Topology and Combinatorics of Hessenberg Varieties

Osaka City University Advanced Mathematical Institute (OCAMI)
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Topology and Combinatorics of Hessenberg Varieties

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Abstract. This workshop held February 20, 2021 to conduct international research exchanges on Hessenberg varieties and related topics.

2020 Mathematics Subject Classification.
14M15, 14M25, 17B22, 05E40, 52B05, 05E10

Key words and Phrases.
Flag varieties, Hessenberg varieties, Peterson varieties, Schubert calculus, toric varieties, polytopes, cohomology rings

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Preface

This volume of OCAMI Reports summarizes the workshop “Hessenberg varieties 2021 in Osaka by Zoom” held from February 20th online because of the COVID-19 pandemic. This workshop was supported by “Osaka city University Advanced Mathematical Institute MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics.” The main focus of this workshop is subvarieties of the flag variety including Hessenberg varieties and torus orbit closures. This workshop consisted of a 100 minutes talk on “Pearson Conjecture: Regular Hessenberg varieties and Toric orbifolds” by Jongbaek Song (KIAS), a 100 minutes talk on “Coxeter matroids and the torus orbit closures in the flag varieties” by Seonjeong Park (KAIST), and a 100 minutes talk on “Bases of the cohomology spaces of regular semisimple Hessenberg varieties” by Jaehyun Hong. There were 25 participants in this workshop. This workshop conducted international research exchanges on Hessenberg varieties and related topics.

May 2021

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Organizers

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Pearson Conjecture: Regular Hessenberg varieties and Toric orbifolds

Jongbaek Song

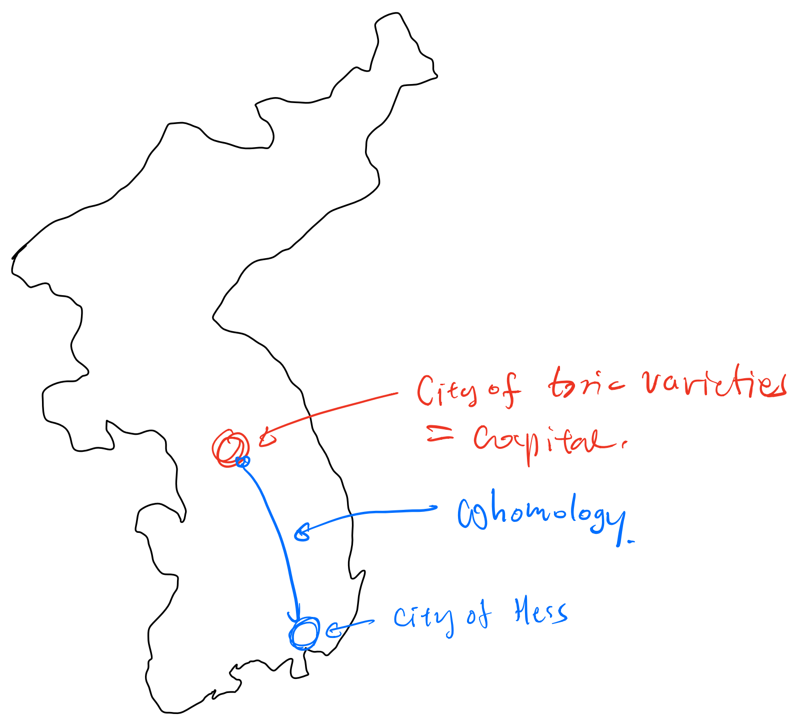
A regular nilpotent Hessenberg variety and a regular semisimple Hessenberg variety with a fixed Hessenberg space have a cohomological relationship, namely the cohomology ring of the former is isomorphic to the ring of invariant of the latter with respect to the Weyl group action. One extreme case of this relationship is given by Peterson varieties and permutohedral varieties, where the latter are smooth toric varieties. In this talk, we introduce a certain family of toric varieties having orbifold singularities and discuss how these objects interact with regular Hessenberg varieties from the cohomological point of view. This is a joint work with T. Horiguchi, M. Masuda and J. Shareshian.

PEARSON CONJECTURE : Regular Hess. Var & Toric Orbifolds

Jongbaek Song (KIAS)

(Jointly with T. Horiguchi, M. Masuda & J. Shareshian)

Hessenberg Varieties 2021. in Osaka. (February 20, 2021)



Pearson conjecture : A question about toric varieties with symmetries by reflections.

$\exists$ As a corollary : "Certain toric varieties" & "certain Hess. var"

"Toric orb. assoc. with "partitioned weight polytope".

regular Hess. var with "minimal Hess sp."

§ Hess. var.

- G : conn. simply conn. alg. gp $/\mathbb{C}$
- $\cup$ B : Borel subgp.
- $\mathfrak{g}, \mathfrak{b}$: their Lie algebras
- $\mathcal{H}$: Hess. space (a B-submodule containing $\mathfrak{b}$)

Def For $x \in \mathfrak{g}$, $\mathcal{H}$: (regular) Hess. sp.

$$\mathrm{Hess}(x, \mathcal{H}) = \{ gB \in G/B \mid g^{-1} \circ x \in \mathcal{H} \}$$

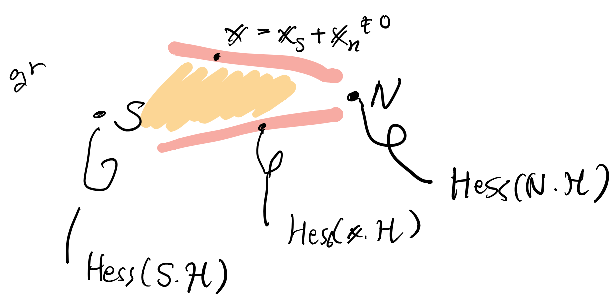
continuous ($x \in \mathfrak{g}^r$) — discrete

In type A_n: $x \in \mathfrak{sl}_n$, $x \sim \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_l \end{pmatrix}$ (Jordan blocks)

x : regular $\Leftrightarrow$ $c_i \neq c_j$ if $i \neq j$

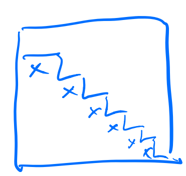
For fixed $\mathcal{H} \subseteq \mathfrak{g}$,

$\mu_{\mathcal{H}} : G \times_B \mathcal{H} \to \mathfrak{g}$, $[g : y] \mapsto g \circ y$. $\rightsquigarrow$ Lemma. $\mathrm{Hess}(x, \mathcal{H}) = \mu_{\mathcal{H}}^{-1}(x)$



Examples ① $\mathcal{H} = \mathfrak{g} \Rightarrow \mathrm{Hess}(x, \mathcal{H}) = G/B \quad \forall x \in \mathfrak{g}$.

② $\mathcal{H}_1 = \mathfrak{b} \oplus \bigoplus_{i=1}^{n} \mathfrak{g}_{-\alpha_i}$. "minimal Hess space"



(i) $\mathrm{Hess}(N, \mathcal{H}_1)$ = Peterson variety

(ii) $\mathrm{Hess}(S, \mathcal{H}_1)$ = toric variety assoc. with <u>Weyl chambers</u> (fan of)

normal fan of "weight polytope".

ex In type A_2.

In type B_2

§2. $H^*(\mathrm{Hess}(X, \mathcal{H}))$.

· For $X = S$: regular semisimple

$W \curvearrowright H^*(\mathrm{Hess}(S, \mathcal{H}))$ $\forall \mathcal{H}$.

"Tymoczko's dot action"

· <u>Theorem</u> [H. Abe – Harada – Horiguchi – Masuda] ← Type A

[T. Abe – Horiguchi – Masuda – Murai – Sato] ← Arbitrary Lie types.

$$H^*(\mathrm{Hess}(S, \mathcal{H}))^W \cong H^*(\mathrm{Hess}(N, \mathcal{H}))$$

· <u>When $\mathcal{H} = \mathcal{H}_1$</u> $\boxed{\mathrm{Hess}(N, \mathcal{H}_1) = \mathrm{Pet}}$

$\boxed{\mathrm{Hess}(S, \mathcal{H}_1) = \text{toric}}$.

Our playground $\mathcal{H}_1$ today!

<u>Theorem</u> [Fukukawa – Harada – Masuda] ← For type A

[Harada – Horiguchi – Masuda] ← ~~Classical~~ Any Lie types.

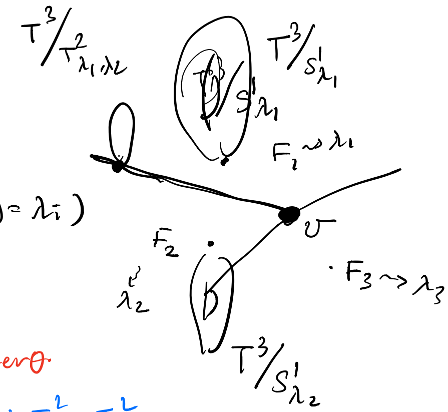
$$H^*(\mathrm{Pet}_{A_n}) = \mathbb{Q}[x_1 \cdots x_n] \Big/ \left\langle x_i\left(\tfrac{1}{2}x_{i-1} - x_i + \tfrac{1}{2}x_{i+1}\right) \,\middle|\, i = 1 \cdots n \right\rangle$$

~~$x_1 + \cdots + x_n = 0$~~

$x_0 = x_{n+1} = 0$.

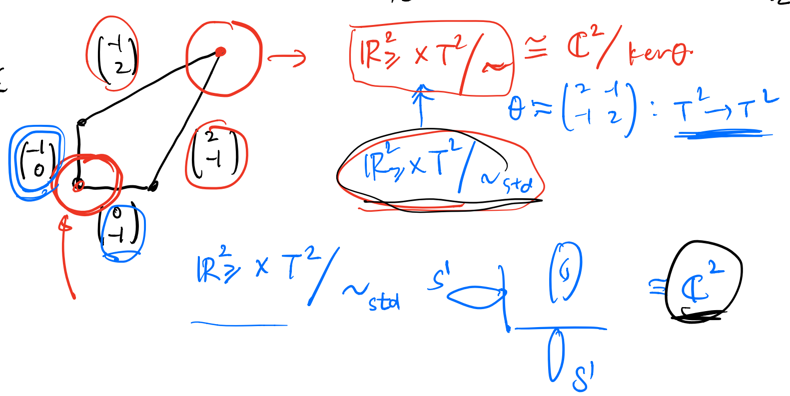
Ex $n = 2$. $\mathbb{Q}[x_1, x_2] \big/ \langle x_1(-2x_1 + x_2),\ x_2(x_1 - 2x_2) \rangle$.

§3. Toric variety (quick review)



- P: convex polytope of dim n.
- $F(P)$: set of facets of P.
- $\lambda : F(P) \to \mathbb{Z}^n \ (\cong \mathrm{Lie}_{\mathbb{Z}}(T^n))$. $\quad (\lambda(F_i) = \lambda_i)$
- $X(P,\lambda) := P \times T^n / \sim_\lambda$

Ex



- P^n: simple polytope, $v = F_{i_1} \cap \cdots \cap F_{i_n} \in V(P)$
- If $\{\lambda_{i_1} \cdots \lambda_{i_n}\}$ form a $\mathbb{Z}$-basis $\Rightarrow$ $X(P,\lambda)$ toric manifold.
  If $\{\lambda_{i_1} \cdots \lambda_{i_n}\}$ lin. indep $\Rightarrow$ $X(P,\lambda)$ toric "orbifold".

Theorem For toric mfd / orb. $\qquad m = |F(P)|$.

$$H^*(X(P,\lambda)) = \mathbb{Q}[x_1 \cdots x_m] / I + J \qquad J = \left\langle \sum_{j=1}^{m} \langle u, \lambda_j \rangle x_j \;\middle|\; u \in \mathrm{Lie}^*(T^n) \right\rangle$$

$$I = \langle x_{i_1} \cdots x_{i_k} \mid F_{i_1} \cap \cdots \cap F_{i_k} = \phi \rangle$$

Ex. ① $P_{A_n} = \mathrm{conv}(\sigma(1 \cdots n+1) \in \mathbb{R}^{n+1} \mid \sigma \in \mathfrak{S}_{n+1})$

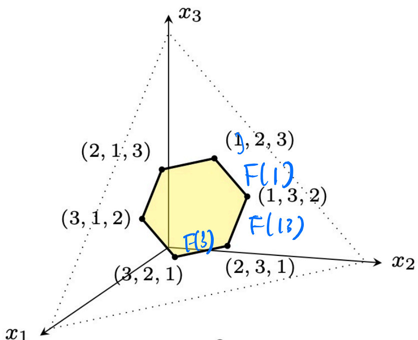
$$\subseteq \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; x_1 + \cdots + x_{n+1} = \frac{n(n+1)}{2} \right\}$$

↑ Parallel translation of the root space of A_n.

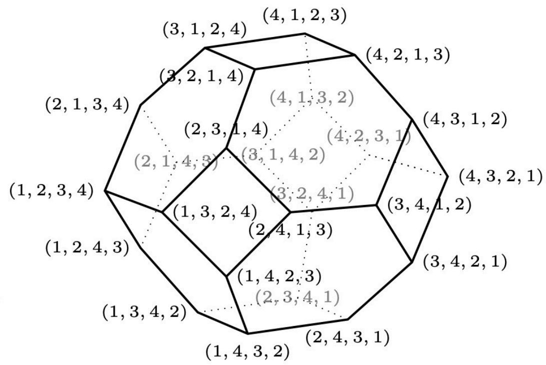
toric. var. of P_{A_n}

$$H^*(X_{A_n}) =$$

$$\big(H^*(\mathrm{Hess}(S, H_1)). \big)$$



$P_{A_2} \subset \mathbb{R}^3$



$P_{A_3} \subset \mathbb{R}^4$

① $F(P_{A_n}) \overset{1:1}{\longleftrightarrow} \emptyset \neq I \subset [n+1]$

② $F(I) \cap F(J) = \emptyset$ iff $I \not\subset J$ & $I \not\supset J$

③ P_{A_n}: flag polytope

④ normal vector of $F(I) = e_I = \sum_{i \in I} e_i$

E = rwt sp

$(0 \cdots 1 \cdots 0) \mapsto e_i$

$\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}/\mathbb{R}(1 \cdots 1)$

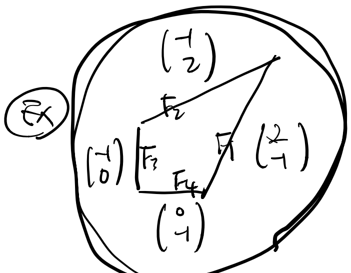
$E^* = L(T_{A_n}^\vee) \otimes \mathbb{R}$

"coweight sp."

