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Citation	manuscripta mathematica. 160(3-4); 359-383
Issue Date	2019-11
Type	Journal Article
Textversion	Author
Right	This is a post-peer-review, pre-copyedit version of an article published in Archive for manuscripta mathematica. The final authenticated version is available online at: https://doi.org/10.1007/s00229-018-01102-9 See Springer Nature terms of use. https://www.springer.com/gp/open-access/publication-policies/aam-terms-of-use .
DOI	10.1007/s00229-018-01102-9

Self-Archiving by Author(s)

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THE NAVIER-STOKES EQUATIONS WITH THE NEUMANN BOUNDARY CONDITION IN AN INFINITE CYLINDER

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ABSTRACT. We prove unique existence of local-in-time smooth solutions of the Navier-Stokes equations for initial data in L^p and $p \in [3, \infty)$ in an infinite cylinder, subject to the Neumann boundary condition.

1. INTRODUCTION

We consider the three-dimensional Navier-Stokes equations subject to the Neumann boundary condition:

$$(1.1) \quad \begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p &= 0, & \operatorname{div} u &= 0 & \text{in } \Pi \times (0, T), \\ \nabla \times u \times n &= 0, & u \cdot n &= 0 & \text{on } \partial\Pi \times (0, T), \\ u &= u_0 & \text{on } \Pi \times \{t = 0\}, \end{aligned}$$

for the infinite cylinder

$$\Pi = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_h = (x_1, x_2), |x_h| < 1\}.$$

Here, n denotes the unit outward normal vector field on $\partial\Pi$. The local well-posedness of the Neumann problem (1.1) is established in [14], [27] for initial data in L^p , when Π is smoothly bounded. See also [28], [21] for the Dirichlet problem. The purpose of this paper is to develop L^p -theory of (1.1) for the infinite cylinder Π . Let L_σ^p denote the L^p -closure of $C_{c,\sigma}^\infty$, the space of all smooth solenoidal vector fields with compact support in Π . The main result of this paper is the following:

Theorem 1.1. *For $u_0 \in L_\sigma^p(\Pi)$ and $p \in [3, \infty)$, there exists $T > 0$ and a unique solution $u \in C([0, T]; L^p) \cap C^\infty(\bar{\Pi} \times (0, T])$ of (1.1) with the associated pressure $p \in C^\infty(\bar{\Pi} \times (0, T])$.*

The Neumann problem plays an important role in the theory of weak solutions to the Euler equations. When Π is a two-dimensional bounded and simply-connected domain, global

Date: December 6, 2018.

2010 Mathematics Subject Classification. 35Q35, 35K90.

Key words and phrases. Navier-Stokes equations, Neumann problem, Infinite cylinder.

weak solutions to the Euler equations are constructed in [5], [38], [29] by taking a vanishing viscosity limit to (1.1). Since vorticity satisfies the homogeneous Dirichlet boundary condition subject to the Neumann boundary condition (1.1), L^p -norms of the vorticity are uniformly bounded independently of viscosity.

For the three-dimensional Cauchy problem, vanishing viscosity methods are applied in [35], [22] to construct unique local-in-time solutions to the Euler equations in \mathbb{R}^3 . It is unknown whether a vanishing viscosity method is applicable for domains with boundary. See [12], [7], [37], [23] for local well-posedness results of the Euler equations. In [1], the author studied vanishing viscosity limits of (1.1) for axisymmetric data based on the main result of this paper.

We outline the proof of Theorem 1.1. We extend the approach for bounded domains [27]. We set the Laplace operator subject to the Neumann boundary condition

$$(1.2) \quad \begin{aligned} Bu &= -\Delta u \quad u \in D(B), \\ D(B) &= \{u \in W^{2,p}(\Pi) \mid \nabla \times u \times n = 0, \quad u \cdot n = 0 \quad \partial\Pi\}. \end{aligned}$$

When Π is smoothly bounded, it is known that the operator $-B$ generates a C_0 -analytic semigroup on L^p for $p \in (1, \infty)$ [27], [4]. We show analyticity of the semigroup for the infinite cylinder by using a solution formula for the resolvent problem. We then define a fractional power $B_0^{1/2}$ for the operator $B_0 = B + \lambda_0$ and $\lambda_0 > 0$. Since the operator B_0 admits a bounded imaginary power [31], [20], the domain of the fractional power $D(B_0^{1/2})$ is continuously embedded to the Sobolev space $W^{1,p}$.

We then define the Stokes operator as a restriction of the Laplace operator

$$(1.3) \quad \begin{aligned} Au &= Bu \quad u \in D(A), \\ D(A) &= D(B) \cap L_\sigma^p. \end{aligned}$$

Since the Laplace operator B is commutable with the Helmholtz projection operator \mathbb{P} , the Stokes operator acts as an operator on the solenoidal vector space L_σ^p . By using the analyticity of the semigroup and boundedness of the Helmholtz projection operator on L^p [33], we construct mild solutions $u \in C([0, T]; L^p)$ for $u_0 \in L_\sigma^p$ and $p \in [3, \infty)$ of the form

$$(1.4) \quad u = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}(\mathbb{P}u \cdot \nabla u)(s)ds.$$

Higher regularity of mild solutions follow from elliptic estimates for the Helmholtz projection and the Stokes operator. We show that all derivatives of solutions belong to the Hölder space $C^\mu((0, T]; L^s)$ for $\mu \in (0, 1/2]$ and $s \in (3, \infty)$.

This paper is organized as follows. In Section 2, we show that the Laplace operator generates an analytic semigroup on L^p for the infinite cylinder. In Section 3, we define a fractional power of the Laplace operator and prove a continuous embedding of the domain of the fractional power. In Section 4, we define the Stokes operator. In Section 5, we construct

mild solutions. In Section 6, we prove higher regularity of mild solutions. In Appendix A, we prove higher regularity estimates for the Laplace operator in the infinite cylinder, used in Section 6. In Appendix B, we estimate the resolvent of the Laplace operator by a multiplier theorem.

2. RESOLVENT ESTIMATES FOR THE LAPLACE OPERATOR

We start with a resolvent estimate for the Laplace operator subject to the Neumann boundary condition in the infinite cylinder. We derive a solution formula for the Neumann problem by using the resolvent of two-dimensional problems.

2.1. A solution formula. We consider the resolvent problem

$$(2.1) \quad \begin{aligned} \lambda u - \Delta u &= f \quad \text{in } \Pi, \\ \nabla \times u \times n &= 0, \quad u \cdot n = 0 \quad \text{on } \partial\Pi, \end{aligned}$$

for $\lambda \in \Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \theta\}$ and $\theta \in (\pi/2, \pi)$. We use a solution formula and estimate the resolvent of (2.1). We consider two-dimensional problems in a unit disk $D = \{x \in \mathbb{R}^2 \mid |x| < 1\}$:

$$(2.2) \quad \begin{aligned} \mu v - \Delta v &= g \quad \text{in } D, \\ \nabla^\perp \cdot v &= 0, \quad v \cdot n = 0 \quad \text{on } \partial D, \end{aligned}$$

$$(2.3) \quad \begin{aligned} \mu w - \Delta w &= h \quad \text{in } D, \\ \partial_n w &= 0 \quad \text{on } \partial D, \end{aligned}$$

for $\mu \in \Sigma_\theta$. Here, $\nabla^\perp = {}^t(\partial_2, -\partial_1)$. We denote by $B_i = -\Delta$ the Laplace operators associated with the boundary conditions in (2.2) and (2.3) for $i = 1, 2$, respectively. We begin with the problem (2.2).

Proposition 2.1. *Let $p \in (1, \infty)$ and $\theta \in (\pi/2, \pi)$. There exists a constant C such that for $g \in L^p(D)$ and $\mu \in \Sigma_\theta$ there exists a unique solution $v \in W^{2,p}(D)$ of (2.2) satisfying*

$$(2.4) \quad \|\mu v\|_{L^p(D)} + |\mu|^{\frac{1}{2}} \|\nabla v\|_{L^p(D)} + \|\nabla^2 v\|_{L^p(D)} \leq C \|g\|_{L^p(D)}.$$

The operator B_1 is invertible on $L^p(D)$.

Proof. The assertion is proved in [27, Theorem 3.10], [29, p.404, 1.8], [4, Theorem 1.2] for $|\mu| \geq \delta$ and arbitrary $\delta > 0$. We shall prove (2.4) for $|\mu| \leq \delta$.

We first prove the a priori estimate

$$(2.5) \quad \|v\|_{W^{1,p}(D)} \leq C\|g\|_{L^p(D)}$$

for solutions of (2.2) for $\mu = 0$ by a contradiction argument. Suppose on the contrary that (2.5) were false. Then there exists a sequence of functions $\{v_m\}$ satisfying (2.2) for $\mu = 0$ and g_m such that

$$\|v_m\|_{W^{1,p}(D)} = 1, \quad \|g_m\|_{L^p(D)} \rightarrow 0 \quad m \rightarrow \infty.$$

Since the estimate (2.4) holds for $\mu = 1$, applying (2.4) for $v_m - \Delta v_m = g_m + v_m$ implies that $\{v_m\}$ is uniformly bounded in $W^{2,p}$. Thus by the Rellich-Kondrachov theorem [13, 5.7 THEOREM 1], there exists a subsequence (still denoted by $\{v_m\}$) such that v_m converges to a limit v in $W^{1,p}$ and the limit v satisfies

$$\begin{aligned} \Delta v &= 0 \quad \text{in } D, \\ \nabla^\perp \cdot v &= 0 \quad v \cdot n = 0 \quad \text{on } \partial D. \end{aligned}$$

Since $\nabla^\perp \cdot v$ is harmonic and vanishes on ∂D , we have $\nabla^\perp \cdot v = 0$ in D . Moreover, by $-\Delta v = \nabla^\perp(-\nabla^\perp \cdot v) - \nabla \operatorname{div} v$ and $v \cdot n = 0$ on ∂D , we see that $\operatorname{div} v = 0$ in D . Since D is simply-connected, there exists a stream function ψ such that $v = \nabla^\perp \psi$. Since ψ is harmonic and constant on ∂D , we have $v \equiv 0$. This contradicts $\|v\|_{W^{1,p}} = 1$. Hence (2.5) holds. By (2.5) and (2.4) for $\mu = 1$, we obtain

$$(2.6) \quad \|v\|_{W^{2,p}(D)} \leq C\|g\|_{L^p(D)}$$

for solutions of (2.2) for $\mu = 0$.

