta)$
  - quasi-isomorphism between tangent complexes:

$$\begin{array}{ccc} 0 & \longrightarrow & TZ|_p \\ \uparrow & & \uparrow \scriptstyle{\mathrm{pr}_2} \\ TZ|_p & \xrightarrow[i_1]{} & T(TZ)|_{0_p} = TZ|_p \oplus TZ|_p \end{array}$$

- $P_Z = (TZ, TZ \times_Z TZ, \delta) \to Z \times Z = (Z \times Z, Z \times Z, 0) : (v_p, (v_p, w_p)) \mapsto ((p, \exp^\nabla v_p), 0)$ is a fibration because the underlying map is a submersion and the linear maps between fibers are onto.

General case:



$$P = P_Z \times_Z Y = (TZ \times_Z Y, TZ \times_Z TZ \times_Z Y, \delta),\ \delta(v_p, y) = (v_p, v_p, y)$$

- base space $TZ \times_Z Y$ is a manifold because $TZ \to Z : v_p \mapsto \exp^\nabla v_p$ is a submersion.
- $P \to Z$ is a fibration: $TZ \times Y \to Z : (v_p, y) \mapsto p$ is a submersion.
- $Y \to P$ is a weak equivalence: $Z(\delta) = \mathrm{Graph}(Y \to Z) \cong Y$, and its tangent map at y is

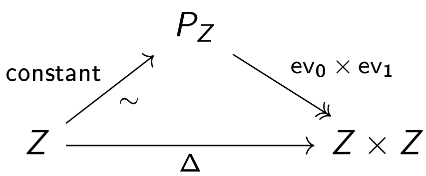
$$\begin{array}{ccc} 0 & \longrightarrow & TZ|_p \\ \uparrow & & \uparrow \scriptstyle{\mathrm{pr}_1} \\ TY|_y & \xrightarrow[i_2]{} & T(TZ \times_Z Y)|_{(0_p, y)} = TZ|_p \oplus TY|_y \end{array}$$

Path space description of P_Z

- Fix a connection ∇ and fix an open interval $I = (c, d) \supset [0, 1]$.
- $P_Z = (P_g Z, P_{\mathrm{con}} TZ\, dt,\ D)$, where
  - $P_g Z := \{a : I \to Z \mid \nabla_{a'} a' = 0\}$ consists of short geodesics.
  - a fiber $P_{\mathrm{con}} TZ\, dt|_a$ over $a \in P_g Z$ is

$$P_{\mathrm{con}} TZ\, dt|_a = \{\alpha\, dt \mid \alpha \in \Gamma(I, a^* TZ),\ (a^*\nabla)(\alpha) = 0\}$$

  - $D : P_g Z \to P_{\mathrm{con}} TZ\, dt : a \mapsto a'\, dt$ is given by derivatives



Introduction Quasi-smooth derived manifolds Homotopy fiber product of manifolds Categories of fibrant objects

Homotopy fiber product of manifolds

Given smooth maps $X \to Z \leftarrow Y$, the homotopy fiber product $X \times^h_Z Y$ is represented by a quasi-smooth derived manifold

$$X \times_Z P_Z \times_Z Y = (X \times_Z T_Z \times_Z Y, X \times_Z T_Z \times_Z T_Z \times_Z Y, \delta),$$

$\delta(x, v_p, y) = (x, v_p, v_p, y)$.

- the classical locus $Z(\delta) \cong X \times_Z Y$ as sets.
- the derived dimension $\dim^h(X \times_Z P_Z \times_Z Y) = \dim X + \dim Y - \dim Z$.
- if one of $X \to Z \leftarrow Y$ is a submersion, then the map $X \times_Z Y \to X \times_Z P_Z \times_Z Y : (x, y) \mapsto (x, 0_p, y)$ is a weak equivalence.

Derived intersections

Let X, Y be submanifolds of a manifold M. The derived intersection $X \cap^h_M Y$ of X and Y in M is understood as

$$X \cap^h_M Y := X \times^h_M Y$$

which is represented by $X \times_M P_M \times_M Y = (N, E, \tilde{D})$, where

- $N = X \times_M TM \times_M Y = X \times_M P_g M \times_M Y$
  = space of short geodesics which start from a point in X and end at a point in Y
  = an open submanifold of $X \times Y$ consisting of $(x, y) \in X \times Y$ such that x and y are sufficiently close to the set-theoretical intersection $X \cap Y$;

- the fiber $E|_a$ over $a \in N$ is

$$E|_a = \{\alpha\, dt \mid \alpha \in \Gamma(a^* TM),\ (a^*\nabla)(\alpha) = 0\} \cong TM|_{a(0)};$$

- the section

$$\tilde{D} : N \to E : a \mapsto a'\, dt$$

  is given by derivatives.

Furthermore,

- classical locus of $X \cap^h_M Y$ = set-theoretical intersection $X \cap Y$;
- $\dim^h(X \cap^h_M Y) = \dim(X) + \dim(Y) - \dim(M)$;
- if X and Y intersect transversally, then $X \cap Y \to (N, E, \tilde{D})$ is a weak equivalence.

1. Quasi-smooth derived manifolds
2. Homotopy fiber product of manifolds
3. Categories of fibrant objects

Introduction Quasi-smooth derived manifolds Homotopy fiber product of manifolds Categories of fibrant objects

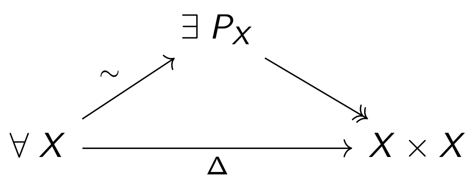
Some problems of quasi-smooth derived manifolds:

- The category of quasi-smooth derived manifolds is NOT closed under homotopy fiber products.
- A weak equivalence is NOT necessarily invertible. To get the expected equivalence relation, we need higher structures.

A solution:
We further extend the category of quasi-smooth derived manifolds to a larger category — the category of derived manifolds — and show that this category is a category of fibrant objects. This structure guarantees a solution to the above problems.

A category of fibrant objects is a category $\mathcal{C}$, together with two subcategories, the category of fibrations and the category of weak equivalences, such that

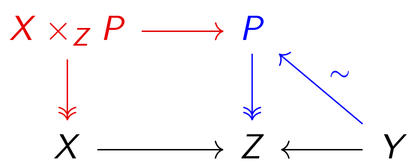
1. weak equivalences satisfy two out of three, all isomorphisms are weak equivalences,
2. all isomorphisms are fibrations,
3. every pullback of a fibration exists, and is again a fibration,
4. every pullback of a trivial fibration (i.e. a fibration which is a weak equivalence) is a trivial fibration,
5. $\mathcal{C}$ has a final object, and all the morphisms ending at the final object are fibrations,
6. there exists a path space object for every object, i.e.



Factorization Lemma (Brown)

Let $\mathcal{C}$ be a category of fibrant objects. Any morphism $X \to Y$ in $\mathcal{C}$ can be factored $X \xrightarrow{\sim} P \twoheadrightarrow Y$, where $X \xrightarrow{\sim} P$ is a section of a trivial fibration, and $P \twoheadrightarrow Y$ is a fibration.

Idea of homotopy fiber products:



In fact, Brown constructed a homotopy fiber product using path space objects: $X \times^h_Z Y = X \times_Z P_Z \times_Z Y$ which is well-defined in the homotopy category.

Derived manifolds

A derived manifold is a triple $\mathcal{M} = (M, L, \lambda)$, where

- M is a manifold,
- $L = L^1 \oplus \cdots \oplus L^n$ is a finite-dimensional positively graded vector bundle over M,
- $\lambda = (\lambda_k)_{k\geq 0}$ is a smooth family of curved $L_\infty[1]$ structures on L.

That is,

$$\lambda_k : S^k L \to L, \qquad k \geq 0,$$

are degree one vector bundle maps such that

$$Q_\lambda \circ Q_\lambda = 0,$$

where $Q_\lambda \in \mathrm{coDer}^1_{C^\infty(M)}(\Gamma(SL))$ is the coderivation generated by $\lambda : SL = \bigoplus_{k\geq 0} S^k L \to L$.

A morphism of derived manifolds $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ is a smooth map $f : M \to M'$ together with a smooth family of morphisms of curved $L_\infty[1]$ algebras $\phi = (\phi_k)_{k\geq 1} : L \rightsquigarrow f^* L'$.

That is,

$$\phi_k : S^k L \to L', \qquad k \geq 1,$$

are degree zero vector bundle maps such that

$$Q_{\lambda'} \circ F_\phi = F_\phi \circ Q_\lambda,$$

where $F_\phi : \Gamma(SL) \to \Gamma(SL')$ is the coalgbra morphism generated by $\phi = \sum_k \phi_k : SL \to L'$.

Remark

The degree restrictions imply that there are only finite nonzero λ_k and ϕ_k.

Explicit $L_\infty[1]$ equations

First few equations in $L_\infty[1]$:

- $\lambda_1(\lambda_0) = 0.$
- $\lambda_2(\lambda_0, x) = \lambda_1^2(x).$
- $\lambda_3(\lambda_0, x, y) + \lambda_2(\lambda_1(x), y) + (-1)^{|x||y|}\lambda_2(\lambda_1(y), x) + \lambda_1(\lambda_2(x, y)) = 0$
- $\phi_1(\lambda_0) = \lambda_0'.$
- $\phi_2(\lambda_0, x) + \phi_1(\lambda_1(x)) = \lambda_1'(\phi_1(x)).$

Special cases:

- Manifold case: $L = M \times 0.$
- Quasi-smooth case: $L = L^1.$

Weak equivalences and fibrations

Assume $p \in M$ is a classical/Maurer-Cartan locus of $\mathcal{M}$, i.e., $\lambda_0(p) = 0 \in L^1|_p$. Define $D_p\lambda_0$ by

$$D_p\lambda_0 : TM|_p \xrightarrow{T\lambda_0|_p} TL^1|_{0_p} \cong TM|_p \oplus L^1|_p \xrightarrow{\text{pr}} L^1|_p$$

The tangent complex of $\mathcal{M}$ at $p \in Z(\lambda_0)$ is

$$T\mathcal{M}|_p \quad := \quad TM|_p \xrightarrow{D_p\lambda_0} L^1|_p \xrightarrow{\lambda_1|_p} L^2|_p \xrightarrow{\lambda_1|_p} \cdots$$

The derived dimension $\dim^h(\mathcal{M})$ of $\mathcal{M}$ = the Euler characteristic of $T\mathcal{M}|_p = \dim(M) - \text{rk}(L^1) + \text{rk}(L^2) - \cdots$.
A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ of derived manifolds induces a cochain map

$$T(f, \phi)|_p : T\mathcal{M}|_p \to T\mathcal{M}'|_{f(p)}$$

Definition

A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ of derived manifolds is called a **weak equivalence** if

- f induces a bijection on classical loci, and
- $T(f, \phi)|_p : T\mathcal{M}|_p \to T\mathcal{M}'|_{f(p)}$ is a quasi-isomorphism at each classical locus $p \in Z(\lambda_0)$.

In particular, if there is a weak equivalence $\mathcal{M} \to \mathcal{M}'$, then $\mathcal{M}$ and $\mathcal{M}'$ have the same derived dimension.

Definition

A morphism $(f, \phi) : \mathcal{M} \to \mathcal{M}'$ of derived manifolds is called a **fibration** if

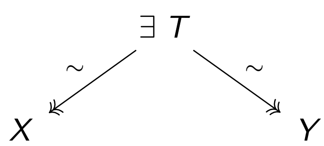
- $f : M \to M'$ is a submersion, and
- $\phi_1 : L \to f^*L'$ is surjective.

Theorem (Behrend, L, Xu)

The category of derived manifolds is a category of fibrant objects.

Back to the 2 problems:

- Existence of homotopy fiber products of derived manifolds is guaranteed: $X \times^h_Z Y = X \times_Z P_Z \times_Z Y$.
  For a derived manifold Z, we construct P_Z explicitly by actual path spaces (short geodesics).
- By a property of categories of fibrant objects, two derived manifolds X and Y are isomorphic in the homotopy category iff there exists the following diagram of derived manifolds:



Thank you!

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Rigid fibers of integrable systems on cotangent bundles

Ryuma Orita

Abstract. We deal with classical integrable systems such as the Lagrangian top and the Kovalevskaya top. Especially, we find a non-displaceable fiber for each of them. This is a joint work with Morimichi Kawasaki (Aoyama Gakuin University).

1 Introduction

Let (M, ω) be a symplectic manifold (i.e., ω is a non-degenerate closed 2-form on M). Then, every smooth function (called *Hamiltonian*) $H\colon [0,1] \times M \to \mathbb{R}$ with compact support defines a time-dependent vector field on M by the formula

$$\omega(X_{H_t}, \cdot) = -dH_t,$$

where $H_t = H(t, \cdot)$ for $t \in [0,1]$. Let $\{\varphi_H^t\}_t$ denote the flow of X_{H_t}. Namely, it satisfies

$$\frac{d\varphi_H^t}{dt} = X_{H_t} \circ \varphi_H^t, \quad \varphi_H^0 = \mathrm{id}_M.$$

The time-one map $\varphi_H = \varphi_H^1$ is called the *Hamiltonian diffeomorphism with compact support* generated by H.

A subset $X \subset M$ is called *displaceable* from a subset $Y \subset M$ if there exists a Hamiltonian $H\colon [0,1] \times M \to \mathbb{R}$ with compact support such that $\varphi_H(X) \cap \overline{Y} = \emptyset$. Otherwise, X is called *non-displaceable* from Y.

Example 1.1. Consider a height function h on the 2-sphere $S^2 \subset \mathbb{R}^3$ equipped with the standard symplectic (i.e., area) form. Then every fiber of h, other than the equator, is displaceable from itself. Indeed, there exists a Hamiltonian circle action which displaces the fiber from itself. On the other hand, the equator is non-displaceable from itself since every diffeomorphism displacing the equator cannot be area-preserving. Note that every Hamiltonian diffeomorphism is a symplectomorphism, and hence in dimension 2, it is area-preserving.

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Let k be a positive integer. We call a smooth map $\Phi = (\Phi_1, \dots, \Phi_k)\colon M \to \mathbb{R}^k$ a *moment map* if $\{\Phi_i, \Phi_j\} = 0$ for all $1 \leq i, j \leq k$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket on (M, ω). Entov and Polterovich [EP] proved the following theorem (compare to Example 1.1).

Theorem 1.2 ([EP, Theorem 2.1])**.** *Let (M, ω) be a closed symplectic manifold and $\Phi = (\Phi_1, \dots, \Phi_k)\colon M \to \mathbb{R}^k$ a moment map. Then, there exists $y_0 \in \Phi(M)$ such that $\Phi^{-1}(y_0)$ is non-displaceable.*

2 Main results

We consider the cotangent bundle (T^*N, ω_0) of a closed smooth n-dimensional manifold N where ω_0 is the standard symplectic form on T^*N. Let (q, p) be canonical coordinates on T^*N. Let $\pi\colon T^*N \to N$ denote the natural projection.

Definition 2.1 ([KO, Definition 1.4])**.** A (time-independent) Hamiltonian $H\colon T^*N \to \mathbb{R}$ satisfies *condition* $(\star)$ if the following conditions hold.

(i) For any $c \in \mathbb{R}$ the sublevel set $H^{-1}\big((-\infty, c]\big) \subset T^*N$ is compact.

(ii) For any $q \in N$,
$$H(q, 0) = \min_{p \in T_q^*N} H(q, p).$$

For a Hamiltonian $H\colon T^*N \to \mathbb{R}$ satisfying condition $(\star)$, we set

$$m_H = \max_{q \in N} \min_{p \in T_q^*N} H(q, p) \quad \text{and} \quad S_H = H^{-1}(m_H) \cap 0_N,$$

where 0_N denotes the zero-section of T^*N.

Typical examples of Hamiltonians satisfying condition $(\star)$ are *convex Hamiltonians*
$$H(q, p) = \frac{1}{2}\|p\|_g^2 + U(q),$$
where $\|\cdot\|_g$ is the dual norm of a Riemannian metric g on N and $U\colon N \to \mathbb{R}$ is a smooth potential. In this case, the value m_H equals the *Mañé critical value* $\max_N U$ (see [Ma]) and
$$S_H = \left\{ (q, 0) \in T^*N \;\middle|\; U(q) = \max_N U \right\}.$$

Now we are in a position to state the main result.

Theorem 2.2 ([KO, Corollary of Theorem 1,7])**.** *Let N be a closed manifold and $\Phi = (\Phi_1, \dots, \Phi_k)\colon T^*N \to \mathbb{R}^k$ a moment map. Assume that Φ_1 satisfies condition $(\star)$ and that the set $\Phi(S_{\Phi_1})$ is a singleton, i.e., $\Phi(S_{\Phi_1}) = \{y_0\}$ for some $y_0 \in \mathbb{R}^k$. Then, the fiber $\Phi^{-1}(y_0)$ of Φ is non-displaceable from itself and from the zero-section 0_N. Moreover, every fiber of Φ, other than $\Phi^{-1}(y_0)$, is displaceable from 0_N.*

One can apply Theorem 2.2 to a broad class of classical Liouville integrable systems. Here we provide a sample example of *spinning tops.* Interested readers are cordially invited to our recent paper [KO].

Corollary 2.3 ([KO, Examples 2.9 and 2.10])**.** *Let* $\Phi = (\Phi_1, \Phi_2, \Phi_3) \colon T^*\mathrm{SO}(3) \to \mathbb{R}^3$ *be the energy moment map of either Lagrange top or Kovalevskaya top, where* Φ_1 *is the Hamiltonian of the system. Then, the fiber* $\Phi^{-1}\big(\Phi(S_{\Phi_1})\big)$ *is non-displaceable from itself and from the zero-section* $0_{\mathrm{SO}(3)}$*. Moreover, every fiber of* Φ*, other than* $\Phi^{-1}\big(\Phi(S_{\Phi_1})\big)$*, is displaceable from* $0_{\mathrm{SO}(3)}$*.*

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Geometric Quantization on CR Manifolds

Chin-Yu Hsiao (Academia Sinica)

We consider a compact connected orientable CR manifold with the action of a connected compact Lie group. Under natural pseudoconvexity assumptions we show that the CR GuilleminSternberg map is Fredholm at the level of Sobolev spaces of CR functions. As an application we study this map for holomorphic line bundles which are positive near the inverse image of zero by the momentum map. We also show that"quantization commutes with reduction" for Sasakian manifolds. This is a joint work with Xiaonan Ma and George Marinescu.

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Geometric quantization on CR manifolds

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Classical Guillemin-Sternberg quantization commutes with reduction theorem

- (L, h^L) : a holomorphic line bundle over a connected compact complex manifold (M, J),
- h^L is a Hermitian fiber metric of L.
- R^L : the curvature of (L, h^L).
- G : a connected compact Lie group with Lie algebra $\mathfrak{g}$. Assume that
  - G acts holomorphically on (M, J),
  - the action lifts to a holomorphic action on L,
  - h^L is preserved by the G-action.
  - $\omega := \frac{i}{2\pi} R^L$ is a G-invariant form.

Classical Guillemin-Sternberg quantization commutes with reduction theorem

- $\mu : M \to \mathfrak{g}^*$: the momentum map induced by ω. Assume that
    - $0 \in \mathfrak{g}^*$ is regular,
    - the action of G on $\mu^{-1}(0)$ is free.
- $M_G := \mu^{-1}(0)/G$: a complex manifold with natural complex structure induced by J (complex reduced space).
- $L_G := L/G$: a holomorphic line bundle over M_G.

Classical Guillemin-Sternberg quantization commutes with reduction theorem

Theorem (Guillemin-Sternberg (1982))

Suppose that $R^L > 0$ on M. We have
$\dim H^0(M, L^m)^G = \dim H^0(M_G, L^m_G)$, *for every $m \in \mathbb{N}^*$.*

- $H^0(M_G, L^m_G)$: *the space of holomorphic sections on M_G with values in L^m_G.*
- $H^0(M, L^m)^G$: *the space of G-invariant holomorphic sections with values in L^m.*

CR viewpoint

- Let X be the circle bundle of (L^*, h^{L^*}), i.e. $X := \left\{v \in L^*;\ |v|^2_{h^{L^*}} = 1\right\}$.
- X is a compact strongly pseudoconvex CR manifold with a group action G.
- X admits a S^1 action $e^{i\theta}$: $e^{i\theta} \circ (z, \lambda) := (z, e^{i\theta}\lambda)$, where λ denotes the fiber coordinate of X.
- $H^0_b(X)^G$: G-invariant L^2 CR functions.

CR viewpoint

- For every $m \in \mathbb{Z}$, let

$$H^0_{b,m}(X)^G := \left\{u \in H^0_b(X)^G;\ (e^{i\theta})^* u = e^{im\theta} u, \text{for every } e^{i\theta} \in S^1\right\},$$
$$H^0_{b,m}(X_G) := \left\{u \in H^0_b(X_G);\ (e^{i\theta})^* u = e^{im\theta} u, \text{for every } e^{i\theta} \in S^1\right\}.$$

- $H^0_b(X)^G := \oplus_{m\in\mathbb{Z}} H^0_{b,m}(X)^G$, $H^0_b(X_G) := \oplus_{m\in\mathbb{Z}} H^0_{b,m}(X_G)$,
- Grauert: For every $m \in \mathbb{Z}$,

$$H^0(M, L^m)^G \cong H^0_{b,m}(X)^G, \quad H^0(M_G, L^m_G) \cong H^0_{b,m}(X_G).$$

CR viewpoint

- From Guillemin-Sternberg theorem,
  - $H^0_{b,m}(X)^G \cong H^0_{b,m}(X_G)$, for every $m \in \mathbb{Z}$.
  - $H^0_b(X)^G \cong H^0_b(X_G)$.

Motivation

- Generalize Guillemin-Sternberg theorem to general compact CR manifolds.
- This generalization is improtant in CR, contact and irregular Sasaki geometry.

Motivation

- $\mu^{-1}(0)$ should determine $H^0(M, L)^G$.
- Does Guillemin-Sternberg theorem holds when L is just positive near $\mu^{-1}(0)$?

Chin-Yu Hsiao Geometric quantization on CR manifolds

Difficulty

- The quantum spaces consist of CR functions and are infinite dimensional.
- $H_b^0(X)$ is infinite dimensional and $H_b^0(X)$ is not a subspace of $\mathcal{C}^\infty(X)$.
- We need new idea and new approach (Szegő kernel method and G-invariant microlocal F.I.O. calculation).