$H^*(X_{A_n}) = \mathbb{Q}[x_I \mid \emptyset \neq I \subset [n+1]] / I + J$

① ②&③ ④

Ex



$$H^*(X(P,\lambda)) = \frac{\mathbb{Q}[x_1, x_2, x_3, x_4]}{\langle x_1x_3,\ x_2x_4\rangle + \langle 2x_1 - x_2 - x_3,\ -x_1 + 2x_2 - x_4\rangle}$$

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \end{pmatrix}$$

$$= \frac{\mathbb{Q}[x_1, x_2]}{\langle x_1(2x_1 - x_2),\ x_2(-x_1 + 2x_2)\rangle}$$

$I^n, (C_n \mid -\mathrm{Id})_{n \times 2n}$

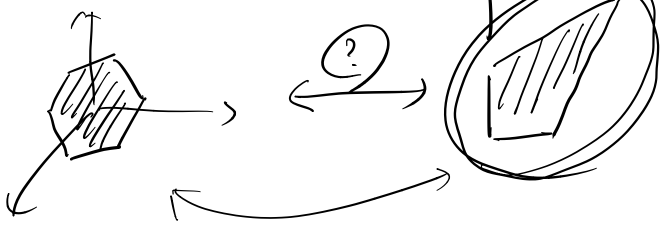
$= H^*(Pet_{A_2})$

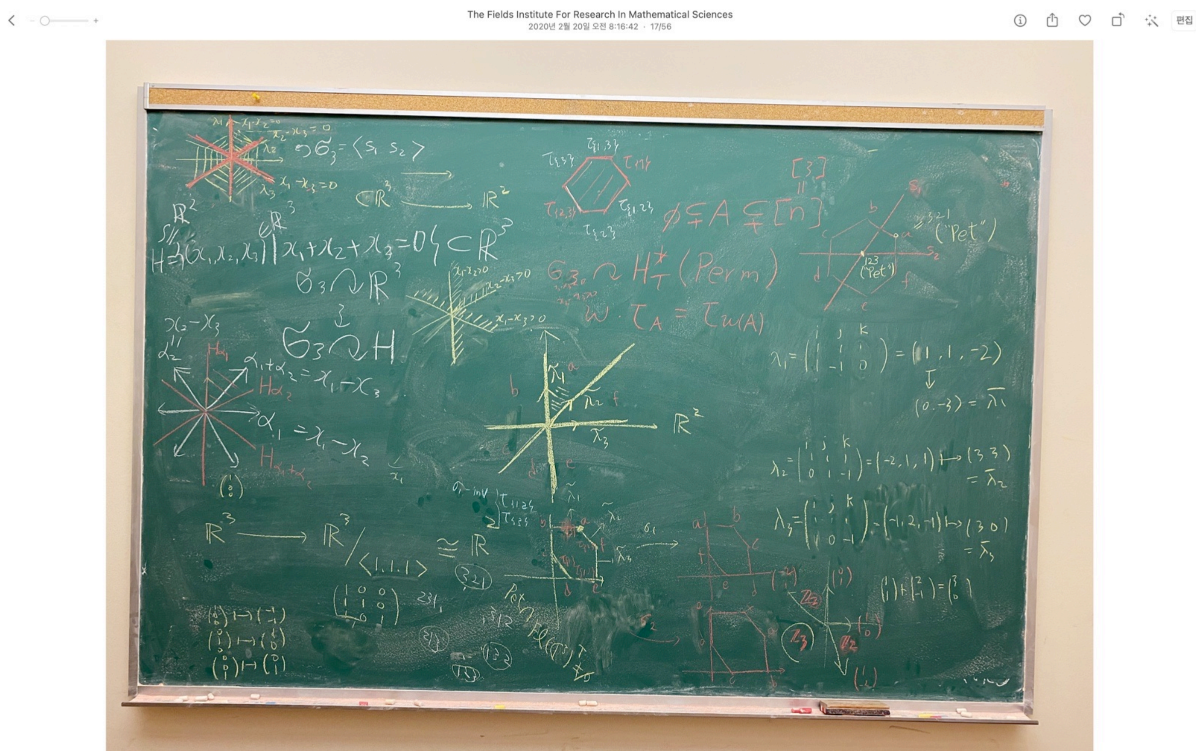
Recall $H^*(Hess(S, H_1))^{S_{n+1}} \cong H^*(Hess(N, H_1)) = H^*(Pet_{A_n})$

$\cup\!|$ $\qquad\qquad\qquad\qquad \|$

$H^*(X_{A_n})^{S_{n+1}} \cong H^*(X(I^n, (C_n \mid -I_n)))$.

Motivational question: Is there any "combinatorial" interpretation of this isom ?



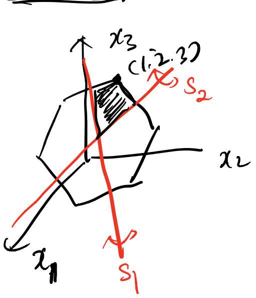


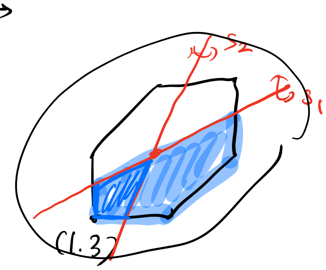
$$H^*(X_{A_2})^{\mathfrak{S}_3} \cong H^*(X(I^2, C_2))$$

$\mathfrak{S}_3 = \langle s_1, s_2 \rangle$

$\mathbb{R}^3 \to \mathbb{R}^3/\langle 1,1,1 \rangle \cong \mathbb{R}^2$

$(x_1, x_2, x_3) \mapsto (x_1, x_1 + x_2)$





Question ① (In toric variety side) What about $H^*(X_{A_2})^{\langle s_1 \rangle} \cong H^*(\,?\,)$ (?)

Thm [Horiguchi – Masuda – Shareshian – S]

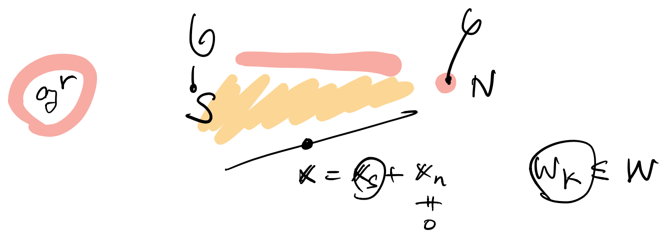
$$\underline{H^*(X_{\Phi_n})^{W_K}} \cong H^*(\text{"Partioned weight polytope"})$$

Toric orb

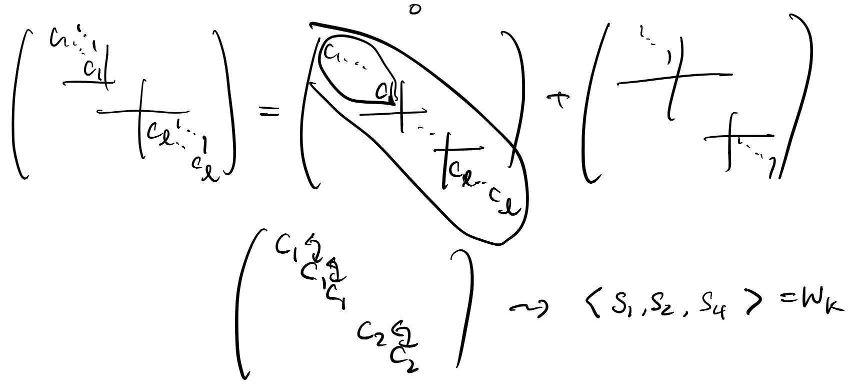
② (In Hess. var. side)

Thm [Balibanu – Crooks '20 arxiv]
[Vilonen – Xue '21, arXiv]

$$H^*(\mathrm{Hess}(S, H_1))^{W_K} \cong H^*(\mathrm{Hess}(x, H_1)).$$



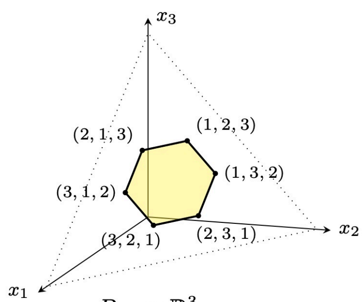
In type A_n.



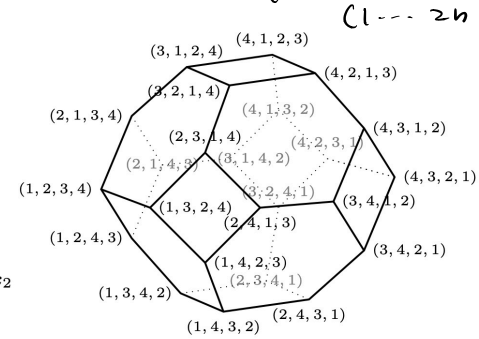
§4. Main result

- Φ_n : rank n root system.
- W : Weyl group $= \langle s_1 \cdots s_n \rangle \subseteq \mathfrak{S}_{2n} = \mathfrak{S}_{[n] \sqcup [\bar{n}]}$. $\left([\bar{n}] = \{\bar{1}, \bar{2} \cdots \bar{n}\},\ \bar{i} = 2n+1-i \right)$ ($= \mathfrak{S}_{n+1}$ in A_n)
  - $s_i = (i . i+1)$ for type A_n
  - $\begin{cases} s_i = (i, i+1) \cdot (\overline{i+1}, \bar{i}) & i = 1 \cdots n-1 \\ s_n = (n . \bar{n}) & i = n \end{cases}$) B_n
  - $\begin{cases} s_i = (i . i+1)\ (\overline{i+1} . \bar{i}) & i = 1 \cdots n-1 \\ s_n = (n-1, \bar{n}) \cdot (n, \overline{n-1}) \end{cases}$) D_n.

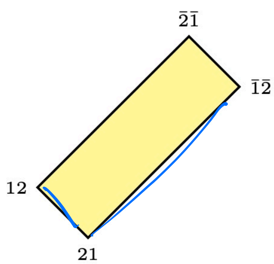
Def. Weight polytope $P_{\Phi_n} = \text{conv hull}\left(u \cdot (1 \cdots n+1) \in \mathbb{R}^{n+1} \mid u \in W \right)$
$(1 \cdots 2n) \in \mathbb{R}^{2n}$

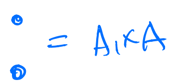


$P_{A_2} \subset \mathbb{R}^3$

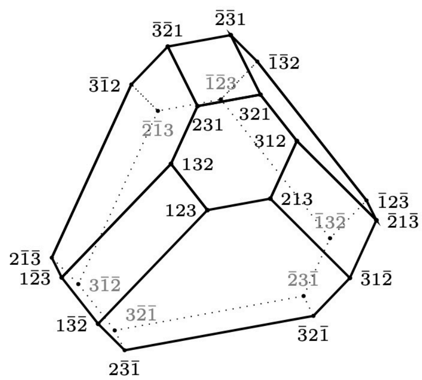


$P_{A_3} \subset \mathbb{R}^4$

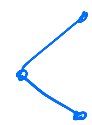




$P_{D_2} \subset \mathbb{R}^4$



$P_{D_3} \subset \mathbb{R}^6$



① • $F(P_{\Phi_n}) \overset{1:1}{\longleftrightarrow} \mathfrak{M} := \begin{cases} \emptyset \neq I \subset [n+1] & A_n \\ \{\emptyset \neq I \subset [n] \cup [\bar{n}] \mid |I \cap \{i, \bar{i}\}| \leq 1\} & B_n \\ \{ \quad " \quad\quad " \quad , |I| \neq n-1\} & D_n \end{cases}$

② • $F(I) = \text{conv}\{ (x_1 \cdots x_{n+1}) \in \mathbb{R}^{n+1} \mid \{x_i \mid i \in I\} = [|I|] \}$ A_n
$(x_1 \cdots x_{2n}) \in \mathbb{R}^{2n} \mid$ " B_n
⋮ D_n

③ • normal vector $F(I) = e_I = \sum_{i \in I} e_i$

④ • $F(I) \cap F(J) = \emptyset$ iff $I \not\subset J$ & $J \not\subset I$. for type A_n

$\Rightarrow H^*(X_{\Phi_n}) = \mathbb{Q}[\underbrace{x_I \mid I \in \mathfrak{M}}_{①}] / \underbrace{I}_{③} + \underbrace{J}_{=②\&④}$

