Quandles and Symmetric Spaces 2021

Osaka City University Advanced Mathematical Institute (OCAMI)
MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics

OCAMI Reports Vol. 9 (2021)
doi: 10.24544/ocu.20220209-001

Quandles and Symmetric Spaces 2021

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November 25 – 26, 2021

Abstract

The workshop “Quandles and Symmetric Spaces” has been held annually since 2018. This volume records the abstracts and the slides of talks presented in this workshop on 2021.

2020 *Mathematics Subject Classification.*
53C35, 57K12

Key words and Phrases.
quandles, quandle colorings, symmetric spaces, antipodal sets

OCAMI. Quandles and Symmetric Spaces 2021. OCAMI Reports. Vol. 9, Osaka City University Advanced Mathematical Institute. 2021, 117pp. doi: 10.24544/ocu.20220209-001

Preface

The workshop "Quandles and Symmetric Spaces" has been held annually since 2018 in order to encourage the cross-pollination among topology (knot theory), differential geometry (symmetric spaces), and other areas through quandles. In this volume the abstracts and the slides of the talks in the conference "Quandles and Symmetric Spaces 2021" are collected. For the talks in 2019 and 2020, one can refer to the previous volume (OCAMI Reports Vol. 4).

A quandle, introduced by Joyce and Matveev independently in 1982, is an abstract algebra whose axioms are motivated by the three Reidemeister moves, and it has played important roles in knot theory. On the other hand, quandles frequently appear in various fields other than knot theory. For instance, there is a relationship between "symmetry" and (involutory) quandles. Indeed, symmetric spaces can be defined as differentiable manifolds equipped with a certain involutory quandle operation. It would be interesting and important to investigate quandles and knots via symmetric space theory.

The series of workshops was organized by experts of knot theory (Kamada and Oshiro) and symmetric spaces (Kubo, Okuda, Tamaru, Tanaka and Tasaki), and the talks consisted of some instructive talks by experts and presentations by young researchers in addition to usual talks. There have been many presenters and participants from various fields, not only topology and differential geometry but also combinatorics, etc. The conference was held in a hybrid format, and during or after the talks, they exchanged their ideas and information, and discussed possible perspectives actively. One of the instructive talks is open to the public via OCAMI YouTube. The organizers are convinced that the workshops would disseminate quandles, and be effective for further developments of the theory of quandles.

January 2022

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Homogeneous quandles arising from finite groups

Hirotake Kurihara

ABSTRACT. The contents of this talk is based on a joint work Akihiro Higashitani. Quandle is an algebraic system with one binary operation, but it is quite different from a group. In this talk, we investigate a special kind of quandles, called generalized Alexander quandles. We develop the quandle invariants for generalized Alexander quandles using "connected components" of quandles. As a results, we can solve some classification problems.

1 introduction

Originally, quandles were introduced by Joyce [3] in the context of the knot theory. We call a set Q a *quandle* if Q is equipped with a binary operation $*$ satisfying the following three axioms:

(Q1) $x * x = x$ for any $x \in Q$;

(Q2) for any $x, y \in Q$, there exists a unique $z \in Q$ such that $z * y = x$;

(Q3) for any $x, y, z \in Q$, we have $(x * y) * z = (x * z) * (y * z)$.

These axioms are derived from the Reidemeister moves appearing in the knot theory. We can rephrase this definition of quandles in terms of "globally defined maps" as follows. For a set Q and each element $x \in Q$, the map $s_x : Q \to Q$ satisfying the following three axioms (Q1')–(Q3') is called a *point symmetry* at x:

(Q1') $s_x(x) = x$ for any $x \in Q$;

(Q2') s_x is a bijection on Q for any $x \in Q$;

(Q3') $s_x \circ s_y = s_{s_x(y)} \circ s_x$ holds for any $x, y \in Q$.

It is straightforward to check that definitions of $*$ and $\{s_x\}_{x \in Q}$ are equivalent by setting $s_x(\cdot) = (\cdot) * x$ for each $x \in Q$. We can see from this definition that every symmetric space has a quandle structure (cf. [3]). There are many studies of quandles from the viewpoint of theory of symmetric spaces (cf. Ishihara–Tamaru [2]). In these studies, homogeneous quandles are the main object.

Let (Q, s) and (Q', s') be quandles. A map $f : Q \to Q'$ is called a *quandle isomorphism* if f is bijective and satisfies

$$f \circ s_x = s'_{f(x)} \circ f \quad \text{for any } x \in Q.$$

If there is a quandle automorphism between Q and Q', then we say that Q and Q' are *isomorphic as quandles*, denoted by $Q \cong_{\mathrm{qu}} Q'$. Let $\mathrm{Aut}(Q, s)$ (or $\mathrm{Aut}(Q)$, for short) be the set of quandle automorphisms from (Q, s) to (Q, s) itself, which is called the *automorphism group* of Q. Remark that $s_x \in \mathrm{Aut}(Q)$ for any $x \in Q$. Let $\mathrm{Inn}(Q, s)$ (or $\mathrm{Inn}(Q)$, for short) be the subgroup of $\mathrm{Aut}(Q)$ generated by $\{s_x : x \in Q\}$, which is called the *inner automorphism group* of Q. We say that Q is *homogeneous* if $\mathrm{Aut}(Q)$ acts transitively on Q, i.e., for any $x, y \in Q$, there exists $f \in \mathrm{Aut}(Q)$ such that $f(x) = y$.

We can obtain many examples of homogeneous quandles using groups and those automorphisms. For a finite group G and an automorphism $\psi \in \mathrm{Aut}(G)$, we define a binary operation $*$ by

$$g * h := h\psi(h^{-1}g) \quad \text{for } g, h \in G.$$

Then $(G, *)$ becomes a homogeneous quandle. This quandle is called a *generalized Alexander quandle*, denoted by $Q(G, \psi)$. Note that in the case G is abelian, $Q(G, \psi)$ is called an *Alexander quandle* (see Nelson [4]).

Given a finite group G, let $\mathcal{Q}(G)$ be the set of quandle isomorphic classes of $Q(G, \psi)$'s, i.e.,

$$\mathcal{Q}(G) := \{Q(G, \psi) : \psi \in \mathrm{Aut}(G)\}/ \cong_{\mathrm{qu}} .$$

The following problem naturally arises.

Problem 1.1. Determine $\mathcal{Q}(G)$ for a given group G.

We write $\psi \equiv_{\mathrm{conj}} \psi'$ when $\psi, \psi' \in \mathrm{Aut}(G)$ are conjugate. If $\psi \equiv_{\mathrm{conj}} \psi'$, then we can show $Q(G, \psi) \cong_{\mathrm{qu}} Q(G, \psi')$. Thus, the following map

$$\mathrm{Aut}(G)/ \equiv_{\mathrm{conj}} \to \mathcal{Q}(G); \quad \psi \mapsto Q(G, \psi)$$

is well-defined and surjective. In general, the map is not injective. In fact, when G is a cyclic group C_n for some n, the map $\mathrm{Aut}(C_n)/ \equiv_{\mathrm{conj}} \to \mathcal{Q}(C_n)$ is not injective (cf. [4]).

2 Main results

Towards a solution of Problem 1.1, we established some quandle invariants for $Q(G, \psi)$.

Theorem 2.1 (Higashitani–Kurihara [1]). *Let $\psi, \psi' \in \mathrm{Aut}(G)$ and let $Q = Q(G, \psi)$ and $Q' = Q(G, \psi')$. If $Q \cong_{\mathrm{qu}} Q'$, then*

(1) $\mathrm{ord}_{\mathrm{Aut}(G)} \psi = \mathrm{ord}_{\mathrm{Aut}(G)} \psi'$;

(2) $[G : \mathrm{Fix}(\psi, G)] = [G : \mathrm{Fix}(\psi', G)]$, *i.e.*, $|\,\mathrm{Fix}(\psi, G)| = |\,\mathrm{Fix}(\psi', G)|$;

(3) $\mathrm{Inn}(Q) \cong_{\mathrm{gr}} \mathrm{Inn}(Q')$;

(4) $Q(G, \psi^i) \cong_{\mathrm{qu}} Q(G, \psi'^i)$ *for any* $i \in \mathbb{Z}_{>0}$,

where $\mathrm{ord}_{\mathrm{Aut}(G)} \psi$ *is the order of* ψ, $\mathrm{Fix}(\psi, G) := \{g \in G : \psi(g) = g\}$ *and* $G \cong_{\mathrm{gr}} G'$ *means that* G *and* G' *are isomorphic as groups.*

Using Theorem 2.1, for a finite group G of small size, we can solve Problem 1.1. On the other hand, for a finite group G of large size, there are generalized Alexander quandles that we cannot distinguish by only Theorem 2.1.

Recently, we obtained results following Higashitani–Kurihara [1]. Let G be a finite group with the identity element e and $\psi \in \mathrm{Aut}(G)$ with $\psi \neq \mathrm{Id}$. For $Q = Q(G, \psi)$, define P_Q by the orbit of e by $\mathrm{Inn}(Q)$, i.e.,

$$P_Q = \{x \in G : \exists z_1, z_2, \ldots, z_p \in G \text{ s.t. } x = s_{z_1} \circ s_{z_2} \circ \cdots \circ s_{z_p}(e)\}.$$

Theorem 2.2. *We have the following:*

(1) P_Q *is a subquandle of* Q*;*

(2) $\psi|_{P_Q}$ *is a quandle automorphism on* P_Q*;*

(3) P_Q *is a normal subgroup of* G*.*

Theorem 2.3. *For* $\psi, \psi' \in \mathrm{Aut}(G)$*, put* $Q = Q(G, \psi)$*,* $Q' = Q(G, \psi')$ *and* $P = P_Q$*,* $P' = P_{Q'}$*. We assume* $Q(G, \psi) \cong_{\mathrm{qu}} Q(G, \psi')$ *and let* $f : Q(G, \psi) \to Q(G, \psi')$ *be a quandle isomorphism with* $f(e) = e$*. then we have the following:*

(1) $f|_P$ *is a quandle isomorphism between* P *and* P'*, i.e.,* $P \cong_{\mathrm{qu}} P'$*;*

(2) $f|_P$ *is a group isomorphism between* P *and* P'*, i.e.,* $P \cong_{\mathrm{gr}} P'$*;*

(3) $\psi'|_{P'} \circ f|_P = f|_P \circ \psi|_P$ *holds.*

Corollary 2.4. *If* G *is a finite simple group, there exists a one-to-one correspondence between* $\mathcal{Q}(G)$ *and* $\mathrm{Aut}(G)/\equiv_{\mathrm{conj}}$*.*

Corollary 2.5. *If* G *is a symmetric group* $\mathfrak{S}_n$*, there exists a one-to-one correspondence between* $\mathcal{Q}(\mathfrak{S}_n)$ *and* $\mathrm{Aut}(\mathfrak{S}_n)/\equiv_{\mathrm{conj}}$*.*

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Homogeneous quandles arising from finite groups

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山口大学

研究集会「カンドルと対称空間」
2021 年 11 月 25 日

東谷 章弘氏（大阪大）との共同研究（arXiv:2005.12057）

1. カンドル速習
2. 一般化アレキサンダーカンドル
3. 最近の話

カンドル 1

カンドルは Joyce によって結び目理論の観点から導入された二項演算をもつ集合である。

Definition 1

集合 Q と二項演算 $*$ が以下の条件を満たすとき、$(Q,*)$ をカンドルという：

(Q1) for $\forall x \in Q$, $x * x = x$;

(Q2) for $\forall x, y \in Q$, $\exists! z \in Q$ s.t. $z * y = x$;

(Q3) for $\forall x, y, z \in Q$, $(x * y) * z = (x * z) * (y * z)$.

この公理はライデマイスター移動に対応している（らしい）。一方、この公理は対称空間の性質と思うこともできる。

カンドルの例

- $\Lambda := \mathbb{Z}[t^{\pm 1}]$：$\mathbb{Z}$ 係数ローラン多項式環
- Q：Λ 加群
- $x, y \in Q$ に対して、$x * y := tx + (1-t)y$ と定める。
- $(Q,*)$ はカンドルであり、このカンドルをアレキサンダーカンドルという。

Remark 2

Q を加法に関する群と思えば、t 倍写像はアーベル群 Q の自己同型写像と思うこともできる。（この話は再び登場する）

カンドル 2

カンドルの定義（再掲）

(Q1) for $\forall x \in Q$, $x * x = x$;

(Q2) for $\forall x, y \in Q$, $\exists! z \in Q$ s.t. $z * y = x$;

(Q3) for $\forall x, y, z \in Q$, $(x * y) * z = (x * z) * (y * z)$.

$s_x(y) := y * x$ とおくと $s_x : Q \to Q$ と思える。

Definition 3

(Q1') for $\forall x \in Q$, $s_x(x) = x$;

(Q2') for $\forall x \in Q$, s_x は全単射;

(Q3') for $\forall x, y \in Q$, $s_x \circ s_y = s_{s_x(y)} \circ s_x$.

Remark 4

対称空間 $(M, \{s_x\}_{x \in M})$ は (Q1')–(Q3') を満たす。従って対称空間はカンドルである。

カンドル準同型

Definition 5

(Q, s) と (Q', s') をカンドルとする。

- 写像 $f : Q \to Q'$ が**カンドル準同型**であるとは次を満たすこと：

$$f \circ s_x = s'_{f(x)} \circ f \quad \textit{for any } x \in Q.$$

  さらに f が全単射のときは、f を**カンドル同型写像**という。
- (Q, s) と (Q', s') の間にカンドル同型写像があるとき、これらを**カンドル同型**といい、$Q \cong_{\mathrm{qu}} Q'$ で表す。
- $\mathrm{Aut}(Q, s)$ *(or* $\mathrm{Aut}(Q)$*)* を**カンドル自己同型群**とする。

Rem: (Q3') $s_x \circ s_y = s_{s_x(y)} \circ s_x$ より $s_x \in \mathrm{Aut}(Q, s)$。

Definition 6

$\{s_x : x \in Q\}$ で生成される $\mathrm{Aut}(Q, s)$ の部分群を**カンドル内部自己同型群**といい、$\mathrm{Inn}(Q, s)$ *(or* $\mathrm{Inn}(Q)$*)* で表す。

等質、連結

Definition 7

(Q,s) をカンドルとする。

- 任意の $x,y\in Q$ に対して、$f(x)=y$ となる $f\in \mathrm{Aut}(Q)$ が存在するとき、Q を**等質**という。
- 任意の $x,y\in Q$ に対して、$f(x)=y$ となる $f\in \mathrm{Inn}(Q)$ が存在するとき、Q を**連結**という。

Definition 8

G を群とし、$\psi\in\mathrm{Aut}(G)$ とする。(G,K,ψ) が**カンドル三つ組**であるとは、満たすこと：$K\subset \mathrm{Fix}(\psi,G):=\{g\in G:\psi(g)=g\}$。

カンドル三つ組と等質カンドル

Example 9

カンドル三つ組 (G,K,ψ) に対して、Q を $G/K=\{[g]:g\in G\}$ とし、

$$s_{[g]}([h]):=[g\psi(g^{-1}h)]$$

と定めると、(Q,s) は等質カンドルになる。これを $Q(G,K,\psi)$ とかく。

Proposition 10

任意の等質カンドル Q について、∃カンドル三つ組 (G,K,ψ) *s.t.* $Q\cong_{\mathrm{qu}} G/K$。

ここまでの話はかなり対称空間の話に近い。なので、カンドルの “リー群” みたいなものが気になる。

1 カンドル速習

2 一般化アレキサンダーカンドル

3 最近の話

一般化アレキサンダーカンドル

G を群とし、e は G の単位元とする。$K = \{e\}$ とすれば、任意の $\psi \in \mathrm{Aut}(G)$ について $(G, \{e\}, \psi)$ はカンドル三つ組になる。

Definition 11

$Q(G, \{e\}, \psi)$ を**一般化アレキサンダーカンドル**といい、今後は $Q(G, \psi)$ とかく。

Rem: つまり、$Q(G, \psi)$ の点対称は $s_g(h) := g\psi(g^{-1}h)$ であり、G がアーベル群の場合はアレキサンダーカンドルである。

Proposition 12

$\psi, \psi' \in \mathrm{Aut}(G)$, $K = \mathrm{Fix}(\psi, G)$, $K' = \mathrm{Fix}(\psi', G)$ とする。もし $Q(G, \psi) \cong_{\mathrm{qu}} Q(G, \psi')$ ならば、$Q(G, K, \psi) \cong_{\mathrm{qu}} Q(G, K', \psi')$ が成り立つ。

したがって $Q(G, \psi)$ は “リー群” みたいなもので、等質カンドルの “親玉” と思える。

$\mathcal{Q}(G)$ の決定問題

群 G に対して、$\mathcal{Q}(G)$ を $Q(G,\psi)$ の同型類とする。すなわち

$$\mathcal{Q}(G) := \{Q(G,\psi) : \psi \in \mathrm{Aut}(G)\}/\cong_{\mathrm{qu}} .$$

Problem 13

群 G ごとに $\mathcal{Q}(G)$ を決定せよ。

すぐに分かること: $\psi, \psi' \in \mathrm{Aut}(G)$ が共役（つまり $\exists\tau \in \mathrm{Aut}(G)$ s.t. $\psi' = \psi^\tau =: \tau^{-1} \circ \psi \circ \tau$、今後 $\psi \equiv_{\mathrm{conj}} \psi'$ と書く）ならば、$Q(G,\psi) \cong_{\mathrm{qu}} Q(G,\psi')$ である。
したがって $\mathrm{Aut}(G)/\equiv_{\mathrm{conj}} \to \mathcal{Q}(G)$: $[\psi] \mapsto Q(G,\psi)$ は well-defined な全射である。しかし、単射かどうか（すなわち $\mathrm{Aut}(G)/\equiv_{\mathrm{conj}}$ と $\mathcal{Q}(G)$ が一対一対応するかどうか）は G に依る。実際単射にならない例が存在する。

C_n の場合

Remark 14 (Nelson, 2003)

$G = C_n$ のときは、$\mathcal{Q}(G) \neq$ ($\mathrm{Aut}(G)$ の共役類) である。

- $\mathrm{Aut}(C_n) \cong_{\mathrm{gr}} U(C_n) := \{a \in C_n : \gcd(a,n) = 1\}$, $x \mapsto ax$.
- $N(n,a) = \frac{n}{\gcd(n,1-a)}$
- $Q(C_n,a) \cong_{\mathrm{qu}} Q(C_n,b)$ の同値条件は以下を満たすこと：
  - $N(n,a) = N(n,b)$
  - $a \equiv b \pmod{N(n,a)}$
- たとえば、$Q(C_9,4) \cong_{\mathrm{qu}} Q(C_9,7)$ である（$N(9,4) = N(9,7) = 3$ かつ $4 \equiv 7 \pmod 3$ だから)。$U(C_n)$ は Abel 群だから、共役類は $U(C_n)$ 自身である。

様々な不変量

Problem 13 を解く戦略：ψ に関するカンドル不変量をたくさん見つけて、$\mathrm{Aut}(G)$ の共役類の相違点を見る。

Theorem 15 (Higashitani–K., arXiv:2005.12057)

G は有限群として、$\psi, \psi' \in \mathrm{Aut}(G)$, $Q = Q(G, \psi)$, $Q' = Q(G, \psi')$ とする。

1. $\psi' \equiv_{\mathrm{conj}} \psi$ ならば、$Q \cong_{\mathrm{qu}} Q'$ である。
2. $Q \cong_{\mathrm{qu}} Q'$ を仮定すると次が成り立つ：
   - $\mathrm{ord}_{\mathrm{Aut}(G)}\, \psi = \mathrm{ord}_{\mathrm{Aut}(G)}\, \psi'$;
   - $[G : \mathrm{Fix}(\psi, G)] = [G : \mathrm{Fix}(\psi', G)]$, *i.e.*, $|\,\mathrm{Fix}(\psi, G)| = |\,\mathrm{Fix}(\psi', G)|$;
   - $\mathrm{Inn}(Q) \cong_{\mathrm{gr}} \mathrm{Inn}(Q')$;
   - $Q(G, \psi^i) \cong_{\mathrm{qu}} Q(G, \psi'^i)$ *for any* $i \in \mathbb{Z}_{>0}$.

Theorem 15 の効用

群 G の位数が小さい場合は、Theorem 15 のカンドル不変量だけで、区別がつき、$\mathcal{Q}(G)$ が決定できる。しかし、群 G の位数が大きくなると、Theorem 15 だけでは区別できなくなる。

Example 1

- $G := \mathfrak{S}_{15}$（15 次対称群）
- $\mathrm{Aut}(G) \cong_{\mathrm{gr}} \mathfrak{S}_{15}$（内部自己同型だけから成る）
- $\mathfrak{S}_{15}$ の共役類 $\leftrightarrow$ 置換の型 $\leftrightarrow$ 箱の数が 15 個のヤング図形
- $\pi_1, \pi_2 \in \mathfrak{S}_{15}$ をそれぞれ型が $(9, 3^2), (9, 3, 1^3)$ である置換とする。
- ψ_i を π_i から定まる内部自己同型写像とする。($i = 1, 2$)
- $Q(G, \psi_1)$ と $Q(G, \psi_2)$ は Theorem 15 の不変量だけでは区別できない。(+ 対称群独自で定義できるカンドル不変量もあるが、それでもダメ)

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カンドル不変量 P_Q

以降 G は有限群を仮定する。

Definition 16

$Q(G,\psi)$ を一般化アレキサンダーカンドルとする。P_Q を $\mathrm{Inn}(Q)$ による単位元 e の軌道とする。すなわち

$$P_Q := \{x \in G : \exists a_1, \ldots, a_p \in G \text{ s.t. } x = s_{a_p} \circ \cdots \circ s_{a_1}(e)\}.$$

Proposition 17

1. $Q(P_Q, \psi|_{P_Q})$ は $Q(G,\psi)$ の部分カンドルになる。
2. $\psi|_{P_Q}$ は P_Q 上のカンドル自己同型写像である。
3. P_Q は G の正規部分群である。

Remark 18

$Q(G,\psi)$ のカンドル構造を用いて、G/P_Q にもカンドル構造が自然に誘導されるが、G/P_Q は自明カンドルになる。

カンドル同型と P_Q

$\psi, \psi' \in \mathrm{Aut}(G)$ に対して、$Q := Q(G, \psi)$、$Q' := Q(G, \psi')$ と置く。また $P := P_Q$、$P' := P_{Q'}$ とかく。

Theorem 19

$Q \cong_{\mathrm{qu}} Q'$ を仮定する。$f : Q \to Q'$ を $f(e) = e$ を満たすカンドル同型写像とする。(Q や Q' は等質カンドルなのでそのような f は必ず存在する)

1. $f(P) = P'$ であり、$f|_P$ は $P \cong_{\mathrm{qu}} P'$ を与える。
2. $f|_P$ は $P \to P'$ の群準同型を与える。従って $P \cong_{\mathrm{gr}} P'$。
3. $f|_P \circ \psi|_P = \psi'|_{P'} \circ f|_P$。特に $P = P'$ だと $\psi|_P$ と $\psi'|_{P'}$ は共役。

Theorems 17, 19 の応用例

Corollary 20

G が(有限)単純群のとき、$\mathcal{Q}(G)$ と $\mathrm{Aut}(G)/\equiv_{\mathrm{conj}}$ は一対一対応する。

Proof.

- $\psi = \mathrm{id}_G$ の場合は、$Q(G, \mathrm{id}_G)$ は自明カンドルであり、$\psi \neq \mathrm{id}_G$ ならば $Q(G, \psi)$ は自明カンドルにならない。したがって $\psi, \psi' \in \mathrm{Aut}(G)$ は $\psi, \psi' \neq \mathrm{id}_G$ を仮定する。
- $Q(G, \psi) \cong_{\mathrm{qu}} Q(G, \psi')$ を仮定する。
- G が単純群なので、定理 17 (3) より $P = P' = G$ である。
- 定理 19 (2) より $f \in \mathrm{Aut}(G)$ であり、定理 19 (3) より $\psi' \circ f = f \circ \psi$ である。したがって $\psi \equiv_{\mathrm{conj}} \psi'$ を得る。

□

Corollary 20 の応用

Corollary 21

$G = \mathfrak{S}_n$ とき、$\mathcal{Q}(G)$ と $\mathrm{Aut}(G)/\equiv_{\mathrm{conj}}$ は一対一対応する。

Proof.

- $n = 3, 4$ のときは定理 15 で示すことができるので、$n \geq 5$ を仮定する。
- $\psi, \psi' \neq \mathrm{id}$ である $\psi, \psi' \in \mathrm{Aut}(\mathfrak{S}_n)$ に対して、$Q := Q(\mathfrak{S}_n, \psi)$, $Q' := Q(\mathfrak{S}_n, \psi')$ とおく。
- $\mathfrak{S}_n$ の非自明な正規部分群は $\mathfrak{A}_n$ しかないので、P と P' は $\mathfrak{A}_n$ になる。
- $Q \cong_{\mathrm{qu}} Q'$ を仮定すると、定理 19 (1) より $Q(\mathfrak{A}_n, \psi|_{\mathfrak{A}_n}) \cong_{\mathrm{qu}} Q(\mathfrak{A}_n, \psi'|_{\mathfrak{A}_n})$ である。
- $\mathfrak{A}_n$ は有限単純群より系 20 から、$\psi|_{\mathfrak{A}_n} \equiv_{\mathrm{conj}} \psi'|_{\mathfrak{A}_n}$。
- $\mathfrak{S}_n$ と $\mathfrak{A}_n$ の関係から、$\psi \equiv_{\mathrm{conj}} \psi'$ を得る。

□

アーベル群の場合

- G をアーベル群とする。
- G はアーベル群なので、$\psi \in \mathrm{Aut}(G)$ に対して、G の準同型写像 $\rho := \mathrm{id} - \psi$ が定まる。
- $Q(G, \psi)$ について、$P = \mathrm{Im}\,\rho$ であることがわかる。

Theorem 22 (Nelson の結果の焼き直し)

G, G' をアーベル群とし、$\psi \in \mathrm{Aut}(G)$、$\psi' \in \mathrm{Aut}(G')$ とする。
$Q = Q(G, \psi)$、$Q = Q(G', \psi')$ に対して次は同値：

1. $Q \cong_{\mathrm{qu}} Q'$
2. $|G| = |G'|$ かつ $P \cong_{\mathrm{qu}} P'$ かつ $P \cong_{\mathrm{gr}} P'$。

以上よりアレキサンダーカンドルにおいては、P の情報がとても大事！
しかし一般化アレキサンダーカンドルでは、P の情報だけでは不十分。

二面体群の場合

- $D_n := \{\tau^\epsilon \sigma^i : \epsilon = 0, 1,\ i \in C_n\}$（二面体群）
- $a \in C_n^\times, b \in C_n$ に対して、$\varphi_{a,b}(\tau^\epsilon \sigma^i) := \tau^\epsilon \sigma^{ai+\epsilon b}$ と定めると、$\mathrm{Aut}(D_n) = \{\varphi_{a,b} : a \in C_n^\times, b \in C_n\}$。
- $g := \gcd(1 - a, n)$ とおく。
- $\mathrm{Aut}(D_n)/\equiv_{\mathrm{conj}}\ = \{(a, b) : a \in C_n^\times, b$ は g の約数 $\}$

Theorem 23 ($\mathcal{Q}(D_n)$ の決定)

$Q = Q(D_n, \varphi_{a,b})$、$Q = Q(D_n, \varphi_{a',b'})$ に対して次は同値：

1. $Q \cong_{\mathrm{qu}} Q'$
2. $P \cong_{\mathrm{qu}} P'$ かつ $P \cong_{\mathrm{gr}} P'$ かつ $|\mathrm{Fix}(\varphi_{a,b}, D_n)| = |\mathrm{Fix}(\varphi_{a',b'}, D_n)|$。

Remark 24

$P \cong_{\mathrm{qu}} P'$ かつ $P \cong_{\mathrm{gr}} P'$ かつ $|\mathrm{Fix}(\varphi_{a,b}, D_n)| = |\mathrm{Fix}(\varphi_{a',b'}, D_n)|$ の条件は、a, b, g, a', b', g' の関係式だけで記述可能。

これからの研究

- もっと一般の群 G に対して、$\mathcal{Q}(G)$ の決定。
- 異なる G に対しても、一般化アレキサンダーカンドルとしては同型となるものが存在する。
  - 例）$Q(C_{2n}, a) \cong_{\mathrm{qu}} Q(D_n, a', -k)$, where $a' = a$ if $1 \leq a < n$ and $a' = a - n$ if $n < a < 2n$.

このような現象も解明していきたい。

ご清聴ありがとうございました

Maximal antipodal sets of E_6 and some compact symmetric spaces

Yuuki Sasaki

Abstract. In this article we give a survey of our studies on Maximal antipodal sets of the exceptional compact Lie group E_6 and the compact symmetric space of EII type. Moreover, we introduce a realization of EII type as a Grassmannian.

1 Introduction

Let M be a compact Riemannian symmetric space and denote the geodesic symmetry at $x \in M$ by s_x. In this paper, we assume that M is connected. If $s_x(y) = y$ for two points $x, y \in M$, we say that x, y are antipodal. A subset S of M is an antipodal set, if any two points of S are antipodal. The 2-number $\#_2 M$ of M is the maximum of the cardinalities of antipodal sets of M. We call an antipodal set S in M great if $\#S = \#_2 M$. An antipodal set S is called maximal if there are no antipodal sets including S properly. These notions were introduced by Chen-Nagano [1]. In general, any antipodal set of any Riemannian symmetric space of noncompact type is a one-point set, so we consider only compact symmetric spaces in this paper. It is known that antipodal sets are finite sets and 2-number is finite [1] [7].

Maximal antipodal sets are considered as an important object in studying symmetric spaces and various mathematicians study maximal antipodal sets. In particular, it is known that maximal antipodal sets are related to the topology of symmetric spaces. Let M be a compact symmetric space and $\chi(M)$ be the Euler number of M. Then, $\chi(M) \leq \#_2 M$ [1]. Moreover, let M be a symmetric R-space (for example, spheres, real or complex or quaternion Grassmannians, $SO(n), U(n), Sp(n)$). Then, $\#_2 M = \sum_i \dim H_i(M; \mathbb{Z}_2)$ [6]. Thus, it is considered that antipodal sets have some informatioms about topologies of compact symmetric spaces.

However, there are compact symmetric spaces whose maximal antipodal sets are not classified and constructed. The exceptional compact Lie group E_6 and the compact symmetric space of EII type are such examples. In E_6, the classfication is finished by Griess by using the method of algebraic groups [3], but the construction is not complete. In the present paper, we observe the classification in E_6 which is based on a different method from Griess and construct by using $Spin(8)$. In EII type, for describing maximal antipodal sets explicitly we give a realization of EII type by using the complex exceptional Jordan algebra and observe the classification and construction of maximal antipodal sets.