By applying (2.6) for solutions of (2.2) for $|\mu| \leq \delta$, we obtain

$$\|v\|_{W^{2,p}(D)} \leq C\|g\|_{L^p(D)} + C\delta\|v\|_{L^p(D)},$$

with the constant C independent of δ . We take a small constant $\delta > 0$ such that $C\delta \leq 1/2$ and obtain $\|v\|_{W^{2,p}} \leq C\|g\|_{L^p}$. Thus (2.4) holds for $|\mu| \leq \delta$. \square

We next estimate the resolvent of the Neumann problem (2.3).

Proposition 2.2. *There exists a constant C such that for $h \in L^p(D)$ and $\mu \in \Sigma_\theta$ there exists a unique solution $w \in W^{2,p}(D)$ of (2.3) satisfying*

$$(2.7) \quad |\mu|\|w\|_{L^p(D)} + |\mu|^{\frac{1}{2}}\|\nabla w\|_{L^p(D)} + \|\nabla^2 w\|_{L^p(D)} \leq C\|h\|_{L^p(D)}.$$

Proof. Since the estimate (2.7) hold for $|\mu| \geq \delta$ and arbitrary $\delta > 0$ [3], [25, Theorem 3.1.2, 3.1.3], we prove it for $|\mu| \leq \delta$.

For $p = 2$, integration by parts yields

$$(2.8) \quad |\mu| \|w\|_2 + |\mu|^{\frac{1}{2}} \|\nabla w\|_2 \leq C \|h\|_2$$

with some constant C , independent of μ . We consider the Neumann problem

$$(2.9) \quad \begin{aligned} -\Delta p &= f & \text{in } D, \\ \partial_n p &= 0 & \text{in } \partial D, \end{aligned}$$

for average-zero functions $f \in L^p$, i.e. $\int_D f dx = 0$. Solutions of (2.9) uniquely exist up to an additive constant and satisfy the estimate

$$(2.10) \quad \|\nabla p\|_{W^{1,p}(D)} \leq C \|f\|_{L^p(D)},$$

by [24]. Applying (2.10) for $-\Delta w = h - \mu w$ yields the estimate (2.7) for $p = 2$.

We next consider the case $p \in (2, \infty)$. We apply the Gagliardo-Nirenberg inequality in \mathbb{R}^2 for an extension of $\varphi \in H^1(D)$ to \mathbb{R}^2 and observe that the inequality

$$\|\varphi\|_{L^p(D)} \leq C \|\varphi\|_{L^2(D)}^{\frac{2}{p}} \|\varphi\|_{H^1(D)}^{1-\frac{2}{p}}$$

holds. Applying the above inequality for $\varphi = w$ and (2.8) imply

$$\|w\|_{L^p(D)} \leq \frac{C'}{|\mu|} \|h\|_{L^2(D)},$$

for $|\mu| \leq \delta$. Since D is bounded, the right-hand side is estimated by the L^p -norm of h . We estimate ∇w by the same way and apply (2.10) to obtain (2.7).

For $p \in (1, 2)$, a duality argument and (2.10) yield the estimate (2.7) for $|\mu| \leq \delta$. We proved (2.7) for $p \in (1, \infty)$. \square

We now derive a solution formula for the problem (2.1). We use the cylindrical coordinate $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, $x_3 = z$ and decompose a vector field $f = f^r e_r(\theta) + f^\theta e_\theta(\theta) + f^z e_z$ by the basis $e_r(\theta) = {}^t(\cos \theta, \sin \theta, 0)$, $e_\theta(\theta) = {}^t(-\sin \theta, \cos \theta, 0)$, $e_z = {}^t(0, 0, 1)$. In the sequel, we write the horizontal component by $f_h = f^r e_r + f^\theta e_\theta$. We define a partial Fourier transform $\hat{u} = \mathcal{F}u$ by

$$(\mathcal{F}u)(x_h, \xi) = \int_{\mathbb{R}} e^{-ix_3 \xi} u(x_h, x_3) dx_3,$$

for functions $u(\cdot, x_3)$ in the Schwartz class $S(\mathbb{R}; X)$ for a Banach space X . See [6, Chapter 6].

Proposition 2.3. For $f \in C_c^\infty(\Pi)$, solutions of (2.1) are represented by $u = u_h + u^z e_z$ and

$$(2.11) \quad \begin{aligned} u_h &= \mathcal{F}^{-1}(\lambda + \xi^2 + B_1)^{-1} \mathcal{F} f_h, \\ u^z &= \mathcal{F}^{-1}(\lambda + \xi^2 + B_2)^{-1} \mathcal{F} f^z, \end{aligned}$$

where \mathcal{F}^{-1} denotes the Fourier inverse transform.

Proof. Let $u = u_h + u^z e_z$ be a solution of (2.1). Since

$$\nabla \times u = \left(-\partial_z u^\theta + \frac{1}{r} \partial_\theta u^z \right) e_r + \left(\partial_z u^r - \partial_r u^z \right) e_\theta + \left(\partial_r u^\theta + \frac{1}{r} u^\theta - \frac{1}{r} \partial_\theta u^r \right) e_z,$$

the Neumann boundary condition in (2.1) implies

$$u^r = 0, \quad \partial_r u^\theta + u^\theta = 0, \quad \partial_r u^z = 0 \quad \text{on } \{r = 1\}.$$

Thus u_h and u^z satisfy

$$\begin{aligned} \lambda u_h - \Delta u_h &= f_h && \text{in } \Pi, \\ \nabla^\perp \cdot u_h &= 0, \quad u_h \cdot n = 0 && \text{on } \partial\Pi, \\ \lambda u^z - \Delta u^z &= f^z && \text{in } \Pi, \\ \partial_n u^z &= 0 && \text{on } \partial\Pi. \end{aligned}$$

We consider the partial Fourier transform for u_h and u^z . Since \hat{u}_h and \hat{u}^z satisfy (2.2) and (2.3) for $\mu = \lambda + \xi^2$, $g = \hat{f}_h$, $h = \hat{f}_z$, we see that $\hat{u}_h = (\mu + B_1)^{-1} \hat{f}_h$ and $\hat{u}^z = (\mu + B_2)^{-1} \hat{f}^z$ by Propositions 2.1 and 2.2. By the Fourier inverse transform, we obtain (2.11). \square

2.2. L^2 -estimates. By using the formula (2.11), we construct unique solutions of (2.1).

Lemma 2.4. There exist constants $\delta > 0$ and $C > 0$ such that for $f \in L^p$ and $\lambda \in \Sigma_\theta$ satisfying $|\lambda| \geq \delta$, there exists a unique solution of (2.1) satisfying

$$(2.12) \quad |\lambda| \|u\|_{L^p(\Pi)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{L^p(\Pi)} + \|\nabla^2 u\|_{L^p(\Pi)} \leq C \|f\|_{L^p(\Pi)}.$$

We first apply a multiplier theorem on a Hilbert space.

Proposition 2.5. The assertion of Lemma 2.4 holds for $p = 2$ and $\delta = 0$.

Proof. We prove the estimate (2.12) for solutions given by the formula (2.11) for $f \in C_c^\infty(\Pi)$. We shall show the estimate

$$(2.13) \quad |\lambda| \|u_h\|_{L^2(\Pi)} + |\lambda|^{\frac{1}{2}} \|\nabla u_h\|_{L^2(\Pi)} + \|\nabla^2 u_h\|_{L^2(\Pi)} \leq C \|f_h\|_{L^2(\Pi)}$$

for $\lambda \in \Sigma_\theta$. We are able to estimate u_z by using (2.7) in the same way. We set

$$m_1(\xi) = \lambda(\lambda + \xi^2 + B_1)^{-1}.$$

By the resolvent estimate (2.4), the operator $m_1(\xi)$ acts as a bounded operator on $L^2(D)$. We set $\lambda = re^{i\eta}$ for $\eta \in (-\theta, \theta)$. We observe that $|\lambda|/|\lambda + \xi^2| \leq 1$ for $|\eta| \leq \pi/2$. For $\pi/2 \leq |\eta| < \theta$, it follows that

$$\frac{|\lambda|}{|\lambda + \xi^2|} = \frac{1}{|e^{i\eta} + \frac{\xi^2}{r}|} \leq \frac{1}{|\sin \theta|}, \quad \theta \in (\pi/2, \pi).$$