$W = \langle s_1 \cdots s_n \rangle$ $K \subseteq [n]$

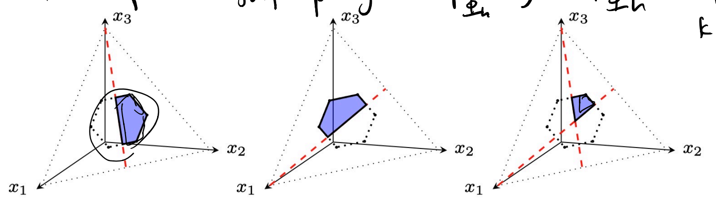
$\cup\!|$

$W_K = \langle s_k \mid k \in K \rangle$

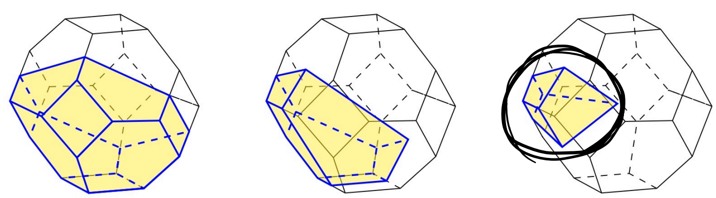
$H(i)$: invariant subsp w.r.t reflection s_i

$H(i)^{\leq}$: half space

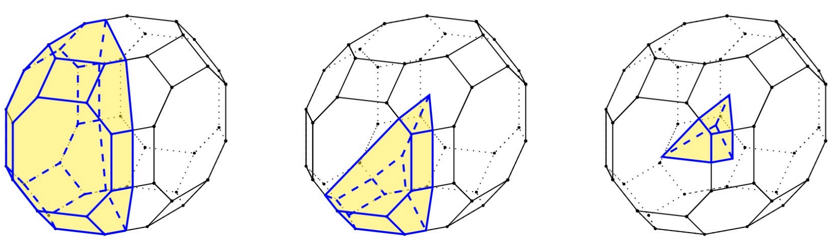
Def "Partioned weight poly" $P_{\Phi_n}(K) = P_{\Phi_n} \cap \bigcap_{k \in K} H(k)^{\leq}$



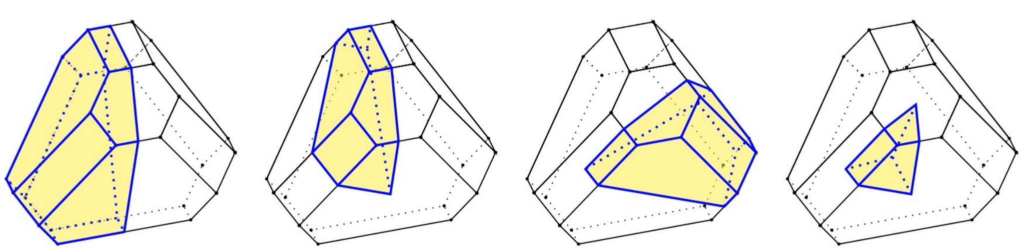
$P_{A_2}(\{1\})$ $P_{A_2}(\{2\})$ $P_{A_2}(\{1,2\})$



$P_{A_3}(\{1\})$ $P_{A_3}(\{1,2\})$ $P_{A_3}(\{1,2,3\})$



$P_{B_3}(\{1\})$ $P_{B_3}(\{1,2\})$ $P_{B_3}(\{1,2,3\})$



$P_{D_3}(\{1\})$ $P_{D_3}(\{1,3\})$ $P_{D_3}(\{2,3\})$ $P_{D_3}(\{1,2,3\})$

$$H^*(X(P_{\Phi_n}(K))) = \mathbb{Q}[x_I \mid x \in \mathcal{M}(K) \cup K] \Big/ I(K) + J(K)$$

Thm. $H^*(X_{\Phi_n})^{W_K} \cong H^*(X(P_{\Phi_n}(K)))$. (Classical Lie type)

proof ① generated by $[x_I] := \sum_{u \in W/W_K^I} x_{u(I)}$

② $H^*(X(P_{\Phi_n}(K))) \to H^*(X_{\Phi_n})^{W_K}$

$x_I \longmapsto [x_I]$

$x_{s_k} \longmapsto$ " certain linear comb. of x_I's.

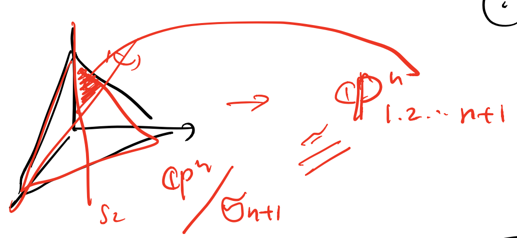
③

Cor. $H^*(\mathrm{Hess}(X, H_1)) \cong H^*(X(P_{\Phi_n}(K)))$

$$H^*(X(P))^W \cong H^*(X(P)/W) \underset{?}{\cong} H^*(X(P/W))$$

?? $\cong$

ex



- [Blume]. $X_{A_n}/\mathfrak{S}_{n+1} \cong X(I^n, C_n)$ $\quad P_{A_n}/\mathfrak{S}_{n+1}$

  $X_{B_n}/W \cong X(I^n, C_n, B_n)$ $\quad P_{B_n}/W$

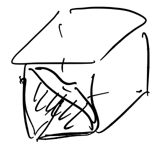
  X_{C_n}/W. $\quad (\quad C_n)$.

- $\mathbb{C}P^n/\mathfrak{S}_{n+1} \cong \mathbb{C}P^n_{(1,\cdots,n+1)}$ $\quad \mathbb{C}P^n \to \mathbb{C}P^n_{(1,\cdots,n+1)}$

  $[x_1 \cdots x_{n+1}] \mapsto [e_1(x) : e_2(x) \cdots e_n(x)]$

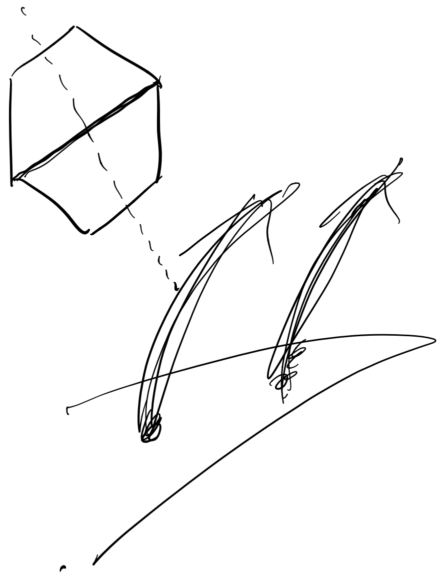
  "Well-known $\quad \Delta^n/\mathfrak{S}_{n+1}$

  $n+1$

$\cdot\ \widetilde{\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1} / \mathfrak{S}_{n+1} =$ 

$\parallel$

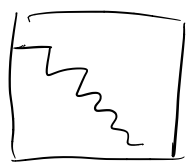
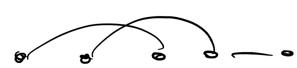
$\mathbb{CP}^{n+1}$



$W \circlearrowright G/B \ni \tilde{\varphi}$ $\quad W \circlearrowright$ Perm

induce?

GKM-graph

$\mathcal{H}$ is given $\Rightarrow$  $\rightsquigarrow$ 

something like that

$\Rightarrow$ We can define a graph & easy to see whether it has certain symmetry or not ...

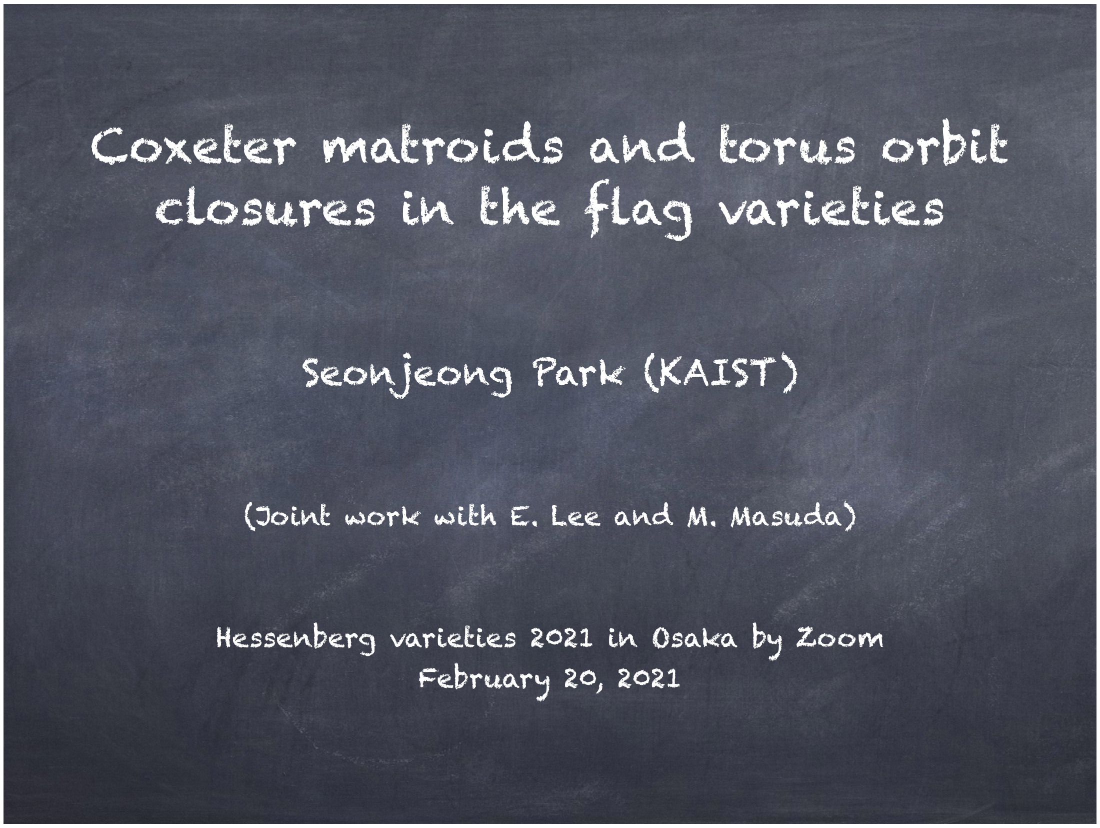
GKM graph depends on $\mathcal{H}$.

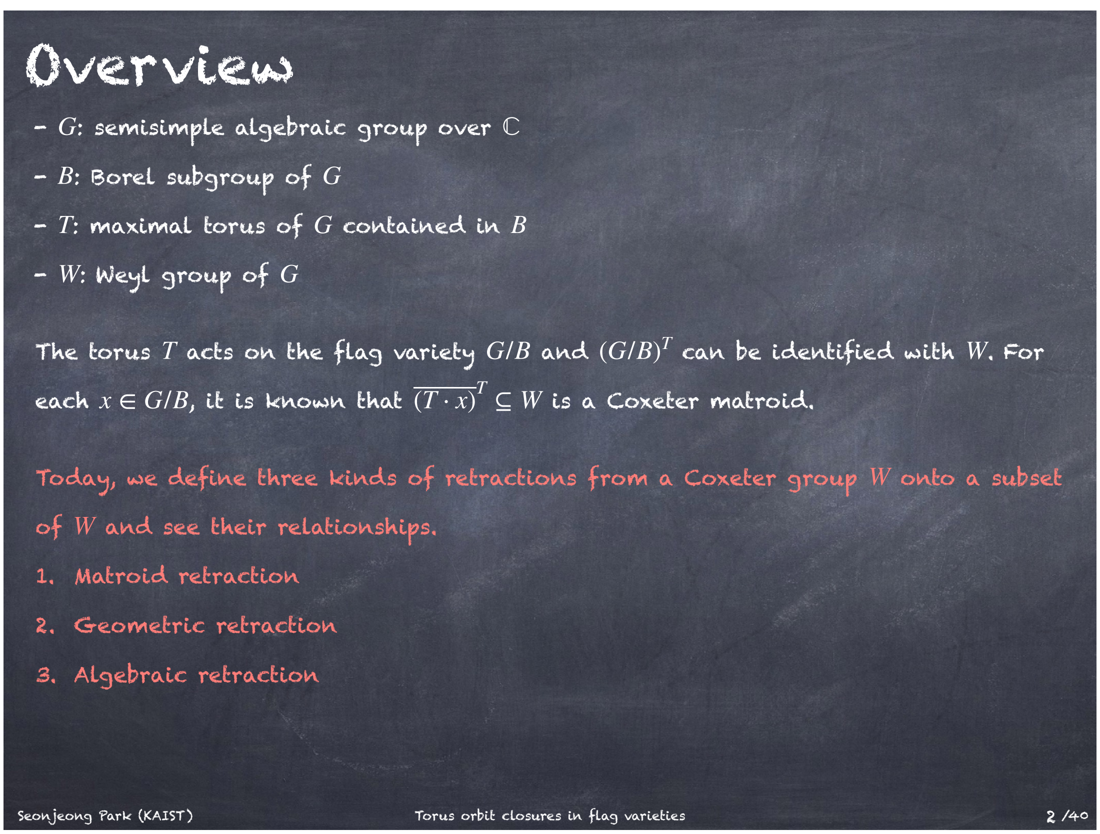
W_k depends on $x \in \mathfrak{g}$

Coxeter matroids and the torus orbit closures in the flag varieties

Seonjeong Park

A subset M of a finite Coxeter group W is a Coxeter matroid if it satisfies the maximality property, that is, for every u in W, there is a unique element in v in M such that $u^{-1}v \geq u^{-1}w$ in Bruhat order. A Coxeter matroid is a generalization of a matroid, and it is known that the torus fixed point set of any torus orbit closure in the flag varieties is a Coxeter matroid. However, not every Coxeter matroid can be realized as a torus orbit closure. In this talk, I will first introduce the notion of matroids and the relation with Coxeter matroids, and then we discuss some geometric and algebraic properties of Coxeter matroids. This talk is based on joint work with Eunjeong Lee and Mikiya Masuda.