2 Preparation

Let M be a compact symmetric space and A, B be subsets of M. If there is an element g contained in the identity component of the isometry group of M such that $g(A) = B$, then we call A and B are congruent. We consider congruent classes of maximal antipodal sets of E_6 and EII type. In this section, we observe some strategies to classify congruent classes of maximal antipodal sets. Let G be a connected compact symmetric space. Then, it is known that G with a biinvariant metric becomes a compact symmetric space and the geodesic symmetry s_g at $g \in G$ is given by $s_g(h) = gh^{-1}g$ $(h \in G)$. A maximal antipodal set of G containing the unit element $e \in G$ become a maximal elementary abelian 2-group. We call such a maximal antipodal set a maximal antipodal subgroup. In G, the classification of congruent classes of maximal antipodal sets reduces to the classfication of conjugate classes of maximal antipodal subgroups. In the following, we consider maximal antipodal subgroups.

Definition 2.1. [2] Let M be a compact symmetric space and $p \in M$. Then, each component of $F(s_p, M) := \{x \in M \ ; \ s_p(x) = p\}$ is called a polar of p. In particular, if a polar is a one-point set, then we call this polar a pole. We call $\{p\}$ a trivial pole.

In G, since $s_e(g) = g^{-1}$, we obtain $F(s_e, G) := \{g \in G \ ; \ g^{-1} = g\}$. Polars of e is given by conjugate orbits in $F(s_e, G)$. Let $M_0^+ = \{e\}$ and M_k^+ $(k = 1, \cdots, n)$ be all polars of e and $\sigma_k \in M_k^+$. Then, $M_k^+ = \bigcup_{g \in G} g\sigma_k g^{-1}$. Set $G_k^\sigma = \{g \in G \ ; \ g\sigma_k = \sigma_k g\}$. Let A be a maximal antipodal subgroup of G. Then,

$$A = \{e\} \sqcup (A \cap M_1^+) \sqcup \cdots \sqcup (A \cap M_n^+).$$

Lemma 2.2. *A is conjugate to a maximal antipodal subgroup of G^{σ_k} if and only if $A \cap M_k^+ \neq \phi$.*

In E_6, we set

$$\begin{aligned} M_1^+ &= \{g \in E_6 \ ; \ g^{-1} = g, \ \text{the multiplicity of the eigenvalue } +1 \text{ of } g \text{ is } 15\}, \\ M_2^+ &= \{g \in E_6 \ ; \ g^{-1} = g, \ \text{the multiplicity of the eigenvalue } +1 \text{ of } g \text{ is } 11\}, \end{aligned}$$

where we consider the complex exceptional Jordan algebra as a representation space of E_6. Then, $F(s_I, E_6) = \{I\} \sqcup M_1^+ \sqcup M_2^+$ for the unit element $I \in E_6$. For each $\sigma_1 \in M_1^+$, it is known that $E_6^{\sigma_1} \cong Sp(1) \times SU(6)/\{\pm(I_1, I_6)\}$, where I_n be the $n \times n$ unit matrix [9]. We denote $Sp(1) \times SU(6)/\{\pm(I_1, I_6)\}$ by $Sp(1) \cdot SU(6)$. In E_6, any maximal antipodal subgroup A satisfies $A \cap M_1^+ \neq \phi$ [4]. Thus, A is conjugate to a maximal antipodal subgroup of $E_6^{\sigma_1}$. Therefore, we consider maximal antipodal subgroups of $Sp(1) \cdot SU(6)$. Let $\pi : Sp(1) \times SU(6) \to Sp(1) \cdot SU(6)$ be the natural projection. Moreover, set subgroups S_1, S_2 of $Sp(1) \times SU(6)$ as follows:

$$S_1 = (\{\pm 1\} \times \Delta_6^+) \cup (\{\pm e_1\} \times i\Delta_6^-)$$

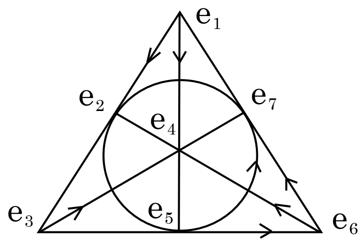
$$S_2 = \begin{array}{l} \left\{ \left(\pm 1, \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix} \right) ; \quad J_i = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \left(\pm e_1, \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix} \right) ; \ J_i = \pm i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \\ \cup \left\{ \left(\pm e_2, \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix} \right) ; \ J_i = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \cup \left\{ \left(\pm e_3, \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix} \right) ; \ J_i = \pm i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \end{array}.$$

Lemma 2.3. *[4] $\pi(S_1), \pi(S_2)$ are maximal antipodal subgroups of $Sp(1) \cdot SU(6)$ and not conjugate to each other. Moreover, any maximal antipodal subgroup of $Sp(1) \cdot SU(6)$ are conjugate to one of them. $\#\pi(S_1) = 64$ and $\#\pi(S_2) = 32$. Thus, $\#_2 Sp(1) \cdot SU(6) = 64$.*

3 Maximal antipodal sets of E_6

In this section, we observe maximal antipodal sets of E_6. Let $\mathbb{O} = \bigoplus_{i=0}^{7} \mathbb{R} e_i$ be the octonions. The multiplicity of $\mathbb{O}$ is defined satisfying following: (1) e_0 is the unit element of this multiplicity, (2) $e_i^2 = -e_0$ and $e_i e_j = -e_j e_i$ for any $1 \leq i \neq j \leq 7$, (3) the multiplicity satisfies the distributive law, (4) the multiplicity among $e_1, \cdots, e_7$ is defined by Figure 1 (for example, $e_1 e_2 = e_3, e_2 e_3 = e_1$ and $e_3 e_1 = e_2$). Remark that the associative law does not follow in the octonions. For

Figure 1:



each $x = \sum_{i=0}^{7} x_i e_i \in \mathbb{O}$ $(x_i \in \mathbb{R})$, we set the conjugation $\overline{x} = x_0 e_0 - \sum_{i=1}^{7} x_i e_i$ of x. Let $y = \sum_{i=0}^{7} y_i e_i \in \mathbb{O}$ $(y_i \in \mathbb{R})$. Then, the standard inner product $(\ , \ )_{\mathbb{O}}$ of $\mathbb{O}$ is defined by $(x, y)_{\mathbb{O}} = \sum_{i=0}^{7} x_i y_i$. Let $SO(\mathbb{O})$ be the set of all isometric linear automorphisms of $\mathbb{O}$ whose determinant is 1. Then, the triality principle of $SO(\mathbb{O})$ is well known.

Proposition 3.1 (The triality principle of $SO(\mathbb{O})$)**.** *For any $g_1 \in SO(\mathbb{O})$ there are $g_2, g_3 \in SO(\mathbb{O})$ such that*

$$\overline{(g_1 x)(g_2 y)} = g_3(\overline{xy}), \quad \overline{(g_2 x)(g_3 y)} = g_1(\overline{xy}), \quad \overline{(g_3 x)(g_1 y)} = g_2(\overline{xy}).$$

Moreover, such (g_2, g_3) are (g_2, g_3) or $(-g_2, -g_3)$.

Definition 3.2 ([9], section 1.16)**.** Set D as follows:

$$D := \{(g_1, g_2, g_3) \in SO(\mathbb{O})^3 \ ; \ g_1, g_2, g_3 \text{ satisfy the triality principle of } SO(\mathbb{O})\}.$$

Then, D is isomorphic to $Spin(8)$, so we denote D to $Spin(8)$.

Let $M(3,\mathbb{O})$ be the set of all 3×3 matrices whose components are octonions. Set $\mathfrak{J} := \{X \in M(3,\mathbb{O})\ ;\ {}^t\overline{X} = X\}$. We call $\mathfrak{J}$ a exceptional Jordan algebra. Each element X of $\mathfrak{J}$ is given by as follows:

$$X = \begin{pmatrix} \xi_1 & x_3 & \bar{x_2} \\ \bar{x_3} & \xi_2 & x_1 \\ x_2 & \bar{x_1} & \xi_3 \end{pmatrix} \ \ (\xi_i \in \mathbb{R},\ x_i \in \mathbb{O},\ i = 1,2,3).$$

The Jordan multiplicity $\circ$ is defined by $X \circ Y = \frac{1}{2}(XY + YX)$ for any $X, Y \in \mathfrak{J}$. The inner product $(\ ,\ )$ is defined by $(X, Y) = \mathrm{tr}(X \circ Y)$ for any $X, Y \in \mathfrak{J}$. We define the Freudenthal multiplicity as follows.

$$X \times Y = \frac{1}{2}\Big(XY + YX - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + \big(\mathrm{tr}(X)\mathrm{tr}(Y) - (X,Y)\big)I_3\Big).$$

for $X, Y \in \mathfrak{J}$. Let $\mathfrak{J}^{\mathbb{C}}$ be the complexification of $\mathfrak{J}$. We call $\mathfrak{J}^{\mathbb{C}}$ the complex exceptional Jordan algebra. Extend the Jordan multiplicity, the inner product and the Freudenthal multiplicity to $\mathfrak{J}^{\mathbb{C}}$ satisfying the complex linearity. These expansions are denoted to the same symbols. Let τ be the complex conjugation with respect to $\mathfrak{J}$. Set $\langle X, Y\rangle = (X, \tau(Y))$ for any $X, Y \in \mathfrak{J}^{\mathbb{C}}$. Then, $\langle\ ,\ \rangle$ is an Hermitian inner product. We see $\tau(X \times Y) = \tau(X) \times \tau(Y)$ for any $X, Y \in \mathfrak{J}^{\mathbb{C}}$. Define $X * Y = \tau(X \times Y)$.

Definition 3.3 ([9], section 3.1)**.** We define the exceptional compact Lie group E_6 as the set of all linear automorphisms f of $\mathfrak{J}^{\mathbb{C}}$ satisfying $f(X * Y) = f(X) * f(Y)$ and $\langle f(X), f(Y)\rangle = \langle X, Y\rangle$ for any $X, Y \in \mathfrak{J}^{\mathbb{C}}$.

Let $\sigma : \mathbb{O}^{\mathbb{C}} \to \mathbb{O}^{\mathbb{C}}\ ;\ \sum_{i=0}^{7} x_i e_i \mapsto \sum_{i=0}^{3} x_i e_i - \sum_{j=4}^{7} x_j e_j\ \ (x_i \in \mathbb{C})$. Define $\sigma \in E_6$ as follows:

$$\sigma\left(\begin{pmatrix} \xi_1 & x_3 & \bar{x_2} \\ \bar{x_3} & \xi_2 & x_1 \\ x_2 & \bar{x_1} & \xi_3 \end{pmatrix}\right) = \begin{pmatrix} \xi_1 & \sigma(x_3) & \overline{\sigma(x_2)} \\ \overline{\sigma(x_3)} & \xi_2 & \sigma(x_1) \\ \sigma(x_2) & \overline{\sigma(x_1)} & \xi_3 \end{pmatrix} \ \ (\xi \in \mathbb{C}, x_i \in \mathbb{O}^{\mathbb{C}}).$$

Then, σ is involutive and the multiplicity of the eigenvalue $+1$ is 15. Thus, $\sigma \in M_1^+$ and $E_6^\sigma \cong Sp(1) \cdot SU(6)$, where E_6^σ is the centralizer of σ in E_6. The isomorphic map $\phi : Sp(1) \cdot SU(6) \to E_6^\sigma$ is constructed explicitly ([9], section 3.11). By Lemma 2.3, $\phi(S_1)$ and $\phi(S_2)$ are maximal antipodal subgroup of E_6^σ. Denote $A_k(E_6) = \phi(S_k)\ \ (k = 1, 2)$. By studying $A_k(E_6) \cap M_1^+\ \ (k = 1, 2)$ we see that there are no $g \in E_6$ such that $gA_2(E_6)g^{-1} = A_1(E_6)$ [4]. Summarizing these arguments we obatin Theorem 3.4.

Theorem 3.4. $A_k(E_6)$ $(k = 1, 2)$ *are maximal antipodal subgroups of* E_6 *and not conjugate to each other. Moreover, any maximal antipodal subgroup of* E_6 *is conjugate to one of them.* $\#A_1(E_6) = 64$ *and* $\#A_2(E_6) = 32$. *In particular,* $\#_2 E_6 = 64$.

Moreover, $A_1(E_6)$ is conjugate to a maximal antipodal subgroup of a maximal torus and $A_2(E_6)$ is conjugate to a maximal antipodal subgroup of $F_4 \subset E_6$. Next, we construct $A_k(E_6)$ by using $Spin(8)$. Set $L_l(x) = e_l x, R_l(x) = xe_l$ $(1 \leq l \leq 7)$ for any $x \in \mathbb{O}^{\mathbb{C}}$ and $T_l = L_l \circ R_l$. The following f_i $(i = 0, \cdots, 3)$ are totally geodesic embeddings.

$$f_0 : Spin(8) \to E_6, \quad f_0(g_1, g_2, g_3)\begin{pmatrix} \xi_1 & x_3 & \bar{x_2} \\ \bar{x_3} & \xi_2 & x_1 \\ x_2 & \bar{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & g_3 x_3 & g_2 \bar{x}_2 \\ g_3 \bar{x}_3 & \xi_2 & g_1 x_1 \\ g_2 x_2 & g_1 \bar{x}_1 & \xi_3 \end{pmatrix},$$

$$f_1 : Spin(8) \to E_6, \quad f_1(g_1, g_2, g_3)\begin{pmatrix} \xi_1 & z_3 & \overline{z_2} \\ \overline{z_3} & \xi_2 & z_1 \\ z_2 & \overline{z_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & ig_3 \circ R_1(z_3) & \overline{ig_2 \circ L_1(z_2)} \\ \overline{ig_3 \circ R_1(z_3)} & -\xi_2 & -g_1 \circ T_1(z_1) \\ ig_2 \circ L_1(z_2) & \overline{-g_1 \circ T_1(z_1)} & -\xi_3 \end{pmatrix},$$

$$f_2 : Spin(8) \to E_6, \quad f_2(g_1, g_2, g_3)\begin{pmatrix} \xi_1 & z_3 & \overline{z_2} \\ \overline{z_3} & \xi_2 & z_1 \\ z_2 & \overline{z_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} -\xi_1 & ig_3 \circ L_1(z_3) & \overline{-g_2 \circ T_1(z_2)} \\ \overline{ig_3 \circ L_1(z_3)} & \xi_2 & ig_1 \circ R_1(z_1) \\ -g_2 \circ T_1(z_2) & \overline{ig_1 \circ R_1(z_1)} & -\xi_3 \end{pmatrix},$$

$$f_3 : Spin(8) \to E_6, \quad f_3(g_1, g_2, g_3)\begin{pmatrix} \xi_1 & z_3 & \overline{z_2} \\ \overline{z_3} & \xi_2 & z_1 \\ z_2 & \overline{z_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} -\xi_1 & -g_3 \circ T_1(z_3) & \overline{ig_2 \circ R_1(z_2)} \\ \overline{-g_3 \circ T_1(z_3)} & -\xi_2 & ig_1 \circ L_1(z_1) \\ ig_2 \circ R_1(z_2) & \overline{ig_1 \circ L_1(z_1)} & \xi_3 \end{pmatrix},$$

where $(g_1, g_2, g_3) \in Spin(8)$ and $\xi_k \in \mathbb{C},\ z_k \in \mathbb{O}^{\mathbb{C}}$ for $k = 1, 2, 3$. Moreover, let $L_{lm} = L_l \circ L_m, R_{lm} = R_l \circ R_m, T_{lm} = T_l \circ T_m$ for $1 \leq l, m \leq 7$. We set for any $\theta \in \mathbb{R}$

$$L_i(\theta) = \cos\theta L_0 + \sin\theta L_i, \quad R_i(\theta) = \cos\theta R_0 + \sin\theta R_i, \quad T_i(\theta) = L_i(-\theta) \circ R_i(-\theta),$$
$$L_{ij}(\theta) = \cos\theta L_0 + \sin\theta L_{ij}, \quad R_{ij}(\theta) = \cos\theta R_0 + \sin\theta R_{ij}, \quad T_{ij}(\theta) = T_i \circ T_{(\cos\theta e_i - \sin\theta e_j)}.$$

Let $g_l(\theta) = (T_l(\theta), L_l(\theta), R_l(\theta))$ and $g_{lm}(\theta) = (T_{lm}(\theta), L_{lm}(\theta), R_{lm}(\theta))$ for any $1 \leq l, m \leq 7,\quad \theta \in \mathbb{R}$. Then, $g_1(\theta_0),\ g_{23}(\theta_1),\ g_{45}(\theta_2),\ g_{67}(\theta_3) \in Spin(8)$ are commutative to each other for any $\theta_k \in \mathbb{R}$ $(k = 0, \cdots, 3)$. The group generated by $g_1(\theta_0),\ g_{23}(\theta_1),\ g_{45}(\theta_2),\ g_{67}(\theta_3)$ is a maximal torus of $Spin(8)$. This maximal torus is denoted to T_1 [4]. Then,

$$A(T_1) = \left\{ \begin{matrix} (g, g, g), & (g, -g, -g), \\ (-g, g, -g), & (-g, -g, g) \end{matrix} \ ;\ j = 0, \cdots, 3, g \in \{I_{\mathbb{O}}, R_{123}, R_{145}, R_{167}\} \right\}$$

is a maximal antipodal subgroup of T_1. Moreover,

$$A(Spin(8)) = \left\{ \begin{matrix} (g, g, g), & (g, -g, -g), \\ (-g, g, -g), & (-g, -g, g) \end{matrix} \ ;\ j = 0, \cdots, 3, g \in \left\{ \begin{matrix} I_{\mathbb{O}}, & R_{123}, & R_{145}, & R_{167}, \\ R_{246}, & R_{257}, & R_{347}, & R_{356} \end{matrix} \right\} \right\}$$

is a maximal antipodal subgroup of $Spin(8)$ and it is known that any maximal antipodal subgroup is congruent to $A(Spin(8))$ [5].

Theorem 3.5. *[4] The following are true.*

$$A_1(E_6) = \bigcup_{i=0}^{3} f_i\Big(A(T_1)\Big), \qquad A_2(E_6) = f_0\Big(A(Spin(8))\Big).$$

4 Maximal antipodal subsets of EII type

It is known that M_1^+ is a compact symmetric space of EII type. The following properties about maximal antipodal sets of polars are obvious.

Lemma 4.1. *Let G be a connected compact Lie group and M^+ be a polar of the unit element e. Then, for any maximal antipodal set A of M^+, there is a maximal antipodal subgroup B such that $A = B \cap M^+$.*

Let $A_k(M^+) = A_k(E_6) \cap M^+$ $(k = 1, 2)$. $\#A_1(M_1^+) = 36$ and $\#A_2(M_1^+) = 28$. Then, we see that there are no $g \in E_6$ such that $gA_2(M^+)g^{-1} \subset A_1(M^+)$ [4].

Theorem 4.2. *[4] $A_k(M_1^+)$ $(k = 1, 2)$ are maximal antipodal sets of M_1^+ and not congruent to each other. Moreover, any maximal antipodal set of M_1^+ is congruent to one of them. $\#A_1(M_1^+) = 36$ and $\#A_2(M_1^+) = 28$. In particular, $\#_2 EI = 36$.*

We construct maximal antipodal sets of EII type by using oriented real Grassmannians. We consider a realization of EII type. Let V be a complex subspace of $\mathbb{O}^{\mathbb{C}}$. If V is a subalgebra with respect to $*$, then we call V a $*$-subspace. For example, $H_3(\mathbb{H})^{\mathbb{C}} := \{X \in M(3, \mathbb{H}) \; ; \; {}^t\overline{X} = X\}^{\mathbb{C}}$ is a $*$-subspace. We denote to $*_{\mathbb{H}}$ and $\langle \; , \; \rangle_{\mathbb{H}}$ the restrictions of $\langle \; , \; \rangle$ and $*$ to $H_3(\mathbb{H})^{\mathbb{C}}$. We call V a $\mathbb{H}$-subspace if there is $f : H_3(\mathbb{H})^{\mathbb{C}} \to V$ such that $\langle X, Y \rangle_{\mathbb{H}} = \langle f(X), f(Y) \rangle$ and $f(X *_{\mathbb{H}} Y) = f(X) * f(Y)$ for any $X, Y \in H_3(\mathbb{H})^{\mathbb{C}}$. We denote to $G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ the set of all $\mathbb{H}$-subspaces. We easily see that E_6 acts on $G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ by the definition of E_6. Then, we obtain Theorem 4.3.

Theorem 4.3. *[4] The action of E_6 on $G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ is transitive. The isotropy subgroup at $H_3(\mathbb{H})^{\mathbb{C}} \in G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ is E_6^{σ} and $G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}}) \cong E_6/Sp(1) \cdot SU(6)$. In particular, $G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ is a compact symmetric space of EII type.*

The following map is an isomorphism as symmetric spaces between M_1^+ and $G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$:

$$M_1 \ni p \mapsto V_p^{+1} := \{X \in \mathfrak{J}^{\mathbb{C}} \; ; \; p(X) = X\} \in G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}}).$$

The image of $A_k(M_1^+)$ by this map is denoted to $A_k(G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}}))$ $(k = 1, 2)$. Next, we consider some Grassmannians. Let $G_n(\mathbb{O})$ be the set of all n-dimensional subspaces of $\mathbb{O}$.

Lemma 4.4. *[4] For any $V_1 \in G_4(\mathbb{O})$, there are $V_2, V_3 \in G_4(\mathbb{O})$ such that*

$$\overline{v_1 v_2} \in V_3, \quad \overline{v_2 v_3} \in V_1, \quad \overline{v_3 v_1} \in V_2 \quad (v_i \in V_i, \; i = 1, 2, 3).$$

Moreover, (V_2, V_3) is (V_2, V_3) or $(V_2^{\perp}, V_3^{\perp})$.

If $(V_1, V_2, V_3) \in G_4(\mathbb{O})^3$ satisfies the condition of Lemma 4.4, we say that (V_1, V_2, V_3) satisfies the triality principle of $G_4(\mathbb{O})$. For exmaple, $(\mathbb{H}, \mathbb{H}, \mathbb{H})$ satisfies the triality principle of $G_4(\mathbb{O})$. We denote to $G_4^t(\mathbb{O})$ the set of all $(V_1, V_2, V_3) \in G_4(\mathbb{O})^3$ which satisfies the triality principle of $G_4(\mathbb{O})$. It is easily seen that $Spin(8)$ acts on $G_4^t(\mathbb{O})$ i.e. for any $(g_1, g_2, g_3) \in Spin(8)$ and $(V_1, V_2, V_3) \in G_4^t(\mathbb{O})$ it is true that $(g_1 V_1, g_2 V_2, g_3 V_3) \in G_4^t(\mathbb{O})$. Let $G_k^{\mathrm{OR}}(\mathbb{R}^n)$ be the set of all oriented k-dimensional subspaces of $\mathbb{R}^8$.

Lemma 4.5. *[4] The action of $Spin(8)$ on $G_4^t(\mathbb{O})$ is transitive. The isotropy subgroup at each point of $G_4(\mathbb{O})$ is isomorphic to $Spin(4) \times Spin(4)/\mathbb{Z}_2$. Thus, $G_4^t(\mathbb{O}) \cong G_4^{\mathrm{OR}}(\mathbb{R}^8)$.*

Let $G_k^{\mathbb{C}}(\mathbb{O}^{\mathbb{C}})$ be the set of all complex k-dimensional subspaces of $\mathbb{O}^{\mathbb{C}}$. Set $X_j^{\pm} = e_{2j} \pm ie_{2j+1}$ $(j = 0, 1, 2, 3)$. We define

$$\begin{aligned}
&G_n^{\tau}(\mathbb{O}^{\mathbb{C}}) = \{V \in G_n^{\mathbb{C}}(\mathbb{O}^{\mathbb{C}}) \ ; \ \tau(V) \subset V\} \cong G_n(\mathbb{O}),\\
&G_4^{\perp}(\mathbb{O}^{\mathbb{C}}) = \{V \in G_4^{\mathbb{C}}(\mathbb{O}^{\mathbb{C}}) \ ; \ \tau(V) = V^{\perp}\},\\
&G_4^{\perp}(\mathbb{O}^{\mathbb{C}})^{+} : \text{the connected component of } G_4^{\perp}(\mathbb{O}^{\mathbb{C}}) \text{ containing } \mathbb{C}X_0^{-} \oplus \sum_{k=2,4,6} \mathbb{C}X_k^{-},\\
&G_4^{\perp}(\mathbb{O}^{\mathbb{C}})^{-} : \text{the connected component of } G_4^{\perp}(\mathbb{O}^{\mathbb{C}}) \text{ containing } \mathbb{C}X_0^{+} \oplus \sum_{k=2,4,6} \mathbb{C}X_k^{-}.
\end{aligned}$$

Lemma 4.6. *[4] For any $V_1 \in G_6^{\tau}(\mathbb{O}^{\mathbb{C}})$, there are $V_2 \in G_4^{\perp}(\mathbb{O}^{\mathbb{C}})$ and $V_3 \in G_4^{\perp}(\mathbb{O}^{\mathbb{C}})$ such that*

$$\tau(\overline{v_1 v_2}) \in V_3, \quad \tau(\overline{v_2 v_3}) \in V_1, \quad \tau(\overline{v_3 v_1}) \in V_2 \quad (v_i \in V_i, \ i = 1, 2, 3).$$

Moreover, (V_2, V_3) is (V_2, V_3) or $(V_2^{\perp}, V_3^{\perp})$.

If $(V_1, V_2, V_3) \in G_6^{\tau}(\mathbb{O}^{\mathbb{C}}) \times G_4^{\perp}(\mathbb{O}^{\mathbb{C}})^{+} \times G_4^{\perp}(\mathbb{O}^{\mathbb{C}})^{-}$ satisfies the condition of Lemma 4.6, we say that (V_1, V_2, V_3) satisfies the triality principle of $G_6(\mathbb{O})$. For example, $(U_{T_1}^{+1}, U_{L_1}^{-i}, U_{R_1}^{+i})$ satisfies the triality principle of $G_6(\mathbb{O})$, where $U_{T_1}^{+1}, U_{L_1}^{-i}, U_{R_1}^{+i}$ are the eigenspaces of T_1, L_1, R_1 corresponding to the eigenvalues $+1, -i, +i$. We denote to $G_6^t(\mathbb{O})$ the set of all $(V_1, V_2, V_3) \in G_6^{\tau}(\mathbb{O}^{\mathbb{C}}) \times G_4^{\perp}(\mathbb{O}^{\mathbb{C}})^{+} \times G_4^{\perp}(\mathbb{O}^{\mathbb{C}})^{-}$ which satisfies the triality principle of $G_6(\mathbb{O})$. It is easily seen that $Spin(8)$ acts on $G_6^t(\mathbb{O})$.

Lemma 4.7. *[4] The action of $Spin(8)$ on $G_6^t(\mathbb{O})$ is transitive. The isotropy subgroup is isomorphic to $Spin(2) \times Spin(6)/\mathbb{Z}_2$. Thus, $G_6^t(\mathbb{O}) \cong G_6^{\mathrm{OR}}(\mathbb{R}^8)$.*

Let E_{ij} be the 3×3 matrix whose (i, j)-component is 1 and the others are 0 and $F_1(x) = xE_{23} + \overline{x}E_{32}, F_2(x) = \overline{x}E_{13} + xE_{31}, F_3(x) = xE_{12} + \overline{x}E_{21}$ for any $x \in \mathbb{O}^{\mathbb{C}}$. Moreover, for any subset $V \subset \mathbb{O}^{\mathbb{C}}$ we define $F_i(V) = \{F_i(v) \ ; \ v \in V\}$ for each $i = 1, 2, 3$. The following $g_0, \cdots, g_3$ are totally geodesic embeddings [4].

$$\begin{aligned}
g_0 \ &: \ G_4^t(\mathbb{O}) \to G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}}) \ ; \ (V_1, V_2, V_3) \mapsto \mathbb{C}E_{11} \oplus \mathbb{C}E_{22} \oplus \mathbb{C}E_{33} \oplus \bigoplus_{k=1}^{3} F_k(V_k^{\mathbb{C}}),\\
g_1 \ &: \ G_6^t(\mathbb{O}) \to G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}}) \ ; \ (V_1, V_2, V_3) \mapsto \mathbb{C}E_{11} \oplus F_1(V_1) \oplus F_2(V_2) \oplus F_3(V_3),\\
g_2 \ &: \ G_6^t(\mathbb{O}) \to G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}}) \ ; \ (V_1, V_2, V_3) \mapsto \mathbb{C}E_{22} \oplus F_1(V_3) \oplus F_2(V_1) \oplus F_3(V_2),\\
g_3 \ &: \ G_6^t(\mathbb{O}) \to G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}}) \ ; \ (V_1, V_2, V_3) \mapsto \mathbb{C}E_{33} \oplus F_1(V_2) \oplus F_2(V_3) \oplus F_3(V_1).
\end{aligned}$$

Then, the following $A(G_4^t(\mathbb{O}))$ is a maximal antipodal set of $G_4^t(\mathbb{O})$ and any maximal antipodal set is congruent to $A(G_4^t(\mathbb{O}))$ [8].

$$A(G_4^t(\mathbb{O})) = \left\{ \begin{array}{l} (V\ \ ,V\ \ ,V\ \ ), \\ (V\ \ ,V^{\perp},V^{\perp}), \\ (V^{\perp},V\ \ ,V^{\perp}), \\ (V^{\perp},V^{\perp},V\ \ ) \end{array} ;\ V \in \left\{ \begin{array}{ll} \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3, & \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5, \\ \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_7, & \mathbb{C}e_0 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_6, \\ \mathbb{C}e_0 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_7, & \mathbb{C}e_0 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_7, \\ \mathbb{C}e_0 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 & \end{array} \right\} \right\}.$$

Moreover, the following $A(G_6^t(\mathbb{O}^{\mathbb{C}}))$ is a maximal antipodal set of $G_6^t(\mathbb{O}^{\mathbb{C}})$ and any maximal antipodal subset is congruent to $A(G_6^t(\mathbb{O}))$ [7].

$$A(G_6^t(\mathbb{O}^{\mathbb{C}})) = \left\{ \begin{array}{l} (V_1, V_2\ \ , V_3\ \ ), \\ (V_1, V_2^{\perp}, V_3^{\perp}) \end{array} ;\ (V_1, V_2, V_3) \in \left\{ \begin{array}{l} (\ (\mathbb{C}e_0 \oplus \mathbb{C}e_1)^{\perp}, U_{L_1}^{i}\ \ , U_{R_1}^{-i}\ ), \\ (\ (\mathbb{C}e_2 \oplus \mathbb{C}e_3)^{\perp}, U_{L_{23}}^{i}, U_{R_{23}}^{-i}), \\ (\ (\mathbb{C}e_4 \oplus \mathbb{C}e_5)^{\perp}, U_{L_{45}}^{i}, U_{R_{45}}^{-i}), \\ (\ (\mathbb{C}e_6 \oplus \mathbb{C}e_7)^{\perp}, U_{L_{67}}^{i}, U_{R_{67}}^{-i}), \end{array} \right\} \right\}.$$

Further, we set $A'(G_4^t(\mathbb{O})) = \{(V_1 \cap U_1, V_2 \cap U_2, V_3 \cap U_3)\ ;\ (V_1, V_2, V_3), (U_1, U_2, U_3) \in A(G_6^t(\mathbb{O})), V_1 \neq U_1\}$. Then, $A'(G_4^t(\mathbb{O})) \subset G_4^t(\mathbb{O})$ and $A'(G_4^t(\mathbb{O}))$ is an antipodal set of $G_4^t(\mathbb{O})$ which is not maximal. In particular, $A'(G_4^t(\mathbb{O}))$ is given by as follows:

$$A'(G_6^t(\mathbb{O}^{\mathbb{C}})) = \left\{ \begin{array}{ll} (V\ \ ,V\ \ ,V\ \ ), & (V\ \ ,V^{\perp},V^{\perp}), \\ (V^{\perp},V\ \ ,V^{\perp}), & (V^{\perp},V^{\perp},V\ \ ) \end{array} ;\ V \in \left\{ \begin{array}{l} \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3, \\ \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5, \\ \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_7, \end{array} \right\} \right\}.$$

Theorem 4.8. *[4] The following are true.*

$$A_1(G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})) = g_0\Big(A'(G_4^t(\mathbb{O}))\Big) \cup \bigcup_{i=1}^{3} g_i\Big(A(G_6^t(\mathbb{O}))\Big),$$

$$A_2(G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})) = g_0\Big(A(G_4^t(\mathbb{O}))\Big).$$

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Maximal antipodal sets of E_6 and some compact symmetric spaces

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2021/11/25 カンドルと対称空間 2021

今日の講演内容

- 本研究では，
  - 例外型コンパクトリー群 E_6
  - $EI, \cdots, EIV$ 型コンパクト対称空間

  における極大対蹠集合の分類・構成を行った．($EIII, EIV$ 型は構成のみ)

- $EI, \cdots, EIV$ 型コンパクト対称空間では，極大対蹠集合を具体的に記述するため，複素例外 Jordan 代数を用いた幾何的実現も構成した．

今日の講演内容

本講演では，

- 例外型コンパクトリー群 E_6 の極大対蹠集合の分類・構成
- EII 型コンパクト対称空間 $E_6/(Sp(1) \times Sp(3)/\mathbb{Z}_2)$ の幾何的実現および極大対蹠集合の分類・構成

の 2 点を紹介したい.