Thus the operators $\{m_1(\xi)\} \subset B(L^2(D))$ are uniformly bounded for $\xi \in \mathbb{R} \setminus \{0\}$ and there exists a constant C , independent of λ such that

$$(2.14) \quad \sup \left\{ \|m_1(\xi)\| \mid \xi \in \mathbb{R} \setminus \{0\} \right\} \leq C, \quad \lambda \in \Sigma_\theta.$$

Here, $\|\cdot\|$ denotes the operator norm on $L^2(D)$. Since the resolvent is holomorphic for $\mu \in \Sigma_\theta$ and $d(\mu + B_1)^{-1}/d\mu = -(\mu + B_1)^{-2}$, the estimate (2.14) implies

$$\sup \left\{ \|\xi m_1'(\xi)\| \mid \xi \in \mathbb{R} \setminus \{0\} \right\} \leq C'.$$

Thus Mikhlin's operator-valued multiplier theorem on a Hilbert space [6, 6.1.6 Theorem], [10, I, 3.20. Corollary] implies that the operator $\mathcal{F}^{-1} m_1(\cdot) \mathcal{F}$ acts as a bounded operator on $L^q(\mathbb{R}; L^2(D))$ for all $q \in (1, \infty)$. Since $L^2(\mathbb{R}; L^2(D)) = L^2(\Pi)$, this in particular implies the resolvent estimate

$$|\lambda| \|u_h\|_{L^2(\Pi)} \leq C \|f_h\|_{L^2(\Pi)}.$$

By a similar way, we are able to estimate the higher order terms and obtain (2.13). We proved (2.12) for $p = 2$ and $f \in C_c^\infty(\Pi)$.

For general $f \in L^2(\Pi)$, we construct solutions by taking a sequence $f_m \in C_c^\infty(\Pi)$ such that $f_m \rightarrow f$ in $L^2(\Pi)$ and using the estimate (2.12). The uniqueness follows from integration by parts. \square

Remark 2.6. By using a multiplier theorem on a UMD-space, we are able to obtain the L^p -estimate

$$(2.15) \quad |\lambda| \|u\|_{L^p(\Pi)} \leq C \|f\|_{L^p(\Pi)}$$

for solutions to (2.1) and $\lambda \in \Sigma_\theta$. We give a proof for (2.15) in Appendix B.

2.3. L^p -estimates. We next prove (2.12) for $p \in (1, \infty)$ and large $|\lambda| \geq \delta$ by a cut-off function argument. We apply L^p -estimates for the resolvent equation in a smoothly bounded domain $G \subset \mathbb{R}^3$:

$$(2.16) \quad \begin{aligned} \lambda u - \Delta u &= f \quad \text{in } G, \\ \nabla \times u \times n &= g, \quad u \cdot n = 0 \quad \text{on } \partial G. \end{aligned}$$

Proposition 2.7. *For $\delta > 0$, there exists a constant $C > 0$ such that*

$$(2.17) \quad |\lambda| \|u\|_{L^p(G)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{L^p(G)} + \|\nabla^2 u\|_{L^p(G)} \leq C(\|f\|_{L^p(G)} + |\lambda|^{\frac{1}{2}} \|g\|_{L^p(G)} + \|\nabla g\|_{L^p(G)})$$

holds for solutions of (2.16) for $|\lambda| \geq \delta$, $f \in L^p(G)$ and $g \in W^{1,p}(G)$ satisfying $g \cdot n = 0$ on ∂G .

Proof. The stronger estimate

$$|\lambda| \|u\|_{L^p(G)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{L^p(G)} + \|\nabla^2 u\|_{L^p(G)} \leq C(\|f\|_{L^p(G)} + |\lambda|^{\frac{1}{2p'}} \|g\|_{L^p(\partial G)})$$

is proved in [4, Theorem 1.2], where p' denotes the conjugate exponent to p . Since the trace of h is estimated by

$$\|g\|_{L^p(\partial G)} \leq C \|g\|_{L^p(G)}^{\frac{1}{p'}} \|g\|_{W^{1,p}(G)}^{\frac{1}{p}},$$

by [19, II.4., Theorem II.4.1], applying the Young's inequality implies (2.17). \square

Proposition 2.8. *The a priori estimate (2.12) holds for solutions of (2.1) for $f \in C_c^\infty(\Pi)$.*

Proof. Let $\{\varphi_j\}_{j=-\infty}^\infty \subset C_c^\infty(\mathbb{R})$ be a partition of the unity such that $0 \leq \varphi_j \leq 1$, $\text{spt } \varphi_j \subset [j-1, j+1]$, $\sum_{j=-\infty}^\infty \varphi_j(x_3) = 1$, $x_3 \in \mathbb{R}$. For a solution u of (2.1), we see that $u_j = u\varphi_j$ satisfies

$$\begin{aligned} \lambda u_j - \Delta u_j &= f_j \quad \text{in } G_j, \\ \nabla \times u_j \times n &= g_j, \quad u \cdot n = 0 \quad \text{on } \partial G_j, \end{aligned}$$

for $G_j = D \times (j - 1, j + 1)$ and

$$\begin{aligned} f_j &= f\varphi_j - 2\partial_{x_3}u\partial_{x_3}\varphi_j - u\partial_{x_3}^2\varphi_j, \\ g_j &= \nabla\varphi_j \times u \times n. \end{aligned}$$

We take a smoothly bounded domain \tilde{G}_j such that $G_j \subset \tilde{G}_j \subset D \times [j - 2, j + 2]$. Since the estimate (2.17) holds in \tilde{G}_j for $|\lambda| \geq \delta$ with $\delta > 0$ by Proposition 2.7, we have

$$|\lambda|\|u_j\|_{L^p(\tilde{G}_j)} + |\lambda|^{\frac{1}{2}}\|\nabla u_j\|_{L^p(\tilde{G}_j)} + \|\nabla^2 u_j\|_{L^p(\tilde{G}_j)} \leq C(\|f_j\|_{L^p(\tilde{G}_j)} + |\lambda|^{\frac{1}{2}}\|g_j\|_{L^p(\tilde{G}_j)} + \|\nabla g_j\|_{L^p(\tilde{G}_j)})$$

with some constant C , independent of j . The above estimate yields

$$|\lambda|\|u\varphi_j\|_{L^p(\Pi)} + |\lambda|^{\frac{1}{2}}\|\nabla u\varphi_j\|_{L^p(\Pi)} + \|\nabla^2 u\varphi_j\|_{L^p(\Pi)} \leq C'(\|f\|_{L^p(G_j)} + \|u\|_{W^{1,p}(G_j)} + |\lambda|^{\frac{1}{2}}\|u\|_{L^p(G_j)}).$$

By summing over j , we obtain

$$|\lambda|\|u\|_{L^p(\Pi)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{L^p(\Pi)} + \|\nabla^2 u\|_{L^p(\Pi)} \leq C''(\|f\|_{L^p(\Pi)} + \delta^{-\frac{1}{2}}(|\lambda|\|u\|_{L^p(\Pi)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{L^p(\Pi)}))$$

for $|\lambda| \geq \delta$ and $\delta \geq 1$. We take $\delta \geq 1$ so that $C''\delta^{-1/2} \leq 1/2$ and obtain (2.12). \square

Proof of Lemma 2.4. We apply the a priori estimate (2.12) for solutions given by the formula (2.11) for $f \in C_c^\infty(\Pi)$. For general $f \in L^p(\Pi)$, we construct solutions by an approximation by elements of $C_c^\infty(\Pi)$ and the estimate (2.12). The uniqueness follows from a duality argument. The proof is now complete. \square

3. FRACTIONAL POWERS

In this section, we see that a domain of a square root of the Laplace operator $B_0 = B + \lambda_0$ for $\lambda_0 > 0$ is continuously embedded to the Sobolev space $W^{1,p}(\Pi)$. We first recall the notion of a bounded H^∞ -calculus for a sectorial operator in an abstract Banach space. In the subsequent section, we apply an abstract theory to the operator B_0 and deduce the continuous embeddings.

3.1. BIP and H^∞ . We recall a bounded H^∞ -calculus [26]. We follow a booklet [10]. We say that a closed linear operator L in a Banach space X is *sectorial* if the domain $D(L)$ and the range $R(L)$ are dense in X , $(-\infty, 0) \subset \rho(L)$ and there exists a constant $C > 0$ such that

$$(3.1) \quad \|(t + L)^{-1}\| \leq \frac{C}{t} \quad t > 0.$$

Here, $\rho(L)$ is the resolvent set of L , i.e., the set of all $\lambda \in \mathbb{C}$ such that $(\lambda - L) : D(L) \subset X \rightarrow X$ is invertible and $(\lambda - L)^{-1}$ acts as a bounded operator on X . If $(-\infty, 0) \subset \rho(L)$, $(-t - L)^{-1} = -(t + L)^{-1}$ is a bounded operator for $t > 0$ and we are able to define the condition (3.1) with the operator norm $\|\cdot\|$ on X . Since $(t + L)^{-1}$ is differentiable and all derivatives are estimated by (3.1), by the Taylor expansion, we are able to extend $(t + L)^{-1}$ as an analytic function to a sector $\Sigma_\theta = \{t \in \mathbb{C} \setminus \{0\} \mid |\arg t| < \theta\}$ for some $\theta \in (0, \pi/2)$ and $t(t + L)^{-1}$ is bounded in Σ_θ , i.e.,

$$(3.2) \quad \sup \{t\|(t + L)^{-1}\| \mid t \in \Sigma_\theta\} < \infty.$$

The infimum of $\phi = \pi - \theta$ for $\theta \in (0, \pi)$ such that (3.2) holds is called *spectral angle* of L , denoted by ϕ_L . If the operator L is sectorial, we are able to define the fractional power L^α for $\alpha \in [0, 1]$. By the notation above, we avoid writing $(-\tilde{L})^\alpha$.

