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MATRIX PRODUCT FORMULA FOR $U_q(A_2^{(1)})$-ZERO RANGE PROCESS

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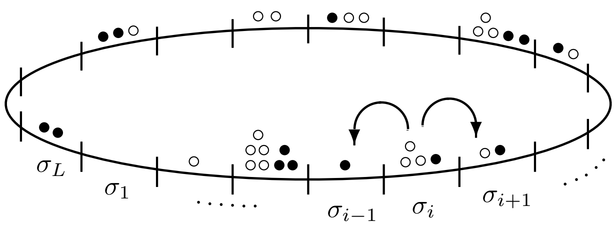
Abstract

The $U_q(A_n^{(1)})$-zero range processes introduced recently by Mangazeev, Maruyama and the authors are integrable discrete and continuous time Markov processes associated with the stochastic R matrix derived from the well-known $U_q(A_n^{(1)})$ quantum R matrix. By constructing a representation of the relevant Zamolodchikov-Faddeev algebra, we present, for $n = 2$, a matrix product formula for the steady state probabilities in terms of q-boson operators.

1. Introduction and main result

Zero range processes [23] are stochastic dynamical models for a variety of systems in biology, chemistry, networks, physics, sociology, traffic flows and so forth. Investigating their rich behaviors like condensation, current fluctuations and hydrodynamic limit, etc has been a prominent theme in mathematical physics of non-equilibrium phenomena. See for example [8, 12, 14] and the references therein.

In the recent work [15], new integrable Markov processes associated with the quantum affine algebra $U_q(A_n^{(1)})$ [7, 13] have been constructed. They are described naturally as discrete and continuous time stochastic dynamics of n-species of particles on a ring obeying a zero range type interaction. We call them $U_q(A_n^{(1)})$-zero range processes (ZRPs) in this paper. Here is a snapshot of the system for the $n = 2$ case on the length L periodic chain.



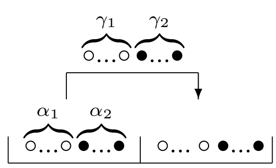
○ particle of species 1

● particle of species 2

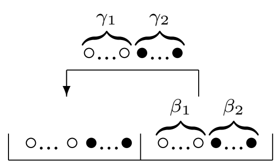
$$\sigma_{i-1} = (0, 1),\quad \sigma_i = (3, 1),\quad \sigma_{i+1} = (1, 1).$$

For n general, a local state at site $i \in \mathbb{Z}_L$ is an array $\sigma_i = (\sigma_{i,1}, \dots, \sigma_{i,n}) \in \mathbb{Z}_{\ge 0}^n$ signifying that there are $\sigma_{i,a}$ particles of species $a\,(1 \le a \le n)$. There is no constraint on the particles occupying a site. In the continuous time version of the model, they can hop either to the right or to the left adjacent sites with a zero range interaction, which means that the local transition rate depends on the occupancy of the departure site only and not on the destination site. For the $U_q(A_2^{(1)})$-ZRP, the rate for the nontrivial hopping (i.e., $\gamma_1 + \gamma_2 \ge 1$ below) is given by

$$a\frac{q^{(\alpha_1-\gamma_1)\gamma_2}\mu^{\gamma_1+\gamma_2-1}(q)_{\gamma_1+\gamma_2-1}}{(\mu q^{\alpha_1+\alpha_2-\gamma_1-\gamma_2})_{\gamma_1+\gamma_2}}\frac{(q)_{\alpha_1}}{(q)_{\gamma_1}(q)_{\alpha_1-\gamma_1}}\frac{(q)_{\alpha_2}}{(q)_{\gamma_2}(q)_{\alpha_2-\gamma_2}} \qquad \text{for}$$



$$b\frac{q^{\gamma_1(\beta_2-\gamma_2)}(q)_{\gamma_1+\gamma_2-1}}{(\mu q^{\beta_1+\beta_2-\gamma_1-\gamma_2})_{\gamma_1+\gamma_2}}\frac{(q)_{\beta_1}}{(q)_{\gamma_1}(q)_{\beta_1-\gamma_1}}\frac{(q)_{\beta_2}}{(q)_{\gamma_2}(q)_{\beta_2-\gamma_2}} \qquad \text{for}$$



in terms of the parameters a, b, μ, q and the symbol $(z)_m$ defined in the end of this section. These are the $n = 2$ cases of (21)–(25) with $\epsilon = 1$.