1. 背景
2. E_6 の極大対蹠集合の分類・構成
3. EII の極大対蹠集合の分類・構成

1, 背景

定義 1.1

リーマン多様体 M について，各点 $x \in M$ に対して次を満たす等長変換 s_x が存在するとき，M を対称空間という.

(1) x は s_x の孤立固定点である.

(2) s_x は対合的である ($s_x^2 = \mathrm{id}_M$).

s_x を x における点対称と呼ぶ.

対蹠集合

定義 1.2 (1988, Chen-Nagano)

M を対称空間とする．

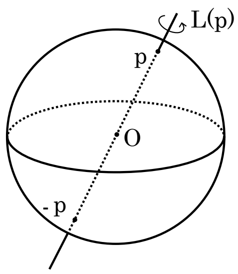
- M の部分集合 S が対蹠集合 $\overset{\text{def}}{\Longleftrightarrow}$ $s_x(y) = y \quad (\forall x, y \in S)$
- 濃度が最大の対蹠集合を大対蹠集合と呼ぶ．

  大対蹠集合の濃度を M の 2-number といい，$\#_2 M$ とかく．

  対蹠集合間の包含関係に関して極大なものを，極大対蹠集合という．

- 以下，対称空間はコンパクトであると仮定する．
- 対蹠集合は常に有限集合になり，2-number は有限である．

例：球面 S^2



- $p \in S^2$ とし，$L(p)$ を中心 o と p を通る直線とする．
- p における点対称 s_p は $L(p)$ を回転軸とした 180 度回転となる．
- $\{x \in S^2 \ ; \ s_p(x) = x\} = \{p, -p\}$ であり，$s_p = s_{-p}$ なので $s_{-p}(p) = p$．
  よって，$\{p, -p\}$ は S^2 の大対蹠集合となり，$\#_2 S^2 = 2$．

例：コンパクトリー群

G をコンパクトリー群とする．

- コンパクトリー群 G には両側不変計量が存在して，コンパクト対称空間になることが知られている．
- このとき，$g \in G$ における点対称は

$$s_g : G \to G\ ;\ h \mapsto gh^{-1}g.$$

- 例えば，ユニタリ群 $U(n)$ においては

$$\left\{ \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \in U(n) \right\}$$

が大対蹠集合となっている．とくに，$\#_2 U(n) = 2^n$ となる．

対蹠集合の性質

対蹠集合について，以下のような性質が知られている．

- 2-number はコンパクト対称空間の不変量である．
すなわち，2-number が異なる 2 つのコンパクト対称空間は同型にならない (Chen-Nagano, 1988).
- M をコンパクト対称空間とし，$\chi(M)$ を M のオイラー数とする．このとき，$\chi(M) \leq \#_2 M$ が成り立つ (Chen-Nagano, 1988).
- M を対称 R 空間とすれば，M の $\mathbb{Z}_2$ 係数ホモロジー群のベッチ数は $\#_2 M$ と一致する (Takeuchi, 1989).

・・・しかしながら，全てのコンパクト対称空間でその極大対蹠集合が分類・構成されているわけではない．

例外型における状況

例外型コンパクト対称空間として次のようなものがある．

- 例外型単純コンパクトリー群

$$G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8$$

- G_2 によるコンパクト対称空間： G 型
- F_4 によるコンパクト対称空間： FI 型， FII 型
- E_6 によるコンパクト対称空間： EI 型， EII 型， $EIII$ 型， EIV 型
- E_7 によるコンパクト対称空間： EV 型， EVI 型， $EVII$ 型
- E_8 によるコンパクト対称空間： $EVIII$ 型， EIX 型

（赤）：分類・構成が未完成，
（黄）：分類は完成しているが，構成は未完成．

本研究では，E_6 および $EI, \cdots, EIV$ 型の極大対蹠集合の分類・構成を行った．
本講演では，E_6 および EII 型における結果を紹介する．

1． E_6 の極大対蹠集合の分類・構成

分類方法

定義 2.1

対称空間 M の部分集合 A, B が互いに合同であるとは，M の等長変換群の単位連結成分の元 g で $g(A) = B$ を満たすものが存在すること．

- 極大対蹠集合の合同類を考える．
- G：連結コンパクトリー群，$e \in G$：単位元
  G における合同変換は，両側移動全体で与えられる．

命題 2.2

G の単位元を含む極大対蹠集合は，各元の位数が 2 のアーベル群になる．

- 単位元を含む極大対蹠集合を極大対蹠部分群という．

極大対蹠集合の合同類の分類 ⟶ 極大対蹠部分群の共役類の分類

定義 2.3 (極地)

対称空間 M の点 p に対して，点対称 s_p の不動点集合 $F(s_p, M)$ の各連結成分を極地という．

- G において単位元 e における点対称は，$s_e(g) = g^{-1}$．よって，

$$F(s_e, G) = \{g \in G\ ;\ g = g^{-1}\}.$$

- $q \in F(s_e, G)$ を含む極地は，$\bigcup_{g \in E_6} gqg^{-1}$ となる．
- $\sigma_i\ (1 \leq i \leq n) \in F(s_e, G)$ をうまく選んで，$M_i^+ = \bigcup_{g \in G} g\sigma_i g^{-1}$ とおくと

$$F(s_e, G) = \ \{e\} \ \sqcup \ M_1^+ \ \sqcup \ M_2^+ \ \sqcup \ \cdots \ \sqcup \ M_n^+$$

- A を G の極大対蹠部分群とする．$A \subset F(s_e, G)$ であるので，

$$A = \{e\} \sqcup (A \cap M_1^+) \sqcup (A \cap M_2^+) \sqcup \cdots \sqcup (A \cap M_n^+)$$

- $G^{\sigma_i} = \{g \in G \ ; \ g\sigma_i = \sigma_i g\} \ (1 \le i \le n)$ とおく．

補題 2.4

A を G の極大対蹠部分群とする．

$$A \cap M_i^+ \neq \phi \iff A \text{ は } G^{\sigma_i} \text{ の極大対蹠部分群と共役}$$

G の極大対蹠部分群の分類 $\longrightarrow$ G^{σ_i} たちの極大対蹠部分群の分類

E_6 の場合

- E_6 において，

$$M_1^+ = \{g \in E_6 \ ; \ g^{-1} = g, \quad g \text{ の固有値 } +1 \text{ の重複度は } 15\} \cong EII \text{ 型}$$
$$M_2^+ = \{g \in E_6 \ ; \ g^{-1} = g, \quad g \text{ の固有値 } +1 \text{ の重複度は } 11\} \cong EIII \text{ 型}$$

のように定めると，単位元 $I \in E_6$ に対して

$$F(s_I, E_6) = \{I\} \sqcup M_1^+ \sqcup M_2^+$$

補題 2.5

E_6 の任意の極大対蹠集合 A は，$A \cap M_1^+ \neq \phi$ を満たす．

- $\sigma \in M_1^+$ に対して，$E_6^\sigma \cong Sp(1) \cdot SU(6)$ となる．

E_6 の極大対蹠部分群の分類 $\longrightarrow$ $Sp(1) \cdot SU(6)$ の極大対蹠部分群の分類

$Sp(1) \cdot SU(6)$ の極大対蹠部分群

- I_n を n 次単位行列とする．次のような自然な射影を考える．

$$\pi : Sp(1) \times SU(6) \to \Big(Sp(1) \times SU(6)\Big)/\{\pm(I_1, I_6)\}$$

π による像を $Sp(1) \cdot SU(6)$ と記す．（Chen-Nagano が導入したドット積）

- 部分群 $S_1, S_2 \subset Sp(1) \times SU(6)$ を次で定める．

$$S_1 = (\{\pm 1\} \times \Delta_6^+) \cup (\{\pm e_1\} \times i\Delta_6^-)$$

$$S_2 = \begin{matrix} \left\{\left(\pm 1, \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix}\right) ; \ J_i = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} & \cup & \left\{\left(\pm e_1, \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix}\right) ; \ J_i = \pm i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \\ \left\{\left(\pm e_2, \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix}\right) ; \ J_i = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\} & \cup & \left\{\left(\pm e_3, \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix}\right) ; \ J_i = \pm i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} \end{matrix}$$

補題 2.6 (S)

$\pi(S_1), \pi(S_2)$ は $Sp(1) \cdot SU(6)$ の極大対蹠部分群になり，任意の極大対蹠部分群はこのいずれかと共役になる．

八元数について

- $\mathbb{O} = \sum_{1=0}^{7} \mathbb{R} e_i$ を八元数とする．積が次のように定義される．

(1) e_0 は積の単位元とする．単に 1 と書く．

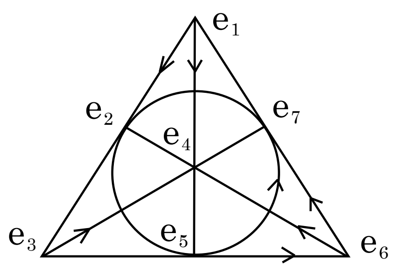
(2) 各 $1 \leq i \neq j \leq 7$ について，

$$e_i^2 = -1, e_i e_j = -e_j e_i.$$

(3) 積は分配法則を満たしている．

(4) 右の図により積を定める．

（例：$e_1 e_2 = e_3, e_1 e_4 = e_5$）



- 八元数 $\mathbb{O}$ では，結合則が成り立たない．
- $x = x_0 e_0 + \sum_{i=1}^{7} x_i e_i$ に対して，共役 $\bar{x} = x_0 e_0 - \sum_{i=1}^{7} x_i e_i$ を定める．
- $x = \sum_{i=0}^{7} x_i e_i$, $\quad y = \sum_{i=0}^{7} y_i e_i$ の内積 $(x, y) = \sum_{i=0}^{7} x_i y_i$ を定める．

$Spin(8)$ について

定理 2.7 ($SO(\mathbb{O})$-3 対原理)
任意の $g_1 \in SO(\mathbb{O})$ に対して，$g_2, g_3 \in SO(\mathbb{O})$ で

$$(g_i x)(g_{i+1} y) = \overline{g_{i+2}(\overline{xy})} \qquad (x, y \in \mathbb{O}, \text{添え字は mod3})$$

を満たすものが存在する．さらに，そのような g_2, g_3 は符号を除いて一意である．

定義 2.8
$SO(\mathbb{O})^3$ の部分群 $Spin(8)$ を次で定める．

$$Spin(8) := \{(g_1, g_2, g_3) \in SO(\mathbb{O})^3 \ ; \ (g_1 x)(g_2 y) = \overline{g_3(\overline{xy})} \qquad (x, y \in \mathbb{O})\}.$$

Jordan 代数について

$M(3, \mathbb{O})$ で各成分が $\mathbb{O}$ の元であるような 3 次正方行列全体とする．

定義 2.9
$M(3, \mathbb{O})$ の実部分空間

$$\mathfrak{J} := \{X \in M(3, \mathbb{O}) \ ; \ {}^t\bar{X} = X\} = \left\{ X(\xi, x) = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \ ; \ \xi_i \in \mathbb{R}, \ x_i \in \mathbb{O} \right\}$$

を例外 Jordan 代数と呼ぶ．

- 内積 $(,)$ を次で定める：$(X, Y) = \frac{1}{2}\mathrm{tr}(XY + YX)$
- Freudenthal 積 $\times$ を次のように定める．

$$X \times Y = \frac{1}{2}(XY + YX - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - (X, Y))E)$$

ただし，E は単位行列．

例外 Jordan 代数 $\mathfrak{J}$ の複素化を複素例外 Jordan 代数 $\mathfrak{J}^{\mathbb{C}}$ という．

- 内積 $(\ ,\ )$，Freudenthal 積を複素線形に拡張する．
- $\mathfrak{J}$ に関する複素共役を τ とおき，積 $*$ を次で定める．

$$X * Y := \tau(X \times Y) = \tau X \times \tau Y \quad (X, Y \in \mathfrak{J}^{\mathbb{C}})$$

- エルミート内積 $\langle\ ,\ \rangle$ を次のように定める．

$$\langle X, Y\rangle = (X, \tau Y) \quad (X, Y \in \mathfrak{J}^{\mathbb{C}})$$

定義 2.10

複素例外 Jordan 代数 $\mathfrak{J}^{\mathbb{C}}$ の線形自己同型 f で，$X, Y \in \mathfrak{J}^{\mathbb{C}}$ に対して

$$(1)\quad f(X) * f(Y) = f(X * Y), \qquad (2)\quad \langle f(X), f(Y)\rangle = \langle X, Y\rangle \qquad (X, Y \in \mathfrak{J}^{\mathbb{C}})$$

を満たすもの全体による群を E_6 と定める．

$Sp(1) \cdot SU(6) \subset E_6$ について

- $\sigma : \mathbb{O}^{\mathbb{C}} \to \mathbb{O}^{\mathbb{C}}$; $\sum_{i=0}^{7} z_i e_i \mapsto \sum_{i=0}^{3} z_i e_i - \sum_{j=4}^{7} z_j e_j \quad (z_i \in \mathbb{C})$ とする．
- 次のように $\sigma \in E_6$ を定める．

$$\sigma\left(\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}\right) = \begin{pmatrix} \xi_1 & \sigma(x_3) & \overline{\sigma(x_2)} \\ \overline{\sigma(x_3)} & \xi_2 & \sigma(x_1) \\ \sigma(x_2) & \overline{\sigma(x_1)} & \xi_3 \end{pmatrix}$$

- $\sigma^2 = I$ となるので対合的である．$E_6^{\sigma} = \{g \in E_6 \ ; \ g\sigma = \sigma g\}$ とおく．
- 全射準同型 $\phi : Sp(1) \times SU(6) \to E_6^{\sigma}$ が存在する．とくに $\mathrm{Ker}\phi = \{\pm(I_1, I_6)\}$ となり，$E_6^{\sigma} \cong Sp(1) \cdot SU(6)$.

- したがって，$Sp(1) \times SU(6)$ の部分群 S_1, S_2 の ϕ による像

$$A_1(E_6) := \phi(S_1), \qquad A_2(E_6) := \phi(S_2)$$

が E_6^σ の極大対蹠部分群になる．

  - E_6 の極大対蹠部分群は，E_6^σ の極大対蹠集合と共役
  - $\#A_1(E_6) = 64$ かつ $\#A_2(E_6) = 32$ となる．
  - $gA_2(E_6)g^{-1} \subset A_1(E_6)$ を満たす $g \in E_6$ は存在しない．

定理 2.11 (S)

$A_i(E_6)$ $(i = 1, 2)$ は E_6 の極大対蹠部分群の共役類の代表元である．とくに，$\#_2 E_6 = 64$ となる．

- $A_1(E_6)$ は E_6 の極大トーラスの極大対蹠部分群
- $A_2(E_6)$ は E_6 の部分群 F_4 の極大対蹠部分群になっている．

定義 2.12

- 対称空間 M の部分集合 S が s 可換である

  $\overset{\text{def}}{\iff}$ 任意の $p, q \in S$ に対して $s_p s_q = s_q s_p$ が成り立つ．

- 対称空間 M の全測地的部分多様体 L の部分集合 S が (L, M)-s 可換である．

  $\overset{\text{def}}{\iff}$ S は L の s 可換集合かつ M の s 可換集合．

- 包含関係に関して極大である (L, M)-s 可換集合を，(L, M)-極大 s 可換集合という．

- 対蹠集合はかならず s 可換集合である．逆は一般には成り立たない．
- 例えば，実射影空間 $\mathbb{R}P^n$ における極大トーラス T に関して，T の s 可換集合は必ずしも $\mathbb{R}P^n$ の s 可換集合になるとは限らない．

$Spin(8)$ からの構成

$A_i(E_6)$ を $Spin(8)$ を用いて構成する.

$Spin(8)$ は次のようにして，E_6 へ全測地的に 4 通りに埋め込まれる.

$$f_0 : Spin(8) \to E_6, \quad f_0(g_1, g_2, g_3)\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & g_3x_3 & g_2\bar{x}_2 \\ g_3\bar{x}_3 & \xi_2 & g_1x_1 \\ g_2x_2 & g_1\bar{x}_1 & \xi_3 \end{pmatrix},$$

$$f_1 : Spin(8) \to E_6, \quad f_1(g_1, g_2, g_3)\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & ig_3x_3 & ig_2\bar{x}_2 \\ ig_3\bar{x}_3 & -\xi_2 & -g_1x_1 \\ ig_2x_2 & -g_1\bar{x}_1 & -\xi_3 \end{pmatrix},$$

$$f_2 : Spin(8) \to E_6, \quad f_2(g_1, g_2, g_3)\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} -\xi_1 & ig_3x_3 & -g_2\bar{x}_2 \\ ig_3\bar{x}_3 & \xi_2 & ig_1x_1 \\ -g_2x_2 & ig_1\bar{x}_1 & -\xi_3 \end{pmatrix},$$

$$f_3 : Spin(8) \to E_6, \quad f_3(g_1, g_2, g_3)\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} -\xi_1 & -g_3x_3 & ig_2\bar{x}_2 \\ -g_3\bar{x}_3 & -\xi_2 & ig_1x_1 \\ ig_2x_2 & ig_1\bar{x}_1 & \xi_3 \end{pmatrix},$$

$Spin(8)$ の中心は次で与えられる.

$$\left\{ \begin{array}{ll} I = (I_{\mathbb{O}}, I_{\mathbb{O}}, I_{\mathbb{O}}), & I_1 = (I_{\mathbb{O}}, -I_{\mathbb{O}}, -I_{\mathbb{O}}), \\ I_2 = (-I_{\mathbb{O}}, I_{\mathbb{O}}, -I_{\mathbb{O}}), & I_3 = (-I_{\mathbb{O}}, -I_{\mathbb{O}}, I_{\mathbb{O}}) \end{array} \right\}$$

- T を $Spin(8)$ の極大トーラスとし，B を単位元を含む $(T, Spin(8))$-極大 s 可換集合とする.
- $B_i := \{g \in B \ ; \ g^2 = I_i\}$ $(0 \le i \le 3)$ とおくと，各 B_i は T の極大対蹠集合で，$B = \sqcup_{i=0}^3 B_i$.
- A を $Spin(8)$ の極大対蹠部分群とする.

定理 2.13 (S)

$\bigcup_{i=0}^3 f_i(B_i)$ は $A_1(E_6)$ と，$f_0(A)$ は $A_2(E_6)$ と E_6-共役である.

2. EII の極大対蹠集合の分類・構成

EII 型に関して

- E_6 の単位元 $\{I\}$ において

$$F(s_I, E_6) = \{I\} \sqcup M_1^+ \sqcup M_2^+$$

であり，M_1^+ は EII 型である．$A_i^{II} = A_i(E_6) \cap M_1^+$ $(i = 1, 2)$ とおく．

命題 3.1

M^+ を連結コンパクトリー群 G の単位元における極地とする．M^+ の任意の極大対蹠集合 A に対して，G のある極大対蹠部分群 B が存在して，$A = B \cap M^+$．

- M^+ における合同変換は G の共役作用である．
- $\#A_1^{II} = 36, \#A_2^{II} = 28$ である．
- $gA_1^{II}g^{-1} \subset A_2^{II}$ を満たす $g \in E_6$ は存在しない．

定理 3.2 (S)

A_i^{II} $(i = 1, 2)$ は M_1^+ の極大対蹠集合の合同類の代表元であり，$\#_2 EII = 36$．

EII の幾何的実現

- $\mathfrak{J}^{\mathbb{C}}$ の部分空間 V が積 $*$ の部分代数であるとき，V を $*$-部分空間という．
- $H_3(\mathbb{H})^{\mathbb{C}} := \{X \in M(3,\mathbb{H}) \ ; \ {}^t\bar{X} = X\}^{\mathbb{C}}$ は $*$-部分空間である．
  積 $*$ およびエルミート内積を $H_3(\mathbb{H})^{\mathbb{C}}$ へ制限したものを $*_{\mathbb{H}}$，$\langle \ , \ \rangle_{\mathbb{H}}$ と記す．

定義 3.3

$*$-部分空間 $V \subset \mathfrak{J}^{\mathbb{C}}$ が $\mathbb{H}$-部分空間であるとは，線形同型 $f : H_3(\mathbb{H})^{\mathbb{C}} \to V$ で

(1) $f : (H_3(\mathbb{H})^{\mathbb{C}}, \langle , \rangle|_{\mathbb{H}}) \to (V, \langle , \rangle|_V)$ は等長的，

(2) $f(X_1 *_{\mathbb{H}} X_2) = f(X_1) * f(X_2) \quad (X_1, X_2 \in H_3(\mathbb{H})^{\mathbb{C}})$

を満たすものが存在すること．

EII の極大対蹠集合の分類・構成

定義 3.4

$\mathfrak{J}^{\mathbb{C}}$ の $\mathbb{H}$-部分空間全体によるグラスマン多様体を，$\mathbb{H}$-グラスマン多様体と呼んで，$G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ と記す．

- E_6 の定義から，

$$E_6 := \left\{ \text{線形同型 } f : \mathfrak{J}^{\mathbb{C}} \to \mathfrak{J}^{\mathbb{C}} \ ; \ \begin{array}{c} f(X * Y) = f(X) * f(Y) \\ f \text{ は等長的} \end{array} \right\}$$

  であるため，任意の $V \in G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}}), g \in E_6$ に対して $g(V) \in G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ となる．
  すなわち，E_6 は $G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ へ作用している．

定理 3.5 (S)

E_6 の $G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ への作用は推移的である．とくに，$G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}}) \cong E_6/Sp(1)\cdot SU(6)$ であり，$G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ は *EII* 型対称空間．

- *EII* 型対称空間の二つの実現 $M_1^+ = \bigcup_{g\in E_6} g\sigma g^{-1}$ と $G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ を得た．
- M_1^+ と $G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ の対応として次がある．

$$M_1^+ \ni p \quad \mapsto \quad V_p^{+1} \in G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}}).$$

  ただし，V_p^{+1} は p の $+1$ 固有空間．
- この対応による M_1^+ の極大対蹠集合 A_i^{II} $(i=1,2)$ の像も同じ記号 A_i^{II} で記す．

$G_4(\mathbb{O})$ の 3 対原理

- $G_k(\mathbb{O})$ で $\mathbb{O}$ の k 次元部分空間全体を表す．

命題 3.6 (S)

任意の $V_1 \in G_4(\mathbb{O})$ に対して，$V_2, V_3 \in G_4(\mathbb{O})$ で次を満たすものが存在する．

$$\overline{v_i v_{i+1}} \in V_{i+2} \qquad (\forall v_i \in V_i,\quad \text{添え字は mod 3})$$

さらに，このような V_2, V_3 は (V_2, V_3) と $(V_2^{\perp}, V_3^{\perp})$ のみである．

- (V_1, V_2, V_3) が $\overline{v_i v_{i+1}} \in V_{i+2}$ を満たすとき，３対原理を満たすという．
- 四元数 $\mathbb{H}$ が八元数 $\mathbb{O}$ の部分代数であることから，$(\mathbb{H}, \mathbb{H}, \mathbb{H})$ は３対原理を満たしている．
- $G_t(\mathbb{O}^3) := \{(V_1, V_2, V_3) \in G_4(\mathbb{O})^3\ ;\ (V_1, V_2, V_3)$ は３対原理を満たす $\}$．
  $G_t(\mathbb{O}^3)$ を３対グラスマン多様体と呼ぶ．

- 任意の $g = (g_1, g_2, g_3) \in Spin(8)$ および $V = (V_1, V_2, V_3) \in G_t(\mathbb{O}^3)$ に関して，$g(V) = (g_1 V_1, g_2 V_2, g_3 V_3)$ と定めると，

$$\overline{(g_1 v_1)(g_2 v_2)} = g_3(\overline{v_1 v_2}) \in g_3(V_3)$$

であるので，$g(V) \in G_t(\mathbb{O}^3)$．とくに，$Spin(8)$ は $G_t(\mathbb{O}^3)$ へ作用している．

命題 3.7 (S)

$Spin(8)$ による $G_t(\mathbb{O}^3)$ への作用は推移的である．とくに，各点におけるイソトロピー群は $Spin(4) \times Spin(4)/\mathbb{Z}_2$ と同型となり，$G_t(\mathbb{O}^3) \cong Spin(8)/(Spin(4) \times Spin(4)/\mathbb{Z}_2)$.

- $G_k^{\mathrm{OR}}(\mathbb{R}^n)$ を $\mathbb{R}^n$ の k 次元向き付き部分空間全体とすると，$G_t(\mathbb{O}^3) \cong G_4^{\mathrm{OR}}(\mathbb{R}^8)$.