Let $H(\Sigma_\phi)$ denote the space of all holomorphic functions in Σ_ϕ . For simplicity, we abbreviate the notation by omitting the symbol Σ_ϕ . Let H^∞ denote the space of all bounded and holomorphic functions in Σ_ϕ . The space H^∞ is equipped with the norm $|f|_\infty^\phi = \sup_{\lambda \in \Sigma_\phi} |f(\lambda)|$. We set $H_0 = \bigcup_{\alpha < 0, \beta < 0} H_{\alpha, \beta}$ by $H_{\alpha, \beta}$, the space of all functions $f \in H$ such that $|f|_{\alpha, \beta}^\infty = \sup_{|\lambda| \leq 1} |\lambda|^\alpha |f(\lambda)| + \sup_{|\lambda| \geq 1} |\lambda|^{-\beta} |f(\lambda)|$ is finite. The space H_0 is smaller than H^∞ and consists of functions vanishing at $\lambda = 0$ and $|\lambda| \rightarrow \infty$.

We define bounded linear operators $f(L)$ for holomorphic functions $f \in H_0$. Here, we take $\phi \in (\phi_L, \pi)$ so that the spectrum $\sigma(L) = \mathbb{C} \setminus \rho(L)$ is included in Σ_ϕ . We take a counter-clockwise integral path $\Gamma = (\infty, 0]e^{i\psi} \cup [0, \infty)e^{-i\psi}$ for $\psi \in (\phi_L, \phi)$ in Σ_ϕ and set a bounded linear operator $f(L)$ by the Dunford integral

$$(3.3) \quad f(L) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - L)^{-1} d\lambda, \quad f \in H_0(\Sigma_\phi).$$

We say that the operator L admits a *bounded H^∞ -calculus* if there exists $\phi \in (\phi_L, \pi)$ and $K > 0$ such that

$$(3.4) \quad \|f(L)\| \leq K|f|_\infty^\phi \quad \text{for } f \in H_0(\Sigma_\phi).$$

The infimum of such ϕ is called *H^∞ -angle*, denoted by ϕ_L^∞ . If the operator L admits a bounded H^∞ -calculus, we are able to define a bounded linear operator $f(L)$ for $f \in H^\infty$ by an approximation. In particular, we are able to define pure imaginary powers L^{is} since $f(\lambda) = \lambda^{is}$ is bounded and holomorphic in Σ_π . Here, λ^{is} takes the principal branch. We say that the operator L admits a *bounded imaginary power* if there exists a constant C such that

$$(3.5) \quad \|L^{is}\| \leq C \quad \text{for } |s| \leq 1.$$

Since L^{is} forms a group, the estimate (3.5) implies that L^{is} is quasi-bounded, i.e., $\|L^{is}\| \leq Ce^{\theta|s|}$ for $s \in \mathbb{R}$ and some constants $\theta, C > 0$. The infimum of such θ is called *power angle* of L , denoted by θ_L . It follows from (3.4) that $0 \leq \phi_L \leq \theta_L \leq \phi_L^\infty < \pi$.

If a sectorial operator admits a bounded imaginary power, it follows that the domain of the fractional power $D(L^\alpha)$ agrees with the complex interpolation space $[X, D(L)]_\alpha$ for $\alpha \in [0, 1]$. Here, $D(L^\alpha)$ is equipped with the graph-norm. See [36], [10] for fractional powers of a sectorial operator.

3.2. A domain of a square root. We now define fractional powers for the operator $B_0 = B + \lambda_0$ for $\lambda_0 > 0$. By Lemma 2.4, the operator B_0 is invertible and sectorial on L^p with spectral angle zero. The boundedness of pure imaginary powers of the operator B_0 is proved by R. Seeley in [31]. More strongly, we have:

Proposition 3.1. *There exists $\lambda_0 > 0$ such that B_0 admits a bounded H^∞ -calculus.*

Proof. The assertion is proved in [20, Theorem 3.1] for general uniformly regular domains. \square

Proposition 3.1 implies that $D(B_0^\alpha) = [L^p, D(B_0)]_\alpha$ for $\alpha \in [0, 1]$. Here, we define the fractional power B_0^α as an inverse of the bounded operator $B_0^{-\alpha}$, i.e.,

$$\begin{aligned} B_0^\alpha u &= (B_0^{-\alpha})^{-1} u, \quad u \in D(B_0^\alpha), \\ D(B_0^\alpha) &= R(B_0^{-\alpha}), \quad \alpha \in (0, 1], \end{aligned}$$

and $B_0^0 = I$. The fractional power $B_0^{-\alpha}$ is defined by the Dunford integral (3.3) by taking $f(\lambda) = \lambda^{-\alpha}$ and the counter-clockwise integral path Γ , consisting of the two half lines $\{\lambda \in \mathbb{C} \mid \arg(\lambda - a) = \pm\psi\}$ for some $a > 0$ and $\psi \in (0, \pi/2)$.

We deduce a continuous embedding of the domain of the square root $B_0^{1/2}$, which is used later in Section 6.

Lemma 3.2. *The domain $D(B_0^{1/2})$ is continuously embedded to $W^{1,p}(\Pi)$.*

Proof. Since there exists a linear extension operator from the complex interpolation space $[L^p(\Pi), W^{2,p}(\Pi)]_{1/2}$ to the Sobolev space $W^{1,p}(\mathbb{R}^3)$ [30, Theorem 6], the interpolation space agrees with the Sobolev space $W^{1,p}(\Pi)$ (see also [2, 7.57]). Thus we have

$$D(B_0^{1/2}) = [L^p(\Pi), D(B_0)]_{1/2} \subset [L^p(\Pi), W^{2,p}(\Pi)]_{1/2} = W^{1,p}(\Pi),$$

with continuous injection. □

4. THE STOKES OPERATOR

We define the Stokes operator as a restriction of the Laplace operator in a solenoidal vector space. Since the Helmholtz projection operator is commutable with the Laplace operator subject to the Neumann boundary condition, a restriction of the semigroup e^{-tB} forms a bounded C_0 -analytic semigroup on L_σ^p .

Lemma 4.1.

$$(4.1) \quad \mathbb{P}B = B\mathbb{P} \quad \text{on } D(B),$$

$$(4.2) \quad \mathbb{P}(\lambda + B)^{-1} = (\lambda + B)^{-1}\mathbb{P} \quad \text{on } L^p.$$

Proof. We prove (4.1). The equality (4.2) follows from (4.1). We take $u \in D(B)$. Since the operator \mathbb{P} acts as a bounded operator on $W^{2,p}$ [33, Theorem 6] (see Lemma 6.2 in Section 6), the function $\mathbb{P}u$ belongs to $W^{2,p}$. By taking the rotation to

$$u = \mathbb{P}u + \mathbb{Q}u = f + \nabla\phi,$$

we see that $\nabla \times f \times n = 0$ on $\partial\Pi$, where $\mathbb{Q} = I - \mathbb{P}$. Thus, $\mathbb{P}u \in D(B) \cap L_\sigma^p$. We shall show that

$$(4.3) \quad \mathbb{Q}B\mathbb{P}u = 0,$$

$$(4.4) \quad \mathbb{P}B\mathbb{Q}u = 0.$$

The property (4.1) follows from (4.3) and (4.4) since

$$\begin{aligned} \mathbb{P}Bu &= \mathbb{P}B(\mathbb{P}u + \mathbb{Q}u) \\ &= (I - \mathbb{Q})B\mathbb{P}u + \mathbb{P}B\mathbb{Q}u \\ &= B\mathbb{P}u. \end{aligned}$$

We prove (4.3). The property (4.4) follows from a duality. We set

$$\begin{aligned} \nabla\Phi &= \mathbb{Q}B\mathbb{P}u, \\ B\mathbb{P}u &= -\Delta f, \quad f = \mathbb{P}u \in D(B) \cap L_\sigma^p. \end{aligned}$$

It is not difficult to see that $\nabla\Phi \equiv 0$ since Φ satisfies the Neumann problem

$$\begin{aligned}\Delta\Phi &= -\operatorname{div}(\Delta f) = 0 \quad \text{in } \Pi, \\ \frac{\partial\Phi}{\partial n} &= -\Delta f \cdot n = \operatorname{div}_{\partial\Pi}(\nabla \times f \times n) = 0 \quad \text{on } \partial\Pi,\end{aligned}$$

in a weak sense. Here, $\operatorname{div}_{\partial\Pi}$ denotes the surface divergence on $\partial\Pi$. Indeed, integration by parts yields

$$\int_{\Pi} \Delta f \cdot \nabla\varphi \, dx = \sum_{i,j=1}^3 \int_{\partial\Pi} (\partial_j f^i - \partial_i f^j) n^j \partial_i \varphi \, d\mathcal{H} = \int_{\partial\Pi} (\nabla \times f \times n) \cdot \nabla\varphi \, d\mathcal{H} = 0,$$

for $\varphi \in C_c^\infty(\overline{\Pi})$. Since $\nabla\varphi$ is orthogonal to solenoidal vector fields, it follows that

$$\int_{\Pi} \nabla\Phi \cdot \nabla\varphi \, dx = - \int_{\Pi} \Delta f \cdot \nabla\varphi \, dx = 0.$$