Chin-Yu Hsiao Geometric quantization on CR manifolds

CR manifolds

- Let X be a smooth and orientable manifold of dimension $2n+1$, $n \geq 1$.
- Let $T^{1,0}X$ be a subbundle of $\mathbb{C}TX$ the complexified tangent bundle of X.

Definition

We say that $T^{1,0}X$ is a CR structure of X if

- (i) $\dim_{\mathbb{C}} T^{1,0}_x X = n$, for every $x \in X$.
- (ii) $T^{1,0}X \bigcap T^{0,1}X = \{0\}$, where $T^{0,1}X := \overline{T^{1,0}X}$.
- (iii) $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, $\mathcal{V} = \mathcal{C}^\infty(X, T^{1,0}X)$.

- For a $2n+1$ dimensional smooth manifold X, if we can find a CR structure $T^{1,0}X$ on X, we call the pair $(X, T^{1,0}X)$ a CR manifold.

CR manifolds

- Take a Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TX$ such that we have the orthogonal decompositions:
  - $\mathbb{C}TX = T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}T$, $T \in \mathcal{C}^\infty(X, TX)$, $\|T\| = 1$,
  - $\mathbb{C}T^*X = T^{*1,0}X \oplus T^{*0,1}X \oplus \mathbb{C}\omega_0$, $\omega_0 \in \mathcal{C}^\infty(X, T^*X)$, $\|\omega_0\| = 1$,
  - $\langle \omega_0 \,, T \rangle = -1$, $T^{*0,1}X = (T^{1,0}X \oplus \mathbb{C}T)^\perp$.
  - ω_0: Reeb one form, T: Reeb vector field, $T^{*0,1}X$: bundle of $(0,1)$ forms.

Definition

For $p \in X$, the Levi form $\mathcal{L}_p$ is the Hermitian quadratic form on $T^{1,0}_p X$ given by $\mathcal{L}_p(U, V) = -\frac{1}{2i} d\omega_0(p)(U, \overline{V})$, $U, V \in T^{1,0}_p X$.

CR manifolds

- We say that X is strongly psudoconvex at $p \in X$ if the Levi form is positive definite at $p \in X$.
- We say that X is strongly psudoconvex if the Levi form is positive definite at each point of X.

CR functions

- Let $\tau : \mathbb{C}T^*X \to T^{*0,1}X$ be the orthogonal projection.
- $\overline{\partial}_b = \tau \circ d : \mathcal{C}^\infty(X) \to \Omega^{0,1}(X)$: tangential Cauchy-Riemann(CR) operator, where $\Omega^{0,1}(X) = \mathcal{C}^\infty(X, T^{*0,1}X)$.
- We extend $\overline{\partial}_b$ to L^2 space: $\overline{\partial}_b : \mathrm{Dom}\,\overline{\partial}_b \subset L^2(X) \to L^2_{(0,1)}(X)$, where $\mathrm{Dom}\,\overline{\partial}_b = \left\{u \in L^2(X);\, \overline{\partial}_b u \in L^2(X)\right\}$.
- For a function $u \in L^2(X)$, we say that u is a CR function if $u \in \mathrm{Ker}\,\overline{\partial}_b$.
- If X is strongly pseudoconvex at some point of X and $\overline{\partial}_b$ has L^2 closed range, then $\dim \mathrm{Ker}\,\overline{\partial}_b = +\infty$ (Boutet de Monvel-Sjöstrand, Hsiao-Marinescu).

CR manifolds with group action

- Let $(X, T^{1,0}X)$ be a compact connected CR manifold of dimension $2n+1$, $n \geq 2$.
- Now, we assume that
  - X admits a d-dim'l connected compact Lie group action G with Lie algebra $\mathfrak{g}$.
  - The Lie group action G preserves ω_0 and CR structure. That is, $g^*\omega_0 = \omega_0$ and $dg(T^{1,0}X) = T^{1,0}X$, for every $g \in G$, $g : X \to X$.
- Goal: Study $H^0_b(X)^G$ the space of global G-invariant L^2 CR functions.

CR momentum map

Definition

The momentum map associated to the form ω_0 is the map $\mu : X \to \mathfrak{g}^*$ such that, for all $x \in X$ and $\xi \in \mathfrak{g}$, we have

$$\langle \mu(x), \xi \rangle = \omega_0(\xi_X(x)), \tag{1}$$

$\xi \in \mathfrak{g}$, ξ_X : the vector field on X induced by ξ.

- We will work under the following natural assumption.

Assumption

0 is a regular value of μ, the action of G on $\mu^{-1}(0)$ is free and the Levi form of X is positive definite near $\mu^{-1}(0)$.

- X is not necessarily strongly pseudoconvex.

CR reduction

- Let $X_G := \mu^{-1}(0)/G$.

Theorem (H/Huang, H/Ma/Marinescu)

X_G is a strongly pseudoconvex CR manifold of dimension $2n-2d+1$ with natural CR structure $T^{1,0}X_G$ induced from $T^{1,0}X$.

- Is $H^0_b(X)^G \cong H^0_b(X_G)$?

canonical map σ_G

- $\iota : \mu^{-1}(0) \to X$: the natural inclusion.
- $\iota^* : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(\mu^{-1}(0))$: the pull-back by ι.
- $\iota_G : \mathcal{C}^\infty(\mu^{-1}(0))^G \to \mathcal{C}^\infty(X_G)$: the natural identification.
- $\sigma_G := \iota_G \circ \iota^* : H^0_b(X)^G \cap \mathcal{C}^\infty(X)^G \to H^0_b(X_G) \cap \mathcal{C}^\infty(X_G)$.

canonical map σ_G

- The map σ_G does not extend to a bounded operator on L^2.
- It necessary to consider its extension to Sobolev spaces.

Sobolev CR functions

For every $s \in \mathbb{R}$, put

- $H^0_b(X)^G_s :=$ $\left\{u \in H^s(X);\ \overline{\partial}_b u = 0,\ h^* u = u, \text{ for every } h \in G\right\}.$
- $H^0_b(X_G)_s := \left\{u \in H^s(X_G);\ \overline{\partial}_b u = 0\right\}.$
- $H^s(X)$: Sobolev space of X of order s.

canonical map σ_G

Theorem 0 (H/Ma/Marinescu)

Suppose that $\overline{\partial}_{b,X_G}$ has L^2 closed range and the Levi form is positive definite near $\mu^{-1}(0)$.

- *σ_G extends by density to a bounded operator*

$$\sigma_G = \sigma_{G,s} : H^0_b(X)^G_s \to H^0_b(X_G)_{s-\frac{d}{4}}, \quad \text{for every } s \in \mathbb{R}. \qquad (2)$$

Geometric quantization on CR manifolds

Theorem I (H/Ma/Marinescu)

Suppose that $\overline{\partial}_{b,X_G}$ has L^2 closed range and the Levi form is positive definite near $\mu^{-1}(0)$.

- *For every $s \in \mathbb{R}$, the map $\sigma_{G,s}$ is Fredholm.*
- *$\mathrm{Ker}\,\sigma_{G,s}$ and $(\mathrm{Im}\,\sigma_{G,s})^\perp$ are finite dimensional subspaces of $\mathcal{C}^\infty(X) \cap H^0_b(X)^G$ and $\mathcal{C}^\infty(X_G) \cap H^0_b(X_G)$, respectively.*
- *$\mathrm{Ker}\,\sigma_{G,s}$ and the index $\dim \mathrm{Ker}\,\sigma_{G,s} - \dim (\mathrm{Im}\,\sigma_{G,s})^\perp$ are independent of s.*

Geometric quantization on CR manifolds

- Theorem I establishes "quantization commutes with reduction" for some contact manifolds.
- If $\dim X_G \geq 5$ or X admits a transversal CR $\mathbb{R}$-action or X is a circle bundle, then $\overline{\partial}_{b,X_G}$ has L^2 closed range (Kohn, Yeganefar-Marinescu).

Geometric quantization on CR manifolds

- There is a Pseudodifferential operator E on X_G of order $-\frac{d}{4}$ such that
- $\tilde{\sigma}_G := S_{X_G} \circ E \circ \sigma_G : H^0_b(X)^G \to H^0_b(X_G)$ is Fredholm.

Applications: Complex manifolds

- Apply Theorem I to circle bundles, we deduce

Theorem

Suppose that R^L is positive near $\mu^{-1}(0)$. Then, for $|m|$ large, we have

$$\dim H^0(M, L^m)^G = \dim H^0(M_G, L_G^m).$$

Examples

- Let $X = \{(z_1, z_2, z_3) \in \mathbb{C}^3;\ |z_1|^4 + |z_2|^2 + |z_3|^2 = 1\}$.
- X is a weakly pseudocovex CR manifold.
- X admits a S^1-action: $e^{i\theta} \circ (z_1, z_2, z_3) = (e^{-i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3)$.
- 0 is a regular value of μ.
- $\mu^{-1}(0) = \{(z_1, z_2, z_3) \in \mathbb{C}^3;\ |z_1|^4 = \frac{1}{3}, |z_2|^2 + |z_3|^2 = \frac{2}{3}\}$.
- X is strongly pseudoconvex near $\mu^{-1}(0)$.

Examples

- Let
  $X = \{z \in \mathbb{C}^6;\ \sum_{j=1}^{6} |z_j|^2 + z_1z_3 + z_2z_4 + \overline{z}_1\overline{z}_3 + \overline{z}_2\overline{z}_4 = 1\}$.
- X is a strongly pseudocovex CR manifold (not come from line bundles).
- X admits a $SU(2)$-action:
  $g \circ z = (g(z_1, z_2)^t, \overline{g}(z_3, z_4)^t, g(z_5, z_6)^t)$, $g \in SU(2)$.
- 0 is a regular value of μ.

Applications: Sasakian manifolds

- Let $(X, T^{1,0}X)$ be a compact strongly pseudoconvex CR manifold.
- We say that X is torsion free if there is a Reeb vector field T on X such that
  - T preserves the CR structure $T^{1,0}X$,
- We call T CR Reeb vector field on X.
- Ornea and Verbitsky: A $(2n+1)$-dimensional smooth manifold X is a Sasakian manifold if and only if X is a torsion free strongly pseudoconvex CR manifold.

Applications: Sasakian manifolds

- X is a quasi-regular (regular) Sasakian manifold if the flow of T induces a locally free (free) S^1-action on X.
- X is an irregular Sasakian manifold if there is an orbit of the flow of T which is non-compact.
- In this case, the flow of T induces a transversal CR $\mathbb{R}$-action on X.
- We now assume that X is an irregular Sasakian manifold with a CR Reeb vector field T and suppose that the Lie group G preserves T and CR structure on X.

Applications: Sasakian manifolds

- Consider the operators

$$-iT : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X),$$
$$-i\widehat{T} : \mathcal{C}^\infty(X_G) \to \mathcal{C}^\infty(X_G),$$

  - $\widehat{T}$ is the CR Reeb vector field on X_G,
- We extend $-iT$ and $-i\widehat{T}$ to L^2 spaces in the standard way.

Applications: Sasakian manifolds

Theorem (Herrmann/H/Li,)

We have that $\mathrm{Spec}\,(-iT)$ *is countable and every element in* $\mathrm{Spec}\,(-iT)$ *is an eigenvalue of* $-iT$*, where* $\mathrm{Spec}\,(-iT)$ *denotes the spectrum of* $-iT$*.*

- Put

$$\mathrm{Spec}\,(-iT) = \{\alpha_1, \alpha_2, \ldots\} \subset \mathbb{R},$$
$$\mathrm{Spec}\,(-i\widehat{T}) = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{R},$$
$$H^0_{b,\alpha}(X)^G := \left\{u \in H^0_b(X)^G;\ -iTu = \alpha u\right\}, \quad \alpha \in \mathrm{Spec}\,(-iT),$$
$$H^0_{b,\beta}(X_G) := \left\{v \in H^0_b(X_G);\ -i\widehat{T}v = \beta u\right\}, \quad \beta \in \mathrm{Spec}\,(-i\widehat{T}).$$

Applications: Sasakian manifolds

- $H^0_{b,\alpha}(X)^G$ and $H^0_{b,\beta}(X_G)$ are finite dimensional subspaces of $\mathcal{C}^\infty(X)^G$ and $\mathcal{C}^\infty(X_G)$ respectively, for every $\alpha \in \mathrm{Spec}\,(-iT)$, $\beta \in \mathrm{Spec}\,(-i\widehat{T})$.
- $H^0_b(X)^G = \oplus_{\alpha\in\mathrm{Spec}\,(-iT)} H^0_{b,\alpha}(X)^G$, $H^0_b(X_G) = \oplus_{\beta\in\mathrm{Spec}\,(-i\hat{T})} H^0_{b,\beta}(X_G)$.

Quantization commutes with reduction for irregular Sasakian manifolds

Theorem II (H/Ma/Marinescu)

There is a $N \in \mathbb{N}$ such that the map

$$\tilde{\sigma}_G : H^0_{b,\alpha_k}(X)^G \to H^0_{b,\alpha_k}(X_G)$$

is an isomorphism, for every $k \geq N$ and if $\beta_k \neq \alpha_k$, where $k \geq N$, then $\dim H^0_{b,\beta_k}(X_G) = 0$.

The outline of the proof of Theorem I

- Let S_G be the orthogonal projection onto G-invariant CR functions (G-invariant Szegő projection).
- Let $S_G(x,y) \in \mathscr{D}'(X \times X)$ be the distribution kernel of S_G (G-invariant Szegő kernel).
- By developing some kind of G-invariant microlocal F.I.O. method,

G-invariant Szegő kernel asymptotics

Theorem III (H/Ma/Marinescu)

- *S_G is smoothing outside $\mu^{-1}(0)$.*
- *In an open set U of $\mu^{-1}(0)$, we have*

$$S_G(x,y) \equiv \int_0^\infty e^{i\Phi(x,y)t} a(x,y,t)dt \quad \text{on } U \times U,$$

  - $a(x,y,t) \sim \sum_{j=0}^\infty a_j(x,y)t^{n-\frac{d}{2}-j}$ *in* $S^{n-\frac{d}{2}}_{1,0}(U \times U \times \mathbb{R}_+)$,
  - $d_x\Phi(x,x) = -d_y\Phi(x,x) = -\omega_0(x)$, $\forall x \in \mu^{-1}(0)$,
  - $\operatorname{Im}\Phi(x,y) \geq 0$, $\operatorname{Im}\Phi(x,x) \approx d(x,\mu^{-1}(0))^2$.

The outline of the proof of Theorem I

- For every $s \in \mathbb{R}$, consider

$$\begin{aligned} \widehat{\sigma}_G : H^s(X) &\to H^0_b(X_G)_s \subset H^s(X_G), \\ u &\to S_{X_G} \circ E \circ \sigma_{G,s} \circ S_G \circ u. \end{aligned}$$

- E: some classical pseudodifferential operator on X_G of order $-\frac{d}{4}$.
- Let $\widehat{\sigma}^*_G : \mathcal{D}'(X_G) \to \mathcal{D}'(X)$ be the adjoint of $\widehat{\sigma}_G$.
- Let $F := \widehat{\sigma}^*_G \widehat{\sigma}_G$.
- $\operatorname{Ker}\sigma_{G,s} \subset \operatorname{Ker} F \cap H^0_b(X)^G_s$.

The outline of the proof of Theorem I

From Theorem III and by developing some kind of complex Fourier integral operators calculation, we can show that

- $F = C_0(I - R)S_G$, C_0 is a constant, R is also the same type of operator as S_G.
- $I - R$ is Fredholm.
- Since $\mathrm{Ker}\,\sigma_{G,s} \subset \mathrm{Ker}\,F \cap H^0_b(X)^G_s \subset \mathrm{Ker}\,(I - R) \cap H^0_b(X)^G_s$, $\mathrm{Ker}\,\sigma_{G,s}$ is a finite dimensional subspace of $\mathcal{C}^\infty(X) \cap H^0_b(X)^G$.

Chin-Yu Hsiao Geometric quantization on CR manifolds

The outline of the proof of Theorem I

- Take some inner products $(\cdot\,|\,\cdot)_{X_G,s}$ on $H^s(X_G)$, for every $s \in \mathbb{R}$.
- $\hat{F} := \sigma_{G,s}\sigma^*_{G,s} = (I - \hat{R})S_{X_G}$, $I - \hat{R}$: Fredholm operator.
- Since $(\mathrm{Im}\,\sigma_{G,s})^\perp \subset \mathrm{Ker}\,(I - \hat{R}) \cap H^0_b(X_G)_{s-\frac{d}{4}}$, $(\mathrm{Im}\,\sigma_{G,s})^\perp$ is a finite dimensional subspace of $\mathcal{C}^\infty(X_G) \cap H^0_b(X_G)$.

Chin-Yu Hsiao Geometric quantization on CR manifolds

Algebraicity of Compact Kähler Manifolds via Dual Positive Cones

Hsueh-Yung Lin (National Taiwan University)

Let X be a compact Kähler manifold. The celebrated Kodaira embedding theorem asserts that if the Kähler cone of X contains a rational cohomology class, then X admits a holomorphic embedding into a projective space. Instead of considering Kähler classes, we will study the algebraicity of X when X carries a 1-dimensional positive rational Hodge class.

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Curvatures and austere property of orbits of path group actions induced by Hermann actions

Masahiro Morimoto

It is known that an isometric action of a Lie group on a compact symmetric space G/K induces an isometric action of a path group on a path space. Let H be a closed subgroup of G acting on G/K isometrically by left translations

$$b \cdot (aK) := (ba)K,$$

where $b \in H$ and $aK \in G/K$. Denote by $\mathcal{G} := H^1([0,1], G)$ the Hilbert Lie group of all Sobolev H^1-paths from $[0,1]$ to G and by $V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g})$ the Hilbert space of all L^2-paths from $[0,1]$ to the Lie algebra $\mathfrak{g}$ of G. $\mathcal{G}$ acts on $V_{\mathfrak{g}}$ via the gauge transformations

$$g * u := gug^{-1} - g'g^{-1},$$

where $g \in \mathcal{G}$, $u \in V_{\mathfrak{g}}$ and g' denotes the weak derivative of g. The subgroup

$$P(G, H \times K) := \{g \in \mathcal{G} \mid g(0) \in H,\ g(1) \in K\}$$

acts on $V_{\mathfrak{g}}$ by the same formula. The $P(G, H \times K)$-action is closely related to the H-action via a natural Riemannian submersion $\Phi_K : V_{\mathfrak{g}} \to G/K$, called the *parallel transport map* ([16]). In fact Φ_K is equivariant with respect to those actions and each $P(G, H \times K)$-orbit is the inverse image of an H-orbit under Φ_K.

The concept of $P(G, H \times K)$-actions (or more generally $P(G, L)$-actions for a closed subgroup L of $G \times G$) was originally introduced by Terng [15] in her attempt to find infinite dimensional analogues of finite dimensional symmetric spaces and related concepts (see also [5]). In fact if H is a symmetric subgroup of G then the $P(G, H \times K)$-action can be thought of the isotropy representation of an *affine Kac-Moody symmetric space* (cf. [4]). Moreover it should be also noted that $P(G, H \times K)$-actions serve as a tool for studying H-actions on G/K (e.g. [2]). It is a fundamental problem to study the submanifold geometry of orbits of $P(G, H \times K)$-actions. Notice that every orbit of the $P(G, H \times K)$-action is a *proper Fredholm* (PF) submanifold of the Hilbert space $V_{\mathfrak{g}}$ ([14]).

The H-action is called a *Hermann action* ([6]) if H is a symmetric subgroup of G, that is, there exists an involutive automorphism τ of G such that H lies between the fixed point subgroup G^τ and its identity component. We know that any orbit of

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a Hermann action is a curvature-adapted submanifold ([1]). Moreover the principal curvatures of orbits of Hermann actions can be described explicitly via the root space decompositions ([12]). Furthermore any Hermann action is *hyperpolar* ([7], [5]), that is, there exists a closed connected totally geodesic submanifold Σ of G/K which is flat in the induced metric and meets every orbit orthogonally.

A submanifold is called *austere* ([3]) if the set of principal curvatures in the direction of each normal vector is invariant under the multiplication by (-1). By definition austere submanifolds are minimal submanifolds. There are many examples of austere submanifolds which are orbits of Hermann actions ([8], [12]). Since the shape operators of PF submanifolds are compact self-adjoint operators, we can similarly define a PF submanifold to be austere. It is an interesting problem to give examples of austere PF submanifolds in Hilbert spaces.

In this talk we introduce the author's recent results on the principal curvatures and the austere property of orbits of $P(G, H \times K)$-actions induced by Hermann actions ([11]). We first show an explicit formula for the principal curvatures of $P(G, H\times K)$-orbits, which unifies and generalizes some results by Terng [14], Pinkall-Thorbergsson [13] and Koike [9]. Then using this explicit formula we show the relation between the following two conditions of austere properties of orbits:

(A) the orbit $H \cdot (\exp w)K$ through $(\exp w)K$ is an austere submanifold of G/K,
(B) the orbit $P(G, H \times K) * \hat{w}$ through $\hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$,

where $w \in \mathfrak{g}$ and $\hat{w}$ denote the constant path with value w. To explain the results we write σ and τ for the involutions of G associated with the symmetric subgroups K and H respectively. Denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ (resp. $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$) the decomposition into the (± 1)-eigenspaces of the differential of σ (resp. τ). Take a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$ and write Δ for the root system of $\mathfrak{t}$ associated to the adjoint representation of $\mathfrak{t}$ on $\mathfrak{g}^{\mathbb{C}}$. We show the following theorem:

Theorem I. *If Δ is a reduced root system then* (A) *and* (B) *are equivalent.*

Without supposing that Δ is reduced we show the following theorem:

Theorem II.

(i) *Suppose that $\sigma = \tau$. Then* (A) *and* (B) *are equivalent.*
(ii) *Suppose that σ and τ commute. Then* (A) *implies* (B).
(iii) *Suppose that G is simple. Then* (A) *implies* (B).

Here we note that (B) does not imply (A) in the cases (ii) and (iii). In fact we show the following counterexample: the triple $(G, K, H) = (SU(p + q), S(U(p) \times U(q)), SO(p + q))$ with the root system $\Delta = \{\pm e_i, \pm 2e_i\}_i \cup \{\pm e_i \pm e_j\}_{i<j}$ of type BC and $w := \frac{\pi}{8} \sum_{i=1}^{q} e_i$ does not satisfy (A) but satisfy (B). Applying examples of austere orbits of Hermann actions to the above theorems we obtain many examples of infinite dimensional austere PF submanifolds in Hilbert spaces.