$G_6^\tau(\mathbb{O}^\mathbb{C}) \times G_4^\perp(\mathbb{O}^\mathbb{C})^+ \times G_4^\perp(\mathbb{O}^\mathbb{C})^-$ の 3 対原理

次のように記号を定める．

- $G_k^\mathbb{C}(\mathbb{O}^\mathbb{C})$: $\mathbb{O}^\mathbb{C}$ の複素 k 次元部分空間全体によるグラスマン多様体
- τ : $\mathbb{O}^\mathbb{C}$ における $\mathbb{O}$ に関する共役

  $G_k^\tau(\mathbb{O}^\mathbb{C}) := \{V \in G_k^\mathbb{C}(\mathbb{O}^\mathbb{C})\ ;\ \tau(V) \subset V\} \cong SO(8)/S(O(k) \times O(8-k))$
- $G_4^\perp(\mathbb{O}^\mathbb{C}) = \{V \in G_4^\mathbb{C}(\mathbb{O}^\mathbb{C})\ ;\ \tau(V) \perp V\}$

  $G_4^\perp(\mathbb{O}^\mathbb{C})^+$: $G_4^\perp(\mathbb{O}^\mathbb{C})$ の L_1 の $\pm i$ 固有空間を含む連結成分 ($\cong SO(8)/U(4)$)

  $G_4^\perp(\mathbb{O}^\mathbb{C})^-$: $G_4^\perp(\mathbb{O}^\mathbb{C})$ の R_1 の $\pm i$ 固有空間を含む連結成分 ($\cong SO(8)/U(4)$)

命題 3.8 (S)

任意の $V_1 \in G_6^{\mathbb{C}}(\mathbb{O}^{\mathbb{C}})$ に対して，$V_2 \in G_4^{\perp}(\mathbb{O}^{\mathbb{C}})^+, V_3 \in G_4^{\perp}(\mathbb{O}^{\mathbb{C}})^-$ で

$$\tau(\overline{v_i v_{i+1}}) \in V_{i+2} \qquad (\forall v_i \in V_i,\quad \text{添え字は mod 3})$$

を満たすものが存在する．さらにこのような V_2, V_3 は (V_2, V_3) と $(V_2^{\perp}, V_3^{\perp})$ のみである．

- 上記を満たす (V_1, V_2, V_3) を複素３対原理を満たすという．
- $(T_1, L_1, R_1) \in Spin(8)$ に関して，

$$V_1 := \{X \in \mathbb{O}^{\mathbb{C}} \ ; \ T_1(X) = X\} = \bigoplus_{k=2}^{7} \mathbb{C}e_k,$$
$$V_2 := \{X \in \mathbb{O}^{\mathbb{C}} \ ; \ L_1(X) = iX\} = \mathbb{C}(e_0 - ie_1) \oplus \bigoplus_{j=2,4,6} \mathbb{C}(e_j - ie_{j+1}),$$
$$V_3 := \{X \in \mathbb{O}^{\mathbb{C}} \ ; \ R_1(X) = -iX\} = \mathbb{C}(e_0 + ie_1) \oplus \bigoplus_{j=2,4,6} \mathbb{C}(e_j - ie_{j+1}),$$

とすれば，$(V_1, V_2, V_3), (V_1, V_2^{\perp}, V_3^{\perp})$ は複素３対原理を満たしている．

- $G_t^6((\mathbb{O}^{\mathbb{C}})^3) := \Big\{(V_1, V_2, V_3) \in G_6^{\tau}(\mathbb{O}^{\mathbb{C}}) \times G_4^{\perp}(\mathbb{O}^{\mathbb{C}})^+ \times G_4^{\perp}(\mathbb{O}^{\mathbb{C}})^-$; (V_1, V_2, V_3) は複素３対原理を満たす $\Big\}$

  $G_t^6((\mathbb{O}^{\mathbb{C}})^3)$ を複素３対グラスマン多様体と呼ぶ．
- 任意の $g = (g_1, g_2, g_3) \in Spin(8)$ および $(V_1, V_2, V_3) \in G_t^6((\mathbb{O}^{\mathbb{C}})^3)$ に関して，$g(V) = (g_1V_1, g_2V_2, g_3V_3)$ と定めると，

$$\tau(\overline{(g_1v_1)(g_2v_2)}) = \tau(g_3(\overline{v_1v_2})) = g_3(\tau(\overline{v_1v_2})) \in g_3(V_3)$$

  となり，$g(V) \in G_t^6((\mathbb{O}^{\mathbb{C}})^3)$．とくに，$Spin(8)$ は $G_t^6((\mathbb{O}^{\mathbb{C}})^3)$ へ作用する．

命題 3.9 (S)

$Spin(8)$ は $G_t^6((\mathbb{O}^{\mathbb{C}})^3)$ へ推移的に作用する．さらに，各点におけるイソトロピー群は $Spin(6) \times Spin(2)/\mathbb{Z}_2$ となり，$G_t^6((\mathbb{O}^{\mathbb{C}})^3) \cong Spin(8)/(Spin(6) \times Spin(2)/\mathbb{Z}_2)$．

- とくに，$G_t^6((\mathbb{O}^{\mathbb{C}})^3) \cong G_6^{\mathrm{OR}}(\mathbb{R}^8)$ となる．

- $G_t(\mathbb{O}^3) \cong G_4^{\mathrm{OR}}(\mathbb{R}^8)$ から *EII* 型対称空間 $G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ への全測地的埋め込み

$$f : G_t(\mathbb{O}^3) \to G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})\ ;\ (V_1, V_2, V_3) \mapsto \left\{ \begin{pmatrix} x_1 & v_3 & \bar{v}_2 \\ \bar{v}_3 & x_2 & v_1 \\ v_2 & \bar{v}_1 & x_3 \end{pmatrix} ;\ x_i \in \mathbb{C},\ \begin{matrix} v_1 \in V_1, \\ v_2 \in V_2, \\ v_3 \in V_3 \end{matrix} \right\}$$

- $G_t^6((\mathbb{O}^{\mathbb{C}})^3) \cong G_6^{\mathrm{OR}}(\mathbb{R}^8)$ から *EII* 型対称空間 $G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ への全測地的埋め込み

$$g_1 : G_t^6((\mathbb{O}^{\mathbb{C}})^3) \to G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})\ ;\ (V_1, V_2, V_3) \mapsto \left\{ \begin{pmatrix} x & v_3 & \bar{v}_2 \\ \bar{v}_3 & 0 & v_1 \\ v_2 & \bar{v}_1 & 0 \end{pmatrix} ;\ x \in \mathbb{C},\ \begin{matrix} v_1 \in V_1, \\ v_2 \in V_2, \\ v_3 \in V_3 \end{matrix} \right\},$$

$$g_2 : G_t^6((\mathbb{O}^{\mathbb{C}})^3) \to G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})\ ;\ (V_1, V_2, V_3) \mapsto \left\{ \begin{pmatrix} 0 & v_2 & \bar{v}_1 \\ \bar{v}_2 & x & v_3 \\ v_1 & \bar{v}_3 & 0 \end{pmatrix} ;\ x \in \mathbb{C},\ \begin{matrix} v_1 \in V_1, \\ v_2 \in V_2, \\ v_3 \in V_3 \end{matrix} \right\},$$

$$g_3 : G_t^6((\mathbb{O}^{\mathbb{C}})^3) \to G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})\ ;\ (V_1, V_2, V_3) \mapsto \left\{ \begin{pmatrix} 0 & v_1 & \bar{v}_3 \\ \bar{v}_1 & 0 & v_2 \\ v_3 & \bar{v}_2 & x \end{pmatrix} ;\ x \in \mathbb{C},\ \begin{matrix} v_1 \in V_1, \\ v_2 \in V_2, \\ v_3 \in V_3 \end{matrix} \right\},$$

- $G_6^{\mathrm{OR}}(\mathbb{R}^8)$ の極大対蹠集合 B を次で定める．

$$B = \left\{ \begin{matrix} \pm\bigoplus_{k=2,4,6}(\mathbb{C}e_k \oplus \mathbb{C}e_{k+1}), & \pm\bigoplus_{k=0,4,6}(\mathbb{C}e_k \oplus \mathbb{C}e_{k+1}), \\ \pm\bigoplus_{k=0,2,6}(\mathbb{C}e_k \oplus \mathbb{C}e_{k+1}), & \pm\bigoplus_{k=0,2,4}(\mathbb{C}e_k \oplus \mathbb{C}e_{k+1}), \end{matrix} \right\}$$

- $B' := \{\pm(V_1 \cap V_2)\ ;\ V_1, V_2 \in B,\ \dim(V_1 \cap V_2) = 4\} \subset G_4^{\mathrm{OR}}(\mathbb{R}^8)$ とおく．
  B' は $G_4^{\mathrm{OR}}(\mathbb{R}^8)$ の対蹠集合となる．（極大ではない）
- $A : G_4^{\mathrm{OR}}(\mathbb{R}^8)$ を極大対蹠集合とする．

定理 3.10 (S)

$G_{\mathbb{H}}(\mathfrak{J}^{\mathbb{C}})$ の極大対蹠集合 A_i^{II} $(i = 1, 2)$ はそれぞれ $f(B') \cup \bigcup_{i=1}^{3} g_i(B), f(A)$ と合同になる．

本日のまとめ

- E_6 および *EII* 型の極大対蹠集合の分類・構成を紹介した.
- E_6 の極大対蹠集合の合同類は 2 種類
  - 極大トーラスの極大対蹠集合
  - 部分群 F_4 の極大対蹠集合
- E_6 では $Spin(8)$ を用いた極大対蹠集合の構成も紹介した.
- *EII* 型の極大対蹠集合の合同類は 2 種類
- *EII* 型では有向実グラスマン多様体を用いた極大対蹠集合の構成を紹介した.

ご清聴ありがとうございました！

Maximal antipodal sets of compact symmetric spaces

Hiroyuki Tasaki

Abstract. We explain our joint study with Makiko Sumi Tanaka on the classification of maximal antipodal sets in compact symmetric spaces.

In this note, we limit our explanation to the achievements related to the unitary group $U(n)$. Refer to [2], [3], [4], [5] and our future publications for the results in the other cases.

1 Definitions and Notations

Let M be a compact Riemannian symmetric space. We denote by s_x the geodesic symmetry at x in M. A subset S of M is called an *antipodal set*, if $s_x(y) = y$ for any $x, y \in S$. We define

$$\#_2 M = \max\{|A| \mid A : \text{antipodal in}\, M\}.$$

If an antipodal set A in M attains $\#_2 M$, we call A a *great* antipodal set. If an antipodal set A is strictly included in no antipodal set, we call A a *maximal* antipodal set. A great antipodal set is a maximal antipodal set, however a maximal antipodal set is not a great antipodal set in general. These are introduced by Chen-Nagano in [1].

Let X be a set and $f : X \to X$ be a map. We define the fixed point set by

$$F(f, X) = \{x \in X \mid f(x) = x\}.$$

In a compact Riemannian symmetric space M we take a point o. We call each connected component of $F(s_o, M)$ a *polar* and a polar consisting of a single point a *pole*. It is known that a polar is a totally geodesic submanifold, in particular it is a compact symmetric space. If p is a pole of o, we denote by $C(o, p)$ the set of the midpoints of geodesics joining o and p. We call $C(o, p)$ the *centrosome* for o and p and each connected component of $C(o, p)$ a *centriole*. It is known that a centriole is a totally geodesic submanifold, so it is a compact symmetric space. Polars and centrioles are orbits of the isotropy action and it is useful for the classification of maximal antipodal sets of compact symmetric spaces realized as polars or centrioles. Taking advantage of these, we have been conducting research on the classification.

This work was supported by JSPS KAKENHI Grant Number JP21K03218.

2 Compact Lie groups

Let G be a compact Lie group. There exists a biinvariant Riemannian metric on G and G is a Riemannian symmetric space with respect to a biinvariant Riemannian metric. In the identity component of G, the geodesic symmetry is described as

$$s_x(y) = xy^{-1}x.$$

The right hand side has meaning not only in the identity component but in the whole G. So this is adopted as the geodesic symmetry in the whole G. We denote by e the identity element of G. If A is a maximal antipodal set containing e, then A is a subgroup of G. Moreover A is isomorphic to $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. It is enough to classify maximal antipodal subgroups in compact Lie groups.

In [2] we classified maximal antipodal subgroups of some compact classical Lie groups. We define some symbols in order to describe the classification of maximal antipodal subgroups. The set of all diagonal matrices of degree n whose diagonal components are ± 1 is denoted by Δ_n. We can see that Δ_n is a maximal antipodal subgroup of $O(n)$. We denote by 1_n the indentity matrix of degree n. We define

$$I_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$D[4] = \{\pm 1_2, \pm I_1, \pm J_1, \pm K_1\} : \text{the dihedral group.}$$

For a natural number n we decompose it into $n = 2^k \cdot l$, where l is an odd number. For $0 \leq s \leq k$ we define

$$\begin{aligned} D(s,n) &= \{d_1 \otimes \cdots \otimes d_s \otimes d_0 \mid d_i \in D[4]\,(1 \leq i \leq s), d_0 \in \Delta_{n/2^s}\}, \\ PD(s,n) &= \{d \in D(s,n) \mid d^2 = 1_n\}, \\ ND(s,n) &= \{d \in D(s,n) \mid d^2 = -1_n\}. \end{aligned}$$

The following theorem is easy to prove.

Theorem 2.1. *A maximal antipodal subgroup of $U(n)$ is $U(n)$-conjugate to Δ_n.*

We take a natural number μ and put $\mathbb{Z}_\mu = \{z 1_n \mid z^\mu = 1\}$. Let θ be a primitive 2μ-th root of 1. We denote by $\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$ the natural projection.

Theorem 2.2 ([2] Theorem 5.1)**.** *For a natural number n we decompose it into $n = 2^k \cdot l$, where l is an odd number. A maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ is $U(n)/\mathbb{Z}_\mu$-conjugate to one of the following.*

(1) In the case where n or μ is odd,

$$\pi_n(\{1, \theta\} D(0,n)) = \pi_n(\{1, \theta\}\Delta_n).$$

(2) In the case where both n and μ are even,

$$\pi_n(\{1, \theta\} D(s,n)) \quad (0 \leq s \leq k),$$

where the case $(s,n) = (k-1, 2^k)$ is excluded.

3 Maximal antipodal sets of polars

Our policy of classifying the maximal antipodal sets in a compact symmetric space realized as a polar in a compact Lie group is as follows. Let G be a compact Lie group and M be a polar of the identity element e in G. We take an antipodal set A of M. Then $A \cup \{e\}$ is an antipodal set of G because $A \subset M \subset F(s_e, G)$. We can take a maximal antipodal subgroup $\tilde{A}$ of G which contains $A \cup \{e\}$. Furthermore, if A is a maximal antipodal set of M, we obtain $A = M \cap \tilde{A}$. We denote by G_0 the identity component of G and let $B_0, \dots, B_k$ be representatives of each G_0-conjugacy class of maximal antipodal subgroups of G. Then $\tilde{A}$ is G_0-conjugate to one of them and there exist $s \in \{0, \dots, k\}$ and $g \in G_0$ such that $\tilde{A} = I_g(B_s)$, where $I_g(x) = gxg^{-1}$. Since M is an orbit of the conjugate action of G_0,

$$A = M \cap \tilde{A} = M \cap I_g(B_s) = I_g(M \cap B_s).$$

It is known that $I_0(M) = \{I_g|_M \mid g \in G_0\}$, so A is $I_0(M)$-congruent to $M \cap B_s$. This implies that a representative of $I_0(M)$-congruence class of maximal antipodal sets of M is one of the following:

$$M \cap B_0, \dots, M \cap B_k.$$

We check which of these are actually maximal antipodal sets.

4 Complex Grassmann manifolds and their quotient spaces

For $1 \leq k \leq n-1$ we denote by $G_k(\mathbb{C}^n)$ the Grassmann manifold consisting of k dimensional subspaces in $\mathbb{C}^n$. We can imbed $G_k(\mathbb{C}^n)$ in $U(n)$ by $G_k(\mathbb{C}^n) \ni V \mapsto 1_V - 1_{V^\perp} \in U(n)$, whose image is a polar.

Theorem 4.1. *A maximal antipodal set of $G_k(\mathbb{C}^n)$ is $U(n)$-congruent to $G_k(\mathbb{C}^n) \cap \Delta_n = \{\langle \boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_k} \rangle \mid 1 \leq i_1 < \cdots < i_k \leq n\}$, where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is the standard unitary basis of $\mathbb{C}^n$ and $\langle \boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_k} \rangle$ is the subspace spanned by $\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_k}$.*

In the quotient group $U(2m)^* = U(2m)/\{\pm 1_{2m}\}$ the quotient space $G_m(\mathbb{C}^{2m})^* = G_m(\mathbb{C}^{2m})/\{\pm 1_{2m}\}$ is a polar. Using the classification of maximal antipodal subgroups of $U(2m)^*$ we can obtain the following theorem.

Theorem 4.2. *For a natural number m we decompose $2m$ into $2m = 2^k \cdot l$, where l is an odd number. A maximal antipodal set of $G_m(\mathbb{C}^{2m})^*$ is $U(2m)^*$-congruent to one of the following.*

$$\begin{aligned} &\pi_{2m}(\{d_1 \otimes \cdots \otimes d_s \otimes d_0 \in PD(s, 2m) \mid \exists d_i \ (0 \leq i \leq s) \ \mathrm{Tr} d_i = 0\} \\ &\quad \cup \sqrt{-1} ND(s, 2m)) \qquad (0 \leq s \leq k), \end{aligned}$$

where the case of $(s, 2m) = (k-1, 2^k)$ is excluded and when $m = 2$, $\pi_4(\{d_0 \in PD(0, 4) \mid \mathrm{Tr} d_0 = 0\})$ is also excluded.

5 A maximal torus of a disconnected compact Lie group

Let G be a connected compact Lie group and T be a maximal torus of G. It is known that

$$G = \bigcup_{g \in G} gTg^{-1}.$$

Let σ be an involutive automorphism of G and denote by $\langle \sigma \rangle$ the subgroup of $\mathrm{Aut}(G)$ generated by σ. The semidirect product $G \rtimes \langle \sigma \rangle$ has two connected components $(G, 1)$ and (G, σ). We take a maximal torus T' of $F(\sigma, G)$. As an application of Hermann action we obtain

$$(G, \sigma) = \bigcup_{g \in G} g(T', \sigma)g^{-1}.$$

This description is useful for treating the connected component (G, σ).

6 $UI(n)$ and its quotient spaces

We define an involutive automorphim σ_I of $U(n)$ by

$$\sigma_I(g) = \bar{g} \qquad (g \in U(n))$$

and define a compact symmetric space $UI(n)$ by

$$UI(n) = \{g \in U(n) \mid \sigma_I(g) = g^{-1}\} \cong U(n)/O(n).$$

$UI(n)$ is not realized in any connected compact Lie group as a polar, however it is realized in the semidirect product $U(n) \rtimes \langle \sigma_I \rangle$ as a polar. We denote by $\hat{e}$ the identity element of $U(n) \rtimes \langle \sigma_I \rangle$. We have

$$F(s_{\hat{e}}, U(n) \rtimes \langle \sigma_I \rangle) = \bigcup_{0 \leq r \leq n} (G_r(\mathbb{C}^n), 1) \cup (UI(n), \sigma_I).$$

Using these we obtain the following theorems.

Theorem 6.1. *A maximal antipodal subgroup of $U(n) \rtimes \langle \sigma_I \rangle$ is $U(n)$-conjugate to $\Delta_n \rtimes \langle \sigma_I \rangle$.*

Theorem 6.2. *A maximal antipodal set of $UI(n)$ is $U(n)$-congruent to Δ_n.*

We denote by $\pi_n : U(n) \rtimes \langle \sigma_I \rangle \to U(n) \rtimes \langle \sigma_I \rangle / \mathbb{Z}_\mu$ the natural projection.

Theorem 6.3. *For a natural number n we decompose it into $n = 2^k \cdot l$, where l is an odd number. A maximal antipodal subgroup of $U(n) \rtimes \langle \sigma_I \rangle / \mathbb{Z}_\mu$ is $U(n)/\mathbb{Z}_\mu$-conjugate to one of the following.*

(1) In the case where μ is odd, $\pi_n(\Delta_n \rtimes \langle \sigma_I \rangle)$.

(2) *In the case where* μ *is even,* $\pi_n(\{1,\theta\}D(s,n) \rtimes \langle\sigma_I\rangle)$ $(0 \leq s \leq k)$, *where the case* $(s,n) = (k-1, 2^k)$ *is excluded.*

Theorem 6.4. *For a natural number* n *we decompose it into* $n = 2^k \cdot l$, *where* l *is an odd number. A maximal antipodal set of* $UI(n)/\mathbb{Z}_\mu$ *is* $U(n)/\mathbb{Z}_\mu$*-congruent to one of the following.*

(1) *In the case where* μ *is odd,* $\pi_n(\Delta_n)$.

(2) *In the case where* μ *is even,* $\pi_n(\{1,\theta\}PD(s,n))$ $(0 \leq s \leq k)$, *where the case* $(s,n) = (k-1, 2^k)$ *is excluded.*

7 $UII(n)$ and its quotient spaces

We set $J_n = J_1 \otimes 1_n \in SO(2n)$. We define an involutive automorphim σ_{II} of $U(2n)$ by

$$\sigma_{II}(g) = J_n \bar{g} J_n^{-1} \qquad (g \in U(2n))$$

and define a compact symmetric space $UII(n)$ by

$$UII(n) = \{g \in U(2n) \mid \sigma_{II}(g) = g^{-1}\} \cong U(2n)/Sp(n).$$

$UII(n)$ is not realized in any connected compact Lie group as a polar, however it is realized in the semidirect product $U(2n) \rtimes \langle\sigma_{II}\rangle$ as a polar. We denote by $\hat{e}$ the identity element of $U(2n) \rtimes \langle\sigma_{II}\rangle$. We have

$$F(s_{\hat{e}}, U(2n) \rtimes \langle\sigma_{II}\rangle) = \bigcup_{0 \leq r \leq 2n} (G_r(\mathbb{C}^{2n}), 1) \cup (UII(n), \sigma_{II}).$$

Using these we obtain the following theorems.

Theorem 7.1. *A maximal antipodal subgroup of* $U(2n) \rtimes \langle\sigma_{II}\rangle$ *is* $U(2n)$*-conjugate to* $(\Delta_{2n}, 1)$ *or* $(1_2 \otimes \Delta_n) \rtimes \langle\sigma_{II}\rangle$.

Theorem 7.2. *A maximal antipodal set of* $UII(n)$ *is* $U(2n)$*-congruent to* $1_2 \otimes \Delta_n$.

We set

$$E(n) = \left\{ \begin{bmatrix} 0 & d_1 \\ d_2 & 0 \end{bmatrix} \middle| d_1, d_2 \in \Delta_n \right\} \subset O(2n).$$

We denote by $\pi_{2n} : U(2n) \rtimes \langle\sigma_{II}\rangle \to U(2n) \rtimes \langle\sigma_{II}\rangle/\mathbb{Z}_\mu$ the natural projection.

Theorem 7.3. *For a natural number* n *we decompose it into* $n = 2^k \cdot l$, *where* l *is an odd number. A maximal antipodal subgroup of* $U(2n) \rtimes \langle\sigma_{II}\rangle/\mathbb{Z}_\mu$ *is* $U(2n)/\mathbb{Z}_\mu$*-conjugate to one of the following.*

(1) *In the case where* μ *is odd,* $\pi_{2n}(\Delta_{2n}, 1)$, $\pi_{2n}((1_2 \otimes \Delta_n) \rtimes \langle\sigma_{II}\rangle)$.

(2) In the case where μ is even,

$$\pi_{2n}((\{1,\theta\}\Delta_{2n},1)\cup(\{1,\theta\}E(n),\sigma_{II})),$$
$$\pi_{2n}(\{1,\theta\}D(s+1,2n)\rtimes\langle\sigma_{II}\rangle)\quad(0\le s\le k),$$

where the case $(s,n)=(k-1,2^k)$ is excluded.

Theorem 7.4. *For a natural number n we decompose it into $n=2^k\cdot l$, where l is an odd number. A maximal antipodal of $UII(n)/\mathbb{Z}_\mu$ is $U(2n)/\mathbb{Z}_\mu$-congruent to one of the following.*

(1) In the case where μ is odd, $\pi_{2n}(1_2\otimes\Delta_n)$.

(2) In the case where μ is even,

$$\pi_{2n}(\{1,\theta\}(1_2\otimes PD(s,n)\cup\{I_1,J_1,K_1\}\otimes ND(s,n)))\quad(0\le s\le k),$$

where the case $(s,n)=(k-1,2^k)$ is excluded.

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コンパクト対称空間の極大対蹠集合

Maximal antipodal sets of compact symmetric spaces

田崎博之
(筑波大学)

田中真紀子さんとの共同研究

カンドルと対称空間
2021 年 11 月 25 日

定義 (Chen-Nagano 1988)

M : コンパクト Riemann 対称空間

s_x : $x \in M$ における点対称

$S \subset M$: **対蹠集合**

$\overset{\text{def}}{\Leftrightarrow}$ 任意の $x, y \in S$ について $s_x(y) = y$

$\#_2 M = \max\{|A| \mid A : \text{対蹠集合} \subset M\}$

A : **大対蹠集合** $\overset{\text{def}}{\Leftrightarrow} |A| = \#_2 M$

A : **極大対蹠集合**

$\overset{\text{def}}{\Leftrightarrow}$ 「$A \subset A'$: 対蹠集合 $\Rightarrow A = A'$」

大対蹠集合 $\Rightarrow$ 極大対蹠集合

極大対蹠集合 $\not\Rightarrow$ 大対蹠集合 (一般には)

写像 $f : X \to X$ に対して

$F(f, X) = \{x \in X \mid f(x) = x\}$

$o \in M$, $F(s_o, M)$ **の各連結成分：極地**

一点から成る極地：極

極地：全測地的部分多様体 (対称空間)

$o \in A$ **：対蹠集合** $\Rightarrow A \subset F(s_o, M)$

o **の極** p **に対して**

　$C(o, p)$:o **と** p **を結ぶ測地線の中点全体**

$C(o, p)$ **：中心体**

$C(o, p)$ **の各連結成分：中心小体**

中心小体：全測地的部分多様体 (対称空間)

3

対称 R **空間の大対蹠集合とトポロジーの関連性**

　：成果がある

一般のコンパクト対称空間の極大対蹠集合

　：情報があまりない

極地、中心小体：イソトロピー部分群の軌道

極地、中心小体に関する情報

　$>$ **対蹠集合に関する情報**

極地、中心小体を利用して対蹠集合を研究する

4

対称 R 空間の大対蹠集合の例

$\boldsymbol{x \in S^n(r)}$ $\boldsymbol{F(s_x, S^n(r)) = \{\pm x\}}$
$\boldsymbol{\{\pm x\}}$ **:** $\boldsymbol{S^n(r)}$ **の大対蹠集合**

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $\boldsymbol{x \in P^n(\mathbb{K})}$ **: 射影空間**
$\boldsymbol{F(s_x, P^n(\mathbb{K})) = \{x\} \cup P^{n-1}(\mathbb{K})}$
$\boldsymbol{e_1, \ldots, e_{n+1}}$ **:** $\mathbb{K}^{n+1}$ **の正規直交** $\mathbb{K}$ **基底**
$\boldsymbol{\{\mathbb{K}e_1, \ldots, \mathbb{K}e_{n+1}\}}$ **:** $\boldsymbol{P^n(\mathbb{K})}$ **の大対蹠集合**

5

コンパクト Lie 群の場合
両側不変 Riemann 計量 $\rightarrow$ 対称空間
点対称 $\boldsymbol{s_x(y) = xy^{-1}x}$ **: 代数的記述**
コンパクト Lie 群を対称空間として考えることにより、その代数的構造を幾何学的観点から調べることができる。

6

G : コンパクト Lie 群、 e : 単位元

A : e を含む極大対蹠集合

　$\Rightarrow$ A : 可換部分群

e 以外の A の各元の位数は 2

　$\Rightarrow$ $A \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$

2 を一般の素数 p に変えた部分群の研究

　: Borel-Serre 1953

$p = 2$ の場合に限って対称空間に一般化

　: Chen-Nagano 1988

古典型コンパクト対称空間の

極大対蹠集合の分類の方針

古典型コンパクト Lie 群

　分類結果の具体的記述に後で言及

古典型コンパクト Lie 群内に中心小体、

極地として実現できるもの

　古典型コンパクト Lie 群の極大対蹠部分群

　の分類を利用して、それらとの共通部分と

　して極大対蹠集合を記述

スピン群、セミスピン群、有向実 Grassmann

多様体を除けば、上記の方針を適用できる

極大対蹠集合の分類結果を記述するための記号

Δ_n : 対角線に ± 1 が並ぶ対角行列 $\subset O(n)$.

$$I_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$D[4] = \{\pm 1_2, \pm I_1, \pm J_1, \pm K_1\}$: 二面体群

自然数 n を $n = 2^k \cdot l$ と 2 の冪と奇数の積に分解する。
$0 \le s \le k$ に対して

$$D(s,n) = \{d_1 \otimes \cdots \otimes d_s \otimes d_0 \mid d_i \in D[4] \, (1 \le i \le s), d_0 \in \Delta_{n/2^s}\}$$

と定める。

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たとえば

$$J_1 \otimes d_0 = \begin{bmatrix} 0 & -d_0 \\ d_0 & 0 \end{bmatrix},$$

$$I_1 \otimes K_1 \otimes d_0 = \begin{bmatrix} 0 & -d_0 & 0 & 0 \\ -d_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_0 \\ 0 & 0 & d_0 & 0 \end{bmatrix}.$$

$d \in D(s,n)$ は $d^2 = \pm 1_n$ を満たす。

$$PD(s,n) = \{d \in D(s,n) \mid d^2 = 1_n\},$$
$$ND(s,n) = \{d \in D(s,n) \mid d^2 = -1_n\}$$

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ユニタリ群に関連したコンパクト対称空間の極大対蹠集合の分類結果

定理 $U(n)$ の極大対蹠部分群は Δ_n に共役

μ : 自然数　$\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$: 自然な射影
θ : 1 の原始 2μ 乗根
n を $n = 2^k \cdot l$ と 2 の冪と奇数の積に分解

定理 $U(n)/\mathbb{Z}_\mu$ の極大対蹠部分群は次のいずれかに共役

(1) n または μ が奇数のとき、$\pi_n(\{1, \theta\}\Delta_n)$.

(2) n と μ がともに偶数のとき、
$\pi_n(\{1, \theta\}D(s, n)) \quad (0 \leq s \leq k)$,
ただし、$(s, n) = (k-1, 2^k)$ の場合は除外.

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G : コンパクト Lie 群
M : G の極地
$\{B_i\}$: G の極大対蹠部分群の分類結果
$\Rightarrow \{M \cap B_i\}$: M の極大対蹠集合の分類結果の候補
各 $M \cap B_i$ が M の極大対蹠集合かどうか確認

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$1 \le k \le n-1$
$G_k(\mathbb{C}^n)$: $\mathbb{C}^n$ 内の k 次元複素部分空間全体
$G_k(\mathbb{C}^n) \ni V \mapsto 1_V - 1_{V^\perp} \in U(n)$: 像は極地

定理 $G_k(\mathbb{C}^n)$ の極大対蹠集合は
$G_k(\mathbb{C}^n) \cap \Delta_n$
$= \{\langle e_{i_1}, \dots, e_{i_k}\rangle \mid 1 \le i_1 < \cdots < i_k \le n\}$
に合同.
ただし、$e_1, \dots, e_n$ は $\mathbb{C}^n$ の標準的ユニタリ基底.

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$U(2m)^* = U(2m)/\{\pm 1_{2m}\}$ 内で
$G_m(\mathbb{C}^{2m})^* = G_m(\mathbb{C}^{2m})/\{\pm 1_{2m}\}$ は極地
$U(2m)^*$ の極大対蹠部分群の分類を利用

定理 商空間 $G_m(\mathbb{C}^{2m})^*$ の極大対蹠集合は次のいずれかに $U(2m)^*$ 合同

$$\begin{aligned}\pi_{2m}(\{d_1 \otimes \cdots \otimes d_s \otimes d_0 \in PD(s, 2m) \mid \\ \exists d_i\ (0 \le i \le s)\ \mathrm{Tr} d_i = 0\} \\ \cup \sqrt{-1} ND(s, 2m)) \qquad (0 \le s \le k),\end{aligned}$$

ただし、$(s, 2m) = (k-1, 2^k)$ の場合は除外、$m = 2$ のときは $\pi_4(\{d_0 \in PD(0,4) \mid \mathrm{Tr} d_0 = 0\})$ も除外.

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$U(n)$ の対合的自己同型写像 σ_I :

$$\sigma_I(g) = \bar{g} \qquad (g \in U(n))$$

$$UI(n) = \{g \in U(n) \mid \sigma_I(g) = g^{-1}\} \cong U(n)/O(n)$$

これは連結コンパクト Lie 群の極地として実現できない

半直積 $U(n) \rtimes \langle\sigma_I\rangle = (U(n), 1) \cup (U(n), \sigma_I)$

：連結成分への分解

$\hat{e}$: $U(n) \rtimes \langle\sigma_I\rangle$ の単位元

$$F(s_{\hat{e}}, U(n) \rtimes \langle\sigma_I\rangle) = \bigcup_{0 \le r \le n} (G_r(\mathbb{C}^n), 1) \cup (UI(n), \sigma_I)$$

定理 $U(n) \rtimes \langle\sigma_I\rangle$ の極大対蹠部分群は $\Delta_n \rtimes \langle\sigma_I\rangle$ に共役

定理 $UI(n)$ の極大対蹠集合は Δ_n に合同

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G : 連結コンパクト Lie 群

T : G の極大トーラス　$G = \bigcup_{g \in G} gTg^{-1}$

σ : G の対合的自己同型写像

T' : σ の不動点部分群の極大トーラス

半直積 $G \rtimes \langle\sigma\rangle$ において

$$(G, \sigma) = \bigcup_{g \in G} g(T', \sigma)g^{-1}$$

Hermann 作用の応用例

これにより、(G, σ) における極地を求められる

簡単な例：$O(2) = SO(2) \cup \bigcup_{g \in SO(2)} g \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} g^{-1}$

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μ : 自然数　$\pi : U(n) \rtimes \langle \sigma_I \rangle \to U(n) \rtimes \langle \sigma_I \rangle / \mathbb{Z}_\mu$
θ : 1 の原始 2μ 乗根
n を $n = 2^k \cdot l$ と 2 の冪と奇数の積に分解

定理 $U(n) \rtimes \langle \sigma_I \rangle / \mathbb{Z}_\mu$ の極大対蹠部分群は次のいずれかに共役

(1) μ が奇数のとき、$\pi_n(\Delta_n \rtimes \langle \sigma_I \rangle)$.