The above equality is extendable for all $\nabla\varphi \in G^{p'}(\Pi)$ since gradients of functions in $C_c^\infty(\overline{\Pi})$ are dense in $G^{p'}(\Pi) = \{\nabla\varphi \in L^{p'}(\Pi) \mid \varphi \in L_{\text{loc}}^{p'}(\Pi)\}$ [33, Lemma 7], where p' is the conjugate exponent to p . It follows that

$$(\nabla\Phi, g) = (\nabla\Phi, \mathbb{P}g + \mathbb{Q}g) = (\nabla\Phi, \mathbb{Q}g) = 0 \quad \text{for } g \in C_c^\infty.$$

Here, (f, g) denotes the integral of $f \cdot g$ in Π for $f \in L^p$ and $g \in L^{p'}$. We proved $\nabla\Phi \equiv 0$. \square

We consider the Stokes operator

$$(4.5) \quad \begin{aligned}Au &= Bu \quad u \in D(A), \\ D(A) &= D(B) \cap L_\sigma^p,\end{aligned}$$

and the Neumann problem

$$(4.6) \quad \begin{aligned}\lambda u - \Delta u &= f, \quad \operatorname{div} u = 0 \quad \text{in } \Pi, \\ \nabla \times u \times n &= 0, \quad u \cdot n = 0 \quad \text{on } \partial\Pi.\end{aligned}$$

The problem (2.1) is equivalent to (4.6) for solenoidal vector fields $f \in L_\sigma^p$.

Lemma 4.2.

$$(4.7) \quad (\lambda + A)^{-1} = (\lambda + B)^{-1} \quad \text{on } L_\sigma^p,$$

for $\lambda \in \rho(-B)$. In particular, the Stokes operator $-A$ generates a bounded C_0 -analytic semigroup on L_σ^p .

Proof. We set $u = (\lambda + B)^{-1}f$ for $f \in L_\sigma^p$. It follows from (4.2) that

$$\mathbb{P}u = \mathbb{P}(\lambda + B)^{-1}f = (\lambda + B)^{-1}f = u.$$

Hence $u \in L_\sigma^p \cap D(B) = D(A)$. Since u is a unique solution of (4.6), $u = (\lambda + A)^{-1}f$. \square

We set the operator

$$A_0 = A + \lambda_0 \quad \lambda_0 > 0,$$

and define fractional powers of the operator by the same way as we did for B_0 in the previous section. Since the resolvent of A_0 agrees with that of B_0 on L_σ^p by (4.7), we have

Proposition 4.3.

$$(4.8) \quad A_0^{-\alpha} = B_0^{-\alpha} \quad \text{on } L_\sigma^p,$$

$$(4.9) \quad R(A_0^{-\alpha}) = R(B_0^{-\alpha}) \cap L_\sigma^p.$$

Proof. The property (4.8) follows from (4.7). We show (4.9). For an arbitrary $f \in R(A_0^{-\alpha})$, there exists $u \in L_\sigma^p$ such that $f = A_0^{-\alpha}u \in L_\sigma^p$. Since $f \in R(B_0^{-\alpha})$ by (4.8), we have $f \in R(B_0^{-\alpha}) \cap L_\sigma^p$. Conversely, for $f \in R(B_0^{-\alpha}) \cap L_\sigma^p$, there exists $u \in L^p$ such that $B_0^{-\alpha}u = f$. Since $f \in L_\sigma^p$, it follows from (4.2) that

$$f = \mathbb{P}f = \mathbb{P}B_0^{-\alpha}u = B_0^{-\alpha}\mathbb{P}u.$$

By multiplying B_0^α by $B_0^{-\alpha}u = B_0^{-\alpha}\mathbb{P}u$, we have $u = \mathbb{P}u \in L_\sigma^p$ and $f = B_0^{-\alpha}u = A_0^{-\alpha}u$ by (4.8). Hence $f \in R(A_0^{-\alpha})$. We proved (4.9). \square

We set the fractional power $A_0^\alpha u$ for $u \in D(A_0^\alpha) = R(A_0^{-\alpha})$ as we did for B_0 . Proposition 4.3 and Lemma 3.2 imply:

Lemma 4.4.

$$A_0^\alpha = B_0^\alpha \quad \text{on } D(A_0^\alpha),$$

$$D(A_0^\alpha) = D(B_0^\alpha) \cap L_\sigma^p.$$

In particular, $D(A_0^{1/2})$ is continuously embedded to $W^{1,p} \cap L_\sigma^p$.

In order to construct mild solutions of (1.4), we prepare an estimate of the composition operator $A_0^{-1/2}\mathbb{P}\text{div}$.

Proposition 4.5. *There exists a constant C such that*

$$(4.10) \quad \|A_0^{-1/2}\mathbb{P}\text{div } F\|_p \leq C\|F\|_p$$

for $F \in C_c^\infty(\Pi)$. The operator $A_0^{-1/2}\mathbb{P}\text{div}$ is uniquely extendable to a bounded operator on L^p .

Proof. We first observe that the operator $A_0 = A_{0,p}$ defined on L_σ^p satisfies

$$(4.11) \quad (A_{0,p}f, g) = (f, A_{0,p'}g), \quad f \in D(A_{0,p}), \quad g \in D(A_{0,p'}).$$

For simplicity, we abbreviate the subscript p . By (4.11), we see that the same property holds for the resolvent of A_0 and we have

$$(A_0^{-1/2}f, g) = (f, A_0^{-1/2}g), \quad f \in L_\sigma^p, \quad g \in L_\sigma^{p'}.$$

For $\varphi \in C_c^\infty$, integration by parts yields

$$(A_0^{-1/2}\mathbb{P}\text{div } F, \varphi) = (A_0^{-1/2}\mathbb{P}\text{div } F, \mathbb{P}\varphi) = (\text{div } F, A_0^{-1/2}\mathbb{P}\varphi) = -(F, \nabla A_0^{-1/2}\mathbb{P}\varphi).$$

Since $D(A_0^{1/2})$ is continuously embedded to $W^{1,p'}$ by Lemma 4.4, there exists a constant C such that

$$\|\nabla A_0^{-1/2}\mathbb{P}\varphi\|_{p'} \leq C\|\varphi\|_{p'}.$$

The estimate (4.10) follows from the duality. □

5. EXISTENCE OF MILD SOLUTIONS

We construct solutions of an integral equation (1.4) by using analyticity of the Stokes semigroup. We first prepare linear estimates for an iterative argument.

Lemma 5.1. *Let $T_0 > 0$, $1 < p \leq q < \infty$, $\alpha \in (0, 1)$ and $|k| \leq 2$. Let p, q satisfy $3(1/p - 1/q) \leq 1$. There exist constants $C_1 - C_4$ such that*

$$(5.1) \quad \|\partial_x^k e^{-tA} f\|_p \leq \frac{C_1}{t^{\frac{|k|}{2}}} \|f\|_p,$$

$$(5.2) \quad \|e^{-tA} f\|_q \leq \frac{C_2}{t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}} \|f\|_p,$$

$$(5.3) \quad \|e^{-tA} \mathbb{P} \operatorname{div} f\|_q \leq \frac{C_3}{t^{\frac{1}{2}+\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}} \|f\|_p,$$

$$(5.4) \quad \|(e^{-\rho A} - 1)e^{-tA} f\|_p \leq \frac{C_4}{t^\alpha} \rho^\alpha \|f\|_p,$$

for $f \in L_\sigma^p$ and $0 < t, \rho \leq T_0$.

Proof. The estimate (5.1) follows from the resolvent estimate (2.12). We prove (5.2). Applying the Gagliardo-Nirenberg inequality in \mathbb{R}^3 for an extension to $\varphi \in W^{1,q}(\Pi)$ [34] implies

$$(5.5) \quad \|\varphi\|_{L^q(\Pi)} \leq C \|\varphi\|_{L^p(\Pi)}^{1-\sigma} \|\varphi\|_{W^{1,p}(\Pi)}^\sigma$$

for $\sigma = 3(1/p - 1/q) \in [0, 1]$. The estimate (5.2) follows from (5.1) by applying (5.5) for $\varphi = e^{-tA} f$.

We prove (5.3). Since the operator $A_0 = A + \lambda_0$ is invertible and generates a bounded analytic semigroup on L_σ^q , we have

$$\|A_0^{\frac{1}{2}} e^{-tA_0} g\|_q \leq \frac{C}{t^{\frac{1}{2}}} \|g\|_q \quad g \in L_\sigma^q.$$

Since $e^{-tA} = e^{-tA_0} e^{-t\lambda_0}$, we have $\|A_0^{1/2} e^{-tA} g\|_q \leq C t^{-1/2} \|g\|_q$ for $t \leq T_0$. It follows from (4.10) and (5.2) that

$$\begin{aligned} \|e^{-tA} \mathbb{P} \operatorname{div} f\|_q &= \|A_0^{\frac{1}{2}} e^{-tA} A_0^{-\frac{1}{2}} \mathbb{P} \operatorname{div} f\|_q \\ &\leq \frac{C}{t^{\frac{1}{2}}} \|e^{-\frac{t}{2}A} A_0^{-\frac{1}{2}} \mathbb{P} \operatorname{div} f\|_q \\ &\leq \frac{C}{t^{\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}} \|f\|_p. \end{aligned}$$

Thus (5.3) holds.