Coxeter matroids and torus orbit closures in the flag varieties
Seonjeong Park (KAIST)
(Joint work with E. Lee and M. Masuda)
Hessenberg varieties 2021 in Osaka by Zoom
February 20, 2021


Overview
- G: semisimple algebraic group over $\mathbb{C}$
- B: Borel subgroup of G
- T: maximal torus of G contained in B
- W: Weyl group of G
The torus T acts on the flag variety G/B and $(G/B)^T$ can be identified with W. For each $x \in G/B$, it is known that $\overline{(T \cdot x)}^T \subseteq W$ is a Coxeter matroid.
Today, we define three kinds of retractions from a Coxeter group W onto a subset of W and see their relationships.
1. Matroid retraction
2. Geometric retraction
3. Algebraic retraction
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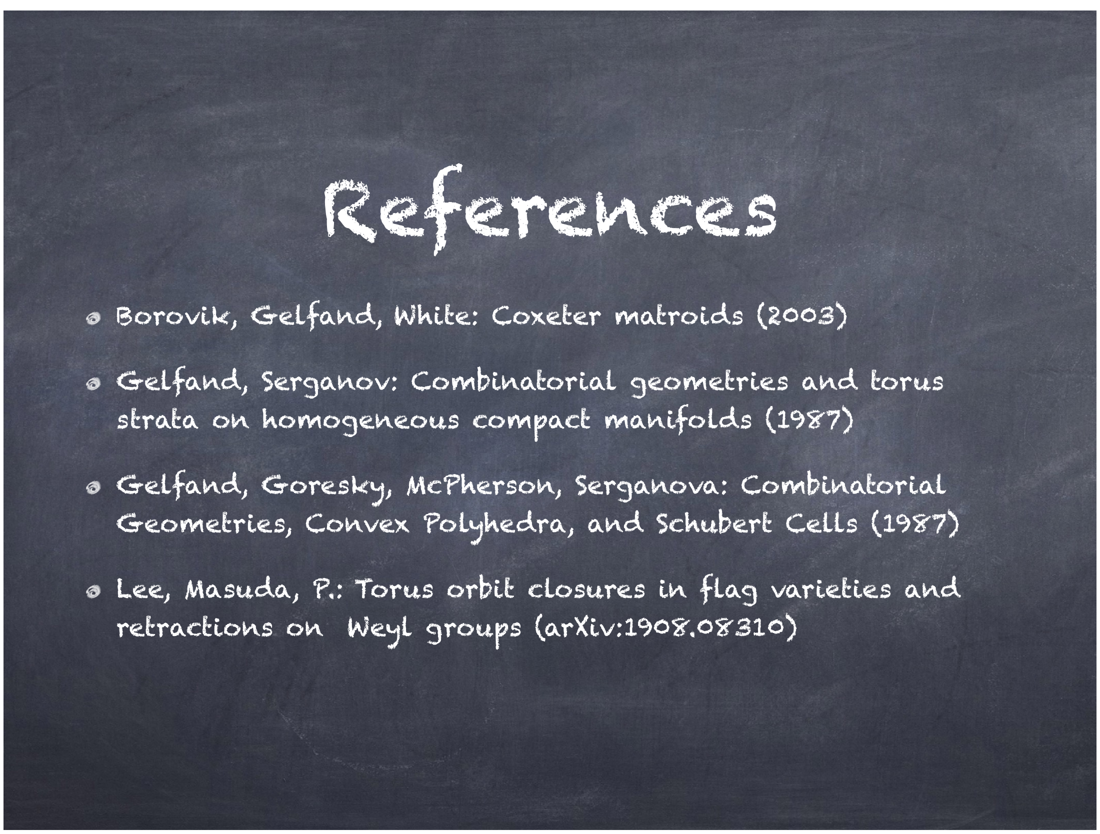
Flag matroids
Coxeter matroids and Matroid retraction
Geometric retraction
Algebraic retraction


References
Borovik, Gelfand, White: Coxeter matroids (2003)
Gelfand, Serganov: Combinatorial geometries and torus strata on homogeneous compact manifolds (1987)
Gelfand, Goresky, McPherson, Serganova: Combinatorial Geometries, Convex Polyhedra, and Schubert Cells (1987)
Lee, Masuda, P.: Torus orbit closures in flag varieties and retractions on Weyl groups (arXiv:1908.08310)

Matroids

Matroids

Let V be the vector space spanned by the columns of

$$[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{v}_4] = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Let $M := \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\}$. Then for each $I \in M$, the set $\{\mathbf{v}_i \mid i \in I\}$ is a basis of V.

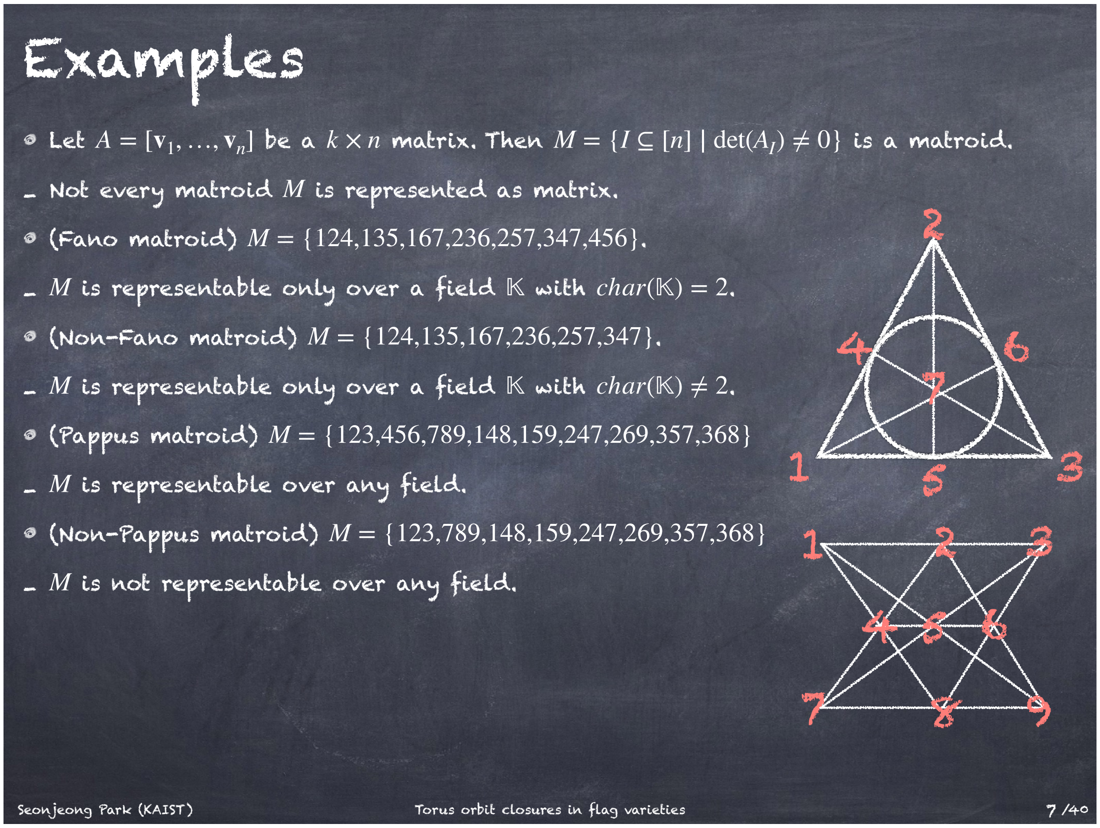
The set M satisfies the exchange property, that is, for all $I, J \in M$ and $i \in I \backslash J$, there exists $j \in J \backslash I$ such that $I \backslash \{i\} \cup \{j\} \in M$.

Definition. Let $[n] := \{1,2,\ldots,n\}$. A nonempty collection M of finite subsets of $[n]$ is called a matroid on $[n]$ if it satisfies the exchange property:

For all A, B in M and $a \in A \backslash B$ there exists $b \in B \backslash A$ such that $A \backslash \{a\} \cup \{b\}$ lies in M.

Every member of M has the same size, $\mathrm{rank}(M)$.

The members of M are called bases of the matroid.


Examples
Let $A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ be a $k \times n$ matrix. Then $M = \{I \subseteq [n] \mid \det(A_I) \neq 0\}$ is a matroid.
- Not every matroid M is represented as matrix.
(Fano matroid) $M = \{124,135,167,236,257,347,456\}$.
- M is representable only over a field $\mathbb{K}$ with $char(\mathbb{K}) = 2$.
(Non-Fano matroid) $M = \{124,135,167,236,257,347\}$.
- M is representable only over a field $\mathbb{K}$ with $char(\mathbb{K}) \neq 2$.
(Pappus matroid) $M = \{123,456,789,148,159,247,269,357,368\}$
- M is representable over any field.
(Non-Pappus matroid) $M = \{123,789,148,159,247,269,357,368\}$
- M is not representable over any field.
2
4
6
7
1
5
3
1
2
3
4
5
6
7
8
9
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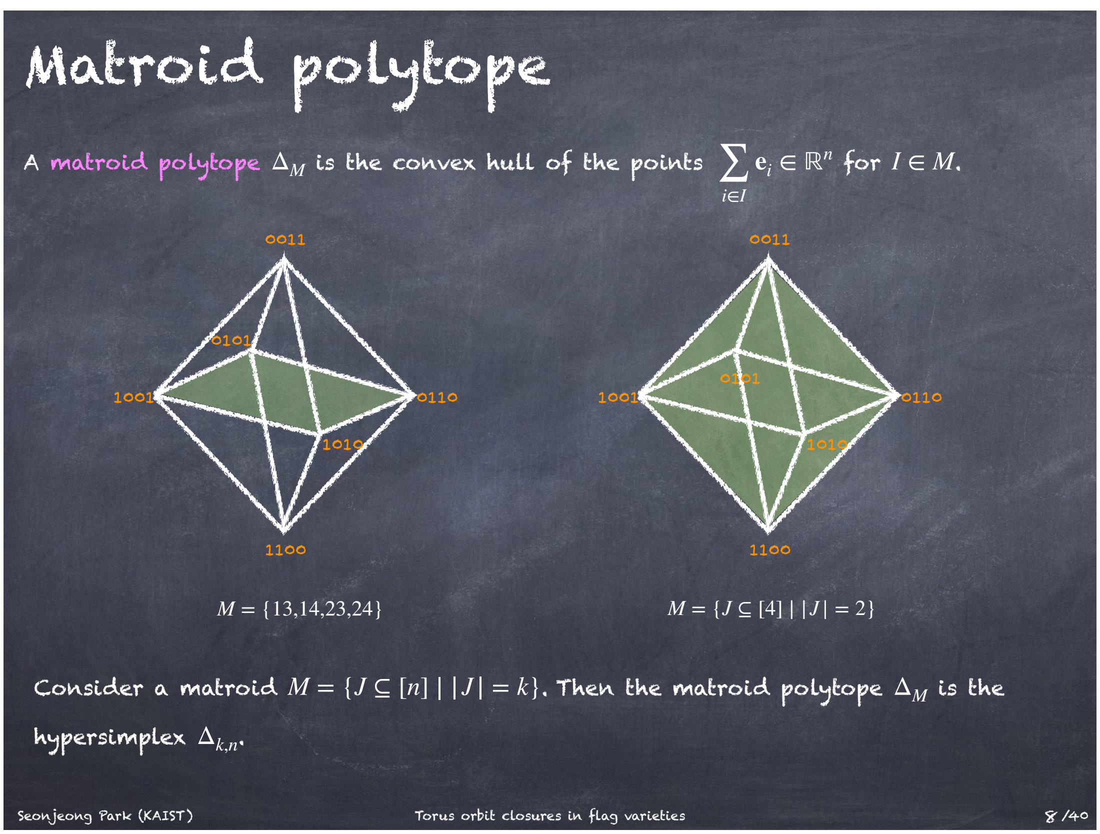
Matroid polytope
A matroid polytope Δ_M is the convex hull of the points $\sum_{i \in I} \mathbf{e}_i \in \mathbb{R}^n$ for $I \in M$.
0011
0101
1001
0110
1010
1100
0011
0101
1001
0110
1010
1100
$M = \{13,14,23,24\}$
$M = \{J \subseteq [4] \mid |J| = 2\}$
Consider a matroid $M = \{J \subseteq [n] \mid |J| = k\}$. Then the matroid polytope Δ_M is the hypersimplex $\Delta_{k,n}$.
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Torus orbit closures in $\mathrm{Gr}_k(\mathbb{C}^n)$

Consider the Grassmannian manifold $\mathrm{Gr}_k(\mathbb{C}^n)$. Set $I_{k,n} = \{J \subseteq [n] \mid |J| = k\}$.

For $x \in \mathrm{Gr}_k(\mathbb{C}^n)$, then the Plücker embedding $\mathrm{Gr}_k(\mathbb{C}^n) \to \mathbb{C}P^{\binom{n}{k}-1}$ maps x to $(\det(A_J))_{J\in I_{k,n}}$, where A is a matrix representing x.

For $x \in \mathrm{Gr}_k(\mathbb{C}^n)$, set $M_x := \{J \in I_{k,n} \mid \det(A_J) \neq 0\}$, where A is a matrix representing x.

For example, $M_x = \{13,14,23,24\}$ for $x \in \mathrm{Gr}_2(\mathbb{C}^4)$ representing by $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

[Gelfand-Goresky-McPherson-Serganova] Define the moment map $\mu\colon \mathrm{Gr}_k(\mathbb{C}^n) \to \mathbb{R}^n$ by $\mu(x) = \dfrac{\sum_{J:\, j\in J} |\det(A_J)|^2}{\sum_J |\det(A_J)|^2}$, where A is a matrix representing x. Then the moment map image of $\overline{T\cdot x}$ is the polytope is the matroid polytope Δ_{M_x}.