The $U_q(A^{(1)}_n)$-ZRPs [15] contain the earlier proposed n species models [17, 18, 25] via various specialization of the parameters. In particular for $n = 1$, they reproduce the single species models studied in [22, 20, 24, 5, 4] up to boundary conditions. A more detailed explanation is available in Section 2.3.

The above transition rate has been chosen so as to guarantee the integrability, or put more practically the Bethe ansatz solvability of the model via the *stochastic R matrix* $\mathcal{S}(\lambda,\mu)$ [15]. It originates in the quantum R matrix for the symmetric tensor representations of $U_q(A^{(1)}_n)$, a basic example of higher-spin representations of higher-rank quantum groups, in the special gauge that allows a probabilistic interpretation. Its nonzero matrix elements $\mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta}$ are described by the function $\Phi_q(\gamma|\beta;\lambda,\mu)$ $(\beta,\gamma \in \mathbb{Z}^n_{\geq 0})$ defined in (4) as

$$q^{\sum_{1\leq i<j\leq n}(\beta_i-\gamma_i)\gamma_j}\left(\frac{\mu}{\lambda}\right)^{\gamma_1+\cdots+\gamma_n}\frac{(\lambda)_{\gamma_1+\cdots+\gamma_n}(\frac{\mu}{\lambda})_{\beta_1+\cdots+\beta_n-\gamma_1-\cdots-\gamma_n}}{(\mu)_{\beta_1+\cdots+\beta_n}}\prod_{i=1}^{n}\frac{(q)_{\beta_i}}{(q)_{\gamma_i}(q)_{\beta_i-\gamma_i}}.$$

For $n = 1$ it reduces to the transition rate in the chipping model [20]. It was also built in the explicit formulas of the R matrix and Q operators for $U_q(A^{(1)}_1)$ [19]. The parameters λ and μ are reminiscents of the degrees of the relevant symmetric tensor representations of $U_q(A^{(1)}_n)$. They play a role analogous to the spectral parameter although the difference property $\mathcal{S}(c\lambda, c\mu) = \mathcal{S}(\lambda,\mu)$ is *absent*.

As the usual vertex models in equilibrium statistical mechanics [2], the stochastic R matrix $\mathcal{S}(\lambda,\mu_i)$ serves as a building block of the commuting *Markov transfer matrix* $T(\lambda|\mu_1,\ldots,\mu_L)$ (15) for the discrete time process governed by the master equation

$$|P(t+1)\rangle = T(\lambda|\mu_1,\ldots,\mu_L)|P(t)\rangle,$$

where μ_i is the inhomogeneity assigned to each lattice site $i \in \mathbb{Z}_L$. The discrete time ZRP covers the continuous time one in the sense that the Markov matrix of the latter is derived from the homogeneous case $T(\lambda|\mu,\ldots,\mu)$ of the former by the logarithmic derivative (20) as in the well-known Baxter's formula for spin chain Hamiltonians [2, Sec.10.14]. A novel feature of the $U_q(A^{(1)}_n)$ Markov transfer matrix $T(\lambda|\mu,\ldots,\mu)$ is the presence of *two* natural "Hamiltonian points" $\lambda = 1$ and $\lambda = \mu$, which yield the right and the left moving particles whose *mixture* is still integrable. These aspects have been demonstrated in detail in [15, Sec.3.4].