It remains to show (5.4). It follows from (5.1) that

$$\begin{aligned}
\|(e^{-\rho A} - 1)e^{-tA}f\|_p &= \left\| \int_0^\rho \frac{d}{ds} e^{-sA} e^{-tA} f ds \right\|_p \\
&= \left\| \int_0^\rho A e^{-(s+t)A} f ds \right\|_p \\
&\leq C\|f\|_p \int_0^\rho \frac{ds}{s+t} = C\|f\|_p \int_0^{\frac{\rho}{r+1}} \frac{dr}{r+1}.
\end{aligned}$$

Since $r+1 \geq r^{1-\alpha}$ for $r \geq 0$, the estimate (5.4) holds. \square

By using the estimates (5.1)-(5.4), we construct mild solutions of (1.4). Let $C^\gamma((0, T]; L^p)$ denote the space of all functions in $C^\gamma([\delta, T]; L^p)$ for all $\delta \in (0, T)$.

Theorem 5.2. *For $u_0 \in L_\sigma^p$ and $p \in [3, \infty)$, there exists $T > 0$ and a unique solution $u \in C([0, T]; L^p)$ of (1.4) satisfying*

$$(5.6) \quad t^{\frac{1}{2}} \nabla u \in C([0, T]; L^p),$$

$$(5.7) \quad u \in C^\gamma((0, T]; L^p),$$

$$(5.8) \quad \nabla u \in C^{\frac{\gamma}{2}}((0, T]; L^p), \quad \gamma \in (0, 1).$$

For $p = 3$, $t^{3/2(1/3-1/q)}u \in C([0, T]; L^q)$ vanishes at time zero for $q \in (3, \infty)$.

Proof. We prove the case $p = 3$. We are able to prove the case $p \in (3, \infty)$ by a similar way. We set a sequence $\{u_j\}$ by

$$\begin{aligned}
u_{j+1} &= e^{-tA}u_0 - \int_0^t e^{-(t-s)A} \mathbb{P} \operatorname{div} u_j u_j ds, \\
u_1 &= e^{-tA}u_0.
\end{aligned}$$

For $q \in (3, \infty)$ and $\gamma = 3/2(1/3 - 1/q)$, we set

$$K_j = \sup_{0 \leq t \leq T} t^\gamma \|u_j\|_q(t).$$

We take $r = q/2$. Since $3(1/r - 1/q) = 3/q < 1$, it follows from (5.1) and (5.3) that

$$\begin{aligned}
\|u_{j+1}\|_q &\leq \|e^{-tA}u_0\|_q + \int_0^t \frac{C}{(t-s)^{\frac{1}{2} + \frac{3}{2}(\frac{1}{r} - \frac{1}{q})}} \|u_j\|_{2r} ds \\
&\leq \|e^{-tA}u_0\|_q + \frac{C'}{t^\gamma} K_j^2.
\end{aligned}$$

We have $K_{j+1} \leq K_1 + C_0 K_j^2$. Since e^{-tA} is strongly continuous on L_σ^3 , we see that $K_1 \rightarrow 0$ as $T \rightarrow 0$. We take small $T > 0$ so that $K_1 \leq (4C_0)^{-1}$ and obtain $K_j \leq 2K_1$ for all $j \geq 1$.

By estimating $u_{j+1} - u_j$ by a similar way, we are able to show that

$$\lim_{j \rightarrow \infty} \sup_{0 \leq t \leq T} t^\gamma \|u_{j+1} - u_j\|_q(t) = 0.$$

Thus a limit u satisfies the integral equation (1.4) such that $t^\gamma u \in C([0, T]; L^q)$ and $t^\gamma u$ vanishes at time zero.

We show continuity at time zero. We set $K = \sup_{0 \leq t \leq T} t^\gamma \|u\|_q(t)$ and $r = q/2$. It follows from (5.3) that

$$\begin{aligned} \|u - e^{-tA} u_0\|_3 &\leq \int_0^t \|e^{-(t-s)A} \mathbb{P} \operatorname{div} uu\|_3 ds \\ &\leq \int_0^t \frac{C}{(t-s)^{\frac{1}{2} + \frac{3}{2}(\frac{1}{r} - \frac{1}{3})} s^{2\gamma}} ds K^2 = C' K^2 \rightarrow 0 \quad \text{as } T \rightarrow 0. \end{aligned}$$

Thus $u \in C([0, T]; L^3)$. By a similar way, $t^{1/2} \nabla u \in C([0, T]; L^3)$ follows. We obtain the Hölder continuity (5.7) and (5.8) by estimating $u(t) - u(\tau)$ by using (5.4). The proof is complete. \square

6. HIGHER REGULARITY

We prove Theorem 1.1. It remains to show that mild solutions constructed in Theorem 5.2 are smooth in $\bar{\Pi} \times (0, T]$. We use the fractional power of $A_0 = A + \lambda_0$, defined in Section 4. By multiplying $e^{-\lambda_0 t}$ by the mild solution u , we see that $v = e^{-tA_0} u$ satisfies

$$\begin{aligned} (6.1) \quad v &= e^{-tA_0} v_0 - \int_0^t e^{-(t-s)A_0} F(s) ds, \\ F &= e^{\lambda_0 t} \mathbb{P} v \cdot \nabla v. \end{aligned}$$

Our goal is to prove:

Theorem 6.1. *All derivatives of v belong to $C^\mu((0, T]; L^p)$ for $\mu \in (0, 1/2)$ and $p \in (3, \infty)$. In particular, v is smooth in $\bar{\Pi} \times (0, T]$.*

Proof of Theorem 1.1. The assertion follows from Theorem 6.1 \square

We prove Theorem 6.1 by using regularity results for linear operators.

Lemma 6.2. (i) Let $f \in L^1(0, T; L^p) \cap C^\mu((0, T]; L^p)$ for $p \in (1, \infty)$ and $\mu \in (0, 1)$. Then,

$$w = \int_0^t e^{-(t-s)A_0} f(s) ds$$

belongs to $C^{1+\mu}((0, T]; L^p) \cap C^\mu((0, T]; D(A_0))$.

(ii) For an integer $m \geq 0$, there exists a constant C such that

$$(6.2) \quad \|\mathbb{P}f\|_{W^{m,p}} + \|\mathbb{Q}f\|_{W^{m,p}} \leq C\|f\|_{W^{m,p}}, \quad f \in W^{m,p}.$$

(iii) There exists a constant C such that

$$(6.3) \quad \|w\|_{W^{m+2,p}} \leq C\|A_0 w\|_{W^{m,p}}$$

for $w \in D(A_0)$ such that $A_0 w \in W^{m,p}$.

Proof. The assertion (i) holds for generators of an analytic semigroup in an abstract Banach space. See [25, 4.3.1 Theorem 4.3.4]. The regularity estimate (ii) is proved in [33, Theorem 6]. The assertion (iii) follows from Lemmas 2.4 and A.1 in Appendix A. \square

We successively apply the regularity results (i)-(iii) and prove that all derivatives of v belong to the Hölder space $C^\mu((0, T]; L^p)$ for $p \in (3, \infty)$. Since solutions of (6.1) for $v_0 \in L^3$ belong to L^p for positive time, by taking some $t_0 > 0$ as an initial time, we may assume that initial data is in L^p . Since $F \in C^\mu((0, T]; L^p)$ for $\mu \in (0, 1/2)$ by (5.8), applying Lemma 6.2 (i) implies

$$(6.4) \quad v \in C^{1+\mu}((0, T]; L^p) \cap C^\mu((0, T]; W^{2,p}).$$

Hence $F \in C^\mu((0, T]; W^{1,p})$ by (6.2). We show that third derivatives of v belong to $C^\mu((0, T]; L^p)$ by applying Lemma 6.2 (i) and (iii).

Proposition 6.3.

$$(6.5) \quad \partial_t v \in C^\mu((0, T]; D(A_0^{1/2})),$$

$$(6.6) \quad v \in C^{1+\mu}((0, T]; W^{1,p}) \cap C^\mu((0, T]; W^{3,p}).$$

Proof. The property (6.6) follows from (6.5) by the continuous injection from $D(A_0^{1/2})$ to $W^{1,p}$ and the elliptic regularity estimate (6.3) for $A_0 v = \partial_t v + F$. We prove (6.5). We differentiate v by the fractional power $A_0^{1/2}$ and apply Lemma 6.2 (i). We use the integral representation for $t \geq \delta > 0$ of the form

$$v(t) = e^{-(t-\delta)A_0}v(\delta) - \int_{\delta}^t e^{-(t-s)A_0}F(s)ds.$$

The first term is smooth for $t > \delta$. We multiply $A_0^{1/2}$ by the second term and observe that

$$\begin{aligned} A_0^{1/2} \int_{\delta}^t e^{-(t-s)A_0}F(s)ds &= \int_{\delta}^t A_0 e^{-(t-s)A_0} A_0^{-1/2} F(s)ds \\ &= \int_{\delta}^t \frac{d}{ds} (e^{-(t-s)A_0} A_0^{-1/2} F(s))ds - \int_{\delta}^t e^{-(t-s)A_0} A_0^{-1/2} F'(s)ds \\ &= A_0^{-1/2} F(t) - e^{-(t-\delta)A_0} A_0^{-1/2} F(\delta) - \int_{\delta}^t e^{-(t-s)A_0} A_0^{-1/2} F'(s)ds. \end{aligned}$$

The first two terms belong to $C^{1+\mu}((0, T]; L^p)$ since $A_0^{-1/2} F' \in C^{\mu}((0, T]; L^p)$ by $\partial_t v v \in C^{\mu}((0, T]; L^p)$ and Proposition 4.5. The last term belongs to $C^{\mu}((\delta, T]; L^p)$ by applying Lemma 6.2 (i). We proved (6.5). \square

The regularity property for third derivatives (6.6) implies that $F \in C^{1+\mu}((0, T]; L^p) \cap C^{\mu}((0, T]; W^{2,p})$. We apply Lemma 6.2 (i) for $v^{(1)} = \partial_t v$ and obtain regularity of fourth order derivatives.