(2) μ が偶数のとき、$\pi_n(\{1, \theta\} D(s, n) \rtimes \langle \sigma_I \rangle)$ $(0 \le s \le k)$. ただし、$(s, n) = (k - 1, 2^k)$ の場合は除外

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$(UI(n), \sigma_I) \subset U(n) \rtimes \langle \sigma_I \rangle$ より
$(UI(n)/\mathbb{Z}_\mu, \sigma_I) \subset U(n) \rtimes \langle \sigma_I \rangle / \mathbb{Z}_\mu$ が定まる
これは連結コンパクト Lie 群の極地として実現できない
$U(n) \rtimes \langle \sigma_I \rangle / \mathbb{Z}_\mu$ の極大対蹠部分群の分類を利用して次を得る

定理 $UI(n)/\mathbb{Z}_\mu$ の極大対蹠集合は次のいずれかに合同

(1) μ が奇数の場合、$\pi_n(\Delta_n)$.

(2) μ が偶数の場合、$\pi_n(\{1, \theta\} PD(s, n))$. ただし、$0 \le s \le k$ であり、$(s, n) = (k - 1, 2^k)$ の場合を除く。

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$J_n = J_1 \otimes 1_n \in SO(2n)$

$U(2n)$ の対合的自己同型写像 σ_{II} ：

$$\sigma_{II}(g) = J_n \bar{g} J_n^{-1} \qquad (g \in U(2n))$$

$UII(n) = \{g \in U(2n) \mid \sigma_{II}(g) = g^{-1}\} \cong U(2n)/Sp(n)$

これは連結コンパクト Lie 群の極地として実現できない

半直積 $U(2n) \rtimes \langle \sigma_{II} \rangle = (U(2n), 1) \cup (U(2n), \sigma_{II})$

：連結成分への分解

$\hat{e}$: $U(2n) \rtimes \langle \sigma_{II} \rangle$ の単位元

$$F(s_{\hat{e}}, U(2n) \rtimes \langle \sigma_{II} \rangle) = \bigcup_{0 \leq r \leq 2n} (G_r(\mathbb{C}^{2n}), 1) \cup (UII(n), \sigma_{II})$$

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定理 $U(2n) \rtimes \langle \sigma_{II} \rangle$ の極大対蹠部分群は $(\Delta_{2n}, 1)$ または $(1_2 \otimes \Delta_n) \rtimes \langle \sigma_{II} \rangle$ に共役

定理 $UII(n)$ の極大対蹠集合は $1_2 \otimes \Delta_n$ に合同

$$E(n) = \left\{ \begin{bmatrix} 0 & d_1 \\ d_2 & 0 \end{bmatrix} \middle| d_1, d_2 \in \Delta_n \right\} \subset O(2n).$$

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$n = 2^k \cdot l$ θ : 1 の原始 2μ 乗根

定理 $(U(2n) \rtimes \langle \sigma_{II} \rangle)/\mathbb{Z}_\mu$ の極大対蹠部分群は次のいずれかに共役

(1) μ が奇数の場合

$$\pi_{2n}(\Delta_{2n}, 1), \quad \pi_{2n}((1_2 \otimes \Delta_n) \rtimes \langle \sigma_{II} \rangle).$$

(2) μ が偶数の場合

$$\pi_{2n}((\{1, \theta\}\Delta_{2n}, 1) \cup (\{1, \theta\}E(n), \sigma_{II})),$$
$$\pi_{2n}(\{1, \theta\}D(s+1, 2n) \rtimes \langle \sigma_{II} \rangle) \quad (0 \leq s \leq k).$$

ただし、$(s, n) = (k-1, 2^k)$ の場合を除く。

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定理 $UII(n)/\mathbb{Z}_\mu$ の極大対蹠集合は次のいずれかに合同

(1) μ が奇数の場合、$\pi_{2n}(1_2 \otimes \Delta_n)$.

(2) μ が偶数の場合、

$$\pi_{2n}(\{1, \theta\}(1_2 \otimes PD(s, n) \cup \{I_1, J_1, K_1\} \otimes ND(s, n)))$$

ただし、$0 \leq s \leq k$ であり、$(s, n) = (k-1, 2^k)$ の場合を除く。

コンパクト対称空間をコンパクト Lie 群に埋め込む方法で研究を進め、多くのコンパクト対称空間の極大対蹠集合の分類が完了している

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f-twisted Alexander matrices of connected quandles

Yuta Taniguchi

Abstract. Ishii and Oshiro introduced the notion of an f-twisted Alexander matrix. They showed that the cokernel of an f-twisted Alexander matrix is an invariant of a pair of a finitely presentable quandle and a quandle homomorphism. We determine the cokernel of an f-twisted Alexander matrix for a connected quandle using a certain Alexander pair.

1 f-twisted Alexander matrices

Let $X = (X, *)$ be a quandle and R be a ring with unity 1. A pair $f = (f_1, f_2)$ of maps $f_1, f_2 : X^2 \to R$ is an *Alexander pair* if f_1 and f_2 satisfy the following conditions ([2]):

- For any $x \in X$, we have $f_1(x,x) + f_2(x,x) = 1$.
- For any $x, y \in X$, $f_1(x,y)$ is a unit of R.
- For any $x, y, z \in X$, we have

$$\begin{aligned} f_1(x*y,z)f_1(x,y) &= f_1(x*z,y*z)f_1(x,z),\\ f_1(x*y,z)f_2(x,y) &= f_2(x*z,y*z)f_1(y,z), \text{and}\\ f_2(x*y,z) &= f_1(x*z,y*z)f_2(x,z) + f_2(x*z,y*z)f_2(y,z). \end{aligned}$$

Let Q be a quandle with a finite presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$, X be a quandle, $\rho : Q \to X$ be a quandle homomorphism and $f = (f_1, f_2)$ be an Alexander pair. We can define the *f-twisted Alexander matrix* with respect to the presentation which is denoted by $A(Q, \rho; f_1, f_2)$ ([2]). The f-twisted Alexander matrix $A(Q, \rho; f_1, f_2)$ is the $m \times n$ matrix whose (i, j) component is an element of R. Let us define a linear map $\psi_{A(Q.\rho;f_1,f_2)} : R^m \to R^n$ by $\psi_{A(Q,\rho;f_1,f_2)}(\boldsymbol{a}) = \boldsymbol{a}A(Q, \rho; f_1, f_2)$ for any $\boldsymbol{a} \in R^m$. We denote the cokernel of $\psi_{A(Q,\rho;f_1,f_2)}$ by $\mathrm{Coker}(A(Q, \rho; f_1, f_2))$.

In [2], Ishii and Oshiro showed the following theorem.

Theorem 1.1 ([2])**.** *Let Q' be a quandle with a finite presentation $\langle x'_1, \ldots, x'_{n'} \mid r'_1, \ldots, r'_{m'} \rangle$. If there exists a quandle isomorphism $\psi : Q' \to Q$, then* $\mathrm{Coker}(A(Q, \rho; f_1, f_2))$ *is R-module isomorphic to* $\mathrm{Coker}(A(Q', \rho \circ \psi; f_1, f_2))$.

Thus, $\mathrm{Coker}(A(Q, \rho; f_1, f_2))$ is an invariant of a pair of a quandle and a quandle homomorphism.

This work was partly supported by JSPS KAKENHI Grant Number 21J21482.

2 Main result

Let X be a quandle and A be an abelian group. Here, the operation of an abelian group is written multiplicatively. A map $\theta : X^2 \to A$ is a *quandle 2-cocycle* if θ satisfies the following conditions ([1]):

- For any $x \in X$, we have $\theta(x, x) = e_A$, where e_A is the identity element of A.
- For any $x, y, z \in X$, we have $\theta(x * y, z)\theta(x, y) = \theta(x * z, y * z)\theta(x, z)$.

Let $\theta : X^2 \to A$ be a quandle 2-cocycle. We denote the linear extension of θ by the same symbol $\theta : \mathbb{Z}[X^2] \to A$, where $\mathbb{Z}[X^2]$ is a free abelian group whose basis is X^2. Then, the linear extension θ is a 2-cocycle of the quandle cochain complex which was introduced in [1]. Using a 2-cocycle, we have the group homomorphism from the *second quandle homology group* $H_2^Q(X; \mathbb{Z})$, which was defined in [1], to A. We also denote this group homomorphism by the same symbol $\theta : H_2^Q(X; \mathbb{Z}) \to A$.

Let $\mathbb{Z}[A]$ be the group ring over the integral domain $\mathbb{Z}$. Let us define maps $f_\theta, 0 : X^2 \to \mathbb{Z}[A]$ by

$$f_\theta(x, y) := 1 \cdot \theta(x, y), \quad 0(x, y) := 0.$$

Lemma 2.1 ([3]). *The pair $(f_\theta, 0)$ is an Alexander pair.*

We determine the $\mathbb{Z}[A]$-module $\mathrm{Coker}(A(Q, \rho; f_\theta, 0))$ for a connected quandle Q.

Theorem 2.2. *Let Q be a connected quandle with a finite presentation, X be a quandle, A be an abelian group, $\rho : Q \to X$ be a quandle homomorphism and $\theta : X^2 \to A$ be a quandle 2-cocycle. Then,* $\mathrm{Coker}(A(Q, \rho; f_\theta, 0))$ *is $\mathbb{Z}[A]$-module isomorphic to the quotient module $\mathbb{Z}[A]$ by the ideal generated by $\{1 \cdot x - 1 \cdot e_A \mid x \in \mathrm{Im}(\theta \circ \rho_*)\}$, where $\rho_* : H_2^Q(Q; \mathbb{Z}) \to H_2^Q(X; \mathbb{Z})$ is the induced homomorphism by ρ.*

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f-twisted Alexander matrices of connected quandles

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研究集会「カンドルと対称空間」
2021 年 11 月 26 日

K: (S^3 内の) **結び目** :⇔ $^\exists f : S^1 \hookrightarrow S^3$: (滑らかな) 埋め込み s.t. $K = f(S^1)$.
K と K' が同値 :⇔ $^\exists \varphi : S^3 \to S^3$: (滑らかな) 向きを保つ同相 s.t. $\varphi(K) = K'$.
今日の講演では結び目は全て向きがついているとする.

カンドル (quandle) . . . 結び目理論と相性の良い代数系. カンドルを用いると結び目の不変量が構成しやすい.
<u>例.</u>

- 基本カンドル $Q(K)$.
- 彩色数.
- カンドルコサイクル不変量.

特にカンドルを用いた不変量は強力な結び目の不変量であることが期待できる.

Quandle

定義 (Joyce, Matveev, 1982)

X: 集合, $* : X^2 \to X$: 2 項演算.
$X = (X, *)$: **カンドル (quandle)**
$\Leftrightarrow$ $*$ は以下の条件を満たす:

1. ${}^\forall x \in X$, $x * x = x$.
2. ${}^\forall y \in X, *y : X \to X; x \mapsto x * y$ は全単射.
3. ${}^\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

Ex.
G: 群, $x * y := y^{-1}xy$ $(x, y \in G)$, $\mathrm{Conj}(G) = (G, *)$: G の 共役カンドル.

X, Y: カンドル, $\rho : X \to Y$ が**カンドル準同型** $\Leftrightarrow$ ${}^\forall x, y \in X, \rho(x * y) = \rho(x) * \rho(y)$.

Alexander pair

定義 (Ishii-Oshiro (cf. Andruskiewitsch-Graña, 2003))

X: カンドル, R: 環, $f_1, f_2 : X^2 \to R$: 写像.
$\boldsymbol{f} = (f_1, f_2)$: **Alexander pair**
$\Leftrightarrow$ f_1 と f_2 は次の条件を満たす:

- ${}^\forall x \in X$, $f_1(x, x) + f_2(x, x) = 1$.
- ${}^\forall x, y \in X$, $f_1(x, y)$ は R の単元.
- ${}^\forall x, y, z \in X$,

$$f_1(x * y, z) f_1(x, y) = f_1(x * z, y * z) f_1(x, z),$$
$$f_1(x * y, z) f_2(x, y) = f_2(x * z, y * z) f_1(y, z) \text{ かつ}$$
$$f_2(x * y, z) = f_1(x * z, y * z) f_2(x, z) + f_2(x * z, y * z) f_2(y, z).$$

今回は次の概念に焦点を当てる.

f-twisted Alexander matrix

$Q = \langle x_1, \dots, x_n \mid r_{11} = r_{12}, \dots, r_{m1} = r_{m2} \rangle$, X: カンドル.
$\rho : Q \to X$: カンドル準同型, R: 環.
$\boldsymbol{f} = (f_1, f_2)$: **Alexander pair** ($f_1, f_2 : X^2 \to R$).
$\rightsquigarrow A(Q, \rho; f_1, f_2) \in M(m, n; R)$: **$\boldsymbol{f}$-twisted Alexander matrix**.
$\rightsquigarrow \mathrm{Coker}(R^m \ni \boldsymbol{a} \mapsto \boldsymbol{a}A(Q, \rho; f_1, f_2) \in R^n)$ は (Q, ρ) の不変量.

定理 (Ishii-Oshiro)

もしカンドル同型 $\psi : Q' \to Q$ (すなわち ψ は全単射かつカンドル準同型) が存在するならば任意のカンドル準同型 $\rho : Q \to X$ に対して
$\mathrm{Coker}(\boldsymbol{a} \mapsto \boldsymbol{a}A(Q, \rho; f_1, f_2)) \cong_{R-\mathrm{mod}} \mathrm{Coker}(\boldsymbol{a} \mapsto \boldsymbol{a}A(Q', \rho \circ \psi; f_1, f_2))$ が成り立つ.

つまり全てのカンドル準同型 $\rho : Q \to X$ を考えれば Q の不変量である.

問題 1

Alexander pair (f_1, f_2) が与えられたとき, $\mathrm{Coker}(\boldsymbol{a} \mapsto \boldsymbol{a}A(Q, \rho; f_1, f_2))$ を決定せよ.

例.
$f_1, f_2 : X^2 \to \mathbb{Z}[t^{\pm 1}]$ を $f_1(x, y) = t, f_2(x, y) = 1 - t$ で定めると組 (f_1, f_2) は Alexander pair である.
K を S^3 内の結び目とすると $\mathrm{Coker}(\boldsymbol{a} \mapsto \boldsymbol{a}A(Q(K), \rho; f_1, f_2))$ は K の補空間の無限巡回被覆の 1 次ホモロジー群 (Alexander 不変量) と $\mathbb{Z}[t^{\pm 1}]$ の直和である.

問題 2

どんな Alexander pair (f_1, f_2) を持ってくると興味深い情報が得られるか.

As : { カンドル } → { 群 } という関手が存在し，これは以下の性質を持つ:

- Conj : { 群 } → { カンドル } の随伴関手である．
- K を S^3 内の有向結び目とすると $\mathrm{As}(Q(K)) = \pi_1(S^3 \backslash K)$.

Fox の free derivative を用いて (有限表示可能な) 群に対する Alexander 行列というのが定義できる．つまり以下の流れで行列 (と不変量) を得ることが出来る．

$$(\text{有限表示可能な})\ \text{カンドル} \xrightarrow{\mathrm{As}} (\text{有限表示可能な})\ \text{群} \xrightarrow{\text{free derivative}} \text{行列}$$

上の手法は表現付きの場合にも拡張できる．
ただしこの方法だと $\mathrm{As}(Q) \cong_{\mathrm{grp}} \mathrm{As}(Q')$ を満たす Q, Q' を区別できない．

問題 2'

$\mathrm{As}(Q) \cong_{\mathrm{grp}} \mathrm{As}(Q')$ を満たす Q, Q' を $\boldsymbol{f}$-twisted Alexander matrix から得られる不変量で区別できるか．

問題 2 に対するアプローチ

"カンドル由来の Alexander pair" を用いると良いのではないだろうか．

今回 $\cdots$ **カンドル 2-コサイクル** $\theta \rightsquigarrow$ Alexander pair $(f_\theta, 0)$.
A: (乗法的) アーベル群．

定義 (Carter-Jelsovsky-Kamada-Lamgford-Saito, 2003)

$\theta : X^2 \to A$: **カンドル 2-コサイクル**
$\Leftrightarrow$ θ は次の条件を満たす:

- ${}^\forall x \in X,\ \theta(x, x) = e_A$.
- ${}^\forall x, y, z \in X,\ \theta(x * y, z)\theta(x, y) = \theta(x * z, y * z)\theta(x, z)$.

注意．
θ を $\mathbb{Z}[X^2]$ (X^2 の元を基底とする自由加群) に線形に拡張した写像 $\theta : \mathbb{Z}[X^2] \to A$ はカンドルコチェイン複体の 2 次のコサイクルになる．
普遍係数定理より群準同型 $H_2^Q(X; \mathbb{Z}) \to A$ が得られる．この準同型も θ と書く．

$\theta : X^2 \to A$: カンドル 2-コサイクル. $\mathbb{Z}[A]$: 群環.
$f_\theta : X^2 \to \mathbb{Z}[A], 0 : X^2 \to \mathbb{Z}[A]$ を次で定める;

$$f_\theta(x, y) := 1 \cdot \theta(x, y), \quad 0(x, y) := 0.$$

補題 (T.)

$(f_\theta, 0)$ は Alexander pair である.

(f_1, f_2): Alexander pair $\Rightarrow {}^\forall x, y, z \in X, f_1(x * y, z) f_1(x, y) = f_1(x * z, y * z) f_1(x, z)$.
θ: カンドル 2-コサイクル $\Rightarrow {}^\forall x, y, z \in X, \theta(x * y, z)\theta(x, y) = \theta(x * z, y * z)\theta(x, z)$.

$Q \curvearrowleft \mathrm{Inn}(Q) := \langle \{ *q : Q \to Q \mid q \in Q \} \rangle \subset \{ \varphi : X \to X \mid \varphi$ はカンドル同型 $\}$.
Q: **連結** $:\Leftrightarrow Q \curvearrowleft \mathrm{Inn}(Q)$ が推移的.

定理 A (T.)

$Q = \langle x_1, \dots, x_n \mid r_{11} = r_{12}, \dots, r_{m1} = r_{m2} \rangle$: 連結なカンドル .
X: カンドル, $\rho : Q \to X$: カンドル準同型.
A: アーベル群, $\theta : X^2 \to A$: カンドル 2-コサイクル.
このとき次が成り立つ:

$$\mathrm{Coker}(\boldsymbol{a} \mapsto \boldsymbol{a} A(Q, \rho; f_\theta, 0)) \cong \mathbb{Z}[A]/(\{ 1 \cdot x - 1 \cdot e_A \mid x \in \mathrm{Im}(\theta \circ \rho_*) \}),$$

ここで ρ_* は ρ から誘導されるカンドルホモロジー群の間の準同型
$\rho_* : H_2^Q(Q; \mathbb{Z}) \to H_2^Q(X; \mathbb{Z})$ である.

応用

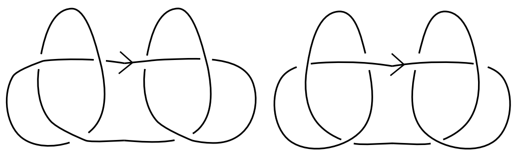
K: S^3 内の非自明な結び目.
$Q(K)$: K の基本カンドル. $H_2^Q(Q(K)) \cong \mathbb{Z} = \langle [K] \rangle$ ($[K]$ は K の基本類).

定理 B (T.)

X: カンドル, $\rho : Q(K) \to X$: カンドル準同型.
A: アーベル群, $\theta : X^2 \to A$: カンドル 2-コサイクル.
このとき次が成り立つ:
$\mathrm{Coker}(\boldsymbol{a} \mapsto \boldsymbol{a}A(Q(K), \rho; f_\theta, 0)) \cong \mathbb{Z}[A]/(1 \cdot \theta \circ \rho_*([K]) - 1 \cdot e_A)\})$

$\theta \circ \rho_*([K])$ を全ての ρ に対して数え上げたものを**カンドルコサイクル不変量**と呼ぶ.
つまり $\boldsymbol{f}$-twisted Alexander matrix からカンドルコサイクル不変量の情報が得られることがわかる.

次の結び目を考える.



K_1 K_2

- $Q(K_1) \ncong_{\mathrm{qdle}} Q(K_2)$
- $\pi_1(S^3 \backslash K_1) \cong_{\mathrm{grp}} \pi_1(S^3 \backslash K_2) (\Leftrightarrow \mathrm{As}(Q(K_1)) \cong \mathrm{As}(Q(K_2)))$

問題 2' (再掲)

$\mathrm{As}(Q) \cong \mathrm{As}(Q')$ を満たす Q, Q' を $\boldsymbol{f}$-twisted Alexander matrix から得られる不変量で区別できるか.

- K_1 と K_2 はロンジチュードを使うと区別できる (Fox, Waldhausen,...).
- カンドルコサイクル不変量は結び目のロンジチュードの値 (Eisermann, 2007).
- $\boldsymbol{f}$-twisted Alexander matrix とカンドルコサイクル不変量は関係する (定理 B).

$\rightsquigarrow$ $\boldsymbol{f}$-twisted Alexander matrix をつかって K_1 と K_2 は区別できるだろう!
$\rightsquigarrow$ 具体的に区別できる Alexander pair $(f_\theta, 0)$ を与えた (arXiv: 2107.0656).
(具体的な計算は arXiv: 2107.06561 を参照)

問題 2′ (再掲)

$\mathrm{As}(Q) \cong \mathrm{As}(Q')$ を満たす Q, Q' を $\boldsymbol{f}$-twisted Alexander matrix から得られる不変量で区別できるか.

A. Yes!

(定理 A の証明の概略) $q \in Q$ を固定する.
次の表示を持つ群を $\mathrm{As}(Q)$ とかく: $\langle x \ (x \in Q) \mid y^{-1}xy(x*y)^{-1} \ (x, y \in Q)\rangle$.
$Q \curvearrowleft \mathrm{As}(Q)$ が自然に存在し $\mathrm{Stab}(q) := \{g \in \mathrm{As}(Q) \mid q \cdot g = q\}$ と置く.
このとき次が成り立つ:

- $H_2^Q(Q; \mathbb{Z}) \cong ([\mathrm{As}(Q), \mathrm{As}(Q)] \cap \mathrm{Stab}(q))_{\mathrm{ab}}$ (Eisermann).
- Q が有限表示可能ならば Q の表示として
  $\langle x_1, \ldots, x_{n-1}, q \mid x_1 \cdot g_1 = q, \ldots, x_{n-1} \cdot g_{n-1} = q, q \cdot g_n = q, \ldots, q \cdot g_m = q\rangle$
  という表示がとれる (ただし $g_i \in F[S]$).

Coker を変えない変形を施すことで上の表示から得られる行列の Coker は

$$\begin{pmatrix} 1 \cdot \theta \circ \rho_*([g_n]) - 1 \cdot e_A \\ \vdots \\ 1 \cdot \theta \circ \rho_*([g_m]) - 1 \cdot e_A \end{pmatrix}$$

の Coker と同型であるということがわかる.

問題 2 (再掲)

どんな Alexander pair (f_1, f_2) を持ってくると興味深い情報が得られるか.

問題 2''

$\boldsymbol{f}$-twisted Alexander matrix からどのようなカンドルの情報が得られるか.

事実.
連結なカンドル Q は $([\mathrm{As}(Q), \mathrm{As}(Q)], [\mathrm{As}(Q), \mathrm{As}(Q)] \cap \mathrm{Stab}(q), \sigma_q)$ という表示を持つ, ただし σ_q は $\mathrm{As}(Q) \ni x \mapsto q^{-1}xq \in \mathrm{As}(Q)$ を制限した写像.

- $([\mathrm{As}(Q), \mathrm{As}(Q)] \cap \mathrm{Stab}(q))_{\mathrm{ab}} \cong H_2^Q(Q; \mathbb{Z})$.
- K を結び目とすると $[\mathrm{As}(Q(K)), \mathrm{As}(Q(K))]$ は無限巡回被覆空間の基本群.
$\rightsquigarrow$ $([\mathrm{As}(Q(K)), \mathrm{As}(Q(K))])_{\mathrm{ab}}$ を σ_q の作用付きで考えたものが Alexander 不変量.

R が可換環ならば「Alexander 不変量」+「$H_2^Q(Q; \mathbb{Z})$」で尽きるかも...?

ご清聴ありがとうございました.

Extended quandle spaces and their applications

Katsumi Ishikawa

ABSTRACT. We show the homotopy equivalence of the extended quandle space $\hat{B}X$ and a covering space of the quandle space B^QX of a quandle X. As applications, we introduce a spectral sequence on quandle homology and examine the relation of quandle homology invariants for knots.

1 Quandles and quandle spaces

A set X equipped with a binary operation $*$ is called a *quandle* if

(Q1) $x * x = x$ for any $x \in X$,

(Q2) $\bullet * x : X \to X$ is bijective for any $x \in X$, and

(Q3) $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$.

For example, the *dihedral quandle* R_k of order k is the cyclic group $\mathbb{Z}_k$ with $*$ defined by $x * y = 2y - x$ $(x, y \in \mathbb{Z}_k)$. For a group G and an automorphism ϕ of G the *generalized Alexander quandle* $Q_{G,\phi} = (G, *)$ is defined by $x * y = \phi(xy^{-1})y$ $(x, y \in G)$. If G is commutative, $Q_{G,\phi}$ is simply called an *Alexander quandle*.

The *associated group* $\mathrm{As}(X)$ of a quandle X is the group which has the presentation $\langle e_x \ (x \in X) \mid e_x e_y = e_y e_{x*y} \ (x, y \in X)\rangle$. A set Y with a right action of $\mathrm{As}(X)$ is called an *X-set*; e.g., X is an X-set by $x \cdot e_y = x * y$ $(x, y \in X)$ and so is $\mathbb{Z}$ by $a \cdot e_x = a + 1$ $(a \in \mathbb{Z}, x \in X)$. If the action of $\mathrm{As}(X)$ on X is transitive, X is said to be *connected*.

For a quandle X, the *rack chain complex* $(C^R_\bullet(X), \partial)$ and the *quandle chain complex* $(C^Q_\bullet(X), \partial)$ are defined, and the *rack space* BX is introduced as a geometric realization of $C^R_\bullet(X)$: $BX = \coprod_{n=0}^{\infty} [0,1]^n \times X^n / \sim_R$ for some relation $\sim_R$. See [8] for definitions. The *quandle space* B^QX is the quotient space $BX/\sim_D$, where $\sim_D$ is the equivalent relation generated by

$$(t_1, \dots, t_n; x_1, \dots, x_n) \sim_D (t_1, \dots, t_{i-1}, t'_i, t'_{i+1}, t_{i+2}, \dots, t_n; x_1, \dots, x_n)$$

if $x_i = x_{i+1}$ and $t_i + t_{i+1} = t'_i + t'_{i+1}$. For an X-set Y we have the covering space $B^Q_Y X$ of B^QX associated to Y since $\pi_1(B^QX) \cong \mathrm{As}(X)$. A well defined action of $S^1 \cong \mathbb{R}/\mathbb{Z}$ on $B^Q_X X$ is given by

$$t \cdot (t_1, \dots, t_n; x; x_1, \dots, x_n) = (t, t_1, \dots, t_n; x; x, x_1, \dots, x_n),$$

where we regard $B^Q_X X$ as a quotient of $\coprod_n [0,1]^n \times X \times X^n$, and then we call the quotient $B^Q_X X/S^1$ the *extended quandle space* and denote it by $\hat{B}X$. By definitions, we have $H_n(BX) \cong H^R_n(X)$, $H_n(B^QX) \cong H^Q_n(X)$, and $H_n(\hat{B}X) \cong H^Q_{n+1}(X)$.

This work was supported by JSPS KAKENHI Grant Number JP2014309.

Remark 1.1. [1] Rack/quandle spaces $BX, B^Q X, \hat{B}X$ were introduced in [4], [7], [9], respectively, though the original definitions of $B^Q X$ and $\hat{B}X$ are slightly different from ours and our definition of $B^Q X$ is due to [6]. Also, we should note that the covering space $B_X X$ of BX associated to X is called the extended rack space, e.g., in [4]; remark that the extended quandle space here is not $B^Q_X X$ but $\hat{B}X$, a quotient of the covering space $B^Q_X X$.

2 Key homeomorphisms

In the definition of $\hat{B}X$, the action of S^1 is free and then $B^Q_X X$ is an S^1-bundle. The following theorem claims that it is trivial.

Theorem 2.1. *Let X be a quandle. Then, we have the following commutative diagram, whose rows are homeomorphic*:

$$\begin{array}{ccc} B^Q_{\mathbb{Z}\times X} X & \xrightarrow{\cong} & \mathbb{R} \times \hat{B}X \\ \downarrow & & \downarrow \\ B^Q_X X & \xrightarrow{\cong} & S^1 \times \hat{B}X \end{array} .$$

Corollary 2.2. $H_n(B^Q_{\mathbb{Z}\times X} X) \cong H^Q_{n+1}(X)$.

Remark 2.3. By this corollary, the covering map induces a degree-(-1) homomorphism $H^Q_{n+1}(X) \to H^Q_n(X)$ of quandle homology groups. In fact, this is equal to the one induced by the shifting chain map algebraically introduced in [5].

3 Spectral sequences on quandle homology

Let us apply the Cartan-Leray spectral sequence (see, e.g., [1]), which is valid for a regular covering, to $B^Q_{\mathbb{Z}\times X} X \to B^Q X$. Unfortunately, this covering is not always regular. However, any generalized Alexander quandle satisfies this condition (in fact, this is necessary and sufficient in the connected cases) and we have

Theorem 3.1. *Let X be a connected generalized Alexander quandle $Q_{G,\phi}$. Then, there exists a spectral sequence $\{E^r_{p,q}\}$ of the form*

$$E^2_{p,q} = H_p(\mathbb{Z} \ltimes_\phi G; H^Q_{q+1}(X)) \Rightarrow H^Q_{p+q}(X),$$

where the actions of $\mathbb{Z} \ltimes_\phi G$ on the coefficient groups are trivial and $E^\infty_{p,q} = 0$ for $p \geq 2$.

For example, quandle homology groups of connected Alexander quandles are calculated as follows:

[1] The author would like to thank Professor Seiichi Kamada for the valuable comments on the history of rack spaces.