In this paper we study the steady states of the $U_q(A^{(1)}_n)$-ZRPs. By definition, steady states are those $|\overline{P}\rangle$ satisfying $|\overline{P}\rangle = T(\lambda|\mu_1,\ldots,\mu_L)|\overline{P}\rangle$. It exists uniquely in each sector specified by the total number of particles of each species, and serves as a basic characteristics of the system analogous to the ground states in the equilibrium spin chain models. Let $\mathbb{P}(\sigma_1,\ldots,\sigma_L)$ be the probability of finding the system in the configuration $(\sigma_1,\ldots,\sigma_L) \in (\mathbb{Z}^n_{\geq 0})^L$ in a steady state up to an overall normalization. For $n = 1$, it is known to become a product of on-site (albeit inhomogeneous) factors [20, 9] as

$$\mathbb{P}(\sigma_1,\ldots,\sigma_L) = \prod_{i=1}^{L} g_{\sigma_i}(\mu_i),$$

where $g_{\sigma_i}(\mu)$ is defined by (11) for general n. Such a factorization, however, is no longer valid in the multispecies case $n \geq 2$ as observed in [15, Example 13], and this is a source of interest in the model even without an interaction with a reservoir. Our main result in this paper is the following matrix product formula for $n = 2$:

$$\begin{aligned}&\mathbb{P}(\sigma_1,\ldots,\sigma_L) = \mathrm{Tr}(X_{\sigma_1}(\mu_1)\cdots X_{\sigma_L}(\mu_L)),\\ &X_\alpha(\mu) = g_\alpha(\mu)Z_\alpha(\mu), \quad Z_\alpha(\mu) = \left(\prod_{m=0}^{\infty}\frac{1-q^m\mathbf{b}_+}{1-q^m\mu^{-1}\mathbf{b}_+}\right)\mathbf{k}^{\alpha_2}\mathbf{b}_-^{\alpha_1}.\end{aligned} \tag{1}$$

Here $\alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}^2_{\geq 0}$ and $\mathbf{b}_+, \mathbf{b}_-, \mathbf{k}$ are q-boson operators (37) acting on the Fock space over which the trace is to be taken. One sees that the $n = 1$ case formally corresponds to setting $Z_\alpha(\mu) = 1$. In this sense $Z_\alpha(\mu)$, which may be viewed as single mode q-boson vertex operator $Z_\alpha(\mu) = \exp\left(\sum_{l\geq 1}\frac{\mu^{-l}-1}{l(1-q^l)}\mathbf{b}_+^l\right)\mathbf{k}^{\alpha_2}\mathbf{b}_-^{\alpha_1}$, is accounting for the first multispecies effect beyond $n = 1$. There are many matrix product formulas in terms of q-bosons known in the literature for similar models like exclusion processes. See [3] for example and the references therein. Our formula (1) is the first example distinct from them involving an *infinite product* of q-bosons.

In order to establish the above result, we invoke the so called *Zamolodchikov-Faddeev (ZF) algebra* [26, 10] having the stochastic R matrix as the structure function. It reads symbolically as

$$X(\mu) \otimes X(\lambda) = \check{\mathcal{S}}(\lambda, \mu)\big[X(\lambda) \otimes X(\mu)\big],$$

where $\check{\mathcal{S}}(\lambda, \mu)$ is defined after (8). See (29) for the concrete description. It is a local version of the stationary condition, and plays a central role in deriving the matrix product formula. Besides the ZF algebra however, we need one further essential ingredient, the *auxiliary condition* (30) on the operator $X_\alpha(\mu)$. Its formulation is another new result in this paper which contrasts with the simpler situation of continuous time models on a ring (cf. [21, 1, 6, 16]) where the ZF algebra alone almost sufficed together with its derivative. We find in the proof of Proposition 6 that the $U_q(A_n^{(1)})$ stochastic R matrix fits the auxiliary condition perfectly by an intriguing mechanism. Consequently the matrix product construction for the arbitrary n and the inhomogeneity $\mu_1, \dots, \mu_L$ is attributed to the task of realizing such an operator $X_\alpha(\mu)$ concretely. The result (1) is an outcome of this exercise for $n = 2$. The general n case is also feasible and will be presented elsewhere.