Proposition 6.4.

$$(6.7) \quad \partial_t^s v \in C^{\mu}((0, T]; W^{4-2s,p}) \quad s = 0, 1, 2.$$

Proof. By differentiating $\partial_t v + A_0 v = F$ by time and integrating for $t \geq \delta > 0$, we see that

$$v^{(1)}(t) = e^{-(t-\delta)A_0}v^{(1)}(\delta) - \int_{\delta}^t e^{-(t-s)A_0}F^{(1)}(s)ds.$$

Since the right-hand side belongs to $C^{1+\mu}((0, T]; L^p) \cap C^{\mu}((0, T]; W^{2,p})$ by applying Lemma 6.2 (i), the desired property (6.7) holds for $s = 1, 2$. The case $s = 0$ follows by applying the higher regularity estimate (6.3) for $A_0 v = -\partial_t v + F$. \square

Proof of Theorem 6.1. We prove

$$(6.8) \quad \partial_t^s v = v^{(s)} \in C^{\mu}((0, T]; W^{2k-2s,p}) \quad s = 0, 1, \dots, k,$$

for any integers k . We prove by induction. For $k = 2$, the assertion holds by Proposition 6.4. Suppose that (6.8) holds for some k . It suffices to show that

$$(6.9) \quad v^{(s)} \in C^{\mu}((0, T]; W^{2k+2-2s,p}) \quad s = 0, 1, \dots, k+1.$$

We prove (6.9) for all s by induction. Since $v^{(k-1)} \in C^{1+\mu}((0, T]; L^p) \cap C^\mu((0, T]; W^{2,p})$ by (6.8), by applying the same argument as in the proof of Propositions 6.3 and 6.4, we obtain (6.9) for $s = k + 1$.

We suppose that (6.9) holds for $l + 1 \leq s \leq k + 1$. Our goal is to prove (6.8) for $s = l$. By the assumption of our induction, we have

$$(6.10) \quad \partial_t v^{(l)} \in C^\mu((0, T]; W^{2k-2l,p}).$$

It follows from (6.8) and (6.2) that

$$\begin{aligned} v^{(l)} &\in C^\mu((0, T]; W^{2k-2l,p}), \\ F^{(l)} &\in C^\mu((0, T]; W^{2k-2l-1,p}). \end{aligned}$$

Applying (6.3) for $A_0 v^{(l)} = -\partial_t v^{(l)} + F^{(l)}$ implies that

$$v^{(l)} \in C^\mu((0, T]; W^{2k-2l+1,p}).$$

Since $F^{(l)} \in C^\mu((0, T]; W^{2k-2l,p})$ by (6.2), we obtain $v^{(l)} \in C^\mu((0, T]; W^{2k-2l+2,p})$ by (6.3). Thus (6.9) holds for $s = l$. We proved (6.9) for all s . The proof is now complete. \square

Remarks 6.5. (i) (Regularity up to time zero) In Theorem 1.1, if in addition that $u_0 \in D(A_q^m)$ for all $m \geq 0$, all derivatives of u belong to $C^\mu([0, T]; L^q)$ for $\mu \in (0, 1/2)$ and $q \in (3, \infty)$.
(ii) (Dirichlet boundary condition) Theorem 1.1 holds also for the Dirichlet boundary condition. It is proved by Farwig-Ri [16] that the Stokes operator subject to the Dirichlet boundary condition generates a bounded C_0 -analytic semigroup on L_σ^p for $p \in (1, \infty)$. Moreover, the operator admits a bounded H^∞ -calculus. See also [15], [17], [18]. The higher regularity estimate (6.3) is also known for the Stokes operator subject to the Dirichlet boundary condition [8], [19, IV]. See also [32, III.1.5.1 Theorem].

APPENDIX A. HIGHER REGULARITY ESTIMATES FOR THE LAPLACE OPERATOR

In Appendix A, we prove higher regularity estimates for the Neumann problem

$$(A.1) \quad \begin{aligned} -\Delta u &= f \quad \text{in } \Pi, \\ \nabla \times u \times n &= g, \quad u \cdot n = 0 \quad \text{on } \partial\Pi. \end{aligned}$$

Let $W_{\tan}^{1,p}(\Pi)$ denote the space of all functions $g \in W^{1,p}(\Pi)$ such that tangential components of g vanish on the boundary $\partial\Pi$.

Lemma A.1. *Let Π be the infinite cylinder. Let $u \in W^{2,p}(\Pi)$ be a solution of (A.1) for $f \in L^p(\Pi)$ and $g \in W_{\tan}^{1,p}(\Pi)$ for $p \in (1, \infty)$. Assume that $f \in W^{m,p}(\Pi)$ and $g \in W^{m+1,p}(\Pi)$ for $m \geq 1$. Then, u belongs to $W^{m+2,p}(\Pi)$ and the estimate*

$$(A.2) \quad \|u\|_{W^{m+2,p}(\Pi)} \leq C(\|f\|_{W^{m,p}(\Pi)} + \|g\|_{W^{m+1,p}(\Pi)} + \|u\|_{W^{1,p}(\Pi)})$$

holds.

We prove Lemma A.1 by a reduction to bounded domains.

Proposition A.2. *For smoothly bounded domains, the assertion of Lemma A.1 holds.*

Proof. For bounded domains, the estimate (A.2) is proved in [4, Theorem 1.2] by a reduction to a bent half space. \square

Proof of Lemma A.1. We prove by a cut-off function argument as we did in the proof of Proposition 2.8. Let $\{\varphi_j\}_{j=-\infty}^{\infty} \subset C_c^{\infty}(\mathbb{R})$ be a partition of the unity such that $0 \leq \varphi_j \leq 1$, $\text{spt } \varphi_j \subset [j-1, j+1]$, $\sum_{j=-\infty}^{\infty} \varphi_j(x_3) = 1$, $x_3 \in \mathbb{R}$. We set $u_j = u\varphi_j$ and $G_j = D \times (j-1, j+1)$. Observe that u_j satisfies

$$\begin{aligned} -\Delta u_j &= f_j \quad \text{in } G_j, \\ \nabla \times u_j \times n &= g_j, \quad u_j \cdot n = 0 \quad \text{on } \partial G_j. \end{aligned}$$

for $f_j = f\varphi_j - 2\partial_{x_3} u \partial_{x_3} \varphi_j - u \partial_{x_3}^2 \varphi_j$ and $g_j = g\varphi_j + \nabla \varphi_j \times u \times n$. We take a smoothly bounded domain \tilde{G}_j such that $G_j \subset \tilde{G}_j \subset \Pi$ and apply Proposition A.2 to estimate

$$\|u_j\|_{W^{m+2,p}(\tilde{G}_j)} \leq C(\|f_j\|_{W^{m,p}(\tilde{G}_j)} + \|g_j\|_{W^{m+1,p}(\tilde{G}_j)} + \|u_j\|_{W^{1,p}(\tilde{G}_j)}).$$

This implies

$$\|\partial_x^k u \varphi_j\|_{L^p(G_j)} \leq C(\|f\|_{W^{m,p}(G_j)} + \|g\|_{W^{m+1,p}(G_j)} + \|u\|_{W^{m+1,p}(G_j)}).$$

for $|k| = m+2$. By summing over j , we have

$$\|u\|_{W^{m+2,p}(\Pi)} \leq C(\|f\|_{W^{m,p}(\Pi)} + \|g\|_{W^{m+1,p}(\Pi)} + \|u\|_{W^{m+1,p}(\Pi)}).$$

We obtain the desired estimate by induction for $m \geq 0$. \square

APPENDIX B. L^p -RESOLVENT ESTIMATES NEAR $\lambda = 0$

In Appendix B, we prove the resolvent estimate (2.15). We apply a multiplier theorem on a UMD-space due to L.Weis [39].

Lemma B.1. *Let $p \in (1, \infty)$ and $\theta \in (\pi/2, \pi)$. There exists a constant C such that*

$$(B.1) \quad |\lambda| \|u\|_{L^p(\Pi)} \leq C \|f\|_{L^p(\Pi)}$$

holds for solutions of (2.1) for $f \in L^p$ and $\lambda \in \Sigma_\theta$.

We prove Lemma B.1 by using the solution formula (2.11). We show that resolvent of the Laplace operators B_i ($i = 1, 2$) are R -bounded. We recall the notion of R -bounded. See [10]. Let X and Y be Banach spaces. Let $B(X, Y)$ denote the space of bounded linear operators from X to Y . We say that a family of bounded linear operators $\tau \subset B(X, Y)$ is R -bounded if there exists a constant C such that for all $T_1, \dots, T_N \in \tau$, $x_1, \dots, x_N \in X$ and $N \geq 1$,

$$\int_0^1 \left\| \sum_{j=1}^N r_j(t) T_j x_j \right\|_Y dt \leq C \int_0^1 \left\| \sum_{j=1}^N r_j(t) x_j \right\|_X dt,$$

holds, where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$ -valued random variables on $[0, 1]$, e.g., the Rademacher functions $r_j(t) = \text{sign}(\sin(2^j \pi t))$. The smallest constant C such that the above inequality holds is denoted by $R(\tau)$. For two families of R -bounded operators $\tau, \kappa \subset B(X, Y)$, the sum and product $\tau + \kappa = \{T + K \mid T \in \tau, K \in \kappa\}$ and $\tau\kappa = \{TK \mid T \in \tau, K \in \kappa\}$ are also R -bounded and satisfies $R(\tau + \kappa) \leq R(\tau) + R(\kappa)$ and $R(\tau\kappa) \leq R(\tau)R(\kappa)$.