Finally we mention the relation between the above theorems and the author's previous result on the austere property of the parallel transport map Φ_K. The author showed:

Theorem ([10])**.** *Let N be a submanifold of G/K. Suppose that G/K is the standard sphere. Then the following conditions are equivalent:*

(i) *N is an austere submanifold of G/K,*

(ii) *$\Phi_K^{-1}(N)$ is an austere PF submanifold of $V_{\mathfrak{g}}$.*

Since each $P(G, H \times K)$-orbits is the inverse image of an H-orbit under Φ_K, Theorems I and II are extensions of this theorem.

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Curvatures and austere property of orbits of path group actions induced by Hermann actions

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Contents

Sec. 1 - The path group actions (1/5)

Setting

- G: connected compact semisimple Lie group with Lie algebra $\mathfrak{g}$,
- K: symmetric subgroup of G with Lie algebra $\mathfrak{k}$.
  i.e. $\exists\sigma : G \to G$: involutive automorphism s.t. $G_0^\sigma \subset K \subset G^\sigma$.
- $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$: ± 1-eigenspace decomposition by $\sigma : \mathfrak{g} \to \mathfrak{g}$.
- Equip G with a bi-inv. Riem. metric induced by Killing form,
  Equip G/K with the normal homogeneous metric.
- Then G/K is a symmetric space of compact type.
- The projection $\pi : G \to G/K$, $a \mapsto aK$ is a Riemannian submersion with totally geodesic fiber.

Sec. 1 - The path group actions (2/5)

Let

H: closed subgroup of G with Lie algebra $\mathfrak{h}$.
(Later H is assumed to be a symmetric subgroup of G.)

Isometric actions

- Then H acts on G/K by left translation, namely

$$b \cdot (aK) := (ba)K, \qquad b \in H,\ aK \in G/K$$

- Moreover the subgroup $H \times K$ acts on G by

$$(b, c) \cdot a := bac^{-1}, \qquad (b, c) \in H \times K,\ a \in G$$

- Furthermore a path group $P(G, H \times K)$ acts on a path space $V_{\mathfrak{g}}$ via the gauge transformations (: next page)

Sec. 1 - The path group actions (3/5)

Definition (Terng 1989, Pinkall-Thorbergsson 1990, Terng 1995)

- $\mathcal{G} := H^1([0,1], G)$: the set of all Sobolev H^1-paths from $[0,1]$ to G with pointwise multiplication.
  $\Rightarrow \mathcal{G}$ is a Hilbert Lie group (i.e. Hilbert mfd with C^∞ group str.)
- $V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g})$: the set of all L^2-paths from $[0,1]$ to $\mathfrak{g}$.
  $\Rightarrow V_{\mathfrak{g}}$ is a separable Hilbert space.
- $\mathcal{G}$ acts on $V_{\mathfrak{g}}$ via the gauge transformations:
  $$g * u := gug^{-1} - g'g^{-1}, \qquad g \in \mathcal{G},\ u \in V_{\mathfrak{g}}$$
- The subgroup
  $$P(G, H \times K) := \{g \in \mathcal{G} \mid g(0) \in H,\ g(1) \in K\}$$
  acts on $V_{\mathfrak{g}}$ by the same formula.

Sec. 1 - The path group actions (4/5)

Proposition (Terng 1989, Palais-Terng 1988)

(1) The action $P(G, H \times K) \curvearrowright V_{\mathfrak{g}}$ is a proper Fredholm (PF) action
i.e. (a) $P(G, H \times K) \times V_{\mathfrak{g}} \to V_{\mathfrak{g}} \times V_{\mathfrak{g}}$, $(g, u) \mapsto (g * u, u)$ is proper,
(b) $\forall u \in V_{\mathfrak{g}}$, the map $P(G, H \times K) \to V_{\mathfrak{g}}$, $g \mapsto g * u$ is Fredholm.

(2) Every orbit of the $P(G, H \times K)$-actions is a proper Fredholm (PF) submanifold of the Hilbert space $V_{\mathfrak{g}}$.
i.e. (a) Infinite dimensional Morse theory can be applied,
(b) The shape operators are compact self-adjoint operators.

Motivations

- Examples of (homogeneous) PF submanifolds in Hilbert spaces.
- $P(G, H \times K)$-actions serve as a tool for studying H-actions.
- $P(G, H \times K)$-actions are the isotropy representations of affine Kac-Moody symmetric spaces (if H is a symmetric subgrp of G)

Sec. 1 - The path group actions (5/5)

Fundamental Problem

Study the submanifold geometry of orbits of $P(G, H \times K)$-actions.

In this talk

We study the principal curvatures and the austere property of orbits of $P(G, H \times K)$-actions.

Here, a submanifold is called austere (Harvey-Lawson 1982) if for each normal vector ξ the set of eigenvalues with multiplicities of the shape operator A_ξ is invariant under the multiplication by (-1).

To do this, we will (later) suppose that H is a symmetric subgroup of G. In this case the H-action is called a Hermann action.

Sec. 2 - The parallel transport map (1/7)

Note

G/K: symmetric space of compact type,
$\pi : G \to G/K$: natural Riemannian submersion.
H: closed subgroup of G acting on G/K by left translation.
Then
(a) π is equivariant with respect to H- and $H \times K$-actions via p_1,
(b) orbits satisfy: $(H \times K) \cdot a = \pi^{-1}(H \cdot aK)$ for each $a \in G$.

There is a natural Riemannian submersion $\Phi : V_{\mathfrak{g}} \to G$,
which has similar equivariant property (: next page)

Sec. 2 - The parallel transport map (2/7)

Definition (Terng 1995, Terng-Thorbergsson 1995)

G: conn. compact Lie group with a bi-invariant Riem. met.
$\mathcal{G} := H^1([0,1], G)$ and $V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g})$: as before.
The parallel transport map is defined by

$$\begin{array}{ccccc} \Phi & : & V_{\mathfrak{g}} & \to & G \\ & & \Cup & & \Cup \\ & & u & \mapsto & \Phi(u) \overset{\text{def}}{:=} g_u(1). \end{array}$$

Here, $g_u \in \mathcal{G}$ is defined by the ODE $\begin{cases} g_u^{-1} g_u' = u, \\ g_u(0) = e \in G. \end{cases}$

Definition

The map $\Psi : \mathcal{G} \to G \times G$ is defined by $\Psi(g) := (g(0), g(1))$ for $g \in \mathcal{G}$

Sec. 2 - The parallel transport map (3/7)

Theorem (Terng-Thorbergsson 1995)

The parallel transport map $\Phi : V_{\mathfrak{g}} \to G$ satisfies:
(1) Φ is a Riemannian submersion,
(2) any two fibers of Φ are congruent under the isometry of $V_{\mathfrak{g}}$,
(3) Φ is a principal $\Omega_e(G)$-bundle. ($\Omega_e(G)$: the based loop group.)
(4) N: closed submanifold of $G \Longrightarrow \Phi^{-1}(N)$: PF submanifold of $V_{\mathfrak{g}}$

Proposition (Terng 1995)

Let H be a closed subgroup of G.
(a) Φ is equivariant with respect to
$P(G, H \times K)$- and $H \times K$-actions via Ψ,
(b) $P(G, H \times K) * u = \Phi^{-1}((H \times K) \cdot a)$ for $u \in V_{\mathfrak{g}}$ and $a := \Phi(u)$.

Sec. 2 - The parallel transport map (4/7)

Generalization (Terng-Thorbergsson 1995)

The composition $\Phi_{G/K} := \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K$ satisfies:
(1) $\Phi_{G/K}$ is a Riemannian submersion,
(2) any two fibers of $\Phi_{G/K}$ are congruent under the isometry of $V_{\mathfrak{g}}$,
(3) $\Phi_{G/K}$ is a principal $P(G, \{e\} \times K)$-bundle.
(4) N: closed submanifold of $G/K \Longrightarrow \Phi_{G/K}^{-1}(N)$: PF submanifold of $V_{\mathfrak{g}}$

Generalization (Terng 1995)

Let H be a closed subgroup of G.
(a) $\Phi_{G/K}$ is equivariant with resp. to $P(G, H \times K)$- and H-actions,
(b) $P(G, H \times K) * u = \Phi_{G/K}^{-1}(H \cdot aK)$ for $u \in V_{\mathfrak{g}}$ and $a = \Phi(u)$.

Sec. 2 - The parallel transport map (5/7)

Theorem (M. 2019) : the second fundamental form

N: closed submanifold of G/K. $\forall X, Y \in T_{\hat{0}}\Phi_{G/K}^{-1}(N)$,

$$\alpha^{\Phi_{G/K}^{-1}(N)}(X,Y) = \alpha^N\left(\int_0^1 X(t)_{\mathfrak{m}}dt, \int_0^1 Y(t)_{\mathfrak{m}}dt\right)$$
$$+\frac{1}{2}\left[\int_0^1 X(t)_{\mathfrak{k}}dt, \int_0^1 Y(t)_{\mathfrak{m}}dt\right]^{\perp} - \frac{1}{2}\left[\int_0^1 X(t)_{\mathfrak{m}}dt, \int_0^1 Y(t)_{\mathfrak{m}}dt\right]^{\perp}$$
$$+\frac{1}{2}\left[\int_0^1 X(t)dt, \int_0^1 Y(t)dt\right]^{\perp} - \left(\int_0^1 \left[\int_0^t X(s)ds, Y(t)\right] dt\right)^{\perp}.$$

Theorem (M. 2019) : the shape operator

N: closed submanifold of G/K. $\forall X \in T_{\hat{0}}\Phi_{G/K}^{-1}(N)$, $\xi \in T_{\hat{0}}^{\perp}\Phi_{G/K}^{-1}(N)$,

$$A_{\xi}^{\Phi_{G/K}^{-1}(N)}(X) =$$
$$A_{\xi}^N\left(\int_0^1 X(t)_{\mathfrak{m}}dt\right) - \frac{1}{2}\left[\int_0^1 X(t)_{\mathfrak{m}}dt, \xi\right]_{\mathfrak{k}} + \frac{1}{2}\left[\int_0^1 X(t)_{\mathfrak{k}}dt, \xi\right]^{\top}$$
$$-\frac{1}{2}\left[\int_0^1 X(t)dt, \xi\right]^{\top} + \left[\int_0^t X(s)ds, \xi\right] - \left[\int_0^1 \int_0^t X(s)dsdt, \xi\right]^{\perp}.$$

Sec. 2 - The parallel transport map (6/7)

Theorem (Koike 2002, M. 2019, M. 2021) : principal curvatures

N : curvature adapted submfd of G/K.
(i.e. $\mathrm{ad}(\xi)^2 : \mathfrak{m} \to \mathfrak{m}$ preserves $T_{eK}N$ and commutes with A^N_ξ)
$\{\lambda\}$: eigenvalue of A^N_ξ, $\{\sqrt{-1}\nu\}$: eigenvalue of $\mathrm{ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$.
Then the principal curvatures of $\Phi^{-1}_{G/K}(N)$ in direction ξ is

$$\left\{0,\ \lambda,\ \frac{\nu}{n\pi},\ \frac{\nu}{\arctan\frac{\nu}{\lambda} + m\pi}\right\}\quad \lambda,\ \nu > 0,\ n \in \mathbb{Z}\backslash\{0\},\ m \in \mathbb{Z}.$$

eigenfunctions and multiplicities are given in the next page:

Sec. 2 - The parallel transport map (7/7)

Theorem (Koike 2002, M. 2019, M. 2021) : principal curvatures

Set $\mu(\nu, \lambda, m) := \dfrac{\nu}{\arctan\frac{\nu}{\lambda} + m\pi}$.

eigenval.	basis of eigenfunctions	multip.
0	$\{x^0_i \sin n\pi t, y^{(0,\lambda)}_j \cos n\pi t, y^{(0,\perp)}_l \cos n\pi t\}_{n\in\mathbb{Z}_{\geq 1}, \lambda, i, j, l}$	∞
λ	$\{y^{(0,\lambda)}_j\}_j$	$m(0,\lambda)$
$\dfrac{\nu}{n\pi}$	$\{x^{(\nu,\perp)}_r \sin n\pi t - y^{(\nu,\perp)}_r \cos n\pi t\}_r$	$m(\nu,\perp)$
$\mu(\nu,\lambda,m)$	$\left\{\sum_{n\in\mathbb{Z}} \frac{\nu}{n\pi\mu+\nu}(x^{(\nu,\lambda)}_k \sin n\pi t + y^{(\nu,\lambda)}_k \cos n\pi t)\right\}_k$	$m(\nu,\lambda)$

Sec. 3 - Hermann actions (1/6)

Suppose

H: symmetric subgroup of G
with involution $\tau : G \to G$ s.t. $G_0^\tau \subset H \subset G^\tau$.
$\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$: ± 1-eigenspace decomposition
$\Rightarrow$ the action $H \curvearrowright G/K$ is called a Hermann action.

Proposition (Heintze-Palais-Terng-Thorbergsson 1995)

Hermann actions are hyperpolar. That is,
$\exists \Sigma$: closed connected totally geodesic submanifold Σ of G/K s.t.
(1) Σ meets every H-orbit orthogonally,
(2) Σ is flat in the induced metric.
(Such a Σ is called a section of the action $H \curvearrowright G/K$.)

In fact, take a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$.
$\Rightarrow \Sigma := \pi(\exp \mathfrak{t})$ is a section of the Hermann action.

Sec. 3 - Hermann actions (2/6)

Two kinds of decompositions

- Root space decomposition with respect to a maximal abelian subspace $\mathfrak{t}$ in $\mathfrak{m} \cap \mathfrak{p}$ $(\subset \mathfrak{m})$

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in \Delta^+} \mathfrak{k}_\alpha, \qquad \mathfrak{m} = \mathfrak{m}_0 + \sum_{\alpha \in \Delta^+} \mathfrak{m}_\alpha,$$

$$\mathfrak{k}_\alpha = \{x \in \mathfrak{k} \mid \forall \eta \in \mathfrak{t},\ \mathrm{ad}(\eta)^2 x = -\langle \alpha, \eta \rangle^2 x\}.$$
$$\mathfrak{m}_\alpha = \{y \in \mathfrak{m} \mid \forall \eta \in \mathfrak{t},\ \mathrm{ad}(\eta)^2 y = -\langle \alpha, \eta \rangle^2 y\}.$$

- The eigenspace decomposition of $\sigma \circ \tau : \mathfrak{g} \to \mathfrak{g}$:

$$\mathfrak{g}^{\mathbb{C}} = \sum_{\epsilon \in U(1)} \mathfrak{g}(\epsilon),$$
$$\mathfrak{g}(\epsilon) = \{z \in \mathfrak{g}^{\mathbb{C}} \mid (\sigma \circ \tau)(z) = \epsilon z\}.$$

Sec. 3 - Hermann actions (3/6)

Proposition (Ohno 2021)

Take $w \in \mathfrak{t}$. Set $a := \exp w$. Consider the orbit $N := H \cdot aK$ through aK. Then

$$T_{aK}N = dL_a\Big(\sum_{\substack{\epsilon \in U(1)_{\geq 0} \\ \epsilon \neq 1}} \mathfrak{m}_{0,\epsilon} + \sum_{\alpha \in \Delta^+} \sum_{\substack{\epsilon \in U(1) \\ \langle \alpha, w\rangle + \frac{1}{2}\arg \epsilon \notin \pi\mathbb{Z}}} \mathfrak{m}_{\alpha,\epsilon} \Big),$$

$$T^{\perp}_{aK}N = dL_a\Big(\mathfrak{t} + \sum_{\alpha \in \Delta^+} \sum_{\substack{\epsilon \in U(1) \\ \langle \alpha, w\rangle + \frac{1}{2}\arg \epsilon \in \pi\mathbb{Z}}} \mathfrak{m}_{\alpha,\epsilon} \Big).$$

Moreover the first decomposition is just the eigenspace decomposition of the family shape operators $\{A^N_{dL_a(\xi)}\}_{\xi \in \mathfrak{t}}$:

$dL_a(\mathfrak{m}_{0,\epsilon})$: the eigenspace of eigenvalue 0,

$dL_a(\mathfrak{m}_{\alpha,\epsilon})$: the eigenspace of eigenvalue $-\langle \alpha, \xi\rangle \cot(\langle \alpha, w\rangle + \frac{1}{2}\arg \epsilon)$.

Sec. 3 - Hermann actions (4/6)

Corollary (Goertsches-Thorbergsson 2007)

Suppose that $\sigma \circ \tau = \tau \circ \sigma$. Then

$$T_{aK}N = dL_a\Big(\mathfrak{m}_0 \cap \mathfrak{h} + \sum_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, w\rangle \notin \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{p} + \sum_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, w\rangle + \pi/2 \notin \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{h} \Big),$$

$$T^{\perp}_{aK}N = dL_a\Big(\mathfrak{t} + \sum_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, w\rangle \in \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{p} + \sum_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, w\rangle + \pi/2 \in \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{h} \Big),$$

$dL_a(\mathfrak{m}_0 \cap \mathfrak{h})$: the eigenspace of eigenvalue 0,

$dL_a(\mathfrak{m}_\alpha \cap \mathfrak{p})$: the eigenspace of eigenvalue $-\langle \alpha, \xi\rangle \cot\langle \alpha, w\rangle$,

$dL_a(\mathfrak{m}_\alpha \cap \mathfrak{h})$: the eigenspace of eigenvalue $\langle \alpha, \xi\rangle \tan\langle \alpha, w\rangle$.

Sec. 3 - Hermann actions (5/6)

Corollary

Suppose that $\sigma = \tau$. Then

$$T_{aK}N = dL_a\Big(\sum_{\substack{\alpha\in\Delta^+ \\ \langle\alpha,w\rangle\notin\pi\mathbb{Z}}} \mathfrak{m}_\alpha \Big),$$

$$T^{\perp}_{aK}N = dL_a\Big(\mathfrak{t} + \sum_{\substack{\alpha\in\Delta^+ \\ \langle\alpha,w\rangle\in\pi\mathbb{Z}}} \mathfrak{m}_\alpha \Big),$$

$dL_a(\mathfrak{m}_\alpha)$: the eigenspace of eigenvalue $-\langle\alpha,\xi\rangle\cot\langle\alpha,w\rangle$.

Sec. 3 - Hermann actions (6/6)

Concerning hyperpolar actions, the following theorem is known:

Theorem (Terng 1995, Heintze-Palais-Terng-Thorbergsson 1995, Gorodski-Thorbergsson 2002)

The following conditions are equivalent:

(1) The action $H \curvearrowright G/K$ is hyperpolar (with section $\pi(\exp\mathfrak{t})$)

(2) The action $H\times K \curvearrowright G$ is hyperpolar (with section $\exp\mathfrak{t}$)

(3) The action $P(G, H\times K) \curvearrowright V_{\mathfrak{g}}$ is hyperpolar (with section $\hat{\mathfrak{t}}$)

($\mathfrak{t}$: maximal abelian subalg. in $\mathfrak{m}\cap\mathfrak{p}$, $\hat{\mathfrak{t}}$: set of constant paths in $\mathfrak{t}$)

Sec. 4 - The principal curvatures (1/3)

Theorem (M. 2021)

H: symmetric subgroup of G. Take $w \in \mathfrak{t}$.
Consider the orbit $P(G, H \times K) * \hat{w}$ through $\hat{w} \in \hat{\mathfrak{t}}$.
Then the principal curvatures in the direction of $\hat{\xi} \in V_{\mathfrak{g}}$ for $\xi \in \mathfrak{t}$ is

$$\{0\} \cup \left\{ \frac{\langle \alpha, \xi \rangle}{-\langle \alpha, w \rangle - \frac{1}{2} \arg \epsilon + m\pi} \;\middle|\; \begin{array}{l} \alpha \in \Delta^+, \ \epsilon \in U(1), \\ \langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon \notin \pi\mathbb{Z}, \ m \in \mathbb{Z} \end{array} \right\}$$
$$\cup \left\{ \frac{\langle \alpha, \xi \rangle}{n\pi} \;\middle|\; \begin{array}{l} \alpha \in \Delta^+, \ n \in \mathbb{Z} \backslash \{0\}, \\ \exists \epsilon \in U(1) \text{ s.t. } \langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon \in \pi\mathbb{Z} \end{array} \right\}$$

The multiplicities are respectively

$$\infty, \quad \dim \mathfrak{m}_{\alpha,\epsilon}, \quad \sum_{\alpha} \dim \mathfrak{m}_{\alpha,\epsilon}$$

Sec. 4 - The principal curvatures (2/3)

Corollary

Suppose that $\sigma \circ \tau = \tau \circ \sigma$. Then the principal curvatures of the orbit $P(G, H \times K) * \hat{w}$ in the direction of $\hat{\xi} \in V_{\mathfrak{g}}$ for $\xi \in \mathfrak{t}$ is

$$\{0\} \cup \left\{ \frac{\langle \alpha, \xi \rangle}{-\langle \alpha, w \rangle + m\pi} \;\middle|\; \alpha \in \Delta^+, \ \langle \alpha, w \rangle \notin \pi\mathbb{Z}, \ m \in \mathbb{Z} \right\}$$
$$\cup \left\{ \frac{\langle \alpha, \xi \rangle}{-\langle \alpha, w \rangle - \frac{1}{2}\pi + m\pi} \;\middle|\; \alpha \in \Delta^+, \ \langle \alpha, w \rangle + \frac{\pi}{2} \notin \pi\mathbb{Z}, \ m \in \mathbb{Z} \right\}$$
$$\cup \left\{ \frac{\langle \alpha, \xi \rangle}{n\pi} \;\middle|\; \begin{array}{l} \alpha \in \Delta^+, \ \langle \alpha, w \rangle \in \pi\mathbb{Z}, \ n \in \mathbb{Z} \backslash \{0\} \\ \text{or } \ \alpha \in \Delta^+, \ \langle \alpha, w \rangle + \frac{\pi}{2} \in \pi\mathbb{Z}, \ n \in \mathbb{Z} \backslash \{0\} \end{array} \right\}.$$

The multiplicities are respectively

$$\infty, \quad \dim(\mathfrak{m}_\alpha \cap \mathfrak{p}), \quad \dim(\mathfrak{m}_\alpha \cap \mathfrak{h}), \quad \dim(\mathfrak{m}_\alpha \cap \mathfrak{p}) + \dim(\mathfrak{m}_\alpha \cap \mathfrak{h})$$

Sec. 4 - The principal curvatures (3/3)

Corollary

Suppose that $\sigma = \tau$. Then the principal curvatures of the orbit $P(G, H \times K) * \hat{w}$ in the direction of $\hat{\xi} \in V_{\mathfrak{g}}$ for $\xi \in \mathfrak{t}$ is

$$\{0\} \cup \left\{ \frac{\langle \alpha, \xi \rangle}{-\langle \alpha, w \rangle + m\pi} \;\middle|\; \alpha \in \Delta^+,\ \langle \alpha, w \rangle \notin \pi\mathbb{Z},\ m \in \mathbb{Z} \right\}$$
$$\cup \left\{ \frac{\langle \alpha, \xi \rangle}{n\pi} \;\middle|\; \alpha \in \Delta^+,\ \langle \alpha, w \rangle \in \pi\mathbb{Z},\ n \in \mathbb{Z} \backslash \{0\} \right\}.$$

The multiplicities are respectively

$$\infty, \quad \dim \mathfrak{m}_\alpha, \quad \dim \mathfrak{m}_\alpha$$

Sec. 5 -The austere property (1/4)

Definition (Harvey-Lawson 1982)

N: a submanifold of Riemannian manifold M
N is called austere
$\underset{\text{def.}}{\Leftrightarrow}$ $\forall p \in N$, $\forall \xi \in T_p^\perp N$, the set of eigenvalues with multiplicities of the shape operator A_ξ is invariant under the multip. by (-1).