The outline of the paper is as follows. In Section 2 we recall the necessary facts on the $U_q(A_n^{(1)})$-ZRPs in this paper. The discrete and continuous time versions are those defined in section 3.3 and section 3.4 in [15], respectively. In Section 3 the matrix product formula for the steady state probabilities are linked with the ZF algebra (Proposition 6). The auxiliary condition (30) or equivalently (34) plays a key role. Until this point all the arguments are valid for general n. In Section 4 we focus on the $n = 2$ case and present a concrete realization of the ZF algebra (Theorem 8) satisfying all the criteria in Proposition 6. It leads to the matrix product formulae (41) and (42), which are the main results of the paper. Section 5 contains a summary and discussion. Systematic applications to the study of physical behaviors is a subject of a future research.

Throughout the paper we use the notation $\theta(\text{true}) = 1, \theta(\text{false}) = 0$, the q-Pochhammer symbol $(z)_m = (z;q)_m = \prod_{j=1}^m (1 - zq^{j-1})$ and the q-binomial $\binom{m}{k}_q = \theta(k \in [0,m]) \frac{(q)_m}{(q)_k (q)_{m-k}}$. The symbols $(z)_m$ appearing in this paper always mean $(z;q)_m$. For integer arrays $\alpha = (\alpha_1, \dots, \alpha_m), \beta = (\beta_1, \dots, \beta_m)$ of *any* length m, we write $|\alpha| = \alpha_1 + \cdots + \alpha_m$. The relation $\alpha \le \beta$ or equivalently $\beta \ge \alpha$ is defined by $\beta - \alpha \in \mathbb{Z}_{\ge 0}^m$. We often denote by 0 to mean $(0, \dots, 0) \in \mathbb{Z}_{\ge 0}^m$ for some m when it is clear from the context.

2. $U_q(A_n^{(1)})$-ZERO RANGE PROCESSES

Let us briefly recall the stochastic R matrix for $U_q(A_n^{(1)})$ and the associated discrete and continuous time ZRPs constructed in [15, Sec. 3.3, 3.4].

2.1. **Stochastic R matrix.**

Set $W = \bigoplus_{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\ge 0}^n} \mathbb{C}|\alpha\rangle$. Define the operator $\mathcal{S}(\lambda, \mu) \in \mathrm{End}(W \otimes W)$ depending on the parameters λ and μ by

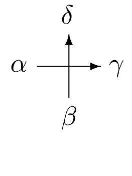
$$\mathcal{S}(\lambda, \mu)(|\alpha\rangle \otimes |\beta\rangle) = \sum_{\gamma, \delta \in \mathbb{Z}_{\ge 0}^n} \mathcal{S}(\lambda, \mu)_{\alpha, \beta}^{\gamma, \delta} |\gamma\rangle \otimes |\delta\rangle, \tag{2}$$

$$\mathcal{S}(\lambda, \mu)_{\alpha, \beta}^{\gamma, \delta} = \theta(\gamma + \delta = \alpha + \beta) \Phi_q(\gamma | \beta; \lambda, \mu), \tag{3}$$

where $\Phi_q(\gamma | \beta; \lambda, \mu)$ is given by

$$\Phi_q(\gamma | \beta; \lambda, \mu) = q^{\varphi(\beta - \gamma, \gamma)} \left(\frac{\mu}{\lambda}\right)^{|\gamma|} \frac{(\lambda)_{|\gamma|} (\frac{\mu}{\lambda})_{|\beta| - |\gamma|}}{(\mu)_{|\beta|}} \prod_{i=1}^n \binom{\beta_i}{\gamma_i}_q, \qquad \varphi(\alpha, \beta) = \sum_{1 \le i < j \le n} \alpha_i \beta_j. \tag{4}$$