- (Alexander quandles $Q_{\mathbb{Q},a}$) For any $a \in \mathbb{Q}\backslash\{0,1\}$, we have $H_0^Q(Q_{\mathbb{Q},a}) \cong H_1^Q(Q_{\mathbb{Q},a}) \cong \mathbb{Z}$ and $H_n^Q(Q_{\mathbb{Q},a}) = 0$ $(n \geq 2)$. In fact, the extended quandle space $\hat{B}Q_{\mathbb{Q},a}$ is contractible and hence the quandle space $B^Q X$ is a $K(\mathbb{Z} \ltimes \mathbb{Q}, 1)$-space.
- (Dihedral quandles R_k) Let $k \geq 1$ be an odd. Then, we have $H_0^Q(R_k) \cong H_1^Q(R_k) \cong \mathbb{Z}$ and $H_n^Q(R_k) \cong \mathbb{Z}_k^{r_n}$ $(n \geq 2)$, where r_n is the delayed Fibonacci sequence defined by $r_1 = r_2 = 0, r_3 = 1, r_n = r_{n-1} + r_{n-3}$ $(n \geq 4)$.

We also have a non-connected version of Theorem 3.1: We consider a connected component X_0 of a quandle X and the covering map $B^Q_{\mathbb{Z}\times X_0} X \to B^Q X$. If this is regular, we can obtain a spectral sequence on quandle homology as above. We here just state results of computation of quandle homology groups:

- (Free quandles FQ_k; cf. [3]) Let FQ_k denote the free quandle of k generators (see, e.g., [8] for definition). Then, we have $H_0^Q(FQ_k) \cong \mathbb{Z}, H_1^Q(FQ_k) \cong \mathbb{Z}^k$, and $H_n^Q(FQ_k) = 0$ $(n \geq 2)$. In fact, the quandle space $B^Q FQ_k$ is a $K(F_k, 1)$ (F_k is the free group of k generators), i.e., homotopy equivalent to the bouquet of k circles.
- (Dihedral quandles R_{2k}) For an odd $k \geq 1$, we have $H_n^Q(R_{2k}) \cong (\mathbb{Z} \oplus \mathbb{Z}_k^{\lfloor 2^n/5 \rfloor})^2$.

4 Relation and efficiency of knot invariants

For an oriented link L in S^3 the *fundamental quandle* $Q(L)$ is defined and a homomorphism $Q(L) \to X$ to a quandle X is called an *X-coloring*. An X-coloring $\mathcal{C}$ can be described by a colored diagram in S^2, and then a homotopy class $\Xi(\mathcal{C}) \in \pi_2(B^Q X)$, called the *quandle homotopy invariant*, is defined and independent of the choice of a diagram; see [7] for details. Specifying a connected component (usually by a "shadow coloring") if necessary, we obtain homology classes $\Phi_2(\mathcal{C}) \in H_2^Q(X) \cong H_2(B^Q X)$ and $\Phi_X(\mathcal{C}) \in H_2^Q(X, X) := H_2(B_X^Q X)$ as the images of $\Xi(\mathcal{C})$ under the Hurewicz homomorphisms. Furthermore, we set $\Phi_3(\mathcal{C}) = \pi_*(\Phi_X(\mathcal{C})) \in H_3^Q(X) \cong H_2(\hat{B}X)$, where $\pi : B_X^Q X \to \hat{B}X$ is the quotient map.

Let $q : H_3^Q(X) \to H_2^Q(X, X)$ denote the composite

$$H_3^Q(X) \cong H_2(\hat{B}X) \cong H_2(B^Q_{\mathbb{Z}\times X} X) \to H_2(B_X^Q X) \cong H_2^Q(X, X)$$

of the homotopy equivalence and the covering map. The following proposition means that the invariants Φ_X and Φ_3 are equivalent.

Proposition 4.1. *For any (shadow) X-coloring $\mathcal{C}$ of any link, $q(\Phi_3(\mathcal{C})) = \Phi_X(\mathcal{C})$.*

Next, let us review a non-abelian invariant briefly. For simplicity, we consider an oriented knot K (with a single component) and assume a quandle X to be connected. Let $\mathcal{C}$ be an X-coloring of K. Since "As" is a functor and $\mathrm{As}(Q(K))$ is isomorphic to the knot group π_K, we have a group homomorphism $\mathrm{As}(\mathcal{C}) : \pi_K \to \mathrm{As}(X)$. Taking a meridian-longitude pair (m, ℓ), we set $x_0 = \mathcal{C}(m) \in X$ and $\Lambda(\mathcal{C}) = (\mathrm{As}(\mathcal{C}))(\ell) \in$

$\mathrm{As}(X)$. In fact, $\Lambda(\mathcal{C})$ is contained in the *fundamental group* $\pi_1(X, x_0)$ of X defined by $\pi_1(X, x_0) = \{g \in \mathrm{As}(X) \mid \varepsilon(g) = 0,\ x_0 \cdot g = x_0\}$, where $\varepsilon : \mathrm{As}(X) \to \mathbb{Z}$ is the group homomorphism taking $e_x \in \mathrm{As}(X)$ to $1 \in \mathbb{Z}$ $(x \in X)$. It is known [2] that the abelianization $\pi_1(X)_{\mathrm{ab}}$ is isomorphic to $H_2^Q(X)$, and under this identification $\Lambda(\mathcal{C})_{\mathrm{ab}} = \Phi_2(\mathcal{C})$.

Thus, both of the invariants Φ_3 and Λ have the universality to Φ_2, i.e., $p_* \circ q(\Phi_3(\mathcal{C})) = \Lambda(\mathcal{C})_{\mathrm{ab}} = \Phi_2(\mathcal{C})$ holds for any coloring $\mathcal{C}$, where $p : B_X^Q X \to B^Q X$ is the covering map. The following theorem examines the relation and possible values of these two invariants. Here, a homology class $\phi \in H_3^Q(X)$ is *realizable* if it is in the kernel of the classifying homomorphism $H_3^Q(X) \cong H_2(\hat{B}X) \to H_2(\pi_1(X))$, where we can easily check that $\pi_1(X) \cong \pi_1(\hat{B}X)$.

Theorem 4.2. *Let X be a connected quandle and take $x_0 \in X$. For $\phi \in H_3^Q(X)$ and $\lambda \in \pi_1(X, x_0)$, there exists a pair $(K, \mathcal{C})$ of an oriented knot K and an X-coloring $\mathcal{C}$ of K such that $\Phi_3(\mathcal{C}) = \phi$ and $\Lambda(\mathcal{C}) = \lambda$ if and only if ϕ is realizable and $p_* \circ q(\phi) = \lambda_{\mathrm{ab}}$.*

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Extended quandle spaces and their applications

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Motivation: Calculation of $H_*^Q(X)$.

$$\begin{array}{rrcl} \text{the quandle space:} & H_n(B^Q X) & \cong & H_n^Q(X) \\ \text{the extended qdle sp.:} & H_n(\hat{B}X) & \cong & H_{n+1}^Q(X) \end{array}$$

Motivation: Calculation of $H_*^Q(X)$.

$$\begin{array}{rrcl} \text{the quandle space:} & H_n(B^QX) & \cong & H_n^Q(X) \\ \text{the extended qdle sp.:} & H_n(\hat{B}X) & \cong & H_{n+1}^Q(X) \end{array}$$

Main result

$$\begin{array}{ccccc} B^Q_{\mathbb{Z}\times X}X & \to & B^Q_X X & \to & B^Q X \quad : \text{cov.maps} \\ \wr\| & & \wr\| & & \\ \mathbb{R}\times \hat{B}X & \to & S^1 \times \hat{B}X & & \end{array}$$

Applications

- CL spectral sequences on quandle homology.
- Relation and efficiency of quandle homology invariants.

Contents

1. Quandles and quandle spaces
   - Definitions
   - Main theorem — key homeomorphisms on quandle spaces
2. Spectral sequences on quandle homology
   - Computations of $H_*^Q(X)$
3. Relation and efficiency of knot invariants
   - 2- vs. 3-cocycle invariants
   - Efficiency of invariants

Quandles

$X = (X, *)$ is a **quandle**

$$\overset{\text{def}}{\Leftrightarrow} \begin{cases} \text{(Q1) } x * x = x, \\ \text{(Q2) the map } s_x : X \ni a \mapsto a * x \in X \text{ is a bijection,} \\ \text{(Q3) } (x * y) * z = (x * z) * (y * z), \\ \qquad\qquad\qquad\qquad\qquad \text{for } {}^\forall x, y, z \in X. \end{cases}$$

Ex.

- **dihedral quandle** $R_k = (\mathbb{Z}/k\mathbb{Z}, *)$.

$$x * y = 2y - x.$$

- **generalized Alexander quandle** $Q_{G,\phi} = (G, *)$ $\begin{pmatrix} G : \text{a group} \\ \phi \in \mathrm{Aut}(G) \end{pmatrix}$.

$$x * y = \phi(xy^{-1})y.$$

Quandle homology

Def X : a quandle.

- The **rack homology** $H^R_\bullet(X) = H_\bullet(C^R_\bullet(X), \partial)$:

  $C^R_n(X) =$ free ab. grp. generated by X^n,

$$\partial \boldsymbol{x} = \sum_{i=1}^{n} (-1)^n \big((x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \\ - (x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n)\big).$$

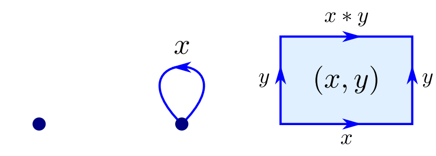
- $C^D_n(X) = \mathrm{span}\{(x_1, \dots, x_n) \in C^R_n(X) \mid x_i = x_{i+1} \text{ for some } i\}$.

- The **quandle homology**:

$$C^Q_n(X) = C^R_n(X)/C^D_n(X), \qquad H^Q_\bullet(X) = H_\bullet(C^Q_\bullet(X), \partial).$$

Quandle spaces

- The **rack space** $BX = \coprod_{n=0}^{\infty} [0,1]^n \times X^n \Big/ \sim_R$.

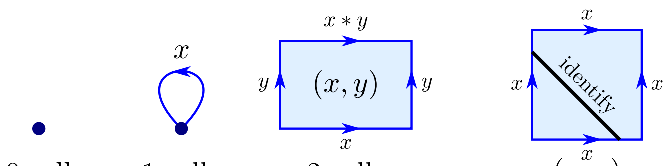


rem $\quad H_\bullet(BX) \cong H_\bullet^R(X)$,

Quandle spaces

- The **rack space** $BX = \coprod_{n=0}^{\infty} [0,1]^n \times X^n \Big/ \sim_R$.
- The **quandle space** $B^Q X = BX / \sim_D$, where

$$(\ldots, t_i, t_{i+1}, \ldots; \ldots, x_i, x_i, \ldots) \sim_D (\ldots, t'_i, t'_{i+1}, \ldots; \ldots, x_i, x_i, \ldots) \quad \text{if } t_i + t_{i+1} = t'_i + t'_{i+1}.$$

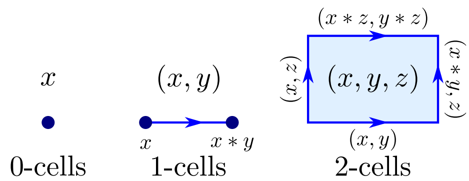


rem $\quad H_\bullet(BX) \cong H_\bullet^R(X), \quad H_\bullet(B^Q X) \cong H_\bullet^Q(X)$.

Quandle spaces

Y : an "X-set". e.g., $X, \mathbb{Z} \times X, \ldots$

- $B_Y^Q X = \bigsqcup_{n=0}^{\infty} [0,1]^n \times Y \times X^n \Big/ \sim_R, \sim_D$. $\leftarrow$ a cov. sp. of $B^Q X$.



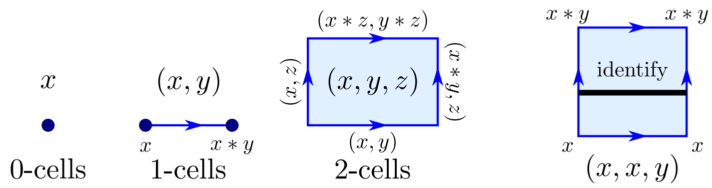
Quandle spaces

Y : an "X-set". e.g., $X, \mathbb{Z} \times X, \ldots$

- $B_Y^Q X = \bigsqcup_{n=0}^{\infty} [0,1]^n \times Y \times X^n \Big/ \sim_R, \sim_D$. $\leftarrow$ a cov. sp. of $B^Q X$.
- $S^1 = (\mathbb{R}/\mathbb{Z}) \curvearrowright B_X^Q X$ is defined by

$$t \cdot (t_1, \ldots, t_n; x; x_1, \ldots, x_n) = (t, t_1, \ldots, t_n; x; x, x_1, \ldots, x_n).$$

$\rightsquigarrow$ The **extended quandle space** $\hat{B}X = B_X^Q X / S^1$.



rem $H_n(\hat{B}X) \cong H_{n+1}^Q(X) \quad (n \geq 0)$.

Key homeomorphisms

Theorem A

There exist homeomorphisms as follows:

$$\begin{array}{ccc} B^Q_{\mathbb{Z}\times X}X & \xrightarrow{\cong} & \mathbb{R}\times \hat{B}X \\ \downarrow & & \downarrow \\ B^Q_X X & \xrightarrow{\cong} & S^1 \times \hat{B}X \end{array}$$

Corollary

$$H_n\big(B^Q_{\mathbb{Z}\times X}(X)\big) \cong H^Q_{n+1}(X) \qquad (n \geq 0).$$

<u>Prop</u> $H^Q_{n+1}(X) \cong H_n(B^Q_{\mathbb{Z}\times X}X) \to H_n(B^Q X) \cong H^Q_n(X)$
is equal to the shifting homomorphism by Hashimoto-Tanaka.

Spectral sequences

X: a connected quandle.

<u>Lem</u> The covering $B^Q_{\mathbb{Z}\times X}X \to B^Q X$ is regular $\Leftrightarrow$ X is gen. Alex.

Theorem B

$X = Q_{G,\phi}$: a connected generalized Alexander quandle.

$\Rightarrow$ There exists a spectral sequence $\{E^r_{p,q}\}$ s.t.

$$E^2_{p,q} = H_p(\mathbb{Z} \ltimes_\phi G; H^Q_{q+1}(X)) \Rightarrow H^Q_{p+q}(X),$$

where $\mathbb{Z} \ltimes_\phi G \curvearrowright H^Q_{q+1}(X)$ is trivial.

Furthermore, we have $E^\infty_{p,q} = 0$ for $p \geq 2$.

Examples

Alexander quandles $Q_{\mathbb{Q},a}$ $(a \in \mathbb{Q} \backslash \{0,1\})$

$H_0^Q(Q_{\mathbb{Q},a}) \cong H_1^Q(Q_{\mathbb{Q},a}) \cong \mathbb{Z}, \qquad H_n^Q(Q_{\mathbb{Q},a}) = 0 \quad (n \geq 2).$

Corollary

$\hat{B}Q_{\mathbb{Q},a}$ is contractible and $B^Q Q_{\mathbb{Q},a}$ is a $K(\mathbb{Z} \ltimes \mathbb{Q}, 1)$.

Dihedral quandles R_k

If $k \geq 1$ is an odd, we have

$$H_0^Q(R_k) \cong H_1^Q(R_k) \cong \mathbb{Z}, \quad H_n^Q(R_k) \cong \mathbb{Z}_k^{r_n} \quad (n \geq 2),$$

where r_n is defined as follows:

$$r_1 = r_2 = 0, \quad r_3 = 1, \quad r_n = r_{n-1} + r_{n-3} \quad (n \geq 4).$$

Spectral sequences

X : a quandle, $\qquad X_0 \subset X$ a connected component.

Def X_0 is **quasi-Alexander** $\overset{\text{def}}{\Leftrightarrow}$ the cov. map $B^Q_{\mathbb{Z}\times X_0} X \to B^Q X$ is regular.

Lem A generalized Alexander quandle is quasi-Alexander.

Theorem B'

X : a quandle, $\qquad X_0$: a quasi-Alexander connected component.

$\Rightarrow$ There exists a spectral sequence $\{E^r_{p,q}\}$ s.t.

$$E^2_{p,q} = H_p(G_0; H^Q_{X_0,q}(X)) \Rightarrow H^Q_{p+q}(X),$$

where $\begin{cases} G_0 : \text{the cov. trfm. gp. of } B^Q_{\mathbb{Z}\times X_0} X \to B^Q X, \\ G_0 \curvearrowright H^Q_{X_0,q}(X): \text{trivial.} \end{cases}$

If ${}^\forall f \in \mathrm{Stab}_{\mathrm{Inn}(X)}(x)$ $(x \in X)$ has a f.p. in X_0, $E^\infty_{p,q} = 0$ for $p \geq 2$.

Examples

Trivial quandle T_k

$$H_n^Q(T_k) \cong \mathbb{Z}^{k(k-1)^n}.$$

Free quandle FQ_k (cf. Farinati)

$$H_0^Q(FQ_k) \cong \mathbb{Z}, \quad H_1^Q(FQ_k) \cong \mathbb{Z}^k, \quad H_n^Q(FQ_k) = 0 \ (n \geq 2).$$

In fact, $B^Q FQ_k$ is a $K(F_k, 1)$, i.e., $B^Q FQ_k \simeq \bigvee^k S^1$.

Alexander quandle $Q_{\mathbb{Z},-1}$

$$H_0^Q(Q_{\mathbb{Z},-1}) \cong \mathbb{Z}, \qquad H_n^Q(Q_{\mathbb{Z},-1}) \cong \mathbb{Z}^2 \quad (n \geq 1).$$

Dihedral quandles R_{2k}

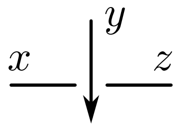
If $k \geq 1$ is an odd, we have

$$H_n^Q(R_{2k}) \cong \left(\mathbb{Z} \oplus \mathbb{Z}_k^{\lfloor 2^n/5 \rfloor}\right)^2.$$

Quandle coloring

X : a quandle $\qquad D$: an oriented link diag.

- A map $\mathcal{C}$: {arcs of D} $\to X$ is an **X-coloring**

$\overset{\text{def}}{\Leftrightarrow} \quad \mathcal{C}(x) * \mathcal{C}(y) = \mathcal{C}(z)$ at  .

<u>Rem</u> $X = G$: conj. quandle $(x * y = y^{-1}xy)$;

$\Rightarrow X$-col. $=$ grp. rep. $\pi_L \to G$.

<u>Rem</u> Reidemeister moves induce bijections of $\mathrm{Col}_X(D) = \{X\text{-col. of } D\}$.

$\rightsquigarrow \#\mathrm{Col}_X(D)$ is an invariant for links.

The **quandle homotopy invariant** (Fenn-Rourke-Sanderson, Nosaka):

$\mathcal{C}$: an X-col. $\rightsquigarrow \Xi(\mathcal{C}) \in \pi_2(B^Q X)$

Fact $\pi_2(B^Q X) \cong \Pi_2(X) := \left\{ \begin{array}{l} \text{link diags.} \\ \text{w/ } X\text{-col.} \end{array} \right\} /\text{RI, RII, RIII}, \sim;$

- Quandle (shadow) cocycle invariants Φ_ψ are recovered from Ξ:

$$\begin{array}{ccccc} \pi_2(B^Q_{(Y)}X) & \xrightarrow{h_2} & H^Q_2(X(,Y)) & \xrightarrow{\langle\psi,\bullet\rangle} & A \\ \Psi & & \Psi & & \Psi \\ \Xi(\mathcal{C}) & \longmapsto & \Phi(\mathcal{C}) & \longmapsto & \Phi_\psi(\mathcal{C}) \end{array}$$

- (S.Y.Yang) We can define shadow homotopy inv. $\hat{\Xi}(\mathcal{C}) \in \pi_2(\hat{B}X)$.

2- vs. 3-cocycle invariants

There are 3 basic quandle homology (cocycle) invs.:

- $\Phi_2(\mathcal{C}) \in H^Q_2(X)$ $\rightsquigarrow$ 2-cocycle invs.
- $\Phi_X(\mathcal{C}) \in H^Q_2(X,X)$ $\rightsquigarrow$ shadow 2-cocycle invs.
- $\Phi_3(\mathcal{C}) \in H^Q_3(X)$ $\rightsquigarrow$ 3-cocycle invs.

Rem The cov. map $p : B^Q_X X \to B^Q X$ induces

$$p_* : H^Q_2(X,X) \ni \Phi_X(\mathcal{C}) \mapsto \Phi_2(\mathcal{C}) \in H^Q_2(X).$$

Proposition

X: quandle

$\Rightarrow \ {}^\exists q : H^Q_3(X) \to H^Q_2(X,X)$ s.t. ${}^\forall K, {}^\forall \mathcal{C},\ q(\Phi_3(\mathcal{C})) = \Phi_X(\mathcal{C})$.

Prf.

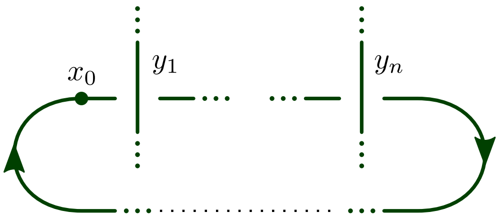
$$q : H^Q_3(X) \cong H_2(\hat{B}X) \overset{\text{Thm}}{\cong} H_2(B^Q_{\mathbb{Z}\times X}X) \xrightarrow{\text{cov.}} H_2(B^Q_X X) \cong H^Q_2(X,X).$$

Def X: a connected quandle.

- The **associated group** $\mathrm{As}(X) := \langle x \in X \mid xy = y(x * y) \rangle$.
- The **fundamental group** $\pi_1(X, x_0) := \mathrm{As}(X)_{x_0} \cap \mathrm{Ker}\, \varepsilon$.
  ($\varepsilon : \mathrm{As}(X) \to \mathbb{Z},\ x \mapsto 1$ for $x \in X$)
- $\mathcal{C}$: an X-col. of a knot $\quad x_0 \in X$: the color of an arc ($\ni$ bs. pt.).

  The **non-abelian cycle invariant**

$$\Lambda(\mathcal{C}) := x_0^{-\sum_i \epsilon_i} \cdot y_1^{\epsilon_1} \cdots y_n^{\epsilon_n} \ \in\ \pi_1(X, x_0)/\text{conj.(change of bs.pt.)}$$



Fact (Eisermann) X : conn. $\Rightarrow$ $\pi_1(X, x_0)_{\mathrm{ab}} \cong H_2^Q(X)$.

Efficiency of invariants

- $\pi_1(X, x_0) \cong \pi_1(\hat{B}X, x_0)$. $\quad \rightsquigarrow \quad \hat{B}X \xrightarrow{\exists c} K(\pi_1(X), 1)$.
- $\phi \in H_3^Q(X)$ is **realizable** $\overset{\text{def}}{\Leftrightarrow}$ $\phi \in \mathrm{Ker}[c_* : H_3^Q(X) \to H_2(\pi_1(X))]$.

Theorem C

X: a conn. qdle, $\quad \phi \in H_3^Q(X), \quad \lambda \in \pi_1(X, x_0)$.

$\Rightarrow$

$$\exists \left\{ \begin{array}{l} K\text{: a knot,} \\ \mathcal{C} \in \mathrm{Col}_X(K) \end{array} \right\} \text{s.t.} \left\{ \begin{array}{l} \Phi_3(\mathcal{C}) = \phi, \\ \Lambda(\mathcal{C}) = \lambda \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \phi \text{ is realizable,} \\ p_* \circ q(\phi) = \lambda_{\mathrm{ab}}. \end{array} \right\}$$

Remind: $p_* \circ q : H_3^Q(X) \to H_2^Q(X)$ is the shifting homomorphism.

Future directions

- Quandle homology of non-Alexander, gen. Alex. quandles.
- Quandle homology of quandles which are not generalized Alexander.
- Relation between BX and $B^Q X$ (or $\hat{B}X$).
- Topological (or homotopical) definition of quandle/rack homology.

Multiple group racks and cocycle invariants of surfaces in the 3-sphere

Shosaku Matsuzaki

Abstract. A multiple group rack was introduced for colorings of oriented compact surfaces in S^3, which yields isotopy invariants. We introduce (co)homology theory of multiple group racks and cocycle invariants of oriented compact surfaces in S^3.

This is a joint work with Tomo Murao (Waseda University).

1 A oriented spatial surface

An *oriented spatial surface* is an oriented compact surface embedded in the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Two oriented spatial surfaces F and F' are *equivalent* if there is an orientation-preserving self-homeomorphism h of S^3 such that $h(F) = F'$ and h respects the orientations of F and F'.

In this manuscript, we suppose the following conditions:

- each component of any oriented spatial surface F has non-empty boundary,
- F has no disk components and has no annuli components.

A *spatial graph* is a finite graph embedded in S^3. Let D be a diagram of a spatial trivalent graph. We obtain an oriented spatial surface F from D by taking a regular neighborhood of D in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and perturbing it around all crossings of D according to its over/under information, and we give an orientation so that the front side of F faces into the positive direction of the z-axis of $\mathbb{R}^3$ (see Fig. 1). Then we call D a *diagram* of F. Any oriented spatial surface is equivalent to an oriented spatial surface obtained by the process (see [5]).

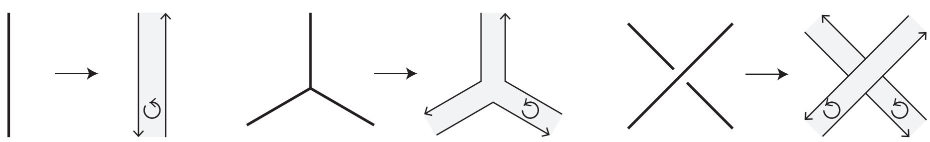


Figure 1: The process for obtaining an oriented spatial surface.

Theorem 1.1 ([5]). *Two oriented spatial surfaces are equivalent if and only if their diagrams are related by finitely many Reidemeister moves on S^2 depicted in Fig. 2.*

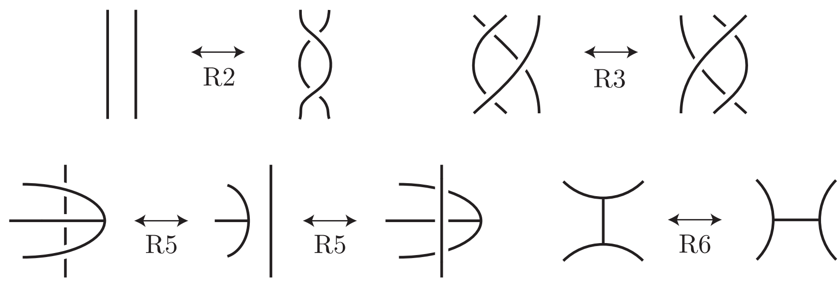


Figure 2: The Reidemeister moves for oriented spatial surfaces.

A *Y-orientation* of a trivalent graph G is a direction of all edges of G satisfying that every vertex of G is both the initial vertex of a directed edge and the terminal vertex of a directed edge (see Fig. 3).

Let D be a diagram of a Y-oriented spatial trivalent graph G. We have the oriented spatial surface F obtained from D by forgetting the Y-orientation. Then we call D a *Y-oriented diagram* of F. It is known that any oriented spatial surface is represented by some Y-oriented diagram. *Y-oriented Reidemeister moves* are the moves depicted in Fig. 2 between two diagrams with Y-orientations.

Theorem 1.2. *Two oriented spatial surfaces are equivalent if and only if their Y-oriented diagrams are related by finitely many Y-oriented Reidemeister moves on* S^2.

2 A multiple group rack and a coloring of oriented spatial surfaces

A *rack* [1] is a non-empty set Q with a binary operation $* : Q^2 \to Q$ satisfying the following axioms:

- For any $a \in Q$, the map $S_a : Q \to Q$ defined by $S_a(x) = x * a$ is bijective.
- For any $a, b, c \in Q$, $(a * b) * c = (a * c) * (b * c)$.

A rack $Q = (Q, *)$ is a *quandle* [4, 6] if $a * a = a$ for any $a \in Q$.

Definition 2.1 ([3]). A *multiple group rack* $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ is a disjoint union of groups G_λ ($\lambda \in \Lambda$) with a binary operation $* : X^2 \to X$ satisfying the following axioms:

(1) For any $x \in X$ and $y_1, y_2 \in G_\lambda$, $x * (y_1 y_2) = (x * y_1) * y_2$ and $x * e_\lambda = x$, where e_λ is the identity of G_λ.

(2) For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

(3) For any $x_1, x_2 \in G_\lambda$ and $y \in X$, $(x_1 x_2) * y = (x_1 * y)(x_2 * y)$, where $x_1 * y$, $x_2 * y \in G_\mu$ for some $\mu \in \Lambda$.

A multiple group rack is a rack with the binary operation $*$. A multiple group rack $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ is called a *multiple conjugation quandle* [2] if $x_1 * x_2 = x_2^{-1}x_1x_2$ for any $\lambda \in \Lambda$ and $x_1, x_2 \in G_\lambda$. A multiple conjugation quandle, which is a multiple group rack, is introduced to define coloring invariants for handlebody-knots.

An *X-set of a multiple group rack* $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ is a non-empty set Y equipped with a map $\star : Y \times X \to Y$ satisfying the following axioms.

(1) For any $\upsilon \in Y$, $\lambda \in \Lambda$ and $x_1, x_2 \in G_\lambda$, $\upsilon * e_\lambda = \upsilon$, $\quad \upsilon \star (x_1x_2) = (\upsilon \star x_1) \star x_2$, where e_λ is the identity of G_λ.

(2) For any $\upsilon \in Y$ and $x, y \in X$, $(\upsilon \star x) \star y = (\upsilon \star y) \star (x * y)$.

Let $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ be a multiple group rack and let D be a Y-oriented diagram of an oriented spatial surface. We denote by $\mathcal{A}(D)$ the set of arcs of D, where an arc is a piece of a curve each of whose endpoints is an undercrossing or a vertex. We denote by $\mathcal{R}(D)$ the set of complementary regions of D.

An *X-coloring* of D is a map $c : \mathcal{A}(D) \to X$ satisfying the conditions (i), (ii) and (iii) depicted in Fig. 3 at each crossing and vertex of D. We denote by $\mathrm{Col}_X(D)$ the set of all X-colorings of D. An *X_Y-coloring* is a map $c : \mathcal{A}(D) \sqcup \mathcal{R}(D) \to X$ such that the restriction $c|_{\mathcal{A}(D)}$ is an X-coloring of D, and that the condition (iv) in Fig. 3 is satisfied at any region of D. We denote by $\mathrm{Col}_X(D)_Y$ the set of all X_Y-colorings of D.

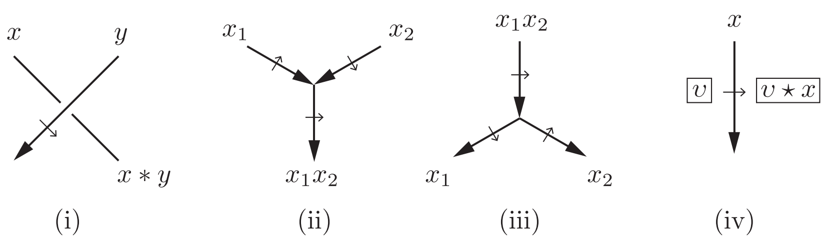


Figure 3: Rules of a coloring, where $\upsilon \in Y$, $x, y \in X$ and $x_1, x_2 \in G_\lambda$.