Remark

Austere submanifolds are minimal submanifolds.

Problem

Give example of austere submanifolds.

Note

We can define a PF submanifold in a Hilbert space to be austere by the similar way

Sec. 5 -The austere property (2/4)

Question

Let $w \in \mathfrak{t}$ $(\subset \mathfrak{m} \cap \mathfrak{p})$.
The relation between the following two conditions? $(w \in \mathfrak{t})$:
(A) the orbit $H \cdot (\exp w)K$ is an austere submanifold of G/K,
(B) the orbit $P(G, H \times K) * \hat{w}$ is an austere PF submanifold of $V_{\mathfrak{g}}$

Sec. 5 -The austere property (3/4)

Theorem I (M. 2021)

Suppose Δ is a reduced root system. Then (A) and (B) are equivalent.

Theorem II (M. 2021)

(1) Suppose that $\sigma = \tau$. Then (A) and (B) are equivalent.
(2) Suppose that $\sigma \circ \tau = \tau \circ \sigma$. Then (A) implies (B).
(3) Suppose that G is simple. Then (A) implies (B).

Counterexample to the converse of Theorem II (ii) and (iii) (M. 2021)

Consider the triple

$$(G, H, K) = (SU(p+q), S(U(p) \times U(q)), SO(p+q))$$

and the orbit through $w := \frac{\pi}{8}(e_1 + \cdots + e_q)$.
(Here the root system $\Delta = \{e_i, 2e_i\}_i \cup \{e_i \pm e_j\}_{i<j}$ is of type BC.)

Sec. 5 -The austere property (4/4)

Remark

- Austere orbits of Hermann actions were classified by Ikawa and Ohno (in the case that G is simple.)
- Applying their results to our theorems, we can obtain austere orbits of $P(G, H \times K)$-actions.
- There exist many austere submanifolds in infinite dimensional Hilbert spaces.

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Thank you very much for your attention !

Rigidity and Symmetry of Cylindrical Handlebody-Knots

Yi-Sheng Wang (Academia Sinica)

The theory of handlebody-knots studies handlebodies in three dimensions; in the case of a genus one handlebody embedded in the 3-sphere, the theory is equivalent to the classical knot theory. The talk concerns symmetries of a genus two handlebody-knot measured by its symmetry group, the path components of the space of self-homeomorphisms of the 3-sphere preserving the handlebody-knot setwise. It follows from a recent result of Funayoshi-Koda that a genus two handlebody-knot has a finite symmetry group if and only if it is hyperbolic—the exterior admits a hyperbolic structure with totally geodesic boundary—or irreducible, atoroidal, cylindrical—the exterior contains no essential disks or tori but contains an essential annulus. Little however is known about the structure of these finite groups. The talk will start with a quick tour through some basics of essential surfaces of non-negative Euler characteristic in a handlebody-knot exterior, and move on from there, I will survey some known results on symmetry groups of cylindrical handlebody-knots.

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Symmetries of Handlebody-Knots

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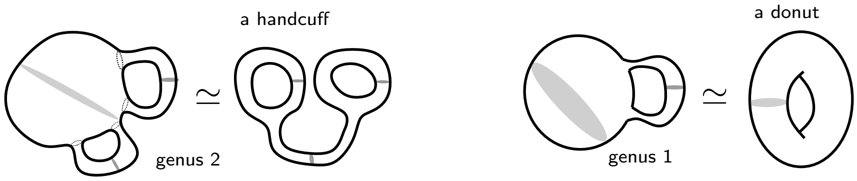
at Japan-Taiwan Joint Conference on Differential Geometry

November 2, 2021

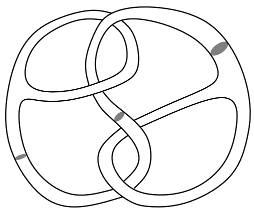
Handlebody-knots

- A handlebody: a 3-ball with some 1-handles attached.



Definition

A handlebody-knot $(\mathbb{S}^3, \mathrm{HK})$ is an embedded handlebody HK in the 3-sphere $\mathbb{S}^3$.



Handlebody-knots

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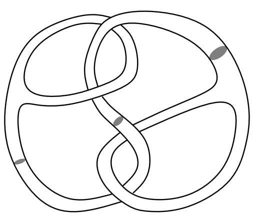
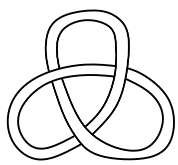
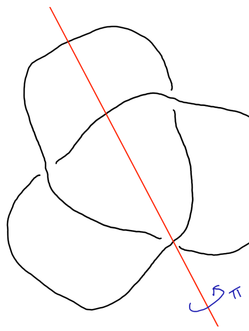


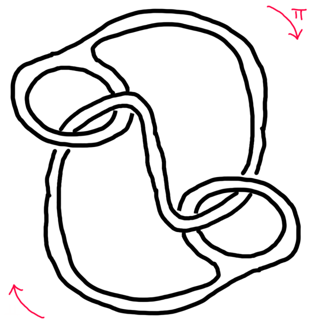


- A knot is a genus one handlebody-knot.
- Today: genus two handlebody-knots.

Symmetries of handlebody-knots

- Examples:





  - A *self-homeomorphism* of $\mathbb{S}^3$ preserving HK is a symmetry.
- *Chiral*: no orientation-reserving homeomorphism of $\mathbb{S}^3$ preserving HK.

- Examples:
- How to compute them?
- Are they all finite groups?

Symmetries of handlebody-knots

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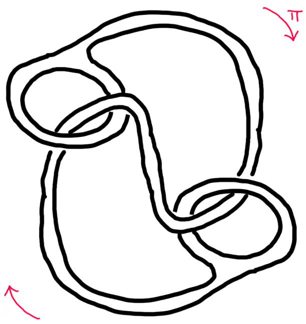
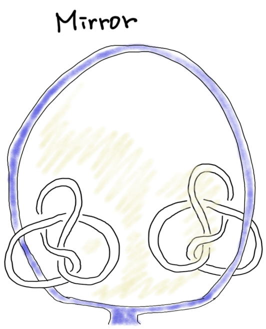


orientation-preserving.



mirror (orientation-reversing).

  - A *self-homeomorphism* of $\mathbb{S}^3$ preserving HK is a symmetry.
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Symmetries of handlebody-knots

Definition (Symmetry Group)

Given $(\mathbb{S}^3, \mathrm{HK})$, the symmetry group $\mathrm{Sym}(\mathbb{S}^3, \mathrm{HK})$ is the group of connected components $\pi_0(\text{Homeo}(\mathbb{S}^3, \mathrm{HK}))$ of the space $\text{Homeo}(\mathbb{S}^3, \mathrm{HK})$ of homeomorphisms of $\mathbb{S}^3$ preserving HK setwise.

- The group of isotopy classes of self-homeomorphisms of $(\mathbb{S}^3, \mathrm{HK})$.
- Orientation-preserving: $\mapsto$ positive symmetry group $\mathrm{Sym}_+(\mathbb{S}^3, \mathrm{HK})$.
- $\mathrm{Sym}(\mathbb{S}^3, \mathrm{HK})/\mathrm{Sym}_+(\mathbb{S}^3, \mathrm{HK})$ is trivial $\Leftrightarrow$ $(\mathbb{S}^3, \mathrm{HK})$ is chiral.
- Examples:
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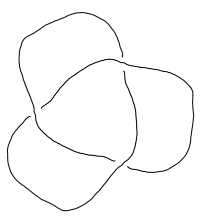
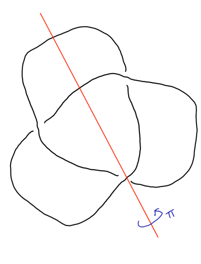


$\mathrm{Sym}(\mathbb{S}^3, \mathrm{HK}) = \mathrm{Sym}_+(\mathbb{S}^3, \mathrm{HK}) = \mathbb{Z}_2.$



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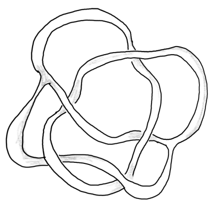

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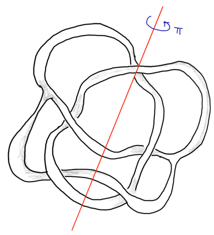

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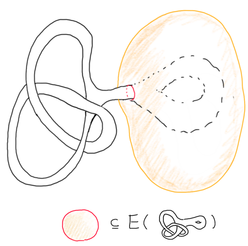
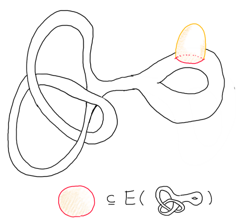
Symmetries and essential surfaces

- Essential surfaces in the knot exterior $E(\mathrm{HK}) := \overline{\mathbb{S}^3 - \mathrm{HK}}$.
- Essential disks in $E(\mathrm{HK})$:



An essential disk.



An inessential disk.

- Essential torus in $E(\mathrm{HK})$:
- $\exists$ essential disk or torus in $E(\mathrm{HK}) \Rightarrow |\mathrm{Sym}_+(\mathbb{S}^3, \mathrm{HK})| = \infty$.

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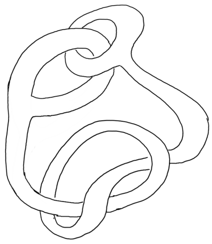

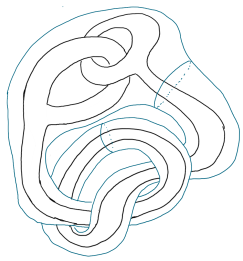

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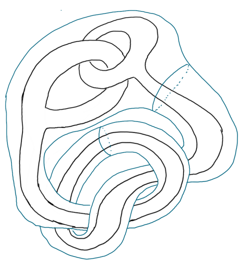

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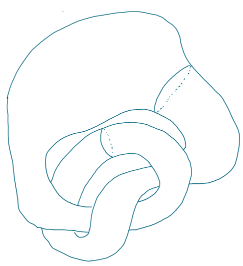

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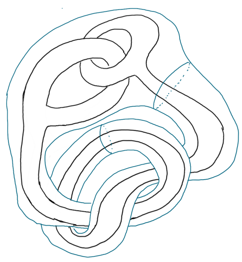

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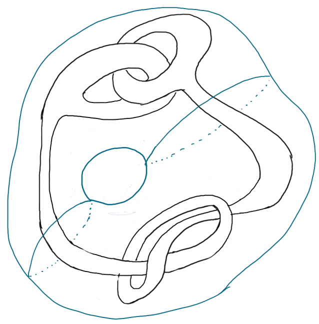

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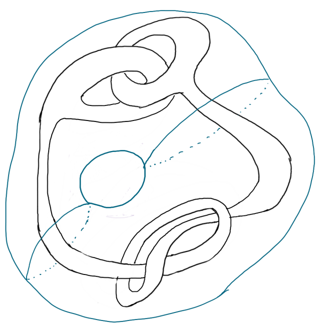

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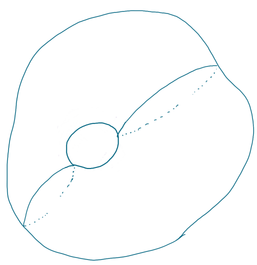

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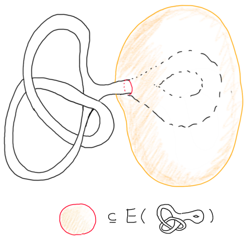

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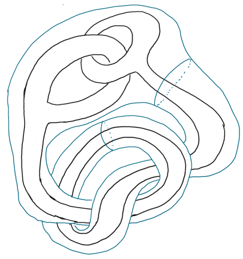

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Symmetries and essential surfaces

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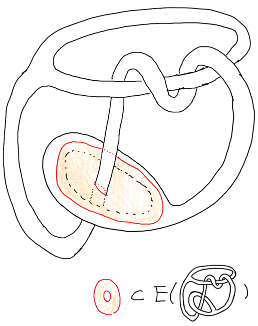
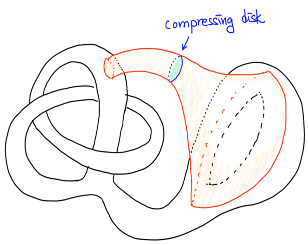


Essential annulus.



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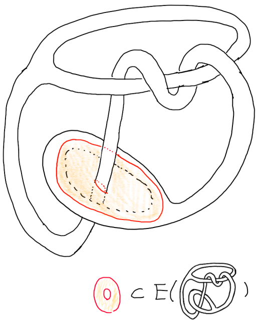

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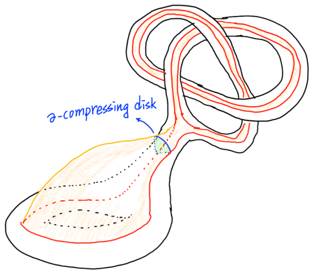

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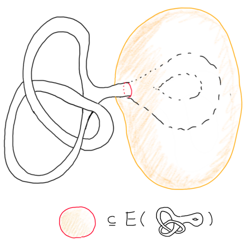

An essential disk.

Definition

A handlebody-knot $(\mathbb{S}^3, \mathrm{HK})$ is called

- irreducible if $E(\mathrm{HK})$ contains no essential disks,
- atoroidal if $E(\mathrm{HK})$ contains no essential tori, and
- acylindrical if $E(\mathrm{HK})$ contains no essential annuli.

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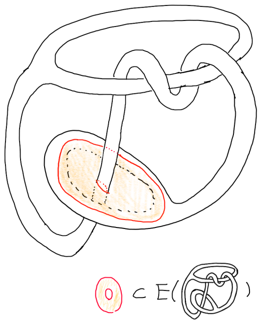

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Why disks, tori and annuli?

Theorem (Hyperbolization)

If $(\mathbb{S}^3, \mathrm{HK})$ *is irreducible, atoroidal and acylindrical,* $E(\mathrm{HK})$ *admits a hyperbolic structure of finite volume with totally geodesic boundary.*

$\Rightarrow$ If $(\mathbb{S}^3, \mathrm{HK})$ irreducible, atoroidal and acylindrical, $\mathrm{Sym}(\mathbb{S}^3, \mathrm{HK})$ is finite.

Theorem (Funayoshi-Koda, '20)

$\mathrm{Sym}(\mathbb{S}^3, \mathrm{HK})$ *is finite* $\Leftrightarrow$ $(\mathbb{S}^3, \mathrm{HK})$ *is hyperbolic or irreducible, atoroidal and cylindrical.*

- Can we classify these finite symmetry groups?

e.g. Genus= 1, a finite symmetry group is either cyclic or dihedral.

- Today: irreducible, atoroidal, cylindrical handlebody-knots.

i.e. $E(\mathrm{HK})$ contains an essential annulus but no essential disks or tori.

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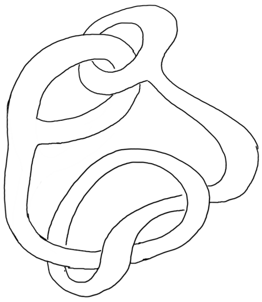
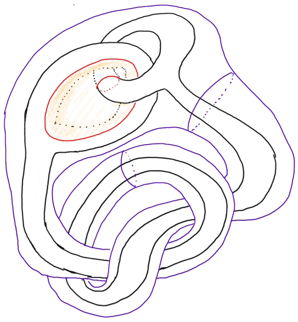
Cylindrical handlebody-knots

- Cylindrical handlebody-knots could be reducible or toroidal.



Toroidal, cylindrical $(\mathbb{S}^3, \mathrm{HK})$.



Essential torus and annulus.

- When is a cylindrical handlebody-knot irreducible and atoroidal (recognition problem)?
- Annular operation:
- Annular operation often simplifies $(\mathbb{S}^3, \mathrm{HK})$:

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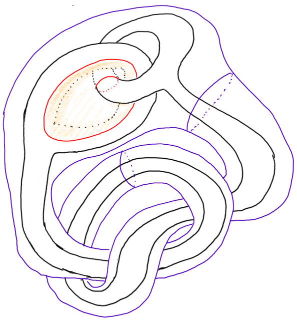

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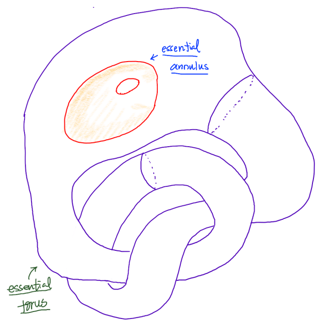

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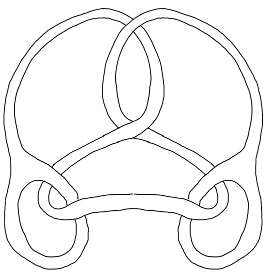
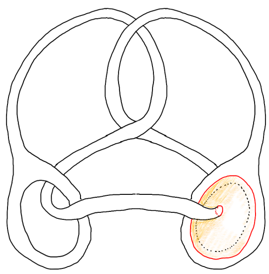

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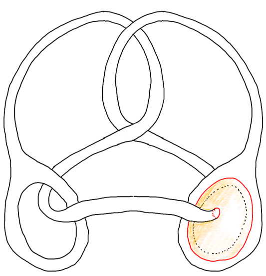
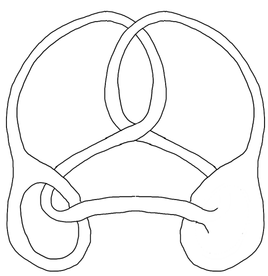

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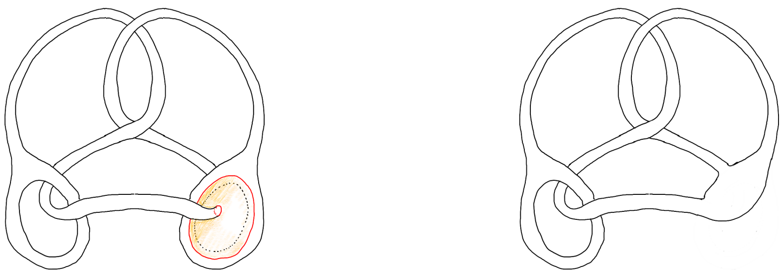

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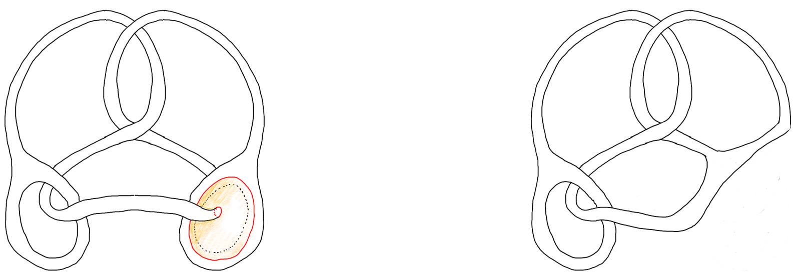

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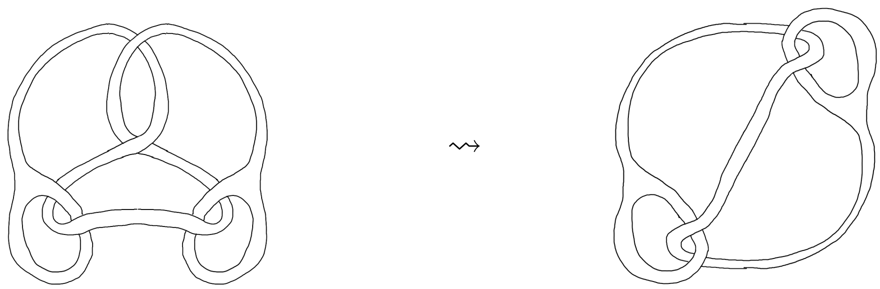

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Theorem (W. '21)

- *$E(\mathrm{HK}_A)$ contains no essential disks $\Rightarrow$ $(\mathbb{S}^3, \mathrm{HK})$ is irreducible.*
- *$E(\mathrm{HK}_A)$ contains no essential tori; ∂A is not an $(2m, 2n)$-torus link, $|m|, |n| > 1 \Rightarrow (\mathbb{S}^3, \mathrm{HK})$ is atoroidal.*

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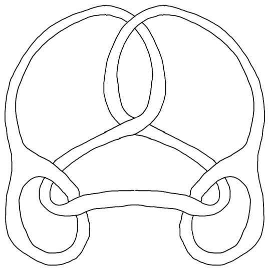
Cylindrical handlebody-knots



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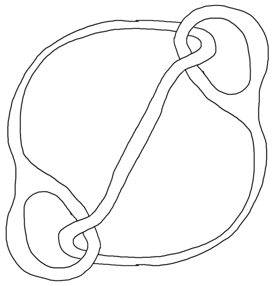

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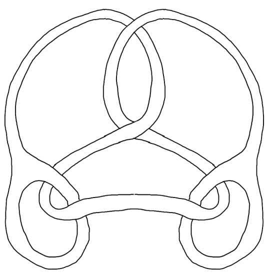

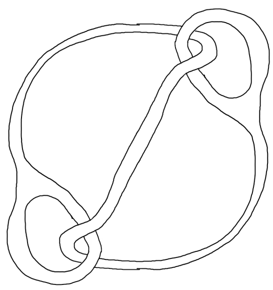

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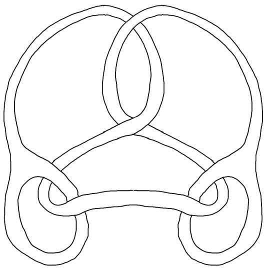

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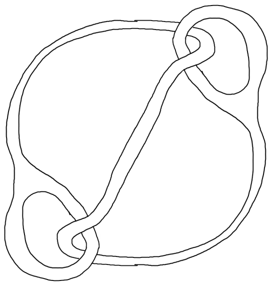

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Classify essential annuli

√ Recognition problem: determine the irreducibility and atoroidality.