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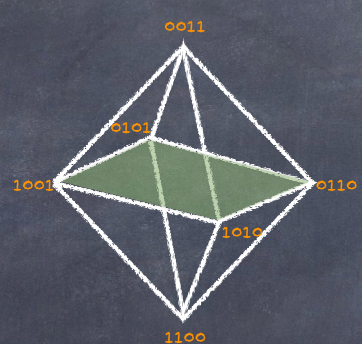
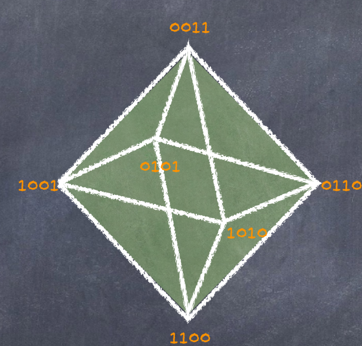
Torus orbit closures in $\mathrm{Gr}_k(\mathbb{C}^n)$

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$



$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$



[Noji-Ogiwara (2019)] A matroid polytope is simple if and only if it is a product of simplices. So, every smooth torus orbit closure in $\mathrm{Gr}_k(\mathbb{C}^n)$ is a product of complex projective spaces.

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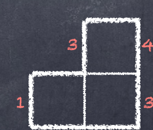
Geometry of matroids

Geometry	Matroids
points in $\mathrm{Gr}_k(\mathbb{C}^n)$	rank-k, $\mathbb{R}$-representable matroids
Schubert varieties	nested matroids
Richardson varieties	lattice path matroids

For example, let $n = 4$, $k = 2$, $S = \{1,3\}$, and $T = \{3,4\}$. For a generic point $x \in X_S^T$, the matroid M_x is

$$\{B \in \binom{[4]}{2} \mid S \leq B \leq T\}.$$



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Flag matroids

Gale ordering

For a permutation $w \in \mathfrak{S}_n$, we give an order $\leq^w$ on $[n]$ as follows:

$$w(1) <^w w(2) <^w < \cdots <^w w(n).$$

Then the Gale ordering $\leq^w$ on $I_{k,n}$ is a partial order defined as follows:

For $A, B \in I_{k,n}$, let $A = \{a_1 <^w a_2 <^w \cdots <^w a_k\}$ and $B = \{b_1 <^w b_2 <^w \cdots <^w b_k\}$. Then

$$A \leq^w B \quad \text{if} \quad a_1 \leq^w b_1, a_2 \leq^w b_2, \ldots, a_k \leq^w b_k.$$

Consider $I_{2,4} = \{12, 13, 14, 23, 24, 34\}$ and $w = 2314 \in \mathfrak{S}_4$. Then we get

$$2 <^w 3 <^w 1 <^w 4.$$

For $A_1 = \{1,3\}, A_2 = \{1,4\}, A_3 = \{2,3\}, A_4 = \{2,4\}$ in $I_{2,4}$, we get

$$A_1 = \{3 <^w 1\},\ A_2 = \{1 <^w 4\},\ A_3 = \{2 <^w 3\}, \text{ and } A_4 = \{2 <^w 4\}.$$

Hence we get

$$A_3 <^w A_1 <^w A_2 \quad \text{and} \quad A_3 <^w A_4 <^w A_2.$$

Note that A_1 and A_4 are not comparable.

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Gale ordering

[Gale (1968)] Let $M \subseteq I_{k,n}$. Then M is a matroid if and only if M satisfies the following Maximality property:

for every $w \in \mathfrak{S}_n$, the collection M contains a unique member $A \in M$ maximal in M with respect to $\leq^w$. i.e., $B \leq^w A$ for all $B \in M$.

Equivalently, M satisfies the Minimality property:

for every $w \in \mathfrak{S}_n$, the collection M contains a unique member $A \in M$ minimal in M with respect to $\leq^w$. i.e., $B \leq^w A$ for all $B \in M$.

Note that $i <^w j$ if and only if $w^{-1}(i) < w^{-1}(j)$. Hence

$$A \leq^w B \text{ if and only if } w^{-1}A \leq w^{-1}B.$$

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Flag matroids

For $w \in \mathfrak{S}_n$, the Gale ordering $\leq^w$ on $\mathfrak{S}_n$ is defined as follows:

For u and v in $\mathfrak{S}_n$, we set $u \leq^w v$ if $\{u(1) <^w \dots <^w u(k)\} \leq^w \{v(1) <^w \dots <^w v(k)\}$ with respect to the Gale order $\leq^w$ on $I_{k,n}$ for all $k = 1, \dots, n$.

Therefore, $u \leq^w v$ if and only if $w^{-1}u \leq w^{-1}v$ in Bruhat order.

A subset $\mathscr{M}$ of $\mathfrak{S}_n$ is called a flag matroid if and only if $\mathscr{M}$ satisfies the Maximality property:

For every $w \in \mathfrak{S}_n$, the subset $\mathscr{M}$ contains a unique element maximal in $\mathscr{M}$ with respect to the ordering $\leq^w$.

Equivalently, $\mathscr{M}$ satisfies the Minimality property:

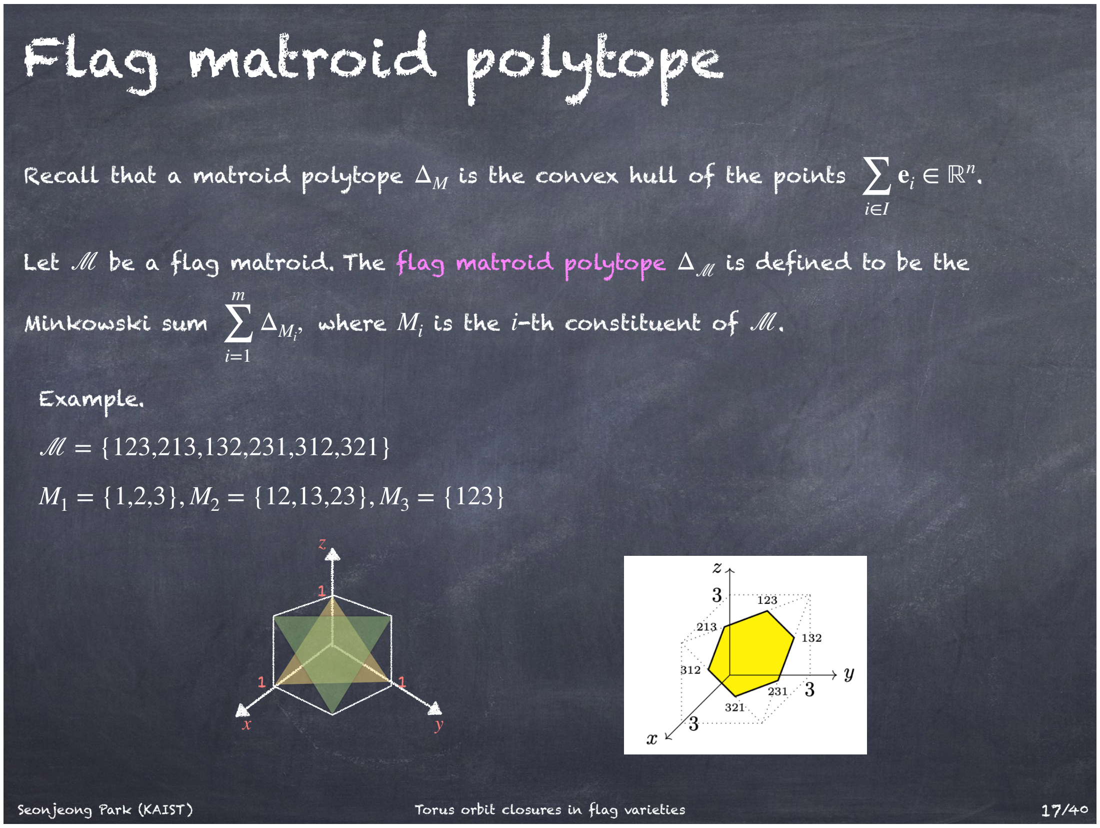
For every $w \in \mathfrak{S}_n$, the subset $\mathscr{M}$ contains a unique element minimal in $\mathscr{M}$ with respect to the ordering $\leq^w$.

Example

- $\{123, 321\}$ is a Coxeter matroid.

w	123	213	132	231	312	321
min	123	123	123	321	321	321

- For a flag matroid $\mathscr{M} \subseteq \mathfrak{S}_n$, the set $\mathscr{M}_k := \{\{v(1), \dots, v(k)\} \mid v \in \mathscr{M}\}$ is a matroid for each $k = 1, \dots, n-1$. We call $\mathscr{M}_k$ the k-th constituent of $\mathscr{M}$.
- $\{213, 132, 231\}$ is not a flag matroid even though $\{1,2,3\}$, $\{12,13,23\}$, and $\{123\}$ are matroids.


Flag matroid polytope
Recall that a matroid polytope Δ_M is the convex hull of the points $\sum_{i \in I} \mathbf{e}_i \in \mathbb{R}^n$.
Let $\mathscr{M}$ be a flag matroid. The flag matroid polytope $\Delta_{\mathscr{M}}$ is defined to be the Minkowski sum $\sum_{i=1}^{m} \Delta_{M_i}$, where M_i is the i-th constituent of $\mathscr{M}$.
Example.
$\mathscr{M} = \{123, 213, 132, 231, 312, 321\}$
$M_1 = \{1,2,3\}, M_2 = \{12,13,23\}, M_3 = \{123\}$
z
1
1
1
x
y
z
3
123
213
132
312
y
231
3
321
3
x
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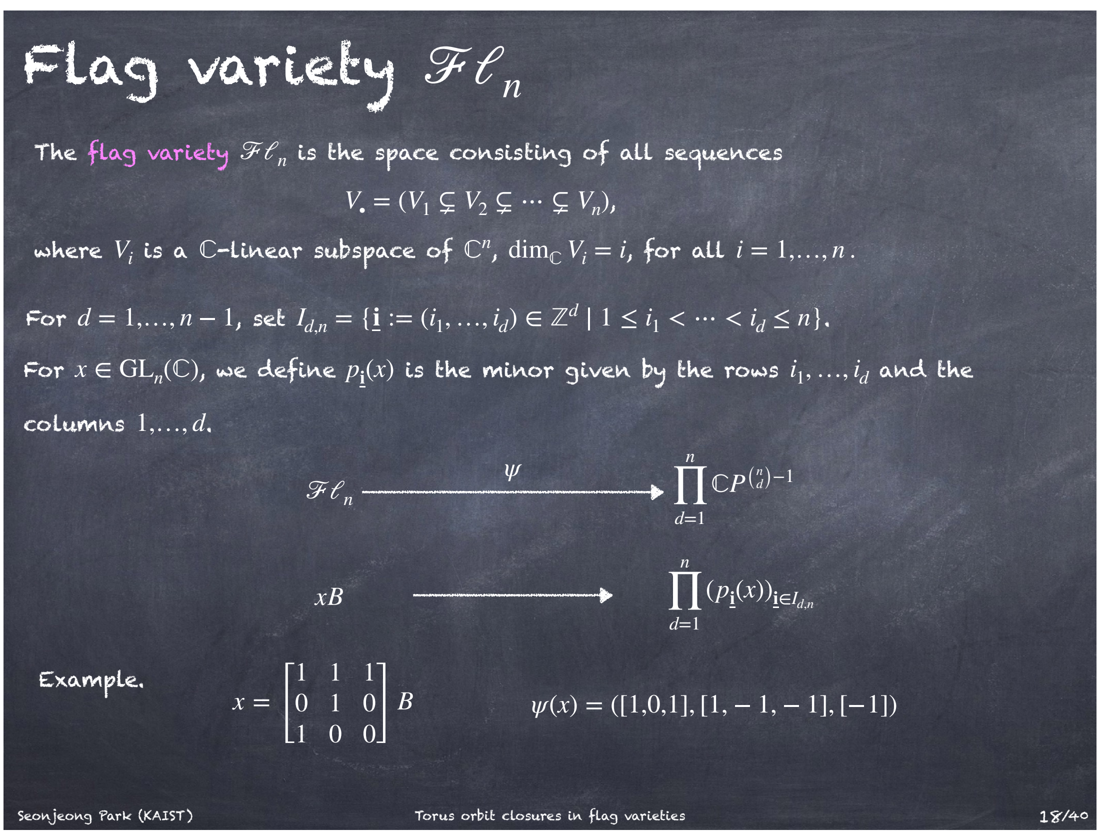
Flag variety $\mathscr{F}\ell_n$
The flag variety $\mathscr{F}\ell_n$ is the space consisting of all sequences
$V_\bullet = (V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n)$,
where V_i is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$, $\dim_{\mathbb{C}} V_i = i$, for all $i = 1, \ldots, n$.
For $d = 1, \ldots, n-1$, set $I_{d,n} = \{\underline{\mathbf{i}} := (i_1, \ldots, i_d) \in \mathbb{Z}^d \mid 1 \le i_1 < \cdots < i_d \le n\}$.
For $x \in \mathrm{GL}_n(\mathbb{C})$, we define $p_{\underline{\mathbf{i}}}(x)$ is the minor given by the rows $i_1, \ldots, i_d$ and the columns $1, \ldots, d$.
$\mathscr{F}\ell_n \xrightarrow{\psi} \prod_{d=1}^{n} \mathbb{C}P^{\binom{n}{d}-1}$
$xB \longrightarrow \prod_{d=1}^{n} (p_{\underline{\mathbf{i}}}(x))_{\underline{\mathbf{i}} \in I_{d,n}}$
Example.
$x = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} B$
$\psi(x) = ([1,0,1], [1,-1,-1], [-1])$
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Torus orbit closures in $\mathcal{F}\ell_n$

The torus $T = (\mathbb{C}^*)^n$ acts on $\mathcal{F}\ell_n$ and

$$(\mathcal{F}\ell_n)^T = \{wB := (\{0\} \subsetneq \langle \mathbf{e}_{w(1)} \rangle \subsetneq \langle \mathbf{e}_{w(1)}, \mathbf{e}_{w(2)} \rangle \subsetneq \cdots \subsetneq \langle \mathbf{e}_{w(1)}, \dots, \mathbf{e}_{w(n)} \rangle) \mid w \in \mathfrak{S}_n\}.$$

[Gelfand-Serganova] For $x \in \mathcal{F}\ell_n$, we set $L_x = \bigcup_{1 \le d \le n-1} \{\underline{\mathbf{i}} \in I_{d,n} \mid p_{\underline{\mathbf{i}}}(x) \neq 0\}$, the list of x.