Theorem 2.2. *Let $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ be a multiple group rack and Y an X-set. If Y-oriented diagram D and D' represent equivalent oriented spatial surfaces, then $\#\,\mathrm{Col}_X(D)_Y = \#\,\mathrm{Col}_X(D')_Y$.*

3 (Co)homology of a multiple group rack

In this section, we suppose that $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ is a multiple group rack, and that Y is an X-set.

We define a free abelian group $C_n(X)_Y$ by

$$C_n(X)_Y = \begin{cases} \mathbb{Z}\left[\displaystyle\bigsqcup_{n_1+\cdots+n_k=n} Y \times \prod_{i=1}^{k}\Big(\bigsqcup_{\lambda\in\Lambda} {G_\lambda}^{n_i}\Big)\right] & (n \in \mathbb{Z}_{\geq 0}),\\ 0 & (n \in \mathbb{Z}_{<0}).\end{cases}$$

Here, $\mathbb{Z}[S]$ means the free abelian group generated by a set S, and the indices $n_1, \dots, n_k$ in the disjoint union run over all the partitions of n. We represent

$$(\upsilon; x_{1,1}, \dots, x_{1,n_1}; \dots; x_{k,1}, \dots, x_{k,n_k}) \in Y \times \bigsqcup_{\lambda\in\Lambda} {G_\lambda}^{n_1} \times \cdots \times \bigsqcup_{\lambda\in\Lambda} {G_\lambda}^{n_k}$$

by the noncommutative multiplication form $\langle\upsilon\rangle\langle x_{1,1}, \dots, x_{1,n_1}\rangle \cdots \langle x_{k,1}, \dots, x_{k,n_k}\rangle$.

Let $\boldsymbol{x}$ be a sequence $x_1, \dots, x_m$ of elements of G_λ. For $x \in X$, the sequence $x_1 * x, \dots, x_n * x$ is denoted by $\boldsymbol{x} * x$. For $\langle\upsilon\rangle\langle\boldsymbol{x}_1\rangle \cdots \langle\boldsymbol{x}_k\rangle \in Y \times \bigsqcup G_\lambda^{n_1} \times \cdots \times \bigsqcup G_\lambda^{n_k}$, the notation $\langle\upsilon\rangle\langle\boldsymbol{x}_1\rangle \cdots \langle\boldsymbol{x}_k\rangle * x$ means $\langle\upsilon \star x\rangle\langle\boldsymbol{x}_1 * x\rangle \cdots \langle\boldsymbol{x}_k * x\rangle$.

For $\langle\boldsymbol{x}\rangle = \langle x_1, \dots, x_m\rangle$, where $x_1, \dots, x_m$ is a sequence of elements of G_λ for some $\lambda \in \Lambda$, we define an operator $\widetilde{\partial}\langle\boldsymbol{x}\rangle$ by

$$\widetilde{\partial}\langle\boldsymbol{x}\rangle = *x_1\langle\boldsymbol{x}^0\rangle + \sum_{i=1}^{m}(-1)^i\langle\boldsymbol{x}^i\rangle.$$

Here, $\boldsymbol{x}^i$ is the sequence $x_1, \dots, x_i x_{i+1}, \dots, x_m$ for any i with $0 < i < m$, $\boldsymbol{x}^0$ is the sequence $x_2, \dots, x_m$, and $\boldsymbol{x}^m$ is the sequence $x_1, \dots, x_{m-1}$. We set $\langle\boldsymbol{x}^0\rangle = \langle\boldsymbol{x}^i\rangle = \langle\rangle$ if $m = 1$.

We denote by $|\boldsymbol{x}|$ the length of $\boldsymbol{x} = x_1, \dots, x_m$, that is, $|\boldsymbol{x}| = m$ and set $|\langle\upsilon\rangle\langle\boldsymbol{x}_1\rangle \cdots \langle\boldsymbol{x}_k\rangle| = |\boldsymbol{x}_1| + \cdots + |\boldsymbol{x}_k|$. We define the boundary homomorphism $\partial_n : C_n(X)_Y \to C_{n-1}(X)_Y$ by

$$\partial_n\left(\langle\upsilon\rangle\langle\boldsymbol{x}_1\rangle \cdots \langle\boldsymbol{x}_k\rangle\right) = \begin{cases} \displaystyle\sum_{i=1}^{k}(-1)^{|\langle\upsilon\rangle\langle\boldsymbol{x}_1\rangle\cdots\langle\boldsymbol{x}_{i-1}\rangle|}\langle\upsilon\rangle\langle\boldsymbol{x}_1\rangle \cdots \widetilde{\partial}\langle\boldsymbol{x}_i\rangle \cdots \langle\boldsymbol{x}_k\rangle & (n \in \mathbb{Z}_{>0}) \\ 0 & (n \in \mathbb{Z}_{\leq 0}), \end{cases}$$

where $\langle\upsilon\rangle\langle\boldsymbol{x}_1\rangle \cdots \langle\boldsymbol{x}_k\rangle$ is a generator of $C_n(X)_Y$.

Proposition 3.1. *$C_*(X)_Y := (C_n(X)_Y, \partial_n)_{n\in\mathbb{Z}}$ is a chain complex.*

We call $C_*(X)_Y$ the *multiple group rack chain complex* of X with Y. We denote by $H_n(X)_Y$ the n-th homology group of $C_*(X)_Y$. For an abelian group A, the cochain complex $C^*(X;A)_Y := (C^n(X;A)_Y, \delta_n)_{n\in\mathbb{Z}}$ is defined in the ordinary way.

For an X_Y-coloring c of a Y-oriented diagram D of an oriented spatial surface, we define the *local weight* $w(\chi; c) \in C_2(X)_Y$ at any $\chi \in U(D) \sqcup V(D)$ of D as depicted in Fig. 4, where $U(D)$ (resp. $V(D)$) is the set of crossings (resp. vertices) of D.

We define a 2-cocycle $W(D; c) \in C_2(X)_Y$ by

$$W(D;c) = \sum_{\chi\in U(D)\sqcup V(D)} w(\chi;c).$$

Proposition 3.2. *Suppose that D is a Y-oriented diagram of an oriented spatial surface, and that D' is a Y-oriented diagram obtained by applying one of Y-oriented Reidemeister moves on S^2 to D once. For any X_Y-coloring c of D and the corresponding X_Y-coloring c' of D', $[W(D;c)] = [W(D';c')]$ in $H_2(X)_Y$, where $[W]$ the homology class that contains W for a 2-cycle $W \in C_2(X)_Y$.*

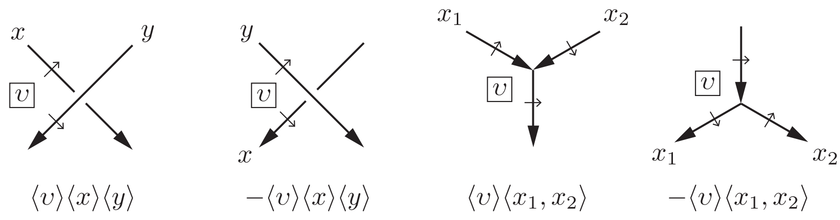


Figure 4: Local chains for a crossing or a vertex χ.

For a Y-oriented diagram D, we define the following multisets:

$$\begin{aligned}\mathcal{H}(D) &= \{[W(D;c)] \in H_2(X)_Y \mid c \in \mathrm{Col}_X(D)_Y\}\,,\\ \Phi_\theta(D) &= \{\theta(W(D;c)) \mid c \in \mathrm{Col}_X(D)_Y\}\,.\end{aligned}$$

Theorem 3.3. *The multisets $\mathcal{H}(D)$ and $\Phi_\theta(D)$ are invariants of oriented spatial surfaces, respectively.*

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Multiple group racks and cocycle invariants of surfaces in the 3-sphere

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joint work with
村尾 智 (早稲田大学)

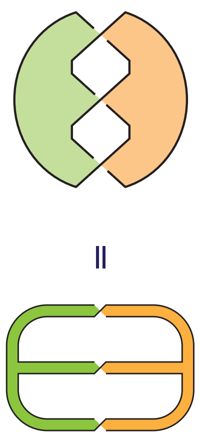
§1 Spatial surfaces, knots and handlebody-knots
§2 Colorings and (Co)homology theory
§3 Constructions of multiple group rack cocycles

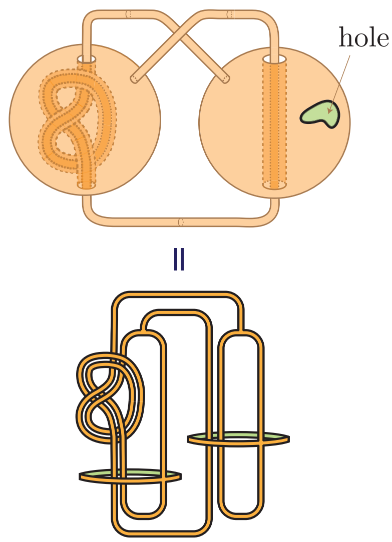
11 月 26 日（金）15:00 – 16:00
カンドルと対称空間 2021

§1 spatial surfaces, knots and handlebody-knots

A *spatial surface* is a compact surface $F \subset S^3$ s.t.

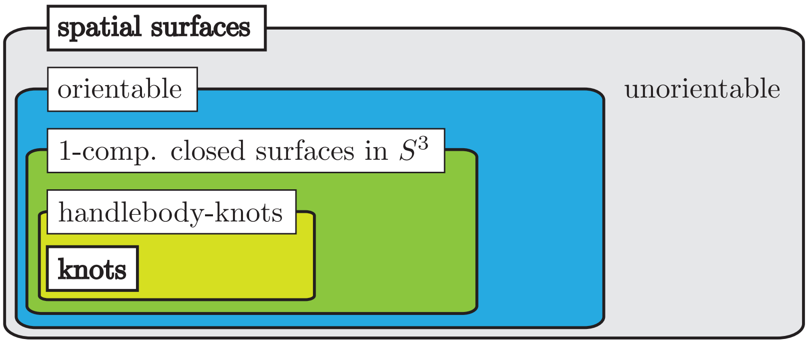
$$\partial C \neq \emptyset \text{ for any connected component } C \text{ of } F.$$





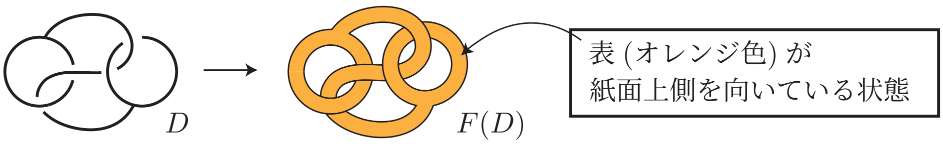
There are injections:

- $\{\text{knots}\} \hookrightarrow \{\text{handlebody-knots}\}$; $K \mapsto Nb(K)$,
- $\{\text{handlebody-knots}\} \hookrightarrow \{\text{1-comp. closed surfaces in } S^3\}$; $H \mapsto \partial H$,
- $\{\text{1-comp. closed surfaces in } S^3\} \hookrightarrow \{\text{spatial surfaces}\}$; $F \mapsto F \setminus (\text{open disk})$.



Today, I will talk about oriented spatial surfaces.

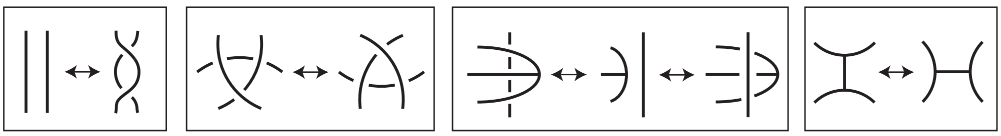
Any oriented spatial surface F is presented
by a diagram D of spatial trivalent graph s.t. $F \overset{\text{a.i.}}{\sim} F(D)$.



Theorem (M, oriented ver)

D, D': diagrams of oriented spatial surfaces F, F'. Then,

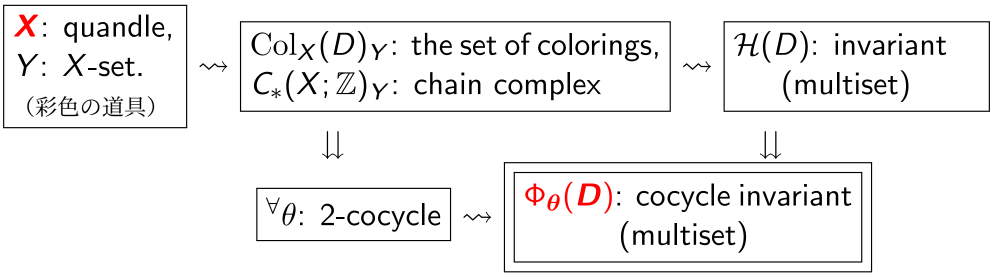
$$F \overset{\text{a.i.}}{\sim} F' \iff D \text{ and } D' \text{ are related by}$$



以上の変形で不変な量が，有向曲面の不変量．

§2 Colorings and (Co)homology theory

Process to obtain cocycle invariants



[Carter-Jelsovsky-Kamada-Langford-Saito '03], [Fenn-Rourke-Sanderson'95]
X: quandle (resp. rack) $\Rightarrow \Phi_\theta(D)$: inv. of links (framed links)

[Carter-Ishii-Saito-Tanaka '17]
X: multiple conjugation quandle $\Rightarrow \Phi_\theta(D)$: inv. of handlebody-links

[M-Murao]
X: *multiple group rack* $\Rightarrow \Phi_\theta(D)$: inv. of oriented spatial surfaces

Definition (Fenn–Rourke '92)
A set $(Q, *)$ with a binary operation $*$ is a *rack* if $\forall a, b, c \in Q$,

- $*a : Q \to Q$; $q \mapsto q * a$: bijective.
- $(a * b) * c = (a * c) * (b * c)$.

A rack $(Q, *)$ is a *quandle* if $a * a = a$ for any $a \in Q$.

A set $(Y, \star)$ with a map $\star : Y \times Q \to Y$ is a *Q-set of a rack Q* if $^\forall a, b \in Q$, $^\forall \upsilon \in Y$

- $\star a : Y \to Y$; $y \mapsto y \star a$: bijective,
- $(\upsilon \star a) \star b = (\upsilon \star b) \star (a * b)$.

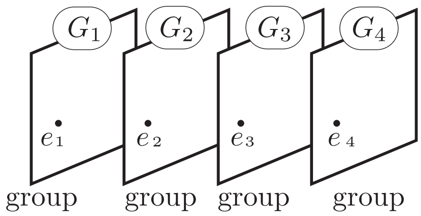
A map c : {arcs of D} $\sqcup$ {regons of D} $\to Q$ is an *Q_Y-coloring* of D if

Definition (Ishii–M–Murao ’20)

$X := \bigsqcup_{\lambda \in \Lambda} G_\lambda$, where $\{G_\lambda\}_{\lambda\in\Lambda}$: a family of groups.

$(X, *)$: *multiple group rack* if $\forall x, y, z \in X$, $\forall x_1, x_2 \in G_\lambda$,

- $x * (x_1 x_2) = (x * x_1) * x_2$, $x * e_\lambda = x$,
- $(x * y) * z = (x * z) * (y * z)$,
- $(x_1 x_2) * x = (x_1 * x)(x_2 * x)$.

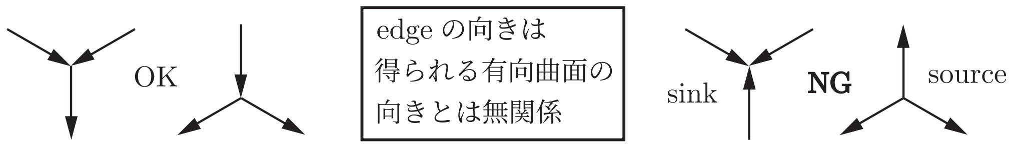


$$X := \bigsqcup_{\lambda\in\{1,2,3,4\}} G_\lambda$$

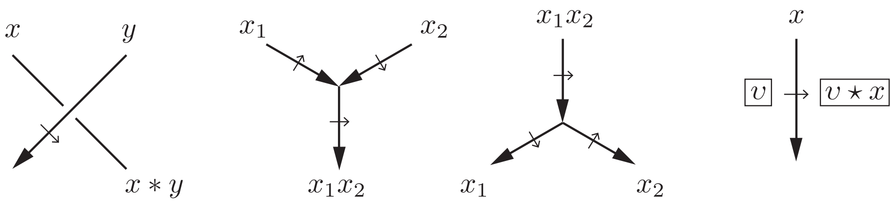
A set $(Y, \star)$ with a map $\star : Y \times X \to Y$ is an *X-set of a multiple group rack* if $^\forall y \in Y$,

1. $y * e_\lambda = y$, $y \star (x_1 x_2) = (y \star x_1) \star x_2$,
2. $(y \star x) \star y = (y \star y) \star (x * y)$.

Hereafter, any diagram of oriented spatial surfaces has *Y-orientation*:



A map c : {arcs of D} ⊔ {regons of D} → Q is an *Q_Y-coloring* of D, if



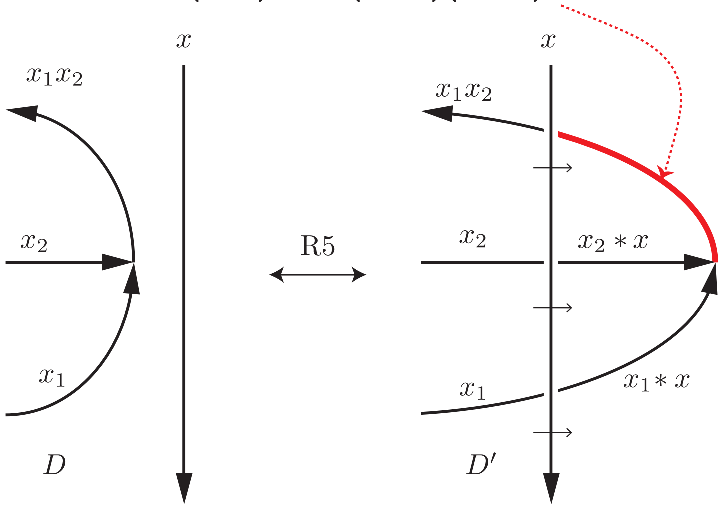
$\mathrm{Col}_X(D)_Y := \{X_Y\text{-colorings of } D\}$.

Proposition (Ishii-M-Murao‘20)

D: a diag, D' a diag. obtained from D by one Reidemeister move.
$\Longrightarrow$ $^\exists$ *a bijection $f : \mathrm{Col}_X(D)_Y \to \mathrm{Col}_X(D')_Y$ s.t.*
Reidemeister move が行われた領域以外では D と D' は同じ彩色.
Therefore, $\#\,\mathrm{Col}_Q(D)_Y$ is an invariant of oriented spatial surfaces.

Sketch of Proof （X-set Y が trivial の 場合）.

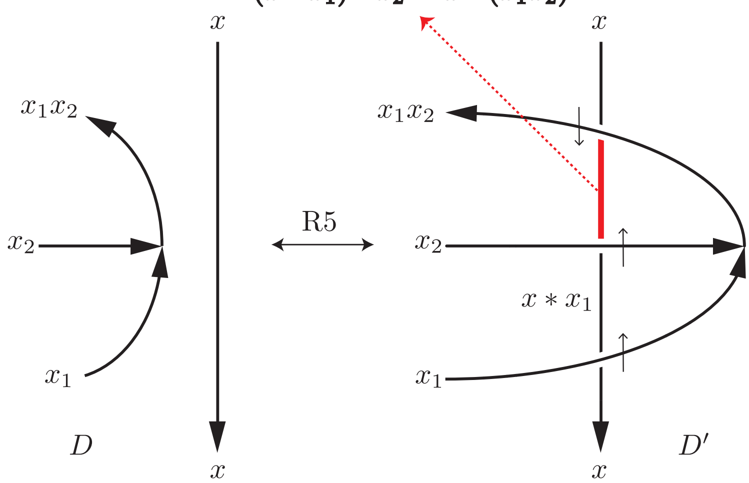
$$(x_1x_2) * x = (x_1 * x)(x_2 * x)$$



（注）x_1，x_2 は同じ群 G_λ に属する.

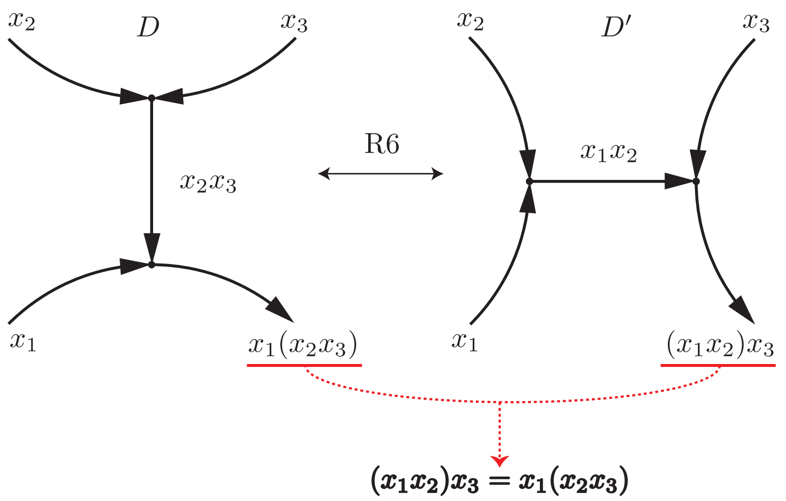
Sketch of Proof （X-set Y が trivial の 場合）.

$$(x * x_1) * x_2 = x * (x_1x_2)$$



（注）x_1，x_2 は同じ群 G_λ に属する.

Sketch of Proof （X-set Y が trivial の 場合）.



（注）x_1，x_2，x_3 は同じ群 G_λ に属する.

Chain complex of multiple group racks

For a multiple group rack $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$, we define $C_n(X)_Y$ by

$$C_n(X)_Y := \begin{cases} \mathbb{Z}\left[\bigsqcup_{n_1+\cdots+n_k=n} Y \times \prod_{i=1}^{k}\left(\bigsqcup_{\lambda\in\Lambda} {G_\lambda}^{n_i}\right)\right] & (n \in \mathbb{Z}_{\geq 0}), \\ 0 & (n \in \mathbb{Z}_{<0}). \end{cases}$$

(Ex.) $C_3(X)_Y$

$$= \mathbb{Z}\Bigg[\left(Y \times \bigsqcup_{\lambda\in\Lambda} G_\lambda \times \bigsqcup_{\lambda\in\Lambda} G_\lambda \times \bigsqcup_{\lambda\in\Lambda} G_\lambda\right) \bigsqcup \left(Y \times \bigsqcup_{\lambda\in\Lambda} G_\lambda \times \bigsqcup_{\lambda\in\Lambda} {G_\lambda}^2\right)$$
$$\bigsqcup \left(Y \times \bigsqcup_{\lambda\in\Lambda} {G_\lambda}^2 \times \bigsqcup_{\lambda\in\Lambda} G_\lambda\right) \bigsqcup \left(Y \times \bigsqcup_{\lambda\in\Lambda} {G_\lambda}^3\right)\Bigg]$$

$$= \mathbb{Z}\Bigg[\left\{ \langle \upsilon\rangle\langle x\rangle\langle y\rangle\langle z\rangle \;\middle|\; \begin{array}{l} \upsilon \in Y, \\ x, y, z \in X \end{array} \right\} \bigsqcup \left\{ \langle \upsilon\rangle\langle x\rangle\langle y_1, y_2\rangle \;\middle|\; \begin{array}{l} \upsilon \in Y,\ x \in X \\ \lambda \in \Lambda,\ y_1, y_2 \in G_\lambda \end{array} \right\}$$
$$\bigsqcup \left\{ \langle \upsilon\rangle\langle x_1, x_2\rangle\langle y\rangle \;\middle|\; \begin{array}{l} \upsilon \in Y,\ y \in X, \\ \lambda \in \Lambda,\ x_1, x_2 \in G_\lambda \end{array} \right\} \bigsqcup \left\{ \langle \upsilon\rangle\langle x_1, x_2, x_3\rangle \;\middle|\; \begin{array}{l} \upsilon \in Y,\ \lambda \in \Lambda, \\ x_1, x_2, x_3 \in G_\lambda \end{array} \right\}\Bigg].$$

We define the boundary hom $\partial_n : C_n(X)_Y \to C_{n-1}(X)_Y$ by

$$\partial_n(\langle \upsilon\rangle\langle \mathbf{x}_1\rangle\cdots\langle \mathbf{x}_k\rangle) = \begin{cases} \displaystyle\sum_{i=1}^{k}(-1)^{|\langle\upsilon\rangle\langle \mathbf{x}_1\rangle\cdots\langle \mathbf{x}_{i-1}\rangle|}\langle\upsilon\rangle\langle \mathbf{x}_1\rangle\cdots\widetilde{\partial}\langle \mathbf{x}_i\rangle\cdots\langle \mathbf{x}_k\rangle & (n\in\mathbb{Z}_{>0})\\ 0 & (n\in\mathbb{Z}_{\le 0})\end{cases}$$

where $\widetilde{\partial}\langle x_1,\dots,x_m\rangle = *x_1\langle x_2,\dots,x_m\rangle + \sum_{i=1}^{m}(-1)^i\langle x_1,\dots,x_ix_{i+1},\dots,x_m\rangle$.

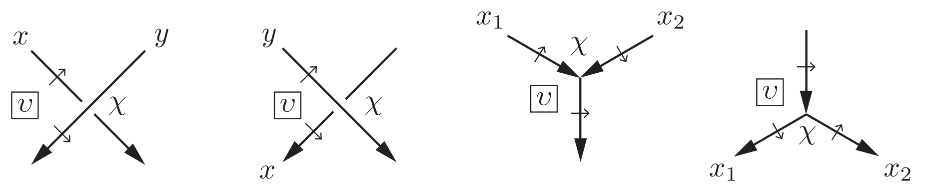
(Ex.)

$$\begin{aligned}\partial_3(\langle\upsilon\rangle\langle x\rangle\langle y\rangle\langle z\rangle) &= \langle\upsilon\star x\rangle\langle y\rangle\langle z\rangle + \langle\upsilon\rangle\langle x\rangle\langle z\rangle + \langle\upsilon\star z\rangle\langle x*z\rangle\langle y*z\rangle\\ &\quad - \langle\upsilon\rangle\langle y\rangle\langle z\rangle - \langle\upsilon\star y\rangle\langle x*y\rangle\langle z\rangle - \langle\upsilon\rangle\langle x\rangle\langle y\rangle.\\ \partial_3(\langle\upsilon\rangle\langle x\rangle\langle y_1,y_2\rangle) &= \langle\upsilon\star x\rangle\langle y_1,y_2\rangle + \langle\upsilon\rangle\langle x\rangle\langle y_1y_2\rangle\\ &\quad - \langle\upsilon\rangle\langle y_1,y_2\rangle - \langle\upsilon\star y_1\rangle\langle x*y_1\rangle\langle y_2\rangle - \langle\upsilon\rangle\langle x\rangle\langle y_1\rangle.\\ \partial_3(\langle\upsilon\rangle\langle x_1,x_2,x_3\rangle) &= \langle\upsilon\star x_1\rangle\langle x_2,x_3\rangle + \langle\upsilon\rangle\langle x_1,x_2x_3\rangle\\ &\quad - \langle\upsilon\rangle\langle x_1x_2,x_3\rangle - \langle\upsilon\rangle\langle x_1,x_2\rangle.\end{aligned}$$

Proposition (M-Murao)

$C_*(X)_Y := (C_n(X)_Y, \partial_n)_{n\in\mathbb{Z}}$ is a chain complex.
The cochain complex $C^*(X;A)_Y := (C^n(X;A)_Y, \delta_n)_{n\in\mathbb{Z}}$ is also defined.

c : X_Y-coloring of a diagram D of an oriented spatial surf F,
we define *local weight* $w(\chi;c)\in C_2(X)_Y$ at any crossing or vertex χ:



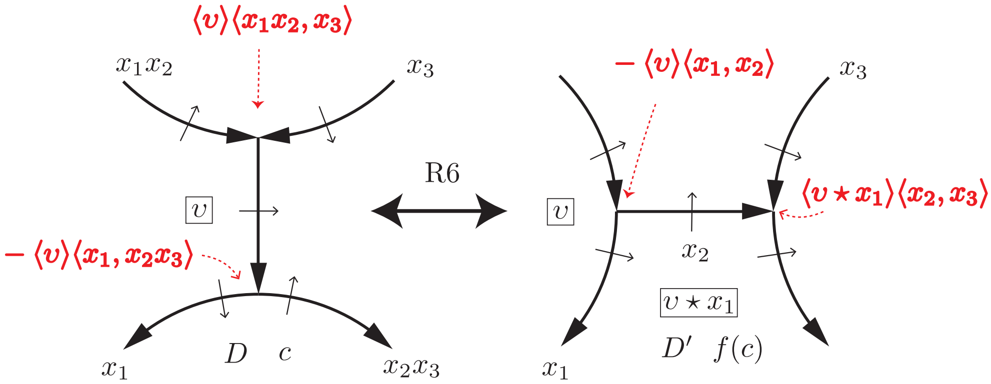
$\langle\upsilon\rangle\langle x\rangle\langle y\rangle$ $\quad -\langle\upsilon\rangle\langle x\rangle\langle y\rangle$ $\quad \langle\upsilon\rangle\langle x_1,x_2\rangle$ $\quad -\langle\upsilon\rangle\langle x_1,x_2\rangle$

And we set $W(D;c) := \sum_{\chi} w(\chi;c)$, which is a cycle.

Theorem (M-Murao)

1. *The multiset $\mathcal{H}(D)$ is an inv. of oriented spatial surfaces.*
   $$\mathcal{H}(D) := \{[W(D;c)]\in H_2(X)_Y \mid c\in \mathrm{Col}_X(D)_Y\}.$$
2. *For $\theta : C_2(X)_Y \to A$ (Abelian group) : a 2-cocycle of $C^*(X;A)_Y$, the multiset $\Phi_\theta(D)$ is inv.of oriented spatial surfaces:*
   $\Phi_\theta(D) := \{\theta(W(D;c)) \mid c\in\mathrm{Col}_X(D)_Y\}$.

D: a diag, D' a diag. obtained from D by one Reidemeister move.
$f : \mathrm{Col}_X(D)_Y \to \mathrm{Col}_X(D')_Y$:a bijection s.t.
Reidemeister move が行われた領域以外では D と D' は同じ彩色.



$$
\begin{aligned}
&W(D'; f(c)) - W(D; c)\\
&= \langle \upsilon \star x_1 \rangle \langle x_2, x_3 \rangle + \langle \upsilon \rangle \langle x_1, x_2x_3 \rangle - \langle \upsilon \rangle \langle x_1x_2, x_3 \rangle - \langle \upsilon \rangle \langle x_1, x_2 \rangle\\
&= \partial_3(\langle \upsilon \rangle \langle x_1, x_2, x_3 \rangle)
\end{aligned}
$$

Then, $[W(D'; f(c))] = [W(D; c)]$.