– Assume $(\mathbb{S}^3, \mathrm{HK})$ is irreducible, atoroidal, and cylindrical.

- Computation problem: determine the structure of $\mathrm{Sym}(\mathbb{S}^3, \mathrm{HK})$.
- Classification of essential annuli.

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Theorem (Koda-Ozawa '15)

Essential annuli A in $E(\mathrm{HK})$ are classified into four types:

I. *Exactly one component of ∂A bounds a disk in* HK.

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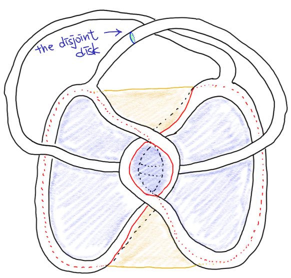
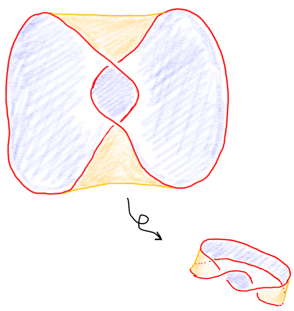
II. ∂A **bound no disks** *in* HK *and* **non-parallel** *in* $\partial\mathrm{HK}$ *and* $\exists$ **a disk** *in* HK *disjoint from A.*





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I. *Exactly one component of ∂A bounds a disk in HK.*
II. *∂A bound no disks in HK and non-parallel in $\partial\mathrm{HK}$ and $\exists$ a disk in HK disjoint from A.*
III. *∂A bound no disks in HK and parallel in $\partial\mathrm{HK}$, and $\exists$ a disk in HK disjoint from A.*
IV. *∂A bound no disks in HK and parallel in $\partial\mathrm{HK}$, and* **no disks** *in HK disjoint from A.*

- HK_A is a handlebody $\Rightarrow$ A is of type I or II.

Why classification?

- Possible configurations of A in relation to HK.
- – Apply spatial graph theory, knot tunnel theory, and mapping class groups of surfaces.

Theorem (W. '21)

If A is a unique unknotting annulus of type I, then $\mathrm{Sym}(\mathbb{S}^3, \mathrm{HK})$ is trivial.

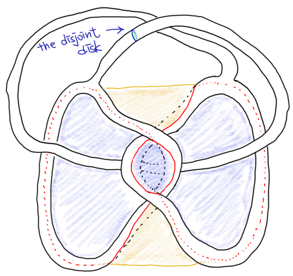
Theorem (W. '21)

If A is a unique annulus of type II with a boundary slope pair (p, p), $p \neq 0$, then $(\mathbb{S}^3, \mathrm{HK})$ is chiral, and $\mathrm{Sym}(\mathbb{S}^3, \mathrm{HK})$ is trivial, $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Boundary slope pair of a type II annulus

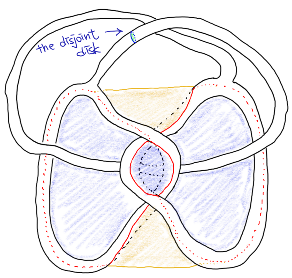
Theorem (W. '21)

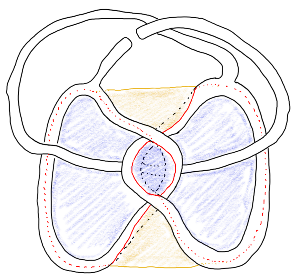
If A is a unique annulus of type II with a **boundary slope pair** (p, p), $p \neq 0$, *then* $(\mathbb{S}^3, \mathrm{HK})$ *is chiral, and* $\mathrm{Sym}(\mathbb{S}^3, \mathrm{HK})$ *is trivial,* $\mathbb{Z}_2$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$.



Boundary slope pair of a type II annulus

- Cut HK along the disjoint disk D.
- $\mathrm{HK} - \mathring{\mathfrak{N}}(D)$ are two tori.
- The slope pair is the slopes of ∂A on the two solid tori.
- Slope pair can only be $(\frac{p}{q}, \frac{q}{p})$ or $(\frac{p}{q}, pq)$, $p, q \in \mathbb{Z}$.





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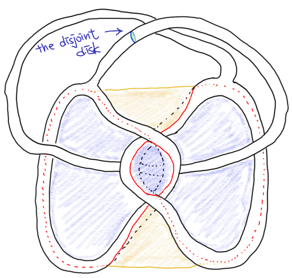

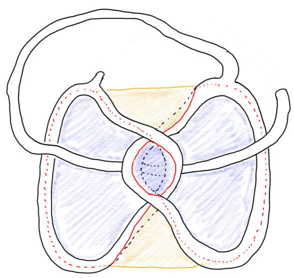

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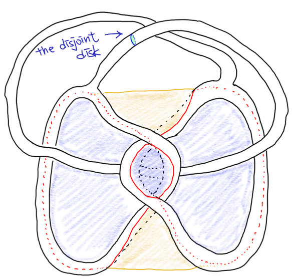

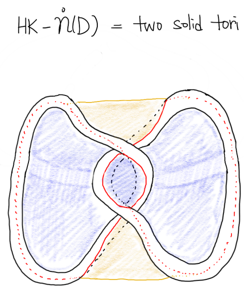

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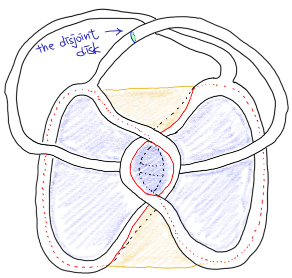

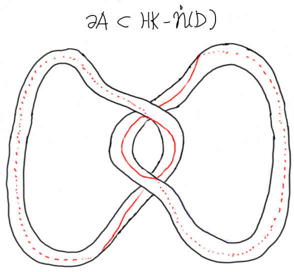

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- HK_A is a handlebody $\Leftrightarrow$ the slope pair is $(\frac{p}{q}, pq)$.

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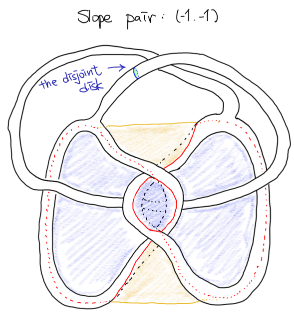

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1. $\mathrm{Sym}_+(\mathbb{S}^3, \mathrm{HK}) \to \mathrm{MCG}(\mathrm{A})$ is injective.
   - $\mathrm{MCG}(\mathrm{A}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.
2. $(\mathbb{S}^3, \mathrm{HK})$ is chiral.

Q. How to show uniqueness?

The Jacobi Spectrum of Null-Torsion Holomorphic Curves in the 6-Sphere

Jesse Madnick (National Center for Theoretical Sciences)

Minimal surfaces are area-minimizing to first order, but not necessarily to second-order. The extent to which a minimal surface is (or isn't) area-minimizing to second-order is encoded by its Jacobi operator. However, for a given minimal surface, computing the spectrum of the Jacobi operator — i.e., the eigenvalues and their multiplicities — is a non-trivial task. In this talk, I will discuss a class of minimal surfaces in the round 6-sphere known as "null-torsion holomorphic curves." These surfaces are of interest to G2 geometry and exist in abundance. Indeed, by a remarkable theorem of Bryant, extended by Rowland, every closed Riemann surface may be conformally embedded into S^6 as a null-torsion holomorphic curve. For nulltorsion holomorphic curves of low genus, we will compute the multiplicity of the first Jacobi eigenvalue. Moreover, for all genera, we will give a simple lower bound for the nullity in terms of the area and genus. We expect that these results will have implications for the deformation theory of asymptotically conical associative 3-folds in euclidean $\mathbb{R}^7$.

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The Jacobi Spectrum of Null-Torsion Holomorphic Curves in $\mathbb{S}^6$

Jesse Madnick
National Center for Theoretical Sciences

3rd Japan-Taiwan Joint Conference on Differential Geometry

Outline

I. Minimal Surfaces; Jacobi Spectra

- Definition
- Context: Some results in $\mathbb{S}^3$, $\mathbb{S}^4$, and $\mathbb{S}^{2k}$

II. The Round 6-Sphere

III. Holomorphic Curves in $\mathbb{S}^6$

- Definition
- Holomorphic Frenet Frame
- Null-Torsion Condition

IV. Null-Torsion Holomorphic Curves in $\mathbb{S}^6$

- Theorem A and Theorem B
- Open Questions
- Ideas of the Proofs: Theorem A′ and Theorem B′

Minimal Surfaces

- Σ^2 closed orientable surface of genus g, area A.
- $(M^n, \langle\cdot,\cdot\rangle)$ Riemannian manifold.

An immersion $u\colon \Sigma^2 \to (M^n, \langle\cdot,\cdot\rangle)$ is a **minimal surface** if: For all variations $u_t\colon \Sigma \to M$ with $u_0 = u$:

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Area}(u_t) = 0.$$

Notation: Let $u\colon \Sigma^2 \to M^n$ immersion.

- $\overline{\nabla}$ = Levi-Civita connection of M.
- $\nabla^{\top}$ = Tangential connection = Levi-Civita connection of Σ.
- $\nabla^{\perp}$ = Normal connection.
- $\mathrm{I\!I}$ = Second fundamental form.

Minimal Surfaces

Second Variation of Area: Let $u\colon \Sigma^2 \to M^n$ be a minimal surface, and $u_t\colon \Sigma^2 \to M^n$ be a variation of u with normal variation vector field $\eta \in \Gamma(N\Sigma)$. Then:

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Area}(u_t) = \int_{\Sigma} \langle -\Delta^{\perp}\eta - \mathcal{B}\eta - \mathcal{R}\eta, \eta\rangle \,\mathrm{vol}_{\Sigma}$$

where

$$\Delta^{\perp}\eta = \nabla^{\perp}_{e_i}\nabla^{\perp}_{e_i}\eta - \nabla^{\perp}_{\nabla^{\top}_{e_i}e_i}\eta \qquad \text{(connection Laplacian of } \nabla^{\perp}\text{)}$$

$$\mathcal{B}\eta = \langle \mathrm{I\!I}(e_i,e_j), \eta\rangle\, \mathrm{I\!I}(e_i,e_j) \qquad \text{(0th-order term)}$$

$$\mathcal{R}\eta = (\overline{R}(\eta,e_i)e_i)^{\perp} \qquad \text{(0th-order term)}$$

Here, (e_i) = (local orthonormal frame on Σ) and $\overline{R}$ = (curvature of M).

The **Jacobi operator** of u is the second-order linear differential operator $\mathcal{L}\colon \Gamma(N\Sigma) \to \Gamma(N\Sigma)$ given by

$$\mathcal{L} = -\Delta^{\perp} - \mathcal{B} - \mathcal{R}.$$

Minimal Surfaces

The **Jacobi operator** of u is the second-order linear differential operator $\mathcal{L}\colon \Gamma(N\Sigma) \to \Gamma(N\Sigma)$ given by

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The eigenvalues of $\mathcal{L}$ form an increasing sequence of real numbers

$$\lambda_1 < \cdots < \lambda_s < 0 = \lambda_{s+1} < \lambda_{s+2} < \cdots \to \infty$$

with finite multiplicities

$$m_1, \ldots, m_s, m_{s+1}, m_{s+2}, \ldots.$$

The **Jacobi spectrum** of u is the set of eigenvalues $\lambda_1, \lambda_2, \ldots$ and their multiplicities $m_1, m_2, \ldots$

The **Morse index** and **nullity** of u are:

$$\textbf{Index}(u) := m_1 + \cdots + m_s$$
$$\textbf{Nullity}(u) := m_{s+1}$$

Notice that u is **stable** iff $\lambda_1 \geq 0$, and **unstable** iff $\lambda_1 < 0$.

Minimal Surfaces in Round Spheres

Suppose $(M^n, \langle \cdot, \cdot \rangle) = (\mathbb{S}^n(1), \text{round})$.

Simons ('68): Every minimal surface $u\colon \Sigma^2 \to \mathbb{S}^n$ satisfies:

- $\lambda = -2$ is a Jacobi eigenvalue.
- $\text{Ind}(u) \geq n-2$. Equality iff $u(\Sigma)$ totally-geodesic 2-sphere.
- $\text{Nullity}(u) \geq 3(n-2)$. Equality iff $u(\Sigma)$ totally-geodesic 2-sphere.

Karpukhin ('19): Suppose $n = 2k$ even and $g = 0$ and u linearly full (but allowing branch points). Let $A = 4\pi d$ denote the area. Then

$$\text{Ind}(u) \geq (n-2)\left(2d + 2 - [\sqrt{8d+1}]_{\text{odd}}\right),$$

where $[x]_{\text{odd}}$ is the largest odd number $\leq x$.

Ejiri ('83): Suppose $n = 2k$ even and $g = 0$. Every minimal 2-sphere $u\colon \mathbb{S}^2 \to \mathbb{S}^{2k}$ of area A has $\lambda_1 = -2$ and

$$m_1 = \frac{A}{\pi} + 2(k-3).$$

Some Results in $\mathbb{S}^3$

Suppose $n = 3$. Let $u\colon \Sigma^2 \to \mathbb{S}^3$ minimal surface, where Σ^2 closed, orientable.

Urbano ('90): If u not totally-geodesic, then

$$\mathrm{Ind}(u) \geq 5.$$

Equality iff $u(\Sigma)$ is the Clifford torus.

Application: Used by Marques-Neves ('12) in their solution of the Willmore Conjecture.

Some Results in $\mathbb{S}^4$

Suppose $n = 4$. Let $u\colon \Sigma^2 \to \mathbb{S}^4$ minimal surface, where Σ^2 closed, orientable, Euler characteristic $\chi(\Sigma) = 2 - 2g$, area A.

Micallef-Wolfson ('93):

$$\mathrm{Ind}(u) \geq \frac{1}{2}\left(\frac{A}{\pi} - \chi(\Sigma)\right).$$

Montiel-Urbano ('97): If u is **infinitesimally holomorphic** (a.k.a. **superminimal**) (i.e.: $\mathrm{I\!I}$ has the same symmetries as a complex curve), then

$$\mathrm{Ind}(u) = m_1 = \frac{A}{\pi} - \chi(\Sigma) \qquad \mathrm{Nullity}(u) = m_2 \geq \frac{A}{\pi} + \chi(\Sigma)$$

Also, for $g = 0, 1$: Equality holds in the nullity bound.

Kusner-Wang ('18): If $g = 1$, then

$$\mathrm{Ind}(u) \geq 6.$$

Equality iff $u(\Sigma)$ is a Clifford torus in a totally-geodesic $\mathbb{S}^3$.

The 6-Sphere

Fact: The n-sphere $\mathbb{S}^n$ admits an almost-complex structure if and only if

$$n = 2 \quad \text{or} \quad n = 6.$$

View $\mathbb{S}^6 \hookrightarrow \mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$. The **standard almost-complex structure** is

$$J_p \colon T_p\mathbb{S}^6 \to T_p\mathbb{S}^6$$
$$J_p(v) = p \times v = \tfrac{1}{2}(pv - vp),$$

where pv and vp denote multiplication in $\mathbb{O}$. Note that J is compatible with the round metric $\langle \cdot, \cdot \rangle$:

$$\langle JX, JY \rangle = \langle X, Y \rangle.$$

Define a **non-degenerate** 2**-form** $\omega \in \Omega^2(\mathbb{S}^6)$ by $\omega(X,Y) := \langle JX, Y \rangle$.

The triple $(\langle \cdot, \cdot \rangle, J, \omega)$ is a $\mathbf{U}(3)$**-structure** on $\mathbb{S}^6$. That is, we are viewing $\mathbb{S}^6$ as an **almost-Hermitian** manifold.

The 6-Sphere: Its Standard U(3)-Structure

The triple $(\langle \cdot, \cdot \rangle, J, \omega)$ on $\mathbb{S}^6$ is a $\mathbf{U}(3)$**-structure** (**almost-Hermitian structure**).

Let $\overline{\nabla}$ the Levi-Civita connection on $\mathbb{S}^6$.

Warnings:

- ω is **not** closed.
- J is **not** integrable.
- J is **not** $\overline{\nabla}$-parallel: $\overline{\nabla} J \not\equiv 0$.

Good News: The U(3)-structure $(\langle \cdot, \cdot \rangle, J, \omega)$ is **nearly-Kähler**, meaning:

$$(\overline{\nabla}_X J)(Y) = -(\overline{\nabla}_Y J)(X), \quad \forall X, Y \in T\mathbb{S}^6.$$

Also: The **unitary connection**

$$\overline{D}_X Y := \overline{\nabla}_X Y + \frac{1}{2}(\overline{\nabla}_X J)(JY)$$

preserves the U(3)-structure $(\langle \cdot, \cdot \rangle, J, \omega)$, e.g.:

$$\overline{D} J = 0 \quad \text{and} \quad \overline{D}\omega = 0.$$

The 6-Sphere: Its Standard SU(3)-Structure

The U(3)-structure $(\langle\cdot,\cdot\rangle, J, \omega)$ on $\mathbb{S}^6$ is nearly-Kähler, but not Kähler. Therefore, $\mathbb{S}^6$ admits an $\mathbb{S}^1$-family of compatible complex volume forms. For concreteness, let's choose one as follows:

View $\mathbb{S}^6 \hookrightarrow \mathbb{R}^7$. The **associative** 3-**form** $\phi_0 \in \Omega^3(\mathbb{R}^7)$ is:

$$\phi_0(X, Y, Z) := \langle X \times Y, Z\rangle_{\mathbb{R}^7}.$$

Let ∂_r = radial vector field on $\mathbb{R}^7$.

Fact: The $(3,0)$-form $\Upsilon \in \Omega^{3,0}(\mathbb{S}^6)$ given by

$$\Upsilon := (\partial_r \lrcorner * \phi_0 + i\phi_0)|_{\mathbb{S}^6}$$

is a **complex volume form**, meaning that

$$\frac{i}{8}\Upsilon \wedge \overline{\Upsilon} = \mathrm{vol}_{\mathbb{S}^6}.$$

The quadruple $(\langle\cdot,\cdot\rangle, J, \omega, \Upsilon)$ is the **standard SU**(3)**-structure** on $\mathbb{S}^6$.

Holomorphic Curves in $\mathbb{S}^6$

An immersion $u\colon \Sigma^2 \to \mathbb{S}^6$ is a **holomorphic curve** if:

$$J_p(T_p\Sigma) = T_p\Sigma, \quad \forall p \in \Sigma.$$

Equivalently:

$$\omega|_\Sigma = \mathrm{vol}_\Sigma.$$

Fact: Holomorphic curves in $\mathbb{S}^6$ are minimal surfaces.

Proof 1:

$$u(\Sigma) \text{ holo curve} \implies \text{Extra symmetries in } \mathrm{I\!I} \implies \mathrm{tr}(\mathrm{I\!I}) = 0.$$

Proof 2:

$$u(\Sigma) \subset \mathbb{S}^6 \text{ holo curve} \iff \mathrm{Cone}(u(\Sigma))^3 \subset \mathbb{R}^7 \text{ is } \textbf{associative}.$$

Associative 3-folds are calibrated submanifolds. So:

$u(\Sigma)$ holo curve $\implies$ Cone$(u(\Sigma))$ homol. vol-minim. $\implies$ $u(\Sigma)$ minimal.

Holomorphic Curves in $\mathbb{S}^6$

Question: What can we say about the Jacobi spectrum of closed holomorphic curves in $\mathbb{S}^6$?

Simons ('68):

- $\lambda = -2$ is a Jacobi eigenvalue.
- $\text{Ind}(u) \geq 4$. Equality iff $u(\Sigma)$ totally-geodesic 2-sphere.
- $\text{Nullity}(u) \geq 12$. Equality iff $u(\Sigma)$ totally-geodesic 2-sphere.

Ejiri ('83): If $g = 0$, then $\lambda_1 = -2$ and

$$m_1 = \frac{A}{\pi}.$$

What if $g \geq 1$?

Observation: Let $u \colon \Sigma^2 \to \mathbb{S}^6$ holomorphic curve of any genus $g \geq 0$.
If u is null-torsion, then Ejiri's argument yields $\lambda_1 = -2$ and

$$m_1 \geq \frac{A}{\pi}.$$

Defining "null-torsion" requires some preparation.

Digression: Curves in $\mathbb{R}^3$

Let $\alpha \colon I \to \mathbb{R}^3$ immersed, oriented, unit-speed curve. Let (e_1, e_2, e_3) oriented orthonormal frame along α. Let's adapt frames:

1st Adaptation: Arrange

$$\begin{cases} e_1 \in TI \\ e_2, e_3 \in NI. \end{cases}$$

2nd Adaptation: At $s \in I$ with $\alpha''(s) \neq 0$, arrange:

$$\begin{cases} e_2 \in \text{span}(\alpha''(s)) \\ e_3 = e_1 \times e_2 \end{cases}$$

Such a local frame $(T, N, B) := (e_1, e_2, e_3)$ is called a **Frenet frame**.

Digression: Curves in $\mathbb{R}^3$

Frenet Equations: For any local Frenet frame (T, N, B) on $U \subset I$, there are functions $\kappa, \tau \colon U \to \mathbb{R}$ s.t.:

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

- Call κ the **curvature** of α. It is a 2nd-order invariant.
- $\kappa = 0 \iff \alpha(I)$ is a line.

- Call τ the **torsion** of α. It is a 3rd-order invariant.
- $\tau = 0 \iff \alpha(I)$ lies in a 2-plane.

The Holomorphic Frenet Frame

Let $u \colon \Sigma^2 \to \mathbb{S}^6$ immersed, oriented, holomorphic curve. Complexify

$$T\mathbb{S}^6 \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}\mathbb{S}^6 \oplus T^{0,1}\mathbb{S}^6.$$

Decompose the $(1,0)$-vectors along Σ into tangent and normal parts:

$$u^*(T^{1,0}\mathbb{S}^6) = T^{1,0}\Sigma \oplus N^{1,0}\Sigma.$$

Let $(e_1, \ldots, e_6)$ special unitary local frame on $U \subset \Sigma$. Set

$$f_1 = \frac{1}{2}(e_1 - ie_2) \qquad f_2 = \frac{1}{2}(e_3 - ie_4) \qquad f_3 = \frac{1}{2}(e_5 - ie_6)$$

so f_1, f_2, f_3 are $(1,0)$-vectors along Σ. Let's adapt to the geometry of the holomorphic curve.