The sum (2) is finite due to the θ factor in (3). In fact the direct sum decomposition $W \otimes W = \bigoplus_{\gamma \in \mathbb{Z}_{\ge 0}^n} \left(\bigoplus_{\alpha + \beta = \gamma} \mathbb{C}|\alpha\rangle \otimes |\beta\rangle \right)$ holds and $\mathcal{S}(\lambda, \mu)$ splits into the corresponding submatrices. Note also that $\mathcal{S}(\lambda, \mu)_{\alpha, \beta}^{\gamma, \delta} = 0$ unless $\gamma \le \beta$ and therefore $\alpha \le \delta$ as well. The difference property $\mathcal{S}(\lambda, \mu) = \mathcal{S}(c\lambda, c\mu)$ is absent. We call $\mathcal{S}(\lambda, \mu)$ the *stochastic R matrix*. Its elements are depicted as

$\mathcal{S}(\lambda, \mu)_{\alpha, \beta}^{\gamma, \delta} =$ 

(5)

The stochastic R matrix originates in the quantum R matrix of the symmetric tensor representation of the quantum affine algebra $U_q(A_n^{(1)})$ [7, 13]. It satisfies the Yang-Baxter equation, the inversion relation and the sum-to-unity condition [15]:

$$\mathcal{S}_{1,2}(\nu_1,\nu_2)\mathcal{S}_{1,3}(\nu_1,\nu_3)\mathcal{S}_{2,3}(\nu_2,\nu_3) = \mathcal{S}_{2,3}(\nu_2,\nu_3)\mathcal{S}_{1,3}(\nu_1,\nu_3)\mathcal{S}_{1,2}(\nu_1,\nu_2), \tag{6}$$

$$\check{\mathcal{S}}(\lambda,\mu)\check{\mathcal{S}}(\mu,\lambda) = \mathrm{id}_{W^{\otimes 2}}, \tag{7}$$

$$\sum_{\gamma,\delta\in\mathbb{Z}^n_{\geq 0}} \mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta} = 1 \qquad (\forall\alpha,\beta\in\mathbb{Z}^n_{\geq 0}), \tag{8}$$

where the checked stochastic R matrix is defined by $\check{\mathcal{S}}(\lambda,\mu)(|\alpha\rangle\otimes|\beta\rangle) = \sum_{\gamma,\delta}\mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta}|\delta\rangle\otimes|\gamma\rangle$.

Let $\mathcal{S}^T(\lambda,\mu)$ be the transpose of $\mathcal{S}(\lambda,\mu)$, i.e.,

$$\mathcal{S}^T(\lambda,\mu)(|\alpha\rangle\otimes|\beta\rangle) = \sum_{\gamma,\delta\in\mathbb{Z}^n_{\geq 0}} \mathcal{S}(\lambda,\mu)^{\alpha,\beta}_{\gamma,\delta}|\gamma\rangle\otimes|\delta\rangle.$$

It satisfies the same Yang-Baxter equation as (6):

$$\mathcal{S}^T_{1,2}(\nu_1,\nu_2)\mathcal{S}^T_{1,3}(\nu_1,\nu_3)\mathcal{S}^T_{2,3}(\nu_2,\nu_3) = \mathcal{S}^T_{2,3}(\nu_2,\nu_3)\mathcal{S}^T_{1,3}(\nu_1,\nu_3)\mathcal{S}^T_{1,2}(\nu_1,\nu_2). \tag{9}$$