Then $(\overline{T \cdot x})^T = \{wB \mid \{w(1), \dots, w(i)\}^{\uparrow} \in L_x$ for all $i = 1, \dots, n-1\}$, and it is a flag matroid, denoted by $\mathcal{M}_x$. Define the moment map $\mu \colon \mathcal{F}\ell_n \to \mathbb{R}^n$ by

$$\mu(x) = \sum_{k=1}^{n} \left\{ \frac{1}{\sum_{\underline{\mathbf{i}} \in I_{k,n}} |p_{\underline{\mathbf{i}}}|^2} \left(\sum_{1 \in \underline{\mathbf{i}}_{k,n}} |p_{\underline{\mathbf{i}}}|^2, \dots, \sum_{n \in \underline{\mathbf{i}}_{k,n}} |p_{\underline{\mathbf{i}}}|^2 \right) \right\}.$$

Then $\mu(\overline{\mathbb{T} \cdot x}) = \Delta_{\mathcal{M}_x}$.

Note that $\mu(wB) = (w^{-1}(n), \dots, w^{-1}(1))$ and $\mu(\mathcal{F}\ell_n)$ is the permutohedron Perm_{n-1}.

♣ A non-realizable flag matroid is obtained from the Fano matroid using Higgs lifts.

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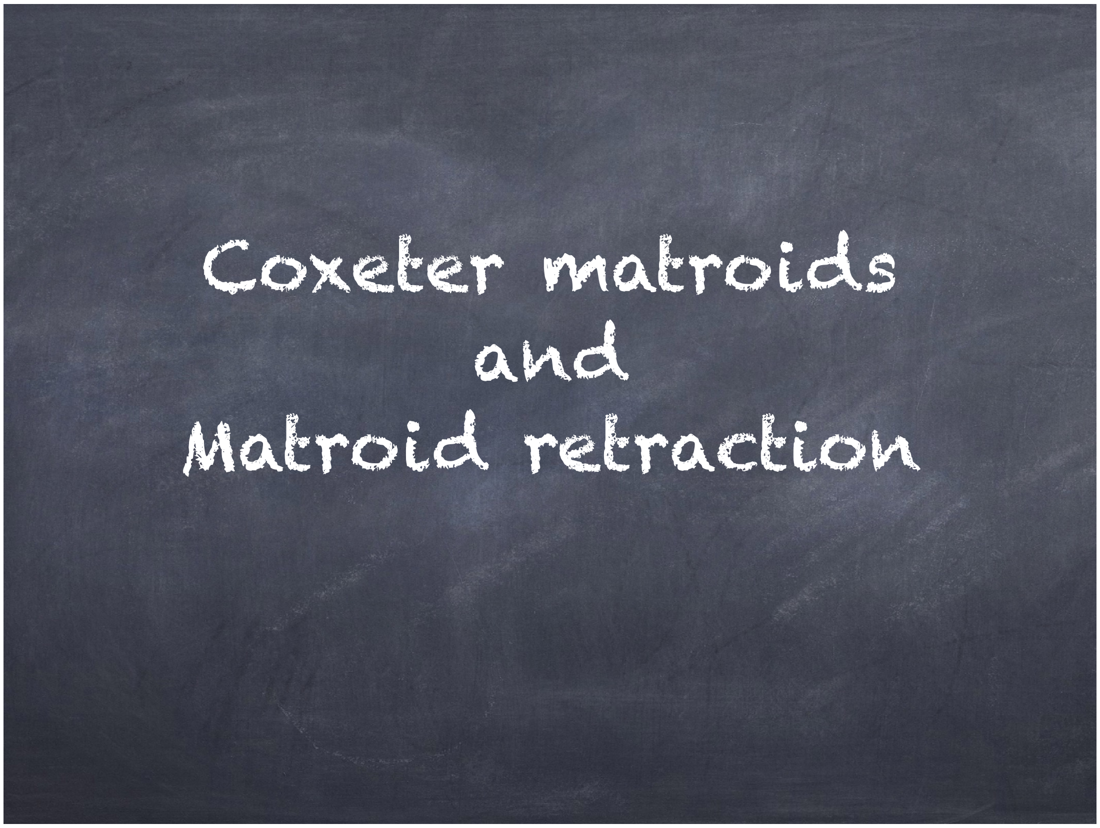
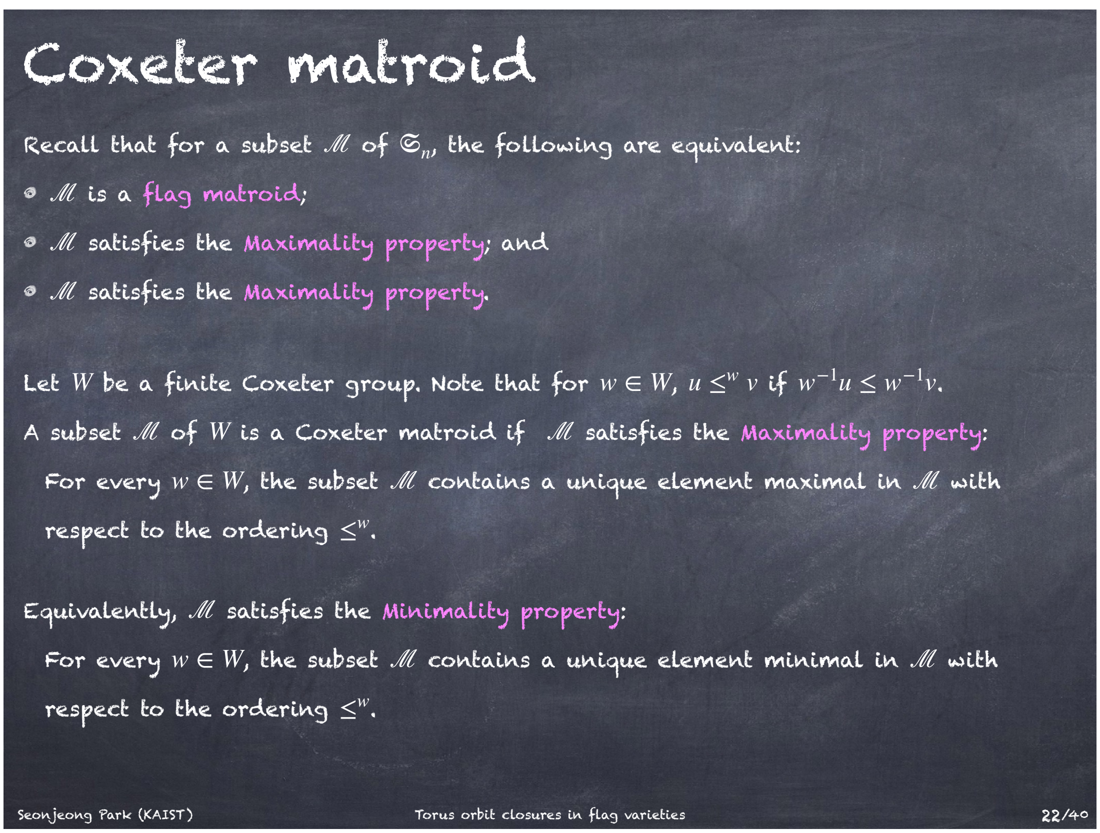
Geometry of flag matroids

Geometry	Matroids
points in $\mathcal{F}\ell_n$	$\mathbb{R}$-representable flag matroids
Schubert varieties	flag of nested matroids
Richardson varieties	flag of lattice path matroids

Seonjeong Park (KAIST) Torus orbit closures in flag varieties 20/40


Coxeter matroids
and
Matroid retraction


Coxeter matroid
Recall that for a subset $\mathscr{M}$ of $\mathfrak{S}_n$, the following are equivalent:
$\mathscr{M}$ is a flag matroid;
$\mathscr{M}$ satisfies the Maximality property; and
$\mathscr{M}$ satisfies the Maximality property.
Let W be a finite Coxeter group. Note that for $w \in W$, $u \leq^w v$ if $w^{-1}u \leq w^{-1}v$.
A subset $\mathscr{M}$ of W is a Coxeter matroid if $\mathscr{M}$ satisfies the Maximality property:
For every $w \in W$, the subset $\mathscr{M}$ contains a unique element maximal in $\mathscr{M}$ with
respect to the ordering $\leq^w$.
Equivalently, $\mathscr{M}$ satisfies the Minimality property:
For every $w \in W$, the subset $\mathscr{M}$ contains a unique element minimal in $\mathscr{M}$ with
respect to the ordering $\leq^w$.
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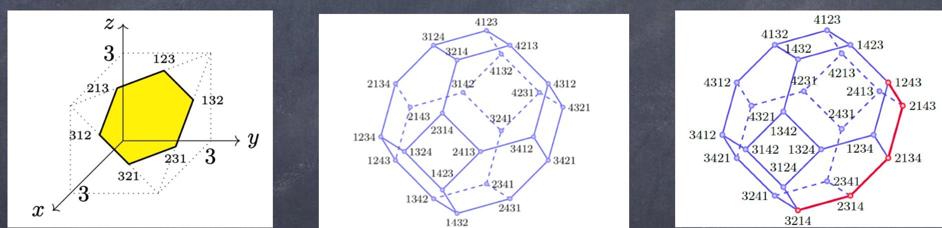
W-permutohedron

Note that W is a reflection group on a Euclidean space V. Choose a point $\nu \in V$ which is not fixed by any reflection in W. The W-permutohedron is

$$\Delta_W := \mathrm{ConvHull}\{w \cdot \nu \mid w \in W\}.$$

We identify $w \cdot \nu$ with w for each $w \in W$. Then the vertices v and w are joined by an edge of Δ_W if and only if $v^{-1}w$ is a simple reflection.



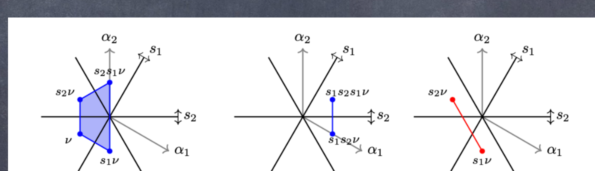
In general, for a subset $\mathscr{M}$ of W, we can construct a convex polytope

$$\Delta_{\mathscr{M}} = \mathrm{ConvHull}\{w \cdot \nu \mid w \in \mathscr{M}\}.$$

Characterization

[Gelfand-Serganova] A subset $\mathscr{M}$ of W is a Coxeter matroid if and only if every edge of $\Delta_{\mathscr{M}}$ is parallel to a root of W.

Example.



(a) $\mathcal{M} = \{123, 213, 132, 312\}$. (b) $\mathcal{M} = \{231, 321\}$. (c) $\mathcal{M} = \{213, 132\}$.

Distance on a Coxeter group

Define a metric d on a finite Coxeter group W by

$$d(v,w) := \ell(v^{-1}w) = \ell(w^{-1}v) \qquad \text{for } v,w \in W.$$

Hence the metric d is the graph metric on the 1-skeleton of Δ_W.

For a subset $\mathscr{M}$ of W, we define

$$d(v,\mathscr{M}) := \min\{d(v,w) \mid w \in \mathscr{M}\}.$$

Example.

- $\mathscr{M}_1 = \{123,321\} \Rightarrow d(213,\mathscr{M}_1) = d(213,123) = 1,\ d(132,\mathscr{M}_1) = d(132,123) = 1,$
  $d(231,\mathscr{M}_1) = d(231,321) = 1,\ d(312,\mathscr{M}_1) = d(312,321) = 1$
- $\mathscr{M}_2 = \{123,231\} \Rightarrow d(213,\mathscr{M}_2) = d(213,123) = d(213,231) = 1,\ d(132,\mathscr{M}_2) = d(132,123) = 1,$
  $d(321,\mathscr{M}_2) = d(321,231) = 1,\ d(312,\mathscr{M}_2) = d(312,123) = d(312,231) = 2$

For a Coxeter matroid $\mathscr{M}$, there is a unique element $q \in \mathscr{M}$ such that $d(v,q) = d(v,\mathscr{M})$ for all $v \in W$.

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Matroid retraction

Define a map $\mathscr{R}^m_{\mathscr{M}} \colon W \to \mathscr{M}$ by sending u to the unique minimal element w.r.t. the order $\leq^w$, which we call a matroid retraction. (Also known as a matroid map.)

Proposition. If $\mathscr{M}$ is a Coxeter matroid of W, then

1. for each $v \in W$, there is a unique $q \in \mathscr{M}$ such that $d(v,q) = d(v,\mathscr{M})$, and
2. $q = \mathscr{R}^m_{\mathscr{M}}(v)$.

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Geometric retraction

Toric variety

A toric variety is an irreducible variety X equipped with an action of T such that there is a Zariski open orbit isomorphic to T. Here, $T \cong (\mathbb{C}^*)^n$.