これまでまとめ：

- multiple group rack の ラックコサイクルさえ見つかれば, oriented spatial surfaces の不変量が得られる.

§3 Constructions of multiple group rack cocycles.

- そもそも multiple group rack の具体例は？
- multiple group rack があったとして, コサイクルはどう作るのか？
  → （一つの答え） rack cocycle から作る.

Q: a finite rack, Y: a finite Q-set, A: an Aberian group,

$$\text{type } Q_Y := \min \left\{ n \in \mathbb{N} \;\middle|\; \begin{array}{ll} a *^n b = a & ({}^\forall a, b \in Q), \\ \upsilon \star^n c = \upsilon & ({}^\forall \upsilon \in Y,\ {}^\forall c \in Q). \end{array} \right\}$$

$\theta : C_2^{\mathrm{R}}(Q; A)_Y \to A$: a 2-cocycle of Q with Y.

Proposition (Ishii-M-Murao)

Put $\bigsqcup_{a \in Q}(\{a\} \times \mathbb{Z}_\kappa) = Q \times \mathbb{Z}_\kappa$, *where* κ: *a multiple of* type Q_Y.

1. $(Q \times \mathbb{Z}_\kappa, \widetilde{*})$: *a multiple group rack if* ${}^\forall i \in \mathbb{Z}$ *and* ${}^\forall a, b \in Q$,
   $(a, [i]) \,\widetilde{*}\, (b, [j]) := (a *^j b, [i]), \quad (a, [i])\,(a, [j]) := (a, [i + j])$.
2. $(Y, \widetilde{\star})$: *an* $(Q \times \mathbb{Z}_\kappa)$*-set with the map* $\widetilde{\star} : Y \times (Q \times \mathbb{Z}_\kappa) \to Y$ *by*
   $\upsilon \,\widetilde{\star}\, (a, [i]) = \upsilon \star^i a$.

Proposition (M-Murao)

The following 2-cochain $\Theta : C_2\,(Q \times \mathbb{Z}_\kappa)_Y \to A$ *is cocycle*

$$\Theta\left(\langle\upsilon\rangle\langle(a, [i])\rangle\langle(b, [j])\rangle\right) = \sum_{p=1}^{i}\sum_{q=1}^{j} \theta\left(\langle\upsilon \star^{p-1} a \star^{q-1} b\rangle\langle a *^{q-1} b\rangle\langle b\rangle\right),$$

$$\Theta\left(\langle\upsilon\rangle\langle(a, [i]), (a, [j])\rangle\right) = 0, \quad \text{where } 1 \le i \le \kappa,\ 1 \le j \le \kappa.$$

if $\Theta\left(\langle\upsilon\rangle\langle(a, [\kappa])\rangle\langle(b, [1])\rangle\right) = \Theta\left(\langle\upsilon\rangle\langle(a, [1])\rangle\langle(b, [\kappa])\rangle\right) = 0$ $({}^\forall \upsilon \in Y,\ {}^\forall a, b \in Q)$.

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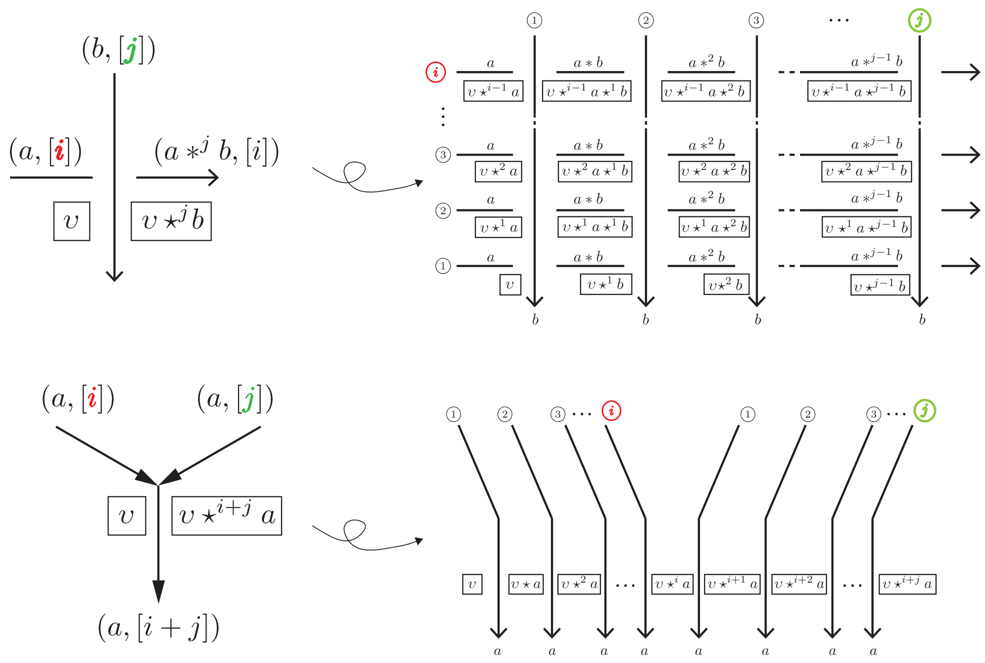

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Corollary

$\mathbb{Z}_p$: *the cyclic group of order* p.
$\kappa := p \cdot \mathrm{type}\, Q_Y$.
The following 2-cochain $\Theta : C_2\left(Q \times \mathbb{Z}_\kappa\right)_Y \to \mathbb{Z}_p$ *is cocycle*

$$\Theta\left(\langle \upsilon\rangle\langle(a,[i])\rangle\langle(b,[j])\rangle\right) = \sum_{p=1}^{i}\sum_{q=1}^{j}\theta\left(\langle \upsilon \star^{p-1} a \star^{q-1} b\rangle\langle a *^{q-1} b\rangle\langle b\rangle\right),$$

$$\Theta\left(\langle \upsilon\rangle\langle(a,[i]),(a,[j])\rangle\right) = 0, \quad \text{where } 1 \le i \le \kappa,\ 1 \le j \le \kappa.$$

Next, we give a way to obtain rack cocycles from "other" rack cocycles.

Q: a finite rack, Y: a finite Q-set A: an Aberian group,
$\theta : C^2_{\mathrm{R}}(Q;A)_Y \to A$: a rach 2-cocycle of Q with Y.

Proposition (Ishii-M-Murao'20)

$e_1,\dots,e_m \in \{-1,1\}$, $i_1,\dots,i_m \in \{1,\dots,n\}$, *where* $m,n \in \mathbb{N}$. *Then,*

Q^n *is a rack with the binary operation defined by*

$(a_1,\dots,a_n) \,\overline{*}\, (b_1,\dots,b_n) = (a_1 *^{e_1} b_{i_1} *^{e_2} \cdots *^{e_m} b_{i_m},\dots,a_n *^{e_1} b_{i_1} *^{e_2} \cdots *^{e_m} b_{i_m})$.

And, Y *is also a* Q^n*-set with the map defined by*

$\upsilon \,\overline{\star}\, (a_1,\dots,a_n) = \upsilon \star^{e_1} b_{i_1} \star^{e_2} \cdots \star^{e_m} b_{i_m}$.

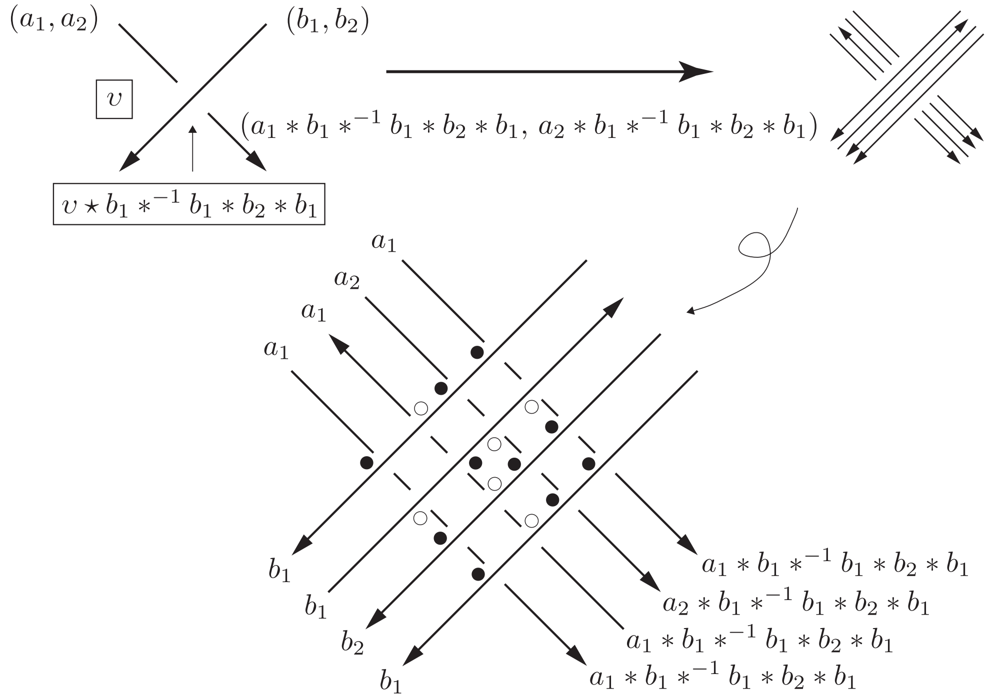
Proposition (M-Murao)

The following 2-cochain $\overline{\theta} : C^2_{\mathrm{R}}(Q^n;\mathbb{Z})_Y \to A$ *is a cocycle.*

$$\overline{\theta}\left(\langle \upsilon\rangle\langle(a_1,\dots,a_n)\rangle\langle(b_1,\dots,b_n)\rangle\right) = \sum_{p=1}^{m}\sum_{q=1}^{m} e_p e_q\, \theta\left(\langle \upsilon_{p,q}\rangle\langle a_{p,q}\rangle\langle b_{i_q}\rangle\right), \text{ where}$$

$$\upsilon_{p,q} := \upsilon \star^{e_1} a_{i_1} \star^{e_2} \cdots \star^{e_{p-1}} a_{i_{p-1}} \star^{-\delta(e_p,-1)} a_{i_p} \star^{e_1} b_{i_1} \star^{e_2} \cdots \star^{e_{q-1}} b_{i_{q-1}} \star^{-\delta(e_q,-1)} b_{i_q},$$

$$a_{p,q} := a_{i_p} *^{e_1} b_{i_1} *^{e_2} \cdots *^{e_{q-1}} b_{i_{q-1}} *^{-\delta(e_q,-1)} b_{i_q}.$$



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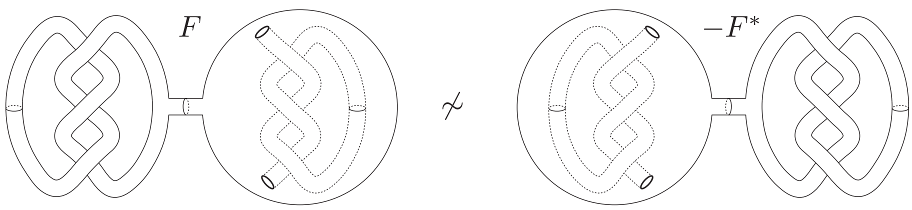

$R_3 := (\mathbb{Z}_3, *)$: a dihedral quandle $(i * j = 2j - i)$,
R_3: R_3-set,
θ: Mochiduki cocycle.

$\downarrow$ (proposition)

$Q := ((R_3)^3, \overline{*})$: a rack by $(a_1, a_2, a_3)\overline{*}(b_1, b_2, b_3) = (a_1 * b_1 * b_2 * b_3,\ a_2 \cdots,\ a_3 \cdots)$,
R_3: Q-set,
$\overline{\theta}$: cocycle of Q

$\downarrow$ (proposition)

$X := Q \times \mathbb{Z}_6$: multiple group rack,
R_3: X-set,
Θ: cocycle of X.

$\Phi_\Theta(\ \ F) = \{0$（46656 個), 1（34992 個）, 2（11664 個)$\}$
$\Phi_\Theta(-F^*) = \{0$（46656 個), 2（34992 個）, 1（11664 個)$\}$

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Quandles from the viewpoint of symmetric spaces — a survey

Hiroshi Tamaru

Abstract. In the conference "Quandles and Symmetric Spaces" held on December 2021, the author gave a survey of our recent studies on quandles from the viewpoint of symmetric spaces. This article records a summary and the slides of that talk.

1 Introduction

The notion of quandles is originated in knot theory, but it now plays important roles in many branches of mathematics. As one aspect, quandles can be regarded as a generalization of symmetric spaces, by extracting the point symmetry, and forgetting about manifold and topological structures.

When studying some mathematical objects, it would be useful to know several fundamental examples. One could ask:

What are the most fundamental nontrivial quandles?

Of course an answer would depend on the purpose of the study or interests of the respondent. In our studies, we suggest some candidates for the most fundamental quandles, from the viewpoint of symmetric spaces.

2 Two-point homogeneous quandles

Among Riemannian symmetric spaces, the rank one spaces can be regarded as the most fundamental examples. Unfortunately the notion of "rank" of quandles has not been defined so far, and hence we need a translation. One of the idea is the notion of two-point homogeneous Riemannian manifolds. A connected Riemannian manifold is said to be *two-point homogeneous* if any equidistant pair of points can be mapped to each other by an isometry. Then the classification result states that, Riemannian manifold is two-point homogeneous if and only if it is isometric to the Euclidean space or a symmetric space of rank one. Motivating by this result, we have introduced the notion of two-point homogeneous quandles in [8]. The classification of finite two-pint homogeneous quandles has been obtained in [14, 15].

This work was partly supported by Osaka City University Advanced Mathematical Institute: MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849. This work was supported by JSPS KAKENHI Grant Number JP19K21831.

3 Flat quandles

Among Riemannian symmetric spaces, the flat ones would be simpler. Recall that a Riemannian manifold is said to be *flat* if the Riemannian curvature tensor vanishes identically. It would be much harder to define curvatures for quandles, but a nice characterisation has been known. Namely, a Riemannian symmetric space is flat if and only if the group generated by $s_x \circ s_y$ is commutative ([7]), where s_x is the symmetry at x. Using this characterization, one can naturally define the notion of flat quandles. For flat quandles, the connectedness gives a strong restriction: a quandle is flat, finite, and connected if and only if it is a "discrete torus" with odd cardinality ([3]). On the other hand, there are many examples of disconnected flat quandles, even if we impose the homogeneity ([2]).

4 Subsets in quandles

The rank of a Riemannian symmetric space is defined as the dimension of a particular submanifold, a maximal flat totally geodesic submanifold. Therefore, it would be natural to study subsets in quandles, in particular, subsets related to the notion of flatness. In our recent study, we define the notion of *s-commutative subsets*, and classify maximal ones in some quandles ([6]). The cardinality of maximal s-commutative subsets, or the number of the conjugate classes of maximal s-commutative subsets, would measure the "complexity" of quandles. This is an effort to define the rank of quandles.

Recall that there are notions of poles and antipodal subsets for symmetric spaces. These notions have been introduced by Chen-Nagano ([1]), and play interesting roles in the theory of symmetric spaces. Note that the notion of s-commutative subsets is a generalization of these two notions, and would be new even for symmetric spaces. Therefore, our idea also contributes to the study on symmetric spaces, namely, it would give a new idea from the viewpoint of quandles.

References

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Quandles from the viewpoint of symmetric spaces - a survey

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研究集会「カンドルと対称空間」
2021/11/26

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概要

- カンドル ＝ 結び目の研究に由来する代数系;
- 対称空間 ⇒ カンドル (対称空間の “離散化”);
- 対称空間論を参考にしてカンドルの構造を調べたい.
- 今回のテーマ: 非自明かつ基本的なカンドルは何か？ (対称空間の立場から)

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1 定義

定義 1.1 (Joyce, Matveev (1982))**.**
集合 Q, 写像 $s : Q \to \mathrm{Map}(Q, Q)$ の組 (Q, s) が カンドル

:⇔ (S1) $\forall x \in Q$, $s_x(x) = x$.
(S2) $\forall x \in Q$, s_x は全単射.
(S3) $\forall x, y \in Q$, $s_x \circ s_y = s_{s_x(y)} \circ s_x$.

注意 1.2.

- 二項演算 $x * y$ $(= s_y(x)$ or $s_x(y))$ あるいは $x \triangleright y$ 等で書く場合も多い.
- $s_x^2 = \mathrm{id}$ となるものを 対合的 あるいは 圭 (**kei**) という.

定義 1.3 (cf. Helgason)**.**
連結リーマン多様体 M が リーマン対称空間
:⇔ • 各点に ∃ 等長変換 $s_x : M \to M$;
• $s_x^2 = \mathrm{id}$;
• x は s_x の孤立固定点.

命題 1.4 (cf. Joyce (1982))**.**

- リーマン対称空間はカンドル.
- Key: $s_x \circ s_y \circ s_x^{-1} = s_{s_x(y)}$.

注意 1.5.

- 多様体が対称空間 (対称多様体) であることも定義できる (cf. Loos). これもカンドル.
- k-対称空間 (s_x 位数 k) もカンドル.

定義 1.6.

- $f : (X, s^X) \to (Y, s^Y)$ が 準同型
  $:\Leftrightarrow f \circ s^X_x = s^Y_{f(x)} \circ f \ (\forall x \in X)$.
- 全単射な準同型を 同型 という.

定義 1.7.

- $\mathrm{Aut}(Q, s)$: 自己同型群;
- (Q, s) 等質 $:\Leftrightarrow \mathrm{Aut}(Q, s) \curvearrowright Q$ 推移的.

定義 1.8 (内部自己同型群)**.**

- $\mathrm{Inn}(Q, s) := \langle \{ s_x \mid x \in Q \} \rangle$;
- (Q, s) 連結 $:\Leftrightarrow \mathrm{Inn}(Q, s) \curvearrowright Q$ 推移的.

注意 1.9.

- 連結 $\Rightarrow$ 等質;
- カンドルは連結/等質とは限らない.

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2 例 (対称対の視点から)

定理 2.1 (cf. Helgason, Loos)**.**
連結対称空間 M は, 対称対 (以下をみたす (G, K, σ)) と対応;

- G 連結 Lie 群, K 閉部分群,
- $\sigma \in \mathrm{Aut}(G)$, $\sigma^2 = \mathrm{id}$,
- $\mathrm{Fix}(\sigma, G)^0 \subset K \subset \mathrm{Fix}(\sigma, G)$.

備考 2.2.

- $\mathrm{Fix}(\sigma, G) := \{ g \in G \mid \sigma(g) = g \}$.
- $\mathrm{Fix}(\sigma, G)^0$: その単位元連結成分.

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定義 2.3.

以下をみたす (G, K, σ) を <u>カンドル組</u>:

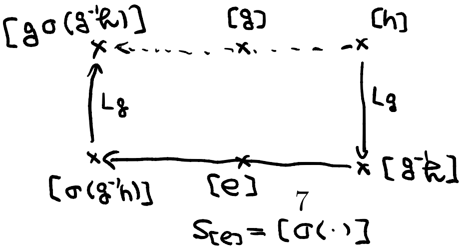
- G 群, K 部分群,
- $\sigma \in \mathrm{Aut}(G)$, $K \subset \mathrm{Fix}(\sigma, G)$.

命題 2.4 (cf. Joyce)**.**

- (G, K, σ) カンドル組 $\Rightarrow$ G/K は等質カンドル (by $s_{[g]}([h]) := [g\sigma(g^{-1}h)]$);
- 等質カンドルはこの方法で得られる.

備考 2.5.

- 上のカンドルを $Q(G, K, \sigma)$ で表す.



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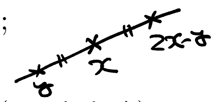
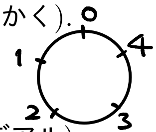
例 2.6 (加法群)**.**

- G 加法群のとき $(G, \{e\}, -\mathrm{id})$ はカンドル組 $(s_x(y) = 2x - y)$;
- $\mathbb{R}^n$ なら通常の点対称;

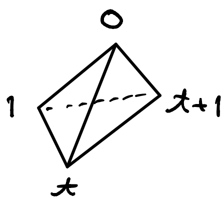


- $\mathbb{Z}_n$ なら二面体カンドル (R_n とかく).



例 2.7 (Alexander カンドル)**.**

- $G := \mathbb{Z}_m[t^{\pm 1}]/J$ (ただし J イデアル);
- $(G, \{0\}, L_t)$ カンドル組 $(L_t(x) = tx)$;
- $s_x(y) = x + t(-x + y) = (1 - t)x + ty$;
- $J = (t + 1)$ のとき二面体カンドル;
- $m = 2$, $J = (t^2 + t + 1)$ なら正四面体.



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3 二点等質性

備考 3.1 (スローガン)**.**

- 特別な (基本的な) カンドルは何か？

備考 3.2 (ここからのあらすじ)**.**

- 二点等質 (→ 階数 1？)
- 平坦 → s-可換 → subset (→ 階数？)

定義 3.3 (二点等質性)**.**

- (Q, s) が 二点等質
:⇔ $\mathrm{Inn}(Q, s) \curvearrowright Q$ 二重推移的 (i.e.,
$\forall (x_1, x_2), (y_1, y_2)$ $(x_1 \neq x_2,\ y_1 \neq y_2)$,
$\exists f \in \mathrm{Inn}(Q, s) :\ f(x_1, x_2) = (y_1, y_2)$)

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備考 3.4 (定義の気持ち)**.**

- 二点等質 ⇔ $\mathrm{Inn}(Q, s)$ が "とても大".

備考 3.5 (元概念)**.**

- リーマン多様体が 二点等質 :⇔ 等距離にある二点の組同士が等長変換で移る.
- FACT: リーマン多様体が二点等質 ⇔ $\mathbb{R}^n$ or "階数 1" 対称空間.

定理 3.6 (T., Wada, Vendramin ('13–'17))**.**
以下は同値:

- (Q, s) 二点等質;
- $(Q, s) \cong Q(\mathbb{F}_q, \{0\}, \sigma)$, ただし $\mathbb{F}_q$ は有限体 (を加法群とみたもの), $\sigma = L_a$ (左作用), a は原始元.

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注意 3.7.

- a ($\in \mathbb{F}_q$) が 原始元
  $:\Leftrightarrow \{a, a^2, \ldots, a^{q-1}\} = \mathbb{F}_q - \{0\}$.

例 3.8.

- $\mathbb{F}_3 = \mathbb{Z}_3$, $Q(\mathbb{F}_3, L_2) = R_3$;
- $\mathbb{F}_4 = \mathbb{Z}_2[t]/(t^2+t+1)$,
  $Q(\mathbb{F}_4, L_t) =$ “正四面体カンドル”;
- $\mathbb{F}_5 = \mathbb{Z}_5$, $Q(\mathbb{F}_5, L_2) \not\cong Q(\mathbb{F}_5, L_3)$.

備考 3.9.

- 各奇素数幕 $q = p^n$ に対し,
  $\exists(Q, s)$: 二点等質 s.t. $\#Q = q$
- 有限体と関係する理由は謎...

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4 平坦性

備考 4.1 (あらすじ)**.**

- 二点等質 ($\to$ 階数 1？)
- 平坦 $\to$ s-可換 $\to$ subset ($\to$ 階数？)

定義 4.2.

- (Q, s) が 平坦
  $:\Leftrightarrow G^0(Q, s) := \langle\{s_x \circ s_y\}\rangle$ が可換.

備考 4.3 (定義の気持ち)**.**

- 平坦 $\Leftrightarrow G^0(Q, s)$ が “とても小”.
- Note: $G^0(Q, s) \subset \mathrm{Inn}(Q, s)$.
- (Q, s) 連結 $\Leftrightarrow G^0(Q, s) \curvearrowright Q$ 推移的.

備考 4.4 (元概念)**.**

- リーマン対称空間の曲率 $\equiv 0$
  $\Leftrightarrow G^0(Q,s)$ が可換.

定理 4.5 (Ishihara-T. ('16))**.**
以下は同値:

- (Q,s) が有限, 連結, 平坦;
- $(Q,s) \cong R_{n_1} \times \cdots \times R_{n_k}$ with n_i 奇数.

備考 4.6.

- $R_{n_1} \times \cdots \times R_{n_k}$ は "離散トーラス".
- 次の定理の離散版: コンパクトリーマン対称空間が平坦 $\Leftrightarrow$ トーラス.
- $\mathrm{Dis}(Q,s) := \langle \{ s_x \circ s_y^{-1} \} \rangle$ が可換なカンドル ("medial") の研究もある.

5 s-可換性

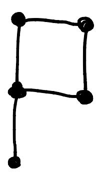
備考 5.1 (あらすじ)**.**

- 二点等質 ($\to$ 階数 1?)
- 平坦 $\to$ s-可換 $\to$ subset ($\to$ 階数?)

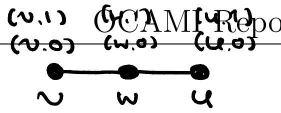
備考 5.2 (考えたこと)**.**

- 非連結 (等質) 平坦カンドルを調べよ.

注意 5.3 (道具の準備)**.**



- $G = (V, E)$ 無向単純グラフ, i.e.,
- V は頂点集合 ($\neq \emptyset$), E は辺集合,
- 辺に向き, 多重辺, ループはない.
- $e : V \times V \to \mathbb{Z}_2$ を隣接写像とする.
  ($e(v,w)$ は vw が辺なら 1, 違えば 0.)



定理 5.4 (Furuki-T.)**.**
無向単純グラフ $G = (V, E)$ に対し,

- $Q_G := V \times \mathbb{Z}_2$;
- $s_{(v,a)}(w, b) := (w, b + e(v, w))$;

と定めると,

- (Q_G, s) は非連結カンドル;
- $\mathrm{Inn}(Q_G, s)$ 可換;
- G が頂点推移的なら (Q_G, s) は等質.

$S_{(v,+1)}(w,b) = (w,b+1)$

$S_{(v,+1)}(u,c) = (u,c)$

備考 5.5.

- $G^0(Q, s) \subset \mathrm{Inn}(Q, s) \subset \mathrm{Aut}(Q, s)$;
- 平坦より “$\mathrm{Inn}(Q, s)$ 可換” は強い;
- $\mathrm{Inn}(Q, s)$ 可換な例が, 等質性を仮定しても豊富に存在.

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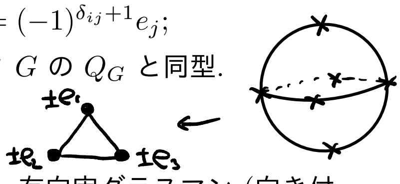
備考 5.6 (どこから思い付いたか)**.**

- この構成は, 対称空間からの着想...

例 5.7.

- S^n 内の $\{\pm e_1, \ldots, \pm e_{n+1}\}$ は s-可換;
- $s_{e_i}(\pm e_j) = (-1)^{\delta_{ij}+1} e_j$;
- 完全グラフ G の Q_G と同型.



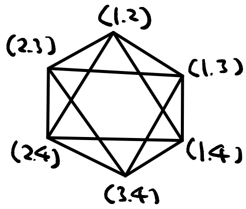
例 5.8.

- $G_k(\mathbb{R}^n)^\sim$: 有向実グラスマン (向き付き k-dim 部分空間の全体);
- s_V は向きも込めた V での折り返し;
- $\{\pm \mathrm{span}\{e_{i_1}, \ldots, e_{i_k}\}\}$ は s-可換.

$G_2(\mathbb{R}^4)^\sim$ のとき:

$S_{(1,2)}(1,3) = (1,-3) = -(1,3)$

$S_{(1,2)}(3,4) = (-3,-4) = (3,4)$



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6 部分集合

備考 6.1 (あらすじ)**.**

- 二点等質 (→ 階数 1？)
- 平坦 → s-可換 → **subset** (→ 階数？)

定義 6.2.

- A ($\subset (Q, s)$) が s-可換
  $:\Leftrightarrow \forall x, y \in A,\ s_x \circ s_y = s_y \circ s_x$.

命題 6.3.

- 極大 s-可換
  ⇒ 部分カンドル, $\mathrm{Inn}(A, s)$ 可換.

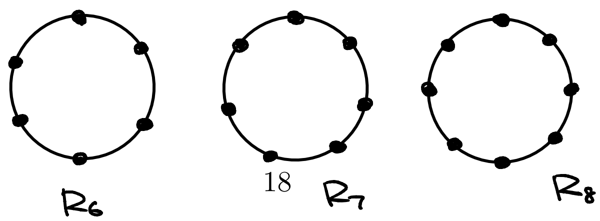
備考 6.4 (動機)**.**

- 新しい例を構成したい (as $G_k(\mathbb{R}^n)^\sim$);
- カンドル/対称空間の性質を引き継ぐ部分集合を探す (デザイン);
- 階数 を定義したい (対称空間の階数 := 極大平坦全測地的部分多様体の次元).

命題 6.5 (二面体カンドル)**.**

- A ($\subset R_n$) 極大 s-可換

$$\Rightarrow \#A = \begin{cases} 1 & (n \text{ 奇数}), \\ 2 & (n \equiv 2 \bmod 4), \\ 4 & (n \equiv 0 \bmod 4). \end{cases}$$



命題 **6.6.**

- $n \neq 2k$ or k 奇数とする;
- このとき, $G_k(\mathbb{R}^n)^\sim$ 内の極大 s-可換は次と合同: $\{\pm \mathrm{span}\{e_{i_1}, \ldots, e_{i_k}\}\}$.

注意 **6.7.**

- いくつかの対称空間/カンドルで調べた限りでは, 極大 s-可換は等質だった. 非等質なものはあるか？
- 多くの場合, 極大 s-可換は合同を除いて一意的. いつ成立するか？
- 極大 s-可換は, 全空間の情報を反映？

注意 **6.8** (補足)**.**

- 対蹠的 $\Rightarrow$ s-可換;
- A $(\subset (Q, s))$ が 対蹠的 (antipodal) $:\Leftrightarrow \forall x, y \in A,\ s_x(y) = y$.

注意 **6.9.**

- 対蹠集合 ... どのくらい大きな自明カンドルを含むか, という性質.
- 極大対蹠の決定は, 対称空間論でも未解決問題. (e.g. Spin(n), $G_k(\mathbb{R}^n)^\sim$)
- 極大 s-可換が決定できると, 極大対蹠の決定は有限カンドルの問題に帰着.

7 まとめ

対称空間の基本的な例:

- 二点等質
- 平坦
- ...

カンドルで定義したもの:

- 二点等質
- 平坦
  - 連結平坦
  - 等質 s-可換
  - s-可換 subset

今後に考えたいこと:

- 然るべく 階数 を定義できるか？

Thank you very much!