To see this, note the identity

$$\mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta} = \mathcal{S}(\lambda,\mu)^{\alpha',\beta'}_{\gamma',\delta'}\frac{\tilde{g}_\gamma(\lambda)\tilde{g}_\delta(\mu)}{\tilde{g}_\alpha(\lambda)\tilde{g}_\beta(\mu)}q^{\varphi(\beta,\alpha)-\varphi(\gamma,\delta)}, \tag{10}$$

$$\tilde{g}_\alpha(\mu) = g_\alpha(\mu)q^{-\varphi(\alpha,\alpha)}, \qquad g_\alpha(\mu) = \frac{\mu^{-|\alpha|}(\mu)_{|\alpha|}}{\prod_{i=1}^n (q)_{\alpha_i}}, \tag{11}$$

where $\alpha' = (\alpha_n,\dots,\alpha_1)$ is the reverse ordered array of $\alpha = (\alpha_1,\dots,\alpha_n)$. One can easily check that the extra factors in the RHS of (10) are gauge freedom not spoiling the Yang-Baxter equation. The factor $g_\alpha(\mu)$, which will appear frequently in the sequel, is a piece of the function $\Phi_q(\gamma|\beta;\lambda,\mu)$ in the following sense:

$$q^{\varphi(\beta,\gamma)}\frac{g_\beta(\mu)g_\gamma(\lambda)}{g_{\beta+\gamma}(\mu)}\mathcal{S}^{0,\alpha}_{\delta,\beta}(\lambda,\mu) = \mathcal{S}^{\gamma,\alpha}_{\delta,\beta+\gamma}(\lambda,\mu). \tag{12}$$

The function $\Phi_q(\gamma|\beta;\lambda,\mu)|_{n=1}$ appeared earlier in [19, 20]. For n general it is zero unless $\gamma\leq\beta$, and satisfies the sum rule [15]:

$$\sum_{\gamma\in\mathbb{Z}^n_{\geq 0}} \Phi_q(\gamma|\beta;\lambda,\mu) = 1 \quad (\forall\beta\in\mathbb{Z}^n_{\geq 0}). \tag{13}$$

The relation (10) is a consequence of the property

$$\frac{g_\gamma(\lambda)g_{\alpha+\beta-\gamma}(\mu)}{g_\alpha(\mu)g_\beta(\lambda)}\Phi_q(\beta|\alpha+\beta-\gamma;\lambda,\mu) = q^{\varphi(\alpha-\gamma,\beta-\gamma)}\Phi_q(\gamma|\alpha;\lambda,\mu). \tag{14}$$

2.2. **Markov transfer matrix and discrete time $U_q(A_n^{(1)})$-ZRP.** Let L be a positive integer. Introduce the operator

$$T(\lambda|\mu_1,\dots,\mu_L) = \mathrm{Tr}_W\left(\mathcal{S}_{0,L}(\lambda,\mu_L)\cdots\mathcal{S}_{0,1}(\lambda,\mu_1)\right)\in\mathrm{End}(W^{\otimes L}). \tag{15}$$

In the terminology of the quantum inverse scattering method, it is the row transfer matrix of the $U_q(A_n^{(1)})$ vertex model of length L with periodic boundary condition whose quantum space is $W^{\otimes L}$ with inhomogeneity parameters $\mu_1,\dots,\mu_L$ and the auxiliary space W with spectral parameter λ. If these spaces are labeled as $W_1\otimes\cdots\otimes W_L$ and W_0, the stochastic R matrix $\mathcal{S}_{0,i}(\lambda,\mu_i)$ acts as $\mathcal{S}(\lambda,\mu_i)$ on $W_0\otimes W_i$ and as the identity elsewhere.

Thanks to the properties (6) and (7), the matrix (15) forms a commuting family (cf. [2]):

$$[T(\lambda|\mu_1,\dots,\mu_L), T(\lambda'|\mu_1,\dots,\mu_L)] = 0. \tag{16}$$

We write the vector $|\alpha_1\rangle\otimes\cdots\otimes|\alpha_L\rangle\in W^{\otimes L}$ representing a state of the system as $|\alpha_1,\dots,\alpha_L\rangle$ and the action of $T = T(\lambda|\mu_1,\dots,\mu_L)$ as

$$T|\beta_1,\dots,\beta_L\rangle = \sum_{\alpha_1,\dots,\alpha_L\in\mathbb{Z}^n_{\geq 0}} T^{\alpha_1,\dots,\alpha_L}_{\beta_1,\dots,\beta_L}|\alpha_1,\dots,\alpha_L\rangle\in W^{\otimes L}.$$