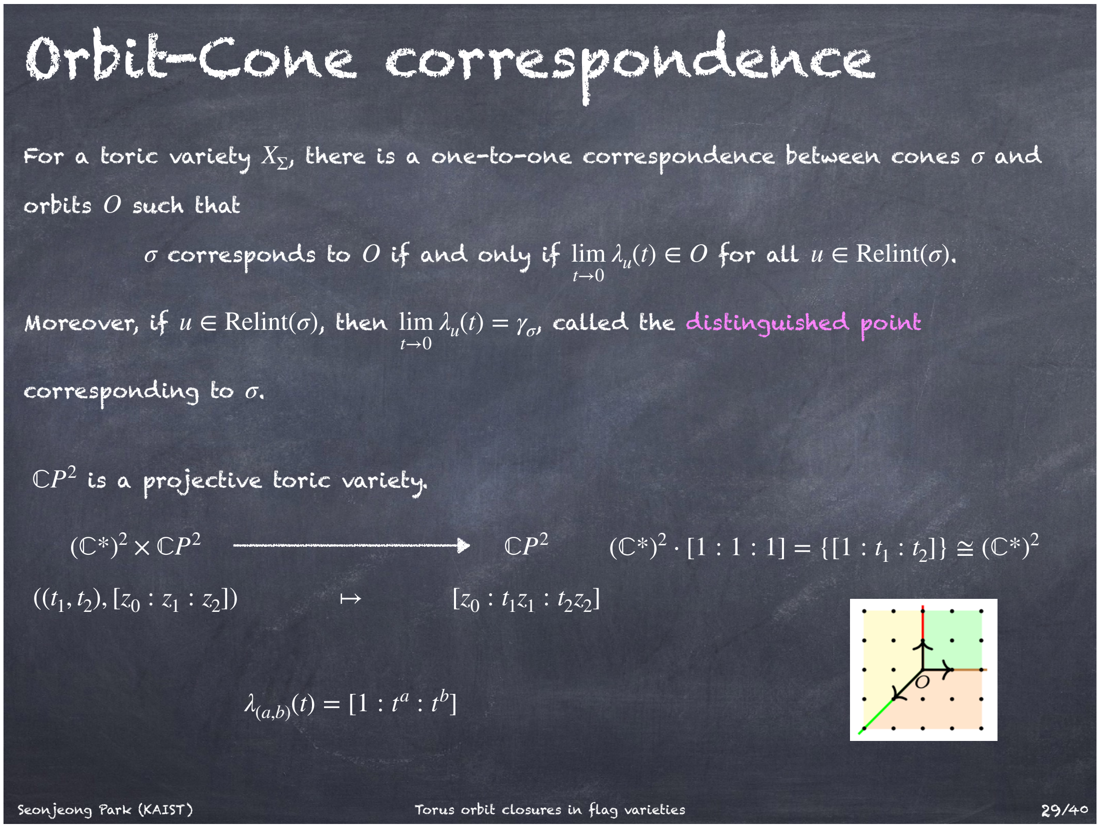
- $\mathbb{C}^n$ is a smooth toric variety.

$$(\mathbb{C}^*)^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n \qquad T \cdot (1,\ldots,1) \cong T$$
$$((t_1,\ldots,t_n),(z_1,\ldots,z_n)) \mapsto (t_1z_1,\ldots,t_nz_n)$$

- $\mathbb{C}P^n$ is a projective smooth toric variety.

$$(\mathbb{C}^*)^n \times \mathbb{C}P^n \longrightarrow \mathbb{C}P^n \qquad T \cdot [1:1:\cdots:1] \cong T$$
$$((t_1,\ldots,t_n),[z_0:z_1:\cdots:z_n]) \mapsto [z_0:t_1z_1:\cdots:t_nz_n]$$

- Torus orbit closures in flag varieties are projective toric varieties.


Orbit-Cone correspondence
For a toric variety X_Σ, there is a one-to-one correspondence between cones σ and orbits O such that
σ corresponds to O if and only if $\lim_{t\to 0}\lambda_u(t) \in O$ for all $u \in \mathrm{Relint}(\sigma)$.
Moreover, if $u \in \mathrm{Relint}(\sigma)$, then $\lim_{t\to 0}\lambda_u(t) = \gamma_\sigma$, called the distinguished point corresponding to σ.
$\mathbb{C}P^2$ is a projective toric variety.
$(\mathbb{C}^*)^2 \times \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2$
$((t_1,t_2),[z_0:z_1:z_2]) \mapsto [z_0 : t_1z_1 : t_2z_2]$
$(\mathbb{C}^*)^2 \cdot [1:1:1] = \{[1:t_1:t_2]\} \cong (\mathbb{C}^*)^2$
$\lambda_{(a,b)}(t) = [1 : t^a : t^b]$
O
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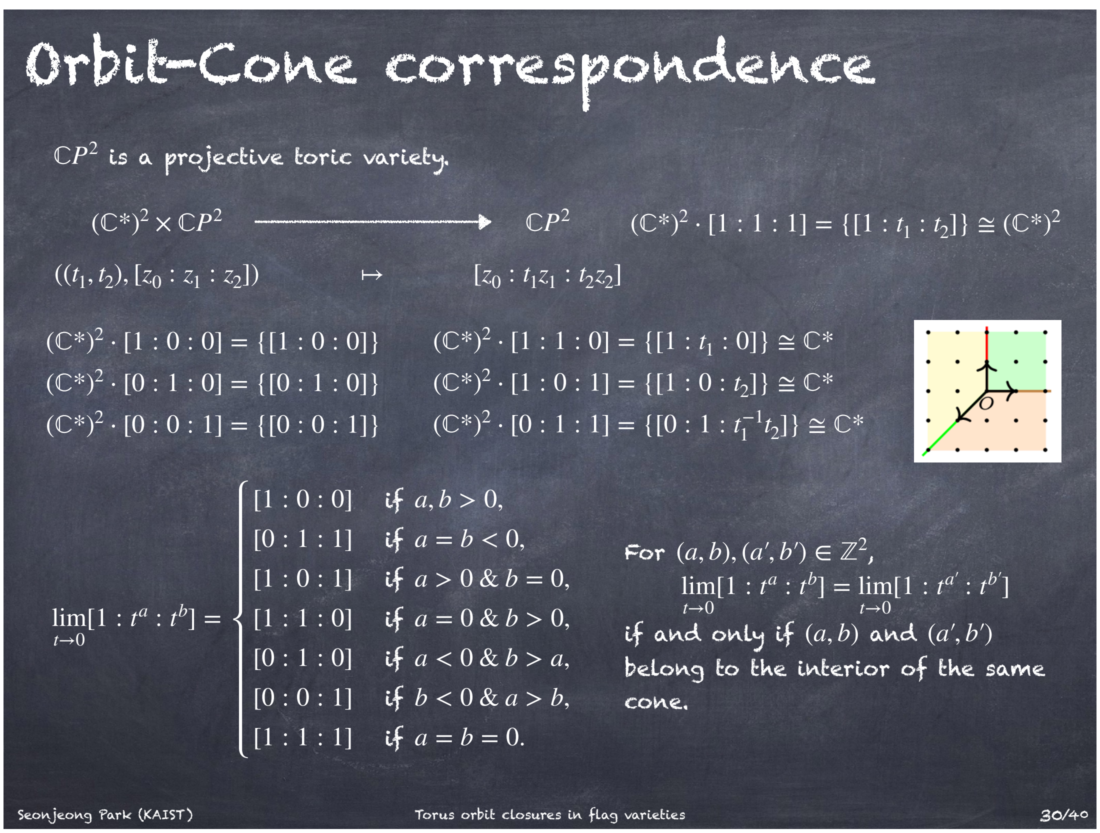
Orbit-Cone correspondence
$\mathbb{C}P^2$ is a projective toric variety.
$(\mathbb{C}^*)^2 \times \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2$
$((t_1,t_2),[z_0:z_1:z_2]) \mapsto [z_0 : t_1z_1 : t_2z_2]$
$(\mathbb{C}^*)^2 \cdot [1:1:1] = \{[1:t_1:t_2]\} \cong (\mathbb{C}^*)^2$
$(\mathbb{C}^*)^2 \cdot [1:0:0] = \{[1:0:0]\}$
$(\mathbb{C}^*)^2 \cdot [0:1:0] = \{[0:1:0]\}$
$(\mathbb{C}^*)^2 \cdot [0:0:1] = \{[0:0:1]\}$
$(\mathbb{C}^*)^2 \cdot [1:1:0] = \{[1:t_1:0]\} \cong \mathbb{C}^*$
$(\mathbb{C}^*)^2 \cdot [1:0:1] = \{[1:0:t_2]\} \cong \mathbb{C}^*$
$(\mathbb{C}^*)^2 \cdot [0:1:1] = \{[0:1:t_1^{-1}t_2]\} \cong \mathbb{C}^*$
O
$\lim_{t\to 0}[1:t^a:t^b] = \begin{cases} [1:0:0] & \text{if } a,b>0, \\ [0:1:1] & \text{if } a=b<0, \\ [1:0:1] & \text{if } a>0 \,\&\, b=0, \\ [1:1:0] & \text{if } a=0 \,\&\, b>0, \\ [0:1:0] & \text{if } a<0 \,\&\, b>a, \\ [0:0:1] & \text{if } b<0 \,\&\, a>b, \\ [1:1:1] & \text{if } a=b=0. \end{cases}$
For $(a,b),(a',b') \in \mathbb{Z}^2$,
$\lim_{t\to 0}[1:t^a:t^b] = \lim_{t\to 0}[1:t^{a'}:t^{b'}]$
if and only if (a,b) and (a',b') belong to the interior of the same cone.
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Geometric retraction

Let G be a semisimple algebraic group, B a Borel subgroup of G, and T a maximal torus of G contained in B. Then G/B is a flag variety.

For each $u \in W$, let $C(u) = \{\lambda \in \mathfrak{t}_{\mathbb{R}} \mid \langle u(\alpha), \lambda \rangle \leq 0 \text{ for all simple roots } \alpha\}$.

Proposition. Let x be a point of G/B and $Y = \overline{T \cdot x}$. For $u \in W$ and $\lambda_u \in \mathrm{Int}(C(u)) \cap \mathfrak{t}_{\mathbb{Z}}$, the limit point $\lim_{t \to 0} \lambda_u(t) \cdot x$ is an element of Y^T depending only on u and Y.

Furthermore, if $u \in Y^T$, then $\lim_{t \to 0} \lambda_u(t) \cdot x = u$.

Let x be a point of G/B and $Y = \overline{T \cdot x}$. A geometric retraction is a map $\mathscr{R}_x^g \colon W \to Y^T$ defined by $\mathscr{R}_Y^g(u) = \lim_{t \to 0} \lambda_u(t) \cdot x$.

Corollary. The maximal cone $C_Y(y)$ corresponding to $y \in Y^T$ in the fan of (the normalization of) Y is given by $\bigcup_{u \in (\mathscr{R}_Y^g)^{-1}(y)} C(u)$.

Geometric retraction

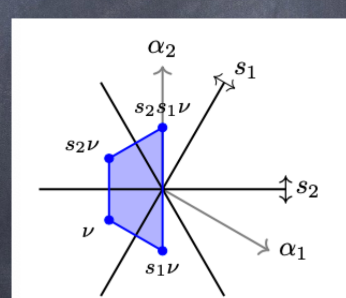
Take $x = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B \in \mathrm{SL}_3(\mathbb{C})$ and consider $Y = \overline{T \cdot x}$. Then $Y^T = \{123, 132, 213, 312\}$.

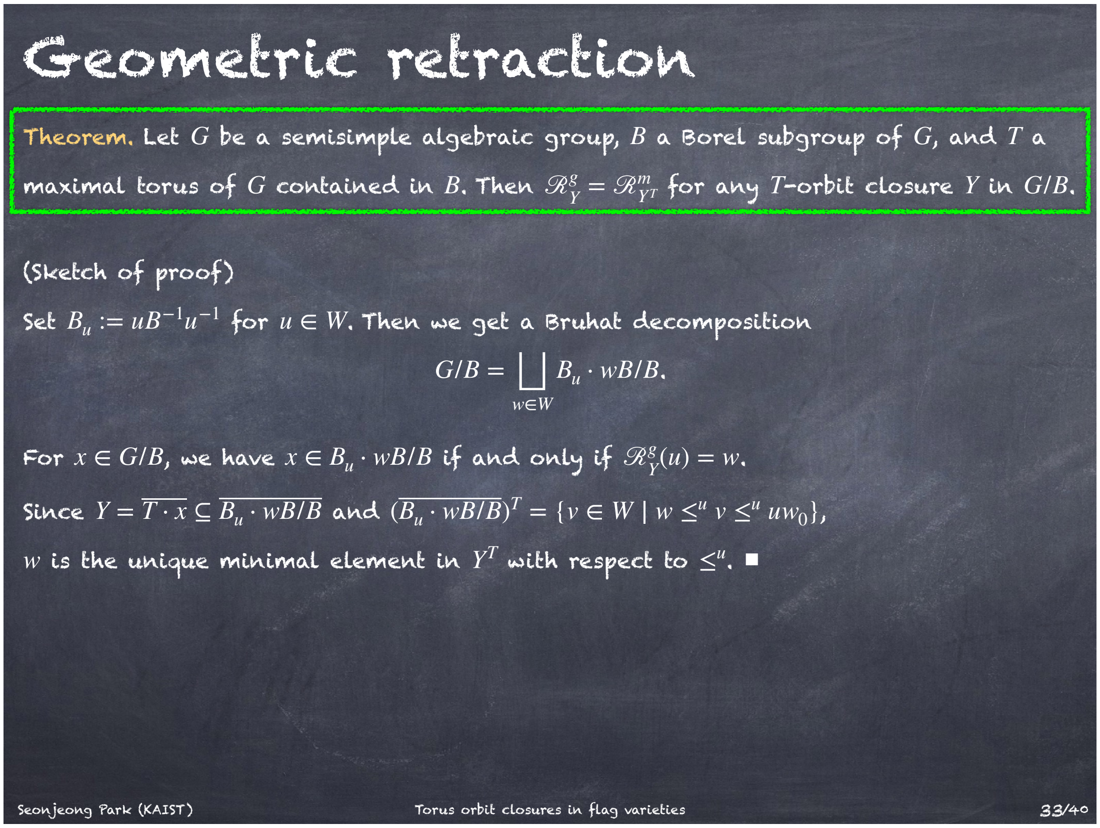
Choose $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Int}(C(231)) \cap \mathfrak{t}_{\mathbb{Z}}$. Then $\lambda_2 < \lambda_3 < \lambda_1$.

$$\lambda(t) \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} t^{\lambda_1} & 0 & 0 \\ 0 & t^{\lambda_2} & 0 \\ 0 & 0 & t^{\lambda_3} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} B = \begin{pmatrix} t^{\lambda_1} & t^{\lambda_1} & t^{\lambda_1} \\ t^{\lambda_2} & 0 & 0 \\ t^{\lambda_3} & 0 & t^{\lambda_3} \end{pmatrix} B = \begin{pmatrix} t^{\lambda_1 - \lambda_2} & 1 & t^{\lambda_1 - \lambda_3} \\ 1 & 0 & 0 \\ t^{\lambda_3 - \lambda_2} & 0 & 1 \end{pmatrix} B$$

$$\longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B \qquad (\text{as } t \to 0)$$

$\mathscr{R}_Y^g(u) = \mathscr{R}_{Y^T}^m(u)$ for all $u \in \mathfrak{S}_3$.