The Holomorphic Frenet Frame

1st Adaptation: Arrange

$$\begin{cases} f_1 \in T^{1,0}\Sigma \\ f_2, f_3 \in N^{1,0}\Sigma. \end{cases}$$

One can show that $\{p \in \Sigma \colon \mathrm{I\!I}_p = 0\}$ is finite or all of Σ.

2nd Adaptation: At $p \in \Sigma$ with $\mathrm{I\!I}_p \neq 0$, arrange:

$$\begin{cases} f_2 \in \mathrm{span}_{\mathbb{C}}(\mathrm{I\!I}(f_1, f_1)) \end{cases}$$

where we extended $\mathrm{I\!I}$ by $\mathbb{C}$-linearity.

Call such a frame (f_1, f_2, f_3) an **adapted frame**.

The Holomorphic Frenet Frame

Frenet Equations (Bryant '82): For any adapted local frame (f_1, f_2, f_3) on $U \subset \Sigma$, there are (local) holomorphic functions $\kappa, \tau \colon U \to \mathbb{C}$ and connection 1-forms $\gamma_{11}, \gamma_{22}, \gamma_{33}$ s.t.:

$$\overline{D}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{pmatrix} \gamma_{11} & \kappa\zeta & 0 \\ -\overline{\kappa}\overline{\zeta} & \gamma_{22} & \tau\zeta \\ 0 & -\overline{\tau}\overline{\zeta} & \gamma_{33} \end{pmatrix} \otimes \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

Here, $\zeta = e^1 + ie^2 \in \Omega^{1,0}(\Sigma)$.

Note: Both κ, τ depend on the choice of adapted frame, but the conditions $\kappa = 0$ and $\tau = 0$ are well-defined.

Analogy:

- $\kappa =$ "curvature" (2nd order).
- $\kappa = 0$ iff $u(\Sigma)$ totally-geodesic.
- $\tau =$ "torsion" (3rd order).

Say u is **null-torsion** if $\tau = 0$ for some (all) adapted frames.

Null-Torsion Holomorphic Curves in $\mathbb{S}^6$

What does "null-torsion" mean?

On $N\Sigma = \text{span}_{\mathbb{R}}(e_3, e_4, e_5, e_6)$, we have:

$$Je_3 = e_4 \qquad\qquad Je_5 = e_6.$$

Define a new complex structure $\widehat{J}$ on $N\Sigma$ via:

$$\widehat{J}e_3 = e_4 \qquad\qquad \widehat{J}e_5 = -e_6.$$

Fact: The following are equivalent:

- u is null-torsion.
- $\nabla^\perp \widehat{J} = 0$.
- $D^\perp \widehat{J} = 0$.
- The **binormal Gauss map**

$$b_u \colon \Sigma^2 \to \mathbb{CP}^6$$
$$b_u(p) := \text{span}_{\mathbb{C}}(e_5 - ie_6)$$

is holomorphic.

Corollary: Null-torsion holomorphic curves have area $A = 4\pi d$, where $d = \deg(b_u) \in \mathbb{Z}^+$. Moreover, $d = 1$ or $d \geq 6$.

Null-Torsion Holomorphic Curves in $\mathbb{S}^6$

Are there any interesting null-torsion holomorphic curves?

Bryant '82:

- Every holomorphic curve of genus $g = 0$ is null-torsion.
- Weierstrass representation formula for null-torsion holomorphic curves.
- Every closed Riemann surface admits a conformal branched immersion into $\mathbb{S}^6$ as a null-torsion holomorphic curve (with arbitrarily many branch points).

Rowland '99: Every closed Riemann surface admits a conformal embedding into $\mathbb{S}^6$ as a null-torsion holomorphic curve.

Results: Closed Null-Torsion Curves

Let $u\colon \Sigma^2 \to \mathbb{S}^6$ immersed null-torsion holomorphic curve.
Let $g = \text{genus}(\Sigma)$ and $A = \text{Area}(\Sigma) = 4\pi d$. Recall: Ejiri's argument gives $\lambda_1 = -2$ and

$$m_1 \geq \frac{A}{\pi} = 4d.$$

Also, if $g = 0$, then equality holds.

Theorem A (M. '21): If $g \leq 6$, then

$$m_1 = \frac{A}{\pi} = 4d \in 4\mathbb{Z}^+.$$

Theorem B (M. '21): For any $g \geq 0$:

$$\text{Nullity}(u) \geq 2d + \chi(\Sigma).$$

Expected Application

Holomorphic curves in $\mathbb{S}^6$ are the links of **associative cones** in $\mathbb{R}^7$.

Lotay ('10): Studied (non-compact) associative 3-folds in $\mathbb{R}^7$ that are **asymptotic to cones**.

Expectation: Theorems A and B likely have consequences for the deformation theory of asymptotically conical associative 3-folds in $\mathbb{R}^7$. This is work in progress.

Open Questions: Closed Holomorphic Curves $\mathbb{S}^6$

- Closed holomorphic curves in $\mathbb{S}^6$: Can one find a lower bound for λ_2?

Simplest case: Compute λ_2 of the **Boruvka sphere**, the unique holomorphic curve with $K = \frac{1}{6}$. Explicitly, the Boruvka sphere is

$$u\colon \mathbb{S}^2 \to \mathbb{S}^6 \subset \mathbb{R}^7$$
$$u(x,y,z) = (p_1(x,y,z), \ldots, p_7(x,y,z))$$

where $\{p_1, \ldots, p_7\}$ is a basis of the harmonic homogeneous cubic polynomials on $\mathbb{R}^3$. It is an orbit of the maximal $\mathrm{SO}(3) \leq \mathrm{G}_2 \circlearrowleft \mathbb{R}^7$.

I have shown that the Boruvka sphere has $\lambda_2 \geq -\frac{5}{3}$, but surely we can be more precise.

Ejiri's result ('83), together with a result of Karpukhin ('19), implies

$$24 + m_2 + \cdots + m_s = \mathrm{Ind}(\text{Boruvka sphere}) \geq 36$$

so the Boruvka sphere has

$$m_2 + \cdots + m_s \geq 12 > 0,$$

and hence $\lambda_2 < 0$.

Open Questions: Closed Superminimal Surfaces in $\mathbb{S}^{2k}$

- In $\mathbb{S}^{2k}$ with $k \geq 2$: If $u\colon \Sigma^2 \to \mathbb{S}^{2k}$ is **superminimal**, then Ejiri's arguments show that:

$$m_1 \geq \frac{A}{\pi} + (k-3)\chi(\Sigma).$$

Supposing u is superminimal, when does equality hold?

- Ejiri '83: Equality if $g = 0$.
- Montiel-Urbano '97: Equality if $k = 2$.
- Theorem A: Equality if $k = 3$, $g \leq 6$ and u holomorphic curve.

Theorems A and B are Special Cases

Let $u\colon \Sigma^2 \to \mathbb{S}^6$ immersed null-torsion holomorphic curve.
Let $g = \text{genus}(\Sigma)$ and $A = \text{Area}(\Sigma) = 4\pi d$.

Theorem A (M. '21): If $g \leq 6$, then

$$m_1 = \frac{A}{\pi} = 4d \in 4\mathbb{Z}^+.$$

Theorem B (M. '21): For any $g \geq 0$:

$$\text{Nullity}(u) \geq 2d + \chi(\Sigma).$$

Theorem A is a special case of a more precise result (called Theorem A′).
Theorem B is a special case of a more precise result (called Theorem B′).

To state Theorems A′ and B′, we need to recast the **holomorphic Frenet frame** as a splitting

$$u^*(T^{1,0}\mathbb{S}^6) = L_T \oplus L_N \oplus L_B$$

into complex line subbundles L_T, L_N, L_B. **To Do:** Define L_T, L_N, L_B.

Defining L_T, L_N, L_B: Step 1 of 2

Let $u\colon \Sigma^2 \to \mathbb{S}^6$ immersed, oriented, holomorphic curve. Decompose the $(1,0)$-vectors along Σ into tangent and normal parts:

$$u^*(T^{1,0}\mathbb{S}^6) = T^{1,0}\Sigma \oplus N^{1,0}\Sigma.$$

Recall the **unitary connection** $\overline{D}$ on $T\mathbb{S}^6$. It yields a connection on $u^*(T^{1,0}\mathbb{S}^6)$. Equip $u^*(T^{1,0}\mathbb{S}^6) \to \Sigma$ with the Koszul-Malgrange holomorphic structure for $\overline{D}$.

Def: Set $L_T := T^{1,0}\Sigma \subset u^*(T^{1,0}\mathbb{S}^6)$. Note that L_T is a holomorphic line subbundle. Define

$$Q_{NB} := \frac{u^*(T^{1,0}\mathbb{S}^6)}{L_T},$$

so that $Q_{NB} \to \Sigma$ inherits a holomorphic structure.

Warning: The subbundle $N^{1,0}\Sigma \subset u^*(T^{1,0}\mathbb{S}^6)$ is **not** a holomorphic vector subbundle unless u is totally-geodesic. The isomorphism $Q_{NB} \simeq N^{1,0}\Sigma$ holds in the smooth (but **not** holomorphic) category.

Defining L_T, L_N, L_B: Step 2 of 2

Let $(e_1, \dots, e_6)$ special unitary local frame on Σ. Set

$$f_1 = \tfrac{1}{2}(e_1 - ie_2) \qquad f_2 = \tfrac{1}{2}(e_3 - ie_4) \qquad f_3 = \tfrac{1}{2}(e_5 - ie_6)$$

so f_1, f_2, f_3 are $(1,0)$-vectors along Σ. Let's adapt frames to the curve.

1st Adaptation: Arrange $f_1 \in T^{1,0}\Sigma = L_T$ and $f_2, f_3 \in N^{1,0}\Sigma$. Then the ($\mathbb{C}$-linearly extended) second fundamental form can be written

$$\mathrm{II}(f_1, f_1) = \kappa f_2 + \mu f_3$$

for some (frame-dependent) functions $\kappa, \mu \colon \Sigma \to \mathbb{C}$.

Fact (Bryant '82): The section $\Phi_{\mathrm{II}} \in H^0(L_T^* \otimes L_T^* \otimes Q_{NB})$ given by

$$\Phi_{\mathrm{II}} := (e^1 + ie^2) \otimes (e^1 + ie^2) \otimes (\kappa[f_2] + \mu[f_3])$$

is a well-defined (frame-independent) holomorphic section.

Def: There exists a unique holomorphic line subbundle $L_N \subset Q_{NB}$ for which $\Phi_{\mathrm{II}} \in H^0(L_T^* \otimes L_T^* \otimes L_N)$. Finally, define

$$L_B := \frac{Q_{NB}}{L_N}.$$

Theorems A′ and B′

Let $u \colon \Sigma^2 \to \mathbb{S}^6$ immersed null-torsion holomorphic curve.
Let $g = \text{genus}(\Sigma)$ and $A = \text{Area}(\Sigma) = 4\pi d$.

Let $K_\Sigma = \Lambda^{1,0}(\Sigma)$ be the canonical line bundle of Σ.

Theorem A′ (M. '21): For any $g \geq 0$, we have

$$\frac{A}{\pi} \leq m_1 \leq \frac{A}{\pi} + h^0(L_B \otimes K_\Sigma^*).$$

Moreover, if $g \leq 6$, then $h^0(L_B \otimes K_\Sigma^*) = 0$.

Theorem B′ (M. '21): For any $g \geq 0$, the null space of the Jacobi operator

$$\text{Null}(u) := \{\eta \in \Gamma(N\Sigma) \colon \mathcal{L}\eta = 0\}$$

contains a vector subspace isomorphic to $H^0(L_N \otimes K_\Sigma^*)$. Consequently,

$$\text{Nullity}(u) \geq \dim_{\mathbb{R}}[H^0(L_N \otimes K_\Sigma^*)] \geq 2d + \chi(\Sigma).$$

(The last inequality is Riemann-Roch.)

Sketch of Theorem A′

Theorem A′ (M. ’21): For any $g \geq 0$, we have

$$\frac{A}{\pi} \leq m_1 \leq \frac{A}{\pi} + h^0(L_B \otimes K_\Sigma^*).$$

Moreover, if $g \leq 6$, then $h^0(L_B \otimes K_\Sigma^*) = 0$.

Sketch: Equip $N\Sigma$ with the complex structure $\widehat{J}$, so $N\Sigma \simeq L_N \oplus L_B^*$. Both $\nabla^\perp$ and $D^\perp$ endow $(N\Sigma, \widehat{J})$ with holomorphic structures, say $\overline{\partial}^\nabla$ and $\overline{\partial}^D$.

Ejiri's argument shows: The first eigenspace $E(\lambda_1)$ of $\mathcal{L}$ is isomorphic to

$$E(\lambda_1) \cong \{\xi \in \Gamma(N\Sigma) \colon \overline{\partial}^\nabla \xi = 0\}.$$

Consider the difference tensor $S(\xi) := \overline{\partial}^\nabla \xi - \overline{\partial}^D \xi$. So:

$$E(\lambda_1) \cong \{\xi \in \Gamma(N\Sigma) \colon \overline{\partial}^D \xi = -S(\xi)\}.$$

Small miracle: The system $\overline{\partial}^D \xi = -S(\xi)$ **decouples** into a system of the form

$$\begin{cases} \overline{\partial}^{L_N} \xi_N = -T(\xi_B) \\ \overline{\partial}^{L_B^*} \xi_B = 0. \end{cases}$$

Solution space has max. $\mathbb{R}$-dim $= 2(h^0(L_N) + h^0(L_B^*)) = \dfrac{A}{\pi} + h^0(L_B \otimes K_\Sigma^*)$.

Sketch of Theorem B′

Theorem B′ (M. ’21): For any $g \geq 0$, the null space of the Jacobi operator

$$\mathsf{Null}(u) := \{\eta \in \Gamma(N\Sigma) \colon \mathcal{L}\eta = 0\}$$

contains a vector subspace isomorphic to $H^0(L_N \otimes K_\Sigma^*)$. Consequently,

$$\mathsf{Nullity}(u) \geq \dim_{\mathbb{R}}[H^0(L_N \otimes K_\Sigma^*)] \geq 2d + \chi(\Sigma).$$

Sketch: Equip $N\Sigma$ with complex structure $\widehat{J}$ and holomorphic structure $\overline{\partial}^\nabla$. Identify $N\Sigma \simeq L_N \oplus L_B^*$, and let $\pi_B \colon N\Sigma \to L_B^*$ denote projection.

For $\xi \in \Gamma(N\Sigma)$, regard $\overline{\partial}^\nabla \xi \in \Gamma(N\Sigma \otimes K_\Sigma^*)$.

Main Claim: The map

$$\begin{aligned} \{\xi \in \mathsf{Null}(u) \colon \pi_B(\overline{\partial}^\nabla \xi) = 0\} &\cong H^0(L_N \otimes K_\Sigma^*) \\ \xi &\mapsto \overline{\partial}^\nabla \xi \end{aligned}$$

is well-defined and an isomorphism. (Injectivity is easy. However, surjectivity and well-definedness are more complicated.) □

Rmk: This argument is a direct analogue of that in Montiel-Urbano ’97.

Fin

Thanks for your attention!

An Example of the Noncompact Yamabe Flow having the Infinitetime Incompleteness

Hikaru Yamamoto (University of Tsukuba)

I explain a recent result on the noncompact Yamabe flow which is joint work with Jin Takahashi at Tokyo Institute of Technology. The noncompact Yamabe flow is complicated compared to the compact case. There are many unexpected phenomena from the viewpoint of the compact Yamabe flow. One of the remaining questions is the following. If each Riemannian metric is complete under the Yamabe flow on a noncompact manifold for all time and the long time limit exists, then is the limit also complete? I give the negative answer to this question by giving a counterexample.

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The 3rd Japan–Taiwan Joint conference on Differential Geometry, Nov. 3rd.

"An Example of the Noncompact Yamabe Flow having the Infinite-time incompleteness"

(joint work with Jin Takahashi at TITech)

§1. What is the Yamabe flow?

(notation)

M : n-dim mfd (without boundary)

$\mathrm{scal}(g) \in C^\infty(M)$: scalar curvature of Riem. met. g on M.

§1.1. Compact case ⇝ Assume M is cpt.

Yamabe functional $E : \{\text{Riem. met. on } M\} \to \mathbb{R}$ is

$$E(g) := \frac{\int_M \mathrm{scal}(g)\, dv_g}{\mathrm{Vol}(M,g)^{\frac{n-2}{2}}}$$

(⇝ scaling inv. $E(cg) = E(g)$) thanks to $\frac{n-2}{2}$-power

The gradient flow eq. of E (restricted to $[g]$) is
↑ some conformal class

$$\frac{\partial}{\partial t} g_t = -\mathrm{scal}(g_t)\cdot g_t + \int_M \mathrm{scal}(g_t)\, dv_{g_t} \cdot g_t$$

constant

"normalized Yamabe flow" ↳ $\mathrm{scal}(g_\infty) = \int_M \mathrm{scal}(g_\infty)\, dv_{g_\infty}$

Remark This PDE can not be defined if $\int_M \mathrm{scal}(g_t)\, dv_{g_t} = \pm\infty$.

(Now, M is cpt, so, this does not occur.)

Remark What is the limit?

Assume g_t exists for all $t \in [0,\infty)$ and $g_t \to {}^\exists g_\infty$ (in some nice sense),

then g_∞ should be a constant scalar curvature metric.

(need more exp.)

The history of compact case

- 1989, Hamilton : long time ex. and uniqueness are OK for $\forall$initial g_0.
- 1994, Ye : $n \geq 3$ and (M, g_0) is LCF $\Rightarrow$ $g_t \xrightarrow{t\to\infty} g_\infty$: const. scal. met.
- 2003, Schwetlich - Struwe : $3 \leq n \leq 5$ $\Rightarrow$ Ye's result is OK. (without LCF)
- 2007, Brendle : $n \geq 6$ and (M, g_0) is spin or satisfying some condition for Wyle tensor $\Rightarrow$ $g_t \xrightarrow{t\to\infty} g_\infty$: const. scal. met.

So, we can roughly think that (in many (almost all?) cases)

[1] long time sol ex. and uniqueness is OK.

[2] convergence $g_t \xrightarrow{t\to\infty} g_\infty$: const. scal. met. OK.

$\rightsquigarrow$ However, if M is noncompact, these do not hold in general!

---

§1.2 Noncompact case $\rightsquigarrow$ Assume M is non cpt.

In this case $\int_M \mathrm{scal}(g_t)\, dv_{g_t}$ is not finite in general.

So, we simply drop this term from the PDE. That is

$$\frac{\partial}{\partial t} g_t = -\mathrm{scal}(g_t) \cdot g_t \quad \text{———} \quad (\text{uYF})$$

"unnormalized Yamabe flow" (In this talk Yamabe flow := uYF)

Some known results

⊙ short time ex. ? $\longrightarrow$ No! in general.

(some sufficient condi. are known).

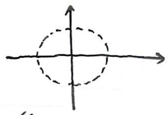
▥ Ma-An (1999) : (M, g_0) is complete, LCF and $\mathrm{Ric} \geq -C$

$\Rightarrow$ short time ex. is OK.

⊙ uniqueness ? ⟶ No! in general.

For example.....

There exist infinitely many Yamabe flows starting from $g_{\mathbb{R}^2}$ on $B^2(1)$!



Moreover, we may find some "interesting" one in those many sol.

Indeed ......

Theorem (Giesen-Topping, 2011)

For $^\forall (M, g_0)$: $\dim M = 2$, connected and incomplete,

$\exists$ Yamabe flow $\{g_t\}_{t\in[0,T)}$ starting from g_0 s.t.

each g_t is complete for $^\forall t \in (0,T)$

↳ This shows the existence of "instanteneously completeness".

⊙ long time sol. ? ⟶ No! in general.

Daskalopoulos - King - Sesum (2013) gave an example.

§2. Motivation and Main result

Completeness is important when we study noncpt Riem mfds.

So, we checked the existence of YF having the following properties.

① g_0 is complete ⟶ ② g_t is also comp. for $t\in[0, {}^\exists T)$ (need to check!)

→ ③ g_T is incomp. $(T<\infty)$ ⟶
- ③-(i) flow blows up at some $T' > T$. ☑ Daskalopoulos-Sesum (2008)
- ③-(ii) flow exists long time. ☐ ?

→ ④ $T=\infty$ and g_t is comp for $^\forall t\in[0,\infty)$ ⟶
- ④-(i) $\lim_{t\to\infty} g_t$ does not exist. ☐ ?
- ④-(ii) $g_\infty := \lim_{t\to\infty} g_t$ exists and it's comp. ☑ $(\mathbb{R}^n, g_{std})$
- ④-(iii) $g_\infty := \lim_{t\to\infty} g_t$ exists and it's incomp. ☑ TY in 2021

Main result

$M := \mathbb{R}^n \setminus \{0\}$ $(n \geq 3)$, fix $\frac{n+2}{2} < \lambda < n+2$.

$u_0(x) := (1 + |x|^{-m\lambda})^{\frac{1}{m}}$ where $m := \frac{n-2}{n+2} \in (0,1)$.

$g_0 := u_0^{\frac{4}{n+2}} g_{\mathbb{R}^n}$ on M.

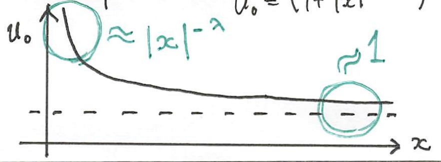
Thm (Takahashi–Y. 2021)

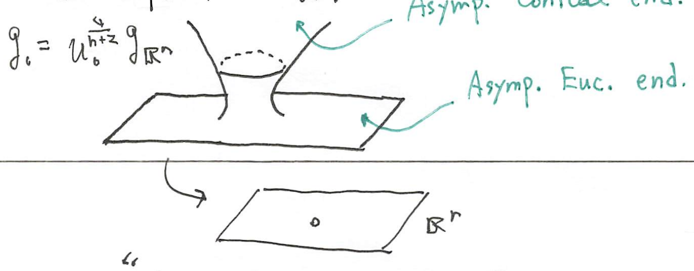
There exists a long time solution $\{g_t\}_{t\in[0,\infty)}$ of Yamabe flow starting from g_0 on M. And the sol. is unique.