Since the R -boundedness is stronger than the uniform boundedness, we are able to define R -sectorial operator and R -angle ϕ_L^R for a sectorial operator L by replacing the uniform bound (3.1) to the R -bound. When X is a UMD-space, it is known that a sectorial operator L with a bounded imaginary powers of power angle θ_L , is R -sectorial for $\phi_L^R \leq \theta_L$ [9]. When $X = Y = L^p(D)$ for $p \in (1, \infty)$, the condition of the R -boundedness is equivalent to the condition

$$(B.2) \quad \left\| \left(\sum_{j=1}^N |T_j x_j|^2 \right)^{1/2} \right\|_{L^p(D)} \leq C \left\| \left(\sum_{j=1}^N |x_j|^2 \right)^{1/2} \right\|_{L^p(D)}.$$

We say that a function $m : \mathbb{R} \setminus \{0\} \rightarrow B(X, Y)$ is a *Fourier Multiplier* on $L^q(\mathbb{R}; X)$ if the operator

$$Kf = \mathcal{F}^{-1} m(\cdot) \mathcal{F} f, \quad \text{for } f \in \mathcal{S}(\mathbb{R}; X),$$

extends to a bounded operator from $L^q(\mathbb{R}; X)$ to $L^q(\mathbb{R}; Y)$. It is known that for UMD-spaces X and Y , a function $m \in C^1(\mathbb{R} \setminus \{0\}; B(X, Y))$ is a Fourier Multiplier on $L^q(\mathbb{R}; X)$ for all $q \in (1, \infty)$ if $m(\xi)$ and $\xi m'(\xi)$ are R -bounded for $\xi \in \mathbb{R} \setminus \{0\}$ [39, 3.4 Theorem].

We apply a multiplier theorem on $L^q(\mathbb{R}; L^p(D))$ and estimate u_h given by the formula (2.11). We use the boundedness of the pure imaginary powers of B_1

Proposition B.2. *The estimate*

$$(B.3) \quad |\lambda| \|u_h\|_{L^p(\Pi)} \leq C \|f_h\|_{L^p(\Pi)}$$

holds for u_h given by the formula (2.11) for $f \in C_c^\infty(\Pi)$.

Proof. We show that the function $m_1(\xi) = \lambda(\lambda + \xi^2 + B_1)^{-1}$ is a Fourier multiplier on $L^q(\mathbb{R}; L^p(D))$ for all $q \in (1, \infty)$. Then, the estimate (B.2) follows from (2.11) by taking $q = p$.

It suffices to show that $m_1(\xi)$ and $\xi m'_1(\xi)$ are R -bounded. Since the operator B_1 on $L^p(D)$ admits a bounded imaginary power of power angle zero by [31], [11], it is an R -sectorial operator of R -angle zero [9]. This means that for $\theta \in (\pi/2, \pi)$ there exists a constant C such that

$$(B.4) \quad R(\{\mu(\mu + B_1)^{-1} \mid \mu \in \Sigma_\theta\}) \leq C.$$

Since $|\lambda|/|\lambda + \xi^2| \leq 1/\sin \theta$ for $\lambda \in \Sigma_\theta$ as in the proof of Proposition 2.5, it follows from (B.2) that

$$R(\{\lambda(\lambda + \xi^2)^{-1} \mid \xi \in \mathbb{R} \setminus \{0\}\}) \leq \frac{C}{\sin \theta}, \quad \lambda \in \Sigma_\theta.$$

Hence we have

$$\begin{aligned} R(\{m_1(\xi)\}) &= R(\{\lambda(\lambda + \xi^2 + B_1)^{-1}\}) \\ &\leq R(\{\lambda(\lambda + \xi^2)^{-1}\}) R(\{(\lambda + \xi^2)(\lambda + \xi^2 + B_1)^{-1}\}) \\ &\leq \frac{C}{\sin \theta}, \quad \lambda \in \Sigma_\theta. \end{aligned}$$

Since the resolvent is holomorphic, we are able to estimate an R -bound of $\xi m'_1(\xi)$ by using (B.4) as we did in the proof of Proposition 2.5. Thus the function m_1 is a Fourier multiplier on $L^q(\mathbb{R}; L^p(D))$ for all $q \in (1, \infty)$. \square

We next estimate u^z . We set the domain of the operator B_2 by $D(B_2) = \{w \in W^{2,p}(D) \mid \partial_n w = 0 \text{ on } \partial D\}$. Since the kernel of the operator B_2 is not empty on $L^p(D)$, it is not a sectorial operator in the sense of Section 3. We thus restrict the operator to a space of average-zero functions $L_0^p(D) = \{h \in L^p(D) \mid \int_D h dx = 0\}$ by setting

$$\begin{aligned}\tilde{B}_2 w &= -\Delta w \quad w \in D(\tilde{B}_2), \\ D(\tilde{B}_2) &= D(B_2) \cap L_0^p.\end{aligned}$$

Since the average of $\tilde{B}_2 w$ in D vanishes by the Neumann boundary condition, the operator \tilde{B}_2 is an invertible sectorial operator acting on L_0^p . Moreover, the operator \tilde{B}_2 admits a bounded imaginary power of power angle zero [33, Theorem 2].

We show the estimate (B.1) for u^z by applying a multiplier theorem for a resolvent of \tilde{B}_2 . We consider functions $f_1 \in C^\infty(\overline{\Pi})$ satisfying

$$(B.5) \quad \begin{aligned} &\text{spt } f_1 \text{ is compact in } \overline{\Pi}, \\ &\int_D f_1(x_h, x_3) dx_h = 0, \quad x_3 \in \mathbb{R}. \end{aligned}$$

We see that $\mathcal{F} f_1$ is average-zero in D and belongs to L_0^p for $\xi \in \mathbb{R}$. Hence we use the resolvent of \tilde{B}_2 and set

$$(B.6) \quad u_1 = \mathcal{F}^{-1}(\lambda + \xi^2 + \tilde{B}_2)^{-1} \mathcal{F} f_1.$$

The function u_1 is a solution of (2.3) for f_1 and its average in D vanishes for each $x_3 \in \mathbb{R}$.

Proposition B.3. *There exists a constant C such that*

$$(B.7) \quad |\lambda| \|u_1\|_{L^p(\Pi)} \leq C \|f_1\|_{L^p(\Pi)}$$

for $\lambda \in \Sigma_\theta$ and $f_1 \in C^\infty(\overline{\Pi})$ satisfying (B.5).

Proof. The assertion follows from a multiplier theorem as in the proof of Proposition B.2. \square

We subtract from f^z the average of f^z in D and apply Proposition B.3.

Proposition B.4. *The estimate*

$$(B.8) \quad |\lambda| \|u^z\|_{L^p(\Pi)} \leq C \|f^z\|_{L^p(\Pi)}$$

holds for u^z given by the formula (2.11) for $f \in C_c^\infty(\Pi)$.

Proof. We set the functions f_1 and f_2 by

$$\begin{aligned} f_1(x_h, x_3) &= f^z - f_2, \\ f_2(x_3) &= \int_D f^z(x_h, x_3) dx_h. \end{aligned}$$

Since $f_1 \in C^\infty(\overline{\Pi})$ satisfies (B.5), the function u_1 defined by (B.6) satisfies the Neumann problem (2.3) for f_1 and the estimate (B.7) holds by Proposition B.3. We set

$$u_2 = \mathcal{F}^{-1}(\lambda + \xi^2)^{-1} \mathcal{F} f_2.$$

Then, u_2 satisfies $\lambda u_2 - \partial_{x_3}^2 u_2 = f_2$. Since $m(\xi) = \lambda(\lambda + \xi^2)^{-1}$ satisfies $|m(\xi)| \leq |\sin \theta|^{-1}$ and $|\xi m'(\xi)| \leq 2|\sin \theta|^{-2}$, the classical Mihlin multiplier theorem [6, 6.1.6 Theorem] implies that

$$|\lambda| \|u_2\|_{L^p(\mathbb{R})} \leq C \|f_2\|_{L^p(\mathbb{R})} \quad \lambda \in \Sigma_\theta.$$

Hence u_2 satisfies $|\lambda| \|u_2\|_{L^p(\Pi)} \leq C \|f_2\|_{L^p(\Pi)}$. Since u^z agrees with $u_1 + u_2$, we obtain (B.8). \square

Proof of Lemma B.1. The estimate (B.1) holds for u given by the formula for $f \in C_c^\infty(\Pi)$ by (B.3) and (B.8). For general $f \in L^p(\Pi)$, we take a sequence $\{f_m\} \subset C_c^\infty(\Pi)$ such that $f_m \rightarrow f$ in $L^p(\Pi)$ and obtain the desired estimate. \square

ACKNOWLEDGEMENTS

The author is partially supported by JSPS through the Grant-in-aid for Young Scientist (B) 17K14217, Scientific Research (B) 17H02853 and Osaka City University Strategic Research Grant 2018 for young researchers.

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