Geometric retraction
Theorem. Let G be a semisimple algebraic group, B a Borel subgroup of G, and T a maximal torus of G contained in B. Then $\mathscr{R}^g_Y = \mathscr{R}^m_{Y^T}$ for any T-orbit closure Y in G/B.
(Sketch of proof)
Set $B_u := uB^{-1}u^{-1}$ for $u \in W$. Then we get a Bruhat decomposition
$$G/B = \bigsqcup_{w \in W} B_u \cdot wB/B.$$
For $x \in G/B$, we have $x \in B_u \cdot wB/B$ if and only if $\mathscr{R}^g_Y(u) = w$.
Since $Y = \overline{T \cdot x} \subseteq \overline{B_u \cdot wB/B}$ and $(\overline{B_u \cdot wB/B})^T = \{v \in W \mid w \leq^u v \leq^u uw_0\}$,
w is the unique minimal element in Y^T with respect to $\leq^u$. ■
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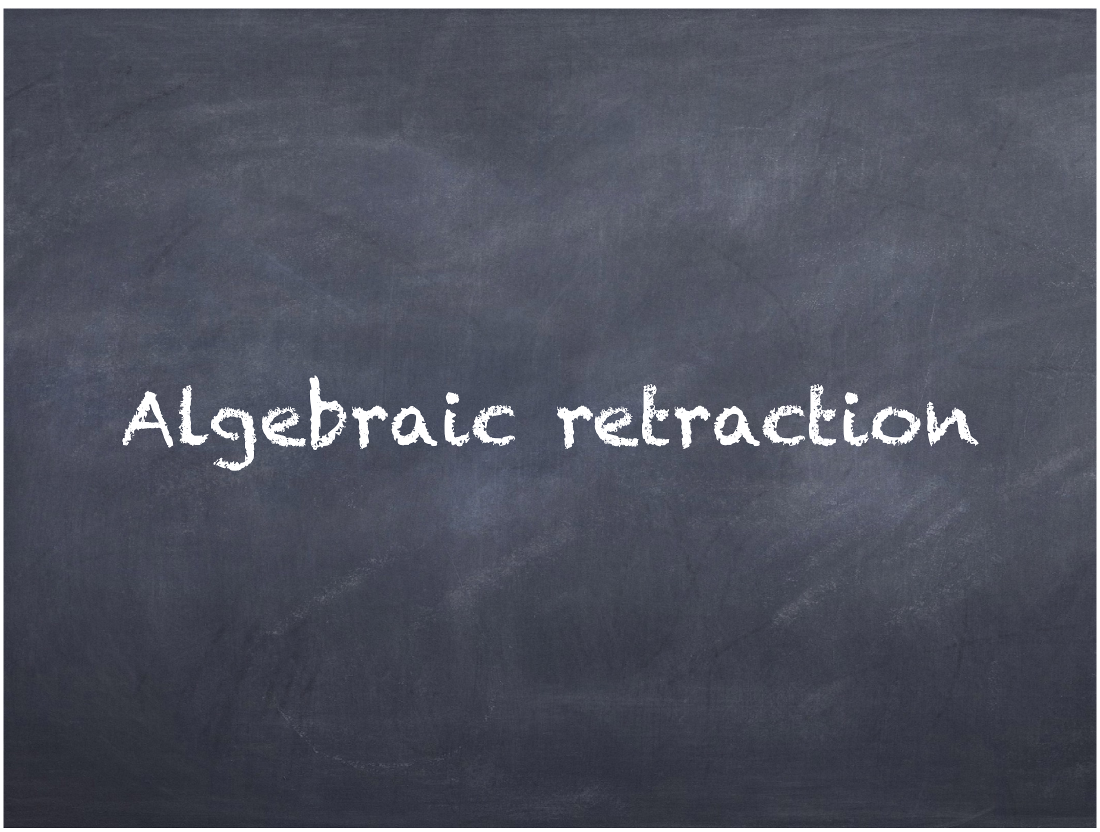

Algebraic retraction

Algebraic retraction

Now we assume that W is the Weyl group of classical Lie type. i.e.,

$$W = \begin{cases} \mathfrak{S}_n & \text{if } W \text{ is of type } A_{n-1}, \\ (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n & \text{if } W \text{ is of type } B_n \text{ or } C_n, \\ (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes \mathfrak{S}_n & \text{if } W \text{ is of type } D_n. \end{cases}$$

We define a linear order $<^u$ on $[n] \cup [\bar{n}]$ by

$$u(1) <^u \cdots <^u u(n) <^u u(\bar{n}) <^u \cdots <^u u(\bar{1}).$$

Define a <u>linear order $<^u$</u> on W as follows:

$$v <^u w \text{ if and only if } v(1)\cdots v(n) <^u_{\text{lex}} w(1)\cdots w(n).$$

Definition. Let W be a Weyl group of classical Lie type and $\mathscr{M}$ an arbitrary subset of W. Then we define a map $\mathscr{R}^a_{\mathscr{M}} \colon W \to \mathscr{M}$ by sending u to the unique minimal element w.r.t. the order $<^u$, which we call an algebraic retraction.

Algebraic retraction

Note that for any $u, v, w \in W$, if $v <^u w$, then $v <^u w$.

Let $W = \prod_{j=1}^{k} W_j$, where each W_j is a Weyl group of classical Lie type.

Let $\mathscr{M} = \prod_{j=1}^{k} \mathscr{M}_j$, where $\mathscr{M}_j$ is an arbitrary subset of W_j. Then we define

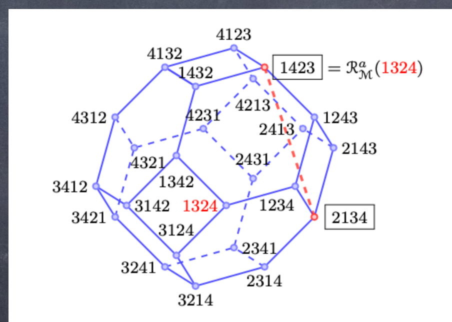
$$\mathscr{R}^a_{\mathscr{M}}(u) := (\mathscr{R}^a_{\mathscr{M}_1}(u_1), \ldots, \mathscr{R}^a_{\mathscr{M}_k}(u_k)) \in \mathscr{M} \text{ for } u = (u_1, \ldots, u_k) \in W.$$

Proposition. Let W and $\mathscr{M}$ be as above. If $\mathscr{M}$ is a Coxeter matroid, then $\mathscr{R}^a_{\mathscr{M}} = \mathscr{R}^m_{\mathscr{M}}$.

Therefore, if Y is a torus orbit closure in G/B, then $\mathscr{R}^g_Y = \mathscr{R}^a_{Y^T} = \mathscr{R}^m_{Y^T}$.

Distance property

Recall that the matroid retraction image is the unique element satisfying $d(u, \mathcal{R}^m_{\mathcal{M}}) = d(u, \mathcal{M})$. However, the algebraic retraction may not give a closest element.



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Characterization of Coxeter matroids

Question. Let W be a Weyl group of classical Lie type. Suppose that a subset $\mathcal{M}$ of W satisfies the following two conditions:

1. for each $u \in W$, there is a unique $q \in \mathcal{M}$ such that $d(u, q) = d(u, \mathcal{M})$, and
2. $q = \mathcal{R}^a_{\mathcal{M}}(u)$.

Is $\mathcal{M}$ a Coxeter matroid?

The question above is true when

1. $\mathcal{M}$ consists of two elements of $\mathfrak{S}_n$. consisting of two elements, or
2. $n \leq 6$.

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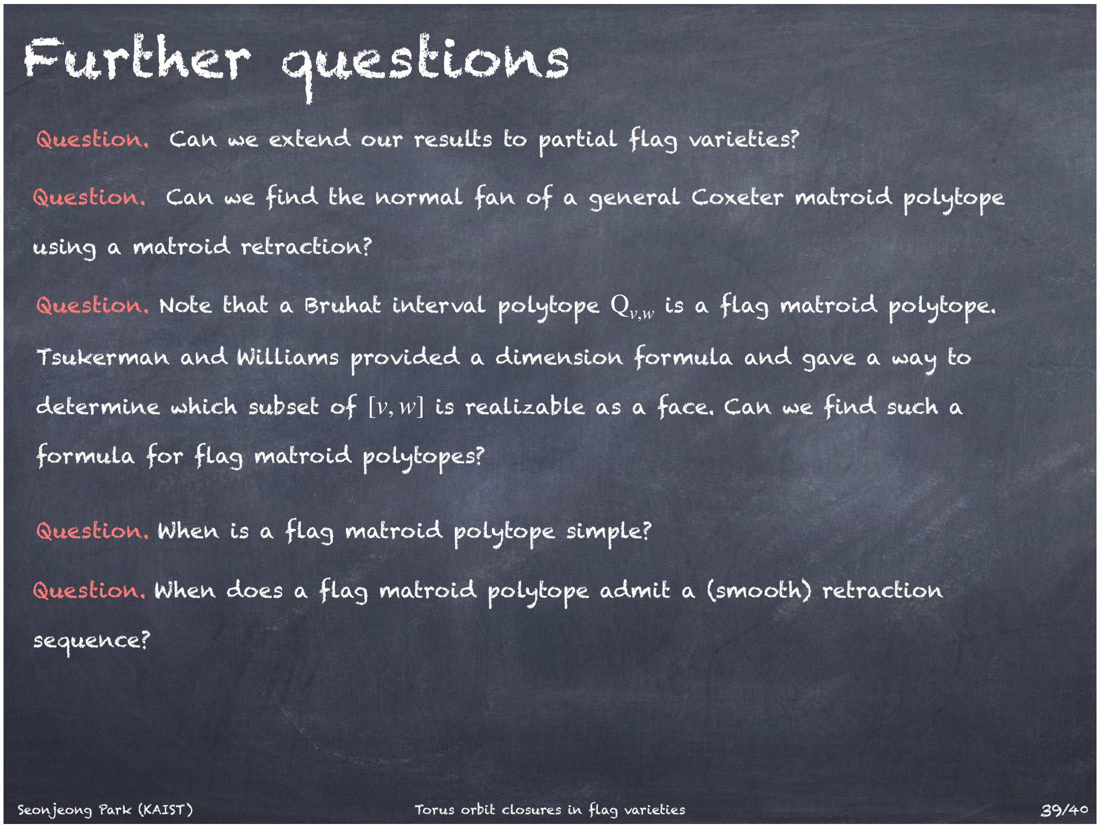
Further questions
Question. Can we extend our results to partial flag varieties?
Question. Can we find the normal fan of a general Coxeter matroid polytope using a matroid retraction?
Question. Note that a Bruhat interval polytope $Q_{v,w}$ is a flag matroid polytope. Tsukerman and Williams provided a dimension formula and gave a way to determine which subset of $[v, w]$ is realizable as a face. Can we find such a formula for flag matroid polytopes?
Question. When is a flag matroid polytope simple?
Question. When does a flag matroid polytope admit a (smooth) retraction sequence?
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Thank you!

Bases of the cohomology spaces of regular semisimple Hessenberg varieties

Jaehyun Hong

Regular semisimple Hessenberg varieties started getting attention in combinatorics after Shareshian and Wachs proposed a conjecture relating their cohomology spaces with chromatic quasi-symmetric functions of the incomparability graphs of (3+1)-free posets, and Brosnan and Chow, and independently Guay-Paquet confirmed it to be true. These works transformed Stanley-Stembridge conjecture on the positivity of chromatic symmetric functions into the decomposability of the cohomology spaces of regular semisimple Hessenberg varieties by permutation submodules.

In this talk, we consider the Bialynicki-Birula decomposition of regular semisimple Hessenberg varieties which induces bases for their equivariant cohomology spaces. For type A, we give an explicit combinatorial description of the support of each class and provide a way to compute the symmetric group action on the classes in our bases. If time permits, we explain how to apply the results to the permutohedral variety to obtain a permutation module decomposition of its cohomology space. This resolves the problem posed by Stembridge on the geometric construction of permutation module decomposition of the cohomology space and the conjecture posed by Chow on the construction of bases for the equivariant cohomology space. This talk is based on joint work with Soojin Cho and Eunjeong Lee.