Moreover, $g_t \xrightarrow{t\to\infty} g_{\mathbb{R}^n}$, so the limit is incomplete.

↳ We want to call this phenomenon the "infinite-time incompleteness".

• the shape of u_0: $u_0 = (1+|x|^{-m\lambda})^{\frac{1}{m}}$ • the shape of (M, g_0)





§3. The proof.

① Reduce the Yamabe flow to the "fast diffusion equation".

We can assume that $g_t = v_t^{\frac{4}{n-2}} g$ $(v_t > 0)$ w.l.g..

Substituting $\mathrm{scal}(g_t) = v_t^{-\frac{n+2}{n-2}} \left(-4\frac{n-1}{n-2} \Delta_g v_t + \mathrm{scal}(g) v_t\right)$

into the PDE $\left(\frac{\partial}{\partial t} g_t = -\mathrm{scal}(g_t) \cdot g_t\right)$, we have

$$\frac{\partial}{\partial t}\left(v_t^{\frac{4}{n-2}}\right) \cdot \cancel{g} = -v_t^{\frac{n+2}{n-2}} (\text{— same —})\, v_t^{\frac{4}{n-2}} \cancel{g}$$

Drop g and put $u_s := v_t^{\frac{n+2}{n-2}}$ (where $s := \frac{n-2}{(n-1)(n+2)} t$). Then,

$$\frac{\partial}{\partial s} u_s = \Delta_g\left(u_s^{\frac{n-2}{n+2}}\right) - \frac{n-2}{4(n-1)} \mathrm{scal}(g)\, u_s^{\frac{n-2}{n+2}}$$

↳ This is the "fast diffusion eq" (with reaction term).

So, if the initial g_0 is conformal to a scalar flat metric $\mathring{g}$, we have .....

[Solving the uYF

$$\begin{cases} \frac{\partial}{\partial t} g = -\mathrm{scal}(g) \cdot g \\ g(\cdot, 0) = g_0 \end{cases}$$

] $\Longleftrightarrow$ [Solving the fast diff. eq.

$$\begin{cases} \frac{\partial}{\partial t} u = \Delta (u^m) & \text{where } m := \frac{n-2}{n+2} \\ u(\cdot, 0) = u_0 \end{cases}$$

]

And, we have done the right hand side. Purely PDE!

Remark

Proving the existence and uniqueness is not sufficient.

We shoud say that $g_t := u_t^{\frac{4}{n+2}} g_{\mathbb{R}^n}$ is complete.

A sufficient condition is $u_t \approx \begin{cases} |x|^{-m\lambda} & (|x| \to 0) \\ 1 & (|x| \to \infty) \end{cases}$ (need more exp.)

Transversal Properties for Period Maps on Moduli Space of Triply Periodic Minimal Surfaces

Toshihiro Shoda (Kansai University)

Triply periodic minimal surfaces are mathematical objects for surfactant, and they have been studied in many fields. We focus on the genus three case and many one-parameter families have been constructed in physics. In the previous work, we computed Morse indices and nullities for the families, and some bifurcation phenomena, that is, the existence of new one-parameter families issuing from the original one-parameter families were pointed out. The key point is the point where the nullity is greater than three. In this talk, we introduce recent works related to classification of nullities from which a new one-parameter family does not issue, in terms of singularities theory. It is a joint work with Norio Ejiri.

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Transversal properties for period maps on Moduli space of triply periodic minimal surfaces

Toshihiro Shoda
Kansai University
(with Norio Ejiri, Nagoya)

Object

$\widetilde{f} : \widetilde{M} \to \mathbb{R}^3$ triply periodic min. surfs.

$\Longleftrightarrow$

$f : M \to \mathbb{R}^3/\Lambda$ cpt. min. surf. in $\mathbb{T}^3$

- mathematical object of surfactant
- method from the viewpoint of classical Moduli theory (Teichmüller theory, Period map, etc.)

Object

$\widetilde{f} : \widetilde{M} \to \mathbb{R}^3$ triply periodic min. surfs.

$\Longleftrightarrow$

$f : M \to \mathbb{R}^3/\Lambda$ cpt. min. surf. in $\mathbb{T}^3$

Main results

Classifying nullities of min. surf. s.t.

- nullity ≥ 4
- $\nexists$New families of triply periodic min. surfs. from original family

Preliminary

$f : M \to \mathbb{R}^3/\Lambda$ immer. of cpt. surf.

$\to A(f) := \int_M dv$ (by induced metric)

Definition

f is minimal $:\Leftrightarrow \frac{d}{dt}A(f_t)|_{t=0} = 0$

(f_t : deformation of f s.t. $f_0 = f$)

- $\exists$isothermal coords. on M
  $\to$ cmplx. str. on M
  $\to$ f is called conf. min. immer.

Preliminary

2nd variational formula

- N : unit normal vector field
- $\frac{\partial f_t}{\partial t}(0) = uN \ (u \in C^\infty(M))$

$$\frac{d^2}{dt^2}A(f_t)|_{t=0} = \int_M \{|\nabla u|^2 + 2Ku^2\}\, dv$$
$$= -\int_M \underbrace{\{(\triangle - 2K)u\}}_{=-\lambda u} u\, dv$$
$$= \lambda \int_M u^2\, dv$$

$\longleftrightarrow$

E. V. prob. $(\triangle - 2K + \lambda)u = 0$

Preliminary

Definition

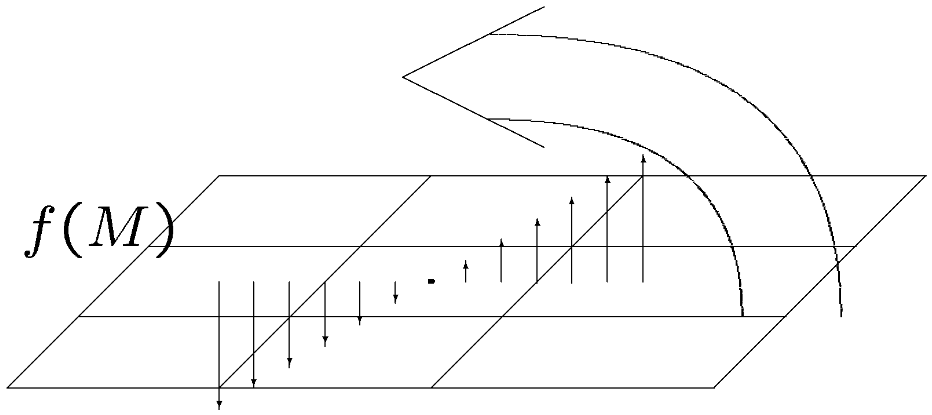
- Morse index $:= \sharp\{\lambda < 0\}$
- nullity $:= \sharp\{\lambda = 0\}$

index: $\sharp$directions of area decreasing

Killing vector fields $\to$ nullity

- translations on $\mathbb{R}^3/\Lambda$ $\to$ nullity
  (We call them trivial Jacobi fields)
- nullity $\geq 3 = \dim \mathbb{R}^3/\Lambda$

Rotations in $\mathbb{R}^3$???

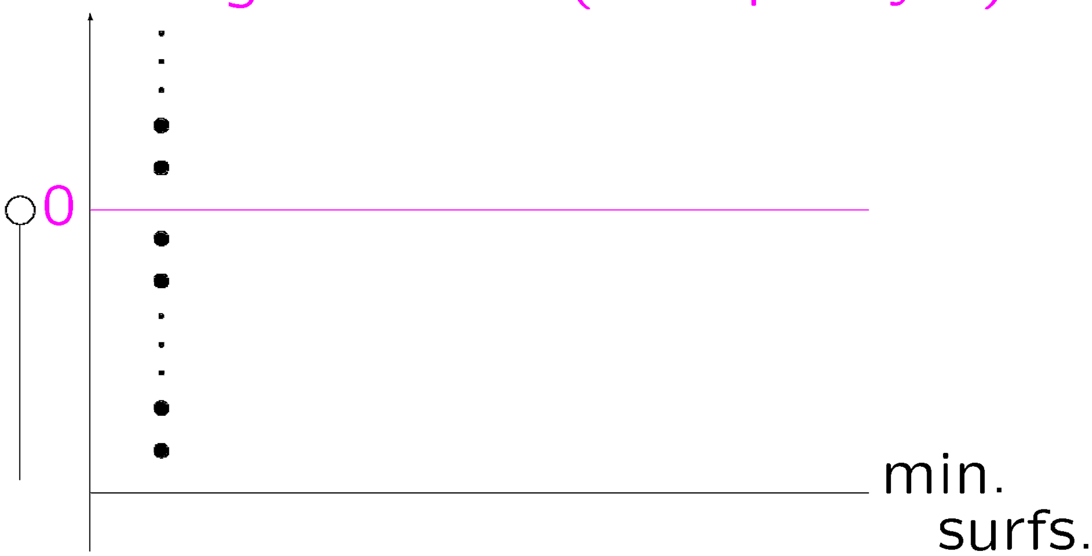


there are no triply periodic Jacobi fields

Point where index changes

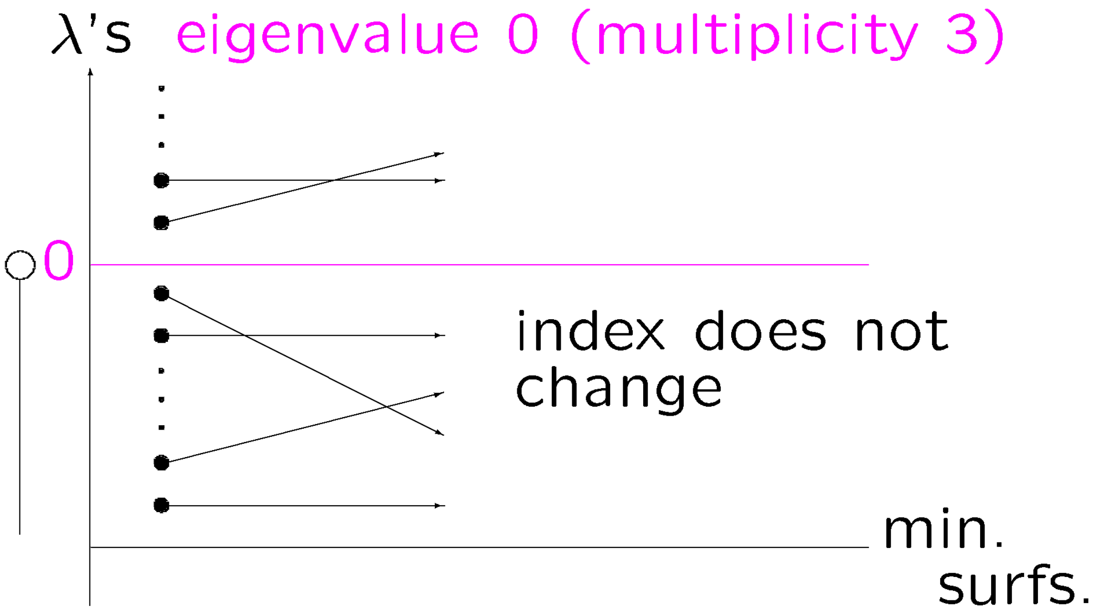
Behavior of λ's

λ's eigenvalue 0 (multiplicity 3)



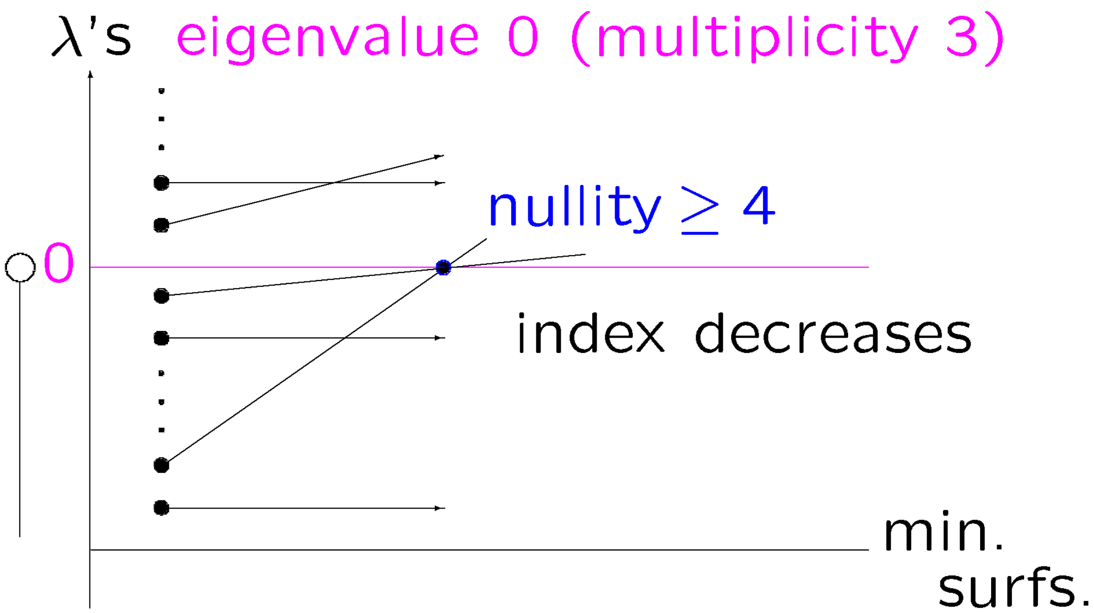
Point where index changes

Behavior of λ's



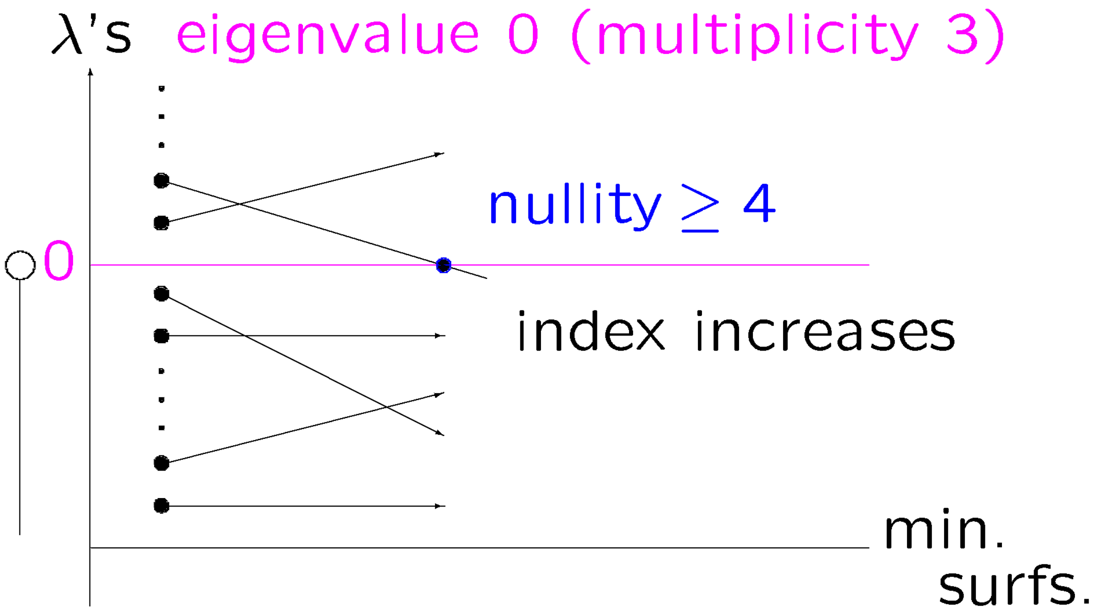
Point where index changes

Behavior of λ's



Point where index changes

Behavior of λ's



Point where index changes

Remark

- the case nullity $=3$ is generic
- nullity ≥ 4 before index changes (the singular case)

Weierstrass representation

$f : M \to \mathbb{R}^3/\Lambda$: conf. min. immer.

up to translations,

$$f(p) = \Re \int_{p_0}^{p} {}^t(\omega_1, \omega_2, \omega_3)$$

$\omega_1, \omega_2, \omega_3$: holo. differentials s.t.

- $\omega_1, \omega_2, \omega_3$: no common zeros
- $\omega_1^2 + \omega_2^2 + \omega_3^2 = 0$
- $\left\{ \Re \int_C {}^t(\omega_1, \omega_2, \omega_3) \mid C \in H_1(M) \right\} \subset \Lambda$

Remark

$\left\{ \Re \int_C {}^t(\omega_1, \omega_2, \omega_3) \mid C \in H_1(M) \right\} \subset \Lambda$

- M has genus $\gamma \geq 3$
- $\{A_j, B_j\}_{j=1}^{2\gamma}$: can. homology basis

the condition

$\Leftrightarrow$

$$\Re \begin{pmatrix} \int_{A_1} \omega_1 & \cdots & \int_{B_{2\gamma}} \omega_1 \\ \int_{A_1} \omega_2 & \cdots & \int_{B_{2\gamma}} \omega_2 \\ \int_{A_1} \omega_3 & \cdots & \int_{B_{2\gamma}} \omega_3 \end{pmatrix} \quad 3 \times 2\gamma\text{-matrix}$$

defines a lattice of $\mathbb{R}^3$ $(\subset \Lambda)$

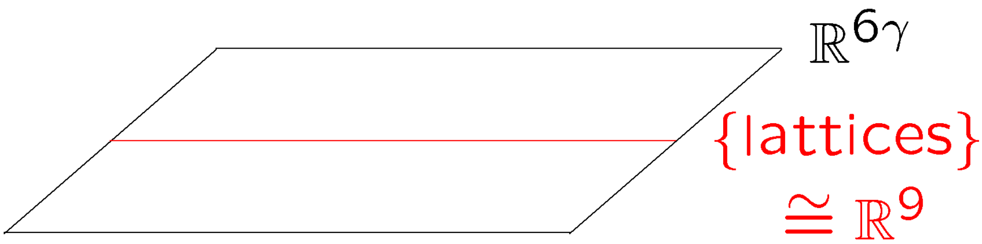
Remark

$$\Re \begin{pmatrix} \int_{A_1}\omega_1 & \cdots & \int_{B_{2\gamma}}\omega_1 \\ \int_{A_1}\omega_2 & \cdots & \int_{B_{2\gamma}}\omega_2 \\ \int_{A_1}\omega_3 & \cdots & \int_{B_{2\gamma}}\omega_3 \end{pmatrix}$$ $3 \times 2\gamma$-matrix

defines a lattice of $\mathbb{R}^3$

3×3 real regular matrix $\cong \mathbb{R}^9$

- lattice $\to$ span$\left\{ \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}, \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \right\}$



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- $\omega_1^2 + \omega_2^2 + \omega_3^2 = 0$
- $\left\{ \Re \int_C {}^t(\omega_1, \omega_2, \omega_3) \mid C \in H_1(M) \right\} \subset \Lambda$

$$\mathcal{M} := \{((M, \{A_j, B_j\}_{j=1}^{2\gamma}), \omega_1, \omega_2, \omega_3)\}$$

Moduli theory

- [1998 Pirola]
- [1999 Arezzo-Pirola]
- [2002 Ejiri], [202? Ejiri]

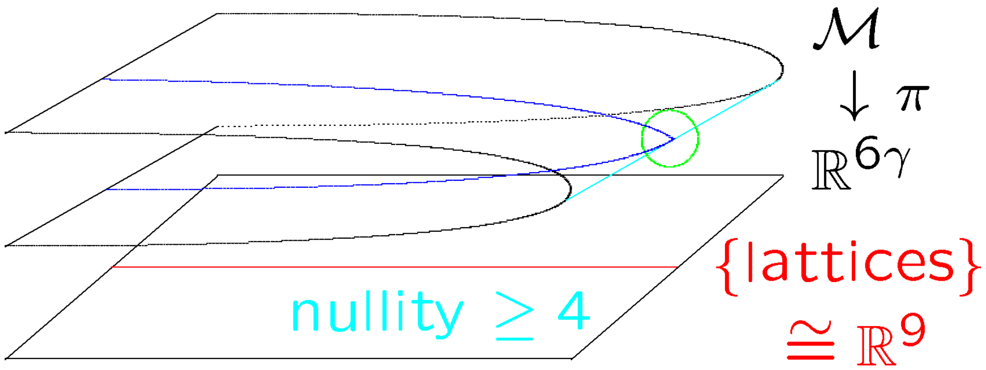
Period map π

$$\mathcal{M} \ni ((M, \{A_j, B_j\}_{j=1}^{2\gamma}), \omega_1, \omega_2, \omega_3) \mapsto \Re \begin{pmatrix} \int_{A_1}\omega_1 & \cdots & \int_{B_{2\gamma}}\omega_1 \\ \int_{A_1}\omega_2 & \cdots & \int_{B_{2\gamma}}\omega_2 \\ \int_{A_1}\omega_3 & \cdots & \int_{B_{2\gamma}}\omega_3 \end{pmatrix} \in \mathbb{R}^{6\gamma}$$

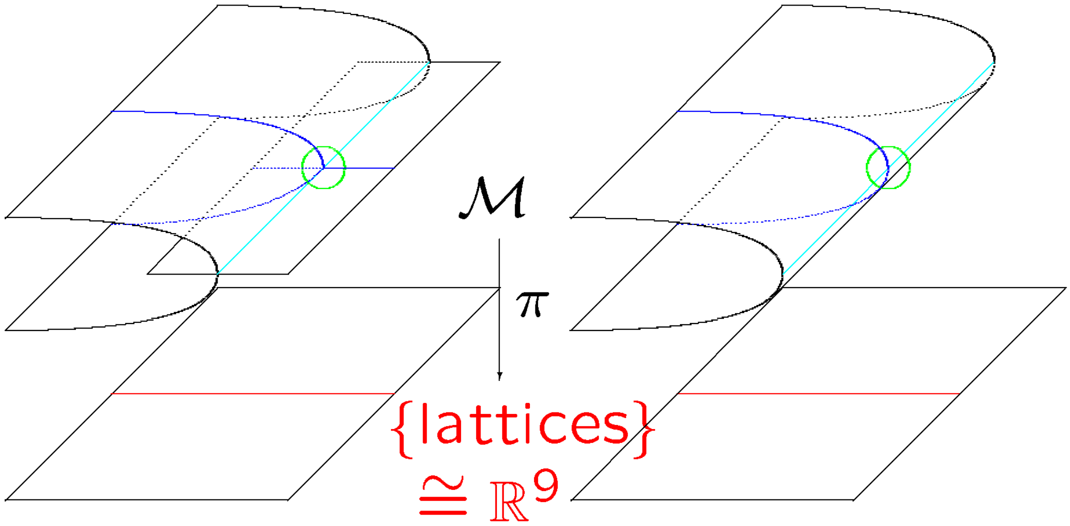
Period map π {3-per. min. surfs.}

$$\mathcal{M} \ni ((M, \{A_j, B_j\}_{j=1}^{2\gamma}), \omega_1, \omega_2, \omega_3) \mapsto \Re \begin{pmatrix} \int_{A_1}\omega_1 & \cdots & \int_{B_{2\gamma}}\omega_1 \\ \int_{A_1}\omega_2 & \cdots & \int_{B_{2\gamma}}\omega_2 \\ \int_{A_1}\omega_3 & \cdots & \int_{B_{2\gamma}}\omega_3 \end{pmatrix} \in \mathbb{R}^{6\gamma}$$

nullity $= 3$
$\Downarrow$
locally graph



Neighborhood around nullity ≥ 4



∃New family → not manifold

∄New family → 9-dim. mfd.

The $\gamma = 3$ case

For triply per. min. surf.

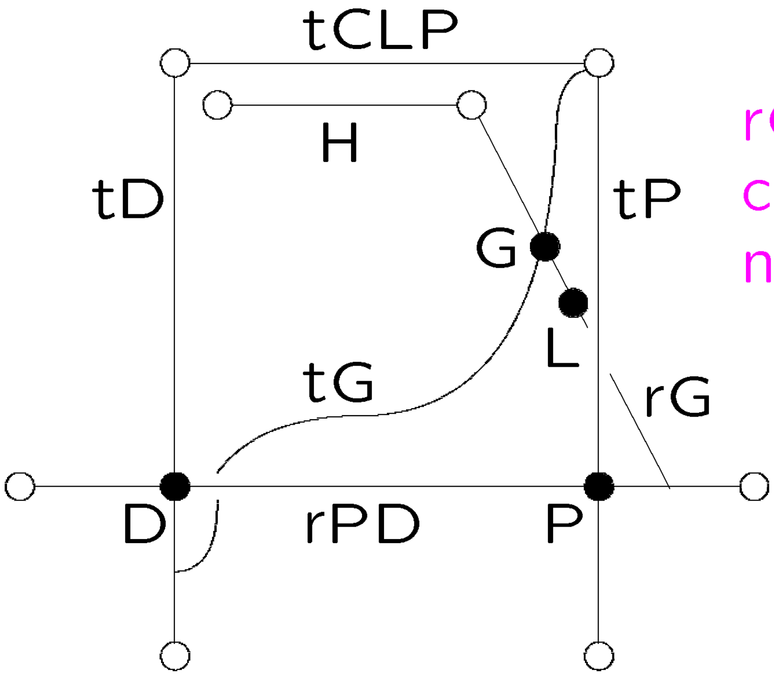
- Morse index ≥ 1, in general

[2006 Ros]

Morse index $= 1 \Rightarrow \gamma = 3$

∃one-parameter families of triply per. min. surfs. in physics

[1990s Fogden, Haeberlein, Hyde, Schröder-Turk]



rG, tG are constructed numerically

P : Primitive D : Diamond
G : Gyroid L : Lidinoid

The $\gamma = 3$ case

[2018, 2020 Ejiri-S.]

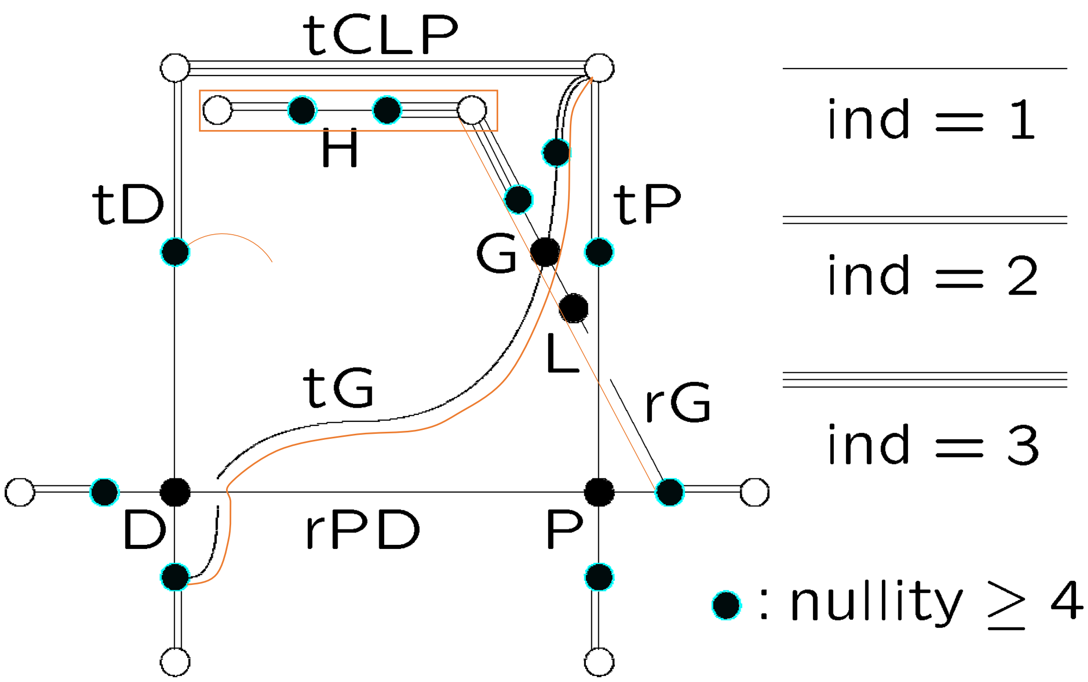
Computing Morse index numerically

[2018 Koiso-Piccione-S.]

Bifurcation phenomena for one-parameter families in physics

[2021 Chen-Weber, Chen]

Mathematical construction of new families



P : Primitive D : Diamond
G : Gyroid L : Lidinoid

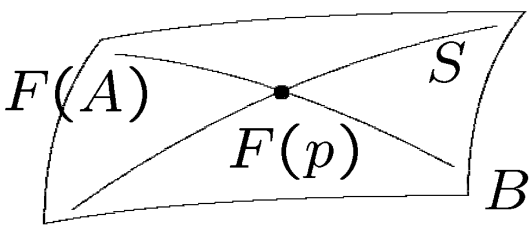
Transversal property for π

Definition

- A, B: manifold • $S \subset B$
- $F : A \to B$: C^∞-map

F is transversal to S at $p \in A$

:$\Leftrightarrow$



(i) $F(p) \notin S$, or

(ii) $F(p) \in S$ and

$$dF_p(T_pA) + T_{F(p)}S = T_{F(p)}B$$

Transversal property for π

Proposition [202? Ejiri] $\mathcal{M}$ is

$F : A \to B$ is transversal to S

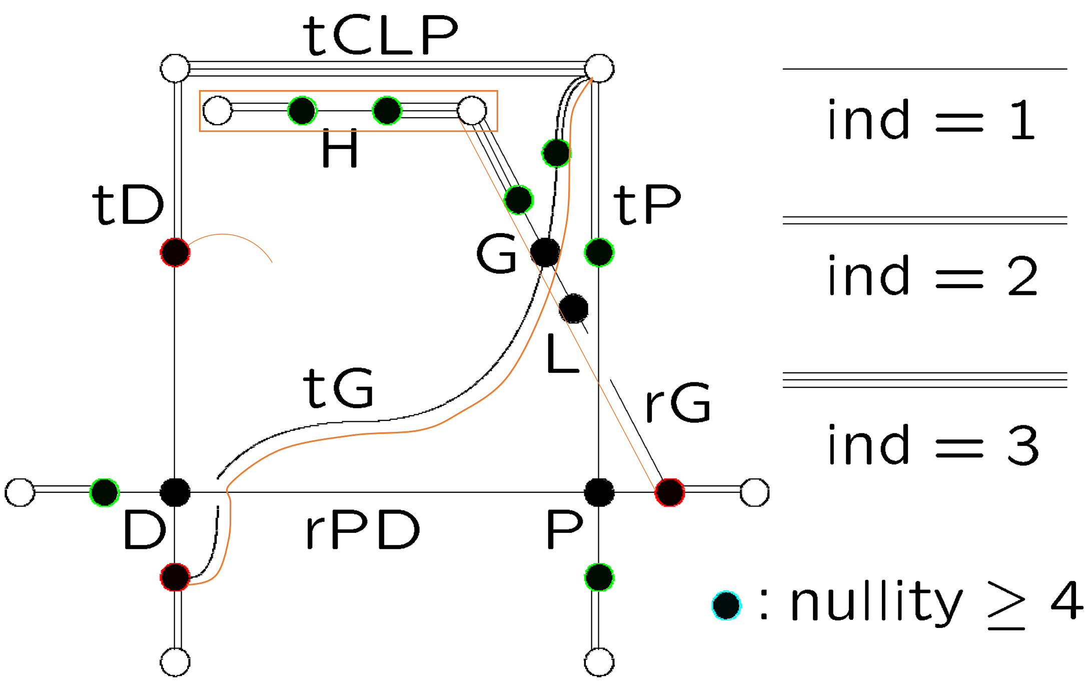
$\Rightarrow$ a cmplx. 9-dim. cmplx. mfd.

- $F^{-1}(S)$ is a submanifold ($\subset A$)
- $\dim A - \dim F^{-1}(S)$

$$= \dim B - \dim S$$

Application to $\pi : \mathcal{M} \to \mathbb{R}^{18}$

- $S = \mathbb{R}^9$ ($= \{$lattices$\}$)
- $\dim \pi^{-1}(S) = 18 - (18 - 9) = 9$



P : Primitive D : Diamond
G : Gyroid L : Lidinoid

The $\gamma = 3$ case

[2021 Ejiri-S.]

- The green points satisfy transversal properties
- The red points do not satisfy transversal properties
- H family is contained in a unique 9-dim. manifold which consists of triply per. min. surfs.

The $\gamma = 3$ case

[2021 Ejiri-S.]

- tP family is contained in a unique 9-dim. manifold which consists of triply per. min. surfs.
- rG family and tG family are contained in a unique 9-dim. manifold which consists of triply per. min. surfs.

First-eigenvalue Maximization and Embedding Optimization

Shin Nayatani (Nagoya University)

Maximization problem for the first eigenvalue of the Laplacian began with the seminal work of Hersch (1970), who proved that on the two-sphere the first eigenvalue (multiplied by area for scale invariance) was maximized by the round metrics (and by them only). Since then, this subject has been studied by many geometers and enriched by many interesting results. Among them, I mention a beautiful theorem of Nadirashvili (1996), which states that a metric maximizing the first eigenvalue of the Laplacian admits an isometric minimal immersion into a round sphere of some dimension. Meanwhile, in graph theory, Fiedler (1989) considered a similar maximization problem, and more recently Göring–Helmberg–Wappler (2008, 2011) formulated a problem which is dual (in the framework of mathematical programming) to Fiedler's problem and concerns embeddings of a graph into Euclidean spaces. In this talk, I will introduce an analogue of GHW formalism in differential geometry. In fact, it turns out that the relevant eigenvalue maximization problem concerns the Bakry–Emery Laplacian on a weighed Riemannian manifold rather than the usual Laplacian. I will discuss examples and an analogue of the above mentioned Nadirashvili theorem.

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First-eigenvalue maximization and embedding optimization

Shin Nayatani

Graduate School of Mathematics, Nagoya University

November 3, 2021

Berger problem

Let M be a compact manifold of dimension n (connected, orientable).

Let g be a Riemannian metric on M.

Denote by $\lambda_1(g)$ the first eigenvalue of the Laplacian

$$-\Delta_g = -\sum_{i,j=1}^{n} g^{ij}\left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial u}{\partial x^k}\right).$$

$$\begin{aligned}\lambda_1(g) &= \min\{\lambda > 0 \mid \exists u \in C^\infty(M), u \neq 0 \text{ s.t. } -\Delta_g u = \lambda u\} \\ &= \inf_{u \neq \text{const.}} \frac{\int_M |du|_g^2\, dv_g}{\int_M (u-\overline{u})^2\, dv_g}, \quad \overline{u} = \int_M u\, dv_g.\end{aligned}$$

Set

$$\Lambda_1(g) := \lambda_1(g)\mathrm{Vol}(g)^{2/n}.$$

Example 1

For the metric g_{S^2} of the unit 2-sphere,

$$\Lambda_1(g_{S^2}) = 2 \times 4\pi = 8\pi.$$

Theorem 1 (Hersch 1970)

For any metric g on S^2, one has $\Lambda_1(g) \leq 8\pi$.

The equality sign holds if and only if $g = g_{S^2}$ up to scaling.

Problem 1 (cf. Berger 1973)

Determine

$$\Lambda_1(M) := \sup_g \Lambda_1(g)$$

and find g such that $\Lambda_1(g) = \Lambda_1(M)$ (if exists).

Known results.

1 (Urakawa 1979)
$$\Lambda_1(S^3) = \infty.$$

2 (Yang-Yau 1980) Let Σ_γ be a compact surface of genus γ. Then
$$\Lambda_1(\Sigma_\gamma) \le 8\pi \left[\frac{\gamma+3}{2}\right].$$

3 (Colbois-Dodziuk 1994) If $n \ge 3$, then
$$\Lambda_1(M) = \infty.$$

Colbois-Dodziuk's proof relies on the results of Urakawa, Tanno, H. Muto for S^n.

4 (Nadirashvili 1996)
$$\Lambda_1(T^2) = 8\pi^2/\sqrt{3}\ (< 16\pi),$$

attained by the flat metric of $\mathbb{R}^2/\mathbb{Z}(1,0) \oplus \mathbb{Z}(1/2, \sqrt{3}/2)$, uniquely up to scaling.

5 (Petrides 2014) If $\Lambda(\Sigma_\gamma) > \Lambda(\Sigma_{\gamma-1})$, then $\Lambda(\Sigma_\gamma)$ is attained by a metric possibly with finitely many conical singularities.

6 (N.-Shoda 2019)
$$\Lambda_1(\Sigma_2) = 16\pi,$$

attained by a certain singular conformal metric on the Bolza Riemann surface.

7 (Ros 2021)
$$\Lambda_1(\Sigma_3) \le 16(4-\sqrt{7})\pi \approx 21.688\pi.$$

Theorem 2 (Nadirashvili 1996)

Let M be a compact surface. Suppose that g attains $\Lambda_1(M)$.
Then there exist first eigenfunctions $\varphi_1, \dots, \varphi_d$ of $-\Delta_g$ such that

$$\varphi = (\varphi_1, \dots, \varphi_d) \colon M \to \mathbb{R}^d$$

is an isometric immersion.
Therefore, φ is a minimal immersion into $S^{d-1}(\sqrt{2/\lambda_1(g)})$ by the Takahashi theorem.

Analougue in weighted Riemannian geometry

Let (M, dv, g) be a weighted Riemannian manifold, where dv is a volume form. Write $dv_g = e^f dv$.

The Bakry-Émery Laplacian $-\Delta_{(dv,g)}$ is given by

$$-\Delta_{(dv,g)}u = -\Delta_g u + g(df, du), \quad u \in C^\infty(M).$$

The first eigenvalue $\lambda_1(dv, g)$ of $-\Delta_{(dv,g)}$ is characterized by

$$\lambda_1(dv, g) = \inf_{u \neq \text{const.}} \frac{\int_M |du|_g^2 \, dv}{\int_M (u - \overline{u})^2 \, dv},$$

where $\overline{u} = \displaystyle\int_M u \, dv$.

Problem 2 (Eigenvalue maximization)

Fix a volume form dv and a metric h on M (e.g. $dv = dv_h$).

Determine

$$\Lambda_1(M; dv, h) := \sup_g \frac{\lambda_1(dv, g)}{\int_M \mathrm{tr}_g h \, dv \ / \ \mathrm{Vol}(dv)}$$

and find g which attains $\Lambda_1(M; dv, h)$.

Note that

$$\mathrm{tr}_g h = \sum_{i,j=1}^{n} g^{ij} h_{ij}.$$

Embedding optimization

Göring-Helmberg-Wappler (2008, 2011) formulated an embedding optimization problem for finite graphs.

Problem 3 (Embedding optimization)

Let dv, h be as in Problem 2.

Consider all C^∞-maps $\varphi\colon M \to \mathbb{R}^N$ (N is arbitrary) such that

$$\varphi^* g_{\mathbb{R}^N} \leq h \quad (\Leftrightarrow \ \varphi \text{ is 1-Lipschitz}).$$

Determine

$$\mathrm{Var}(M; dv, h) := \sup_\varphi \frac{1}{\mathrm{Vol}(dv)} \int_M \|\varphi - \overline{\varphi}\|^2 \, dv,$$

where $\overline{\varphi} = \dfrac{1}{\mathrm{Vol}(dv)} \displaystyle\int_M \varphi \, dv$, and find φ which attains $\mathrm{Var}(M; dv, g)$.

Duality

Proposition 3

Problems 2, 3 are dual to each other: There exists a function

$$L\colon \mathcal{RM}(M) \times C^\infty(M, \mathbb{R}^\infty)_{1\text{-}Lip} \to \mathbb{R},$$

where $\mathbb{R}^\infty = \varinjlim \mathbb{R}^N$, *such that*

$$\inf_g \sup_\varphi L(g, \varphi) \quad \Leftrightarrow \quad \text{Problem 2},$$
$$\sup_\varphi \inf_g L(g, \varphi) \quad \Leftrightarrow \quad \text{Problem 3}.$$

Since

$$\sup_\varphi \inf_g L(g, \varphi) \leq \inf_g \sup_\varphi L(g, \varphi),$$

we obtain

Corollary 4 (Weak duality)

$$\mathrm{Var}(M; dv, g) \leq \frac{1}{\Lambda_1(M; dv, h)}.$$

The corollary can be proved directly:

$$\int_M \|\varphi - \overline{\varphi}\|^2 \, dv \leq \frac{1}{\lambda_1(dv, g)} \int_M \|d\varphi\|_g^2 \, dv \leq \frac{\int_M \mathrm{tr}_g h \, dv}{\lambda_1(dv, g)},$$

since

$$\|d\varphi\|_g^2 = \mathrm{tr}_g \varphi^* g_{\mathbb{R}^N} \leq \mathrm{tr}_g h.$$

Therefore,

$$\frac{1}{\mathrm{Vol}(dv)} \int_M \|\varphi - \overline{\varphi}\|^2 \, dv \leq \frac{\int_M \mathrm{tr}_g h \, dv / \mathrm{Vol}(dv)}{\lambda_1(dv, g)}.$$

The equality sign holds if and only if

$$(\#) \quad \begin{cases} -\Delta_{(dv,g)}(\varphi - \overline{\varphi}) = \lambda_1(dv, g)(\varphi - \overline{\varphi}), \\ \varphi^* g_{\mathbb{R}^N} = h. \end{cases}$$

Riemannian inequality

Let g be a metric on M, and let $\varphi\colon M \to \mathbb{R}^N$ be a C^∞-map such that $\varphi^* g_{\mathbb{R}^N} \leq g$. Then

$$\int_M \|\varphi - \overline{\varphi}\|^2 \, dv_g \leq \frac{1}{\lambda_1(g)} \int_M \|d\varphi\|_g^2 \, dv_g \leq \frac{n\mathrm{Vol}(g)}{\lambda_1(g)},$$

since

$$\|d\varphi\|_g^2 = \mathrm{tr}_g \varphi^* g_{\mathbb{R}^N} \leq \mathrm{tr}_g g = n.$$

Therefore,

$$\frac{1}{\mathrm{Vol}(g)} \int_M \|\varphi - \overline{\varphi}\|^2 \, dv_g \leq \frac{n}{\lambda_1(g)}.$$

The right-hand side depends only on g, but the left-hand side depends on both φ and g.

Examples

Observation 1

Let

$$\varphi = (\varphi_1, \ldots, \varphi_d)\colon (M, h) \to S^{d-1} \subset \mathbb{R}^d$$

be an isometric minimal immersion by first eigenfunctions.

Consider Problems 2 and 3 by choosing $(dv, h) = (dv_h, h)$.

Then by $(\#)$, $g = h$ and φ are optimal solutions to Problems 2 and 3, respectively, and

$$\mathrm{Var}(M; dv_h, h) = \frac{1}{\Lambda_1(M; dv_h, h)}$$

holds.

Example 2

- Isotropy irreducible Riemannian homogeneous spaces.
  E.g. Symmetric spaces of compact type.
- Many compact minimal hypersurfaces in the unit spheres by the Yau conjecture.
- But still rare... The flat metrics of

$$\mathbb{R}^2/\mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1) \quad \text{and} \quad \mathbb{R}^2/\mathbb{Z}(1,0) \oplus \mathbb{Z}(1/2, \sqrt{3}/2)$$

  are the only metrics on T^2 which admit such an isometric immersion.

Example 3

Let h be the flat metric of $\mathbb{R}^2/\mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1)$. Then the map

$$\varphi\colon (x,y) \in \mathbb{R}^2/\mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1) \mapsto \frac{1}{2\pi}(e^{2\pi i x}, e^{2\pi i y}) \in \mathbb{C}^2$$

is an isometric immersion by first eigenfunctions.

For $c > 0$, $c \neq 1$, let

$$\varphi_c(x,y) = \frac{1}{2\pi}(e^{2\pi i x}, c^2 e^{2\pi i y}).$$

Then $h_c = {\varphi_c}^* g_{\mathbb{C}^2}$ is isometric to the flat metric of $\mathbb{R}^2/\mathbb{Z}(1,0) \oplus \mathbb{Z}(0,c)$ and $dv_{h_c} = c^2\, dv_h$.

By $(\#)$, $g = h$ and φ_c are optimal solutions to Problems 2 and 3 for (dv_{h_c}, h_c).

Nadirashvili-type theorem

Theorem 5

Suppose that g is an optimal solution to Problem 2.

Then there exist first eigenfunctions $\varphi_1, \dots, \varphi_d$ of $-\Delta_{(dv,g)}$ such that

$$\varphi = (\varphi_1, \dots, \varphi_d) \colon M \to \mathbb{R}^d$$

is an isometric immersion with respect to h.

In particular, φ is an optimal solution to Problem 3, and

$$\mathrm{Var}(M; dv, h) = \frac{1}{\Lambda_1(M; dv, h)} \qquad \text{“Strong duality”}$$

holds.

Thank you for your attention.

謝謝.

有難うございます.