Then the matrix element is depicted by the concatenation of (5) as

$$T^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L} = \sum_{\gamma_1,\ldots,\gamma_L\in\mathbb{Z}^n_{\geq 0}} \gamma_L \overset{\alpha_1}{\underset{\beta_1}{\longrightarrow}} \gamma_1 \overset{\alpha_2}{\underset{\beta_2}{\longrightarrow}} \gamma_2 \;\cdots\; \gamma_{L-1} \overset{\alpha_L}{\underset{\beta_L}{\longrightarrow}} \gamma_L. \tag{17}$$

By the construction it satisfies the weight conservation:

$$T^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L} = 0 \;\text{ unless }\; \alpha_1+\cdots+\alpha_L = \beta_1+\cdots+\beta_L \in \mathbb{Z}^n_{\geq 0}. \tag{18}$$

Let t be a time variable and consider the evolution equation

$$|P(t+1)\rangle = T(\lambda|\mu_1,\ldots,\mu_L)|P(t)\rangle \in W^{\otimes L}. \tag{19}$$

Although this is an equation in an infinite-dimensional vector space, the property (18) lets it split into finite-dimensional subspaces which we call *sectors*. In terms of the array $m = (m_1,\ldots,m_n) \in \mathbb{Z}^n_{\geq 0}$ and the set

$$S(m) = \{(\sigma_1,\ldots,\sigma_L) \in (\mathbb{Z}^n_{\geq 0})^L \mid \sigma_1+\cdots+\sigma_L = m\},$$

the corresponding sector, which will also be referred to as m, is given by $\oplus_{(\sigma_1,\ldots,\sigma_L)\in S(m)}\mathbb{C}|\sigma_1,\ldots,\sigma_L\rangle$. We interpret a vector $|\sigma_1,\ldots,\sigma_L\rangle \in W^{\otimes L}$ with $\sigma_i = (\sigma_{i,1},\ldots,\sigma_{i,n}) \in \mathbb{Z}^n_{\geq 0}$ as a state of the system in which the i th site from the left is populated with $\sigma_{i,a}$ particles of the a th species. Thus $m = (m_1,\ldots,m_n)$ is the multiplicity meaning that there are m_a particles of species a in total in the corresponding sector.

In order to interpret (19) as the master equation of a discrete time Markov process, the matrix $T = T(\lambda|\mu_1,\ldots,\mu_L)$ should fulfill the following conditions:

(i) Non-negativity; all the elements (17) belong to $\mathbb{R}_{\geq 0}$,
(ii) Sum-to-unity property; $\sum_{\alpha_1,\ldots,\alpha_L\in\mathbb{Z}^n_{\geq 0}} T^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L} = 1$ for any $(\beta_1,\ldots,\beta_L) \in (\mathbb{Z}^n_{\geq 0})^L$.

The property (i) holds if $\Phi_q(\gamma|\beta;\lambda,\mu_i) \geq 0$ for all $i \in \mathbb{Z}_L$. This is achieved by taking $0 < \mu_i^\epsilon < \lambda^\epsilon < 1, q^\epsilon < 1$ in the either alternative $\epsilon = \pm 1$. The property (ii) means the total probability conservation and can be shown by using (13) as in [15, Sec.3.2].

Henceforth we call the $T(\lambda|\mu_1,\ldots,\mu_L)$ *Markov transfer matrix* assuming $0 < \mu_i^\epsilon < \lambda^\epsilon < 1, q^\epsilon < 1$ always. The choice of $\epsilon = \pm 1$ may be viewed as specifying one of the two physical regimes of the system. The equation (19) represents a stochastic dynamics of n-species of particles hopping to the right periodically via an extra lane (horizontal arrows in (17)) which particles get on or get off when they leave or arrive at a site. The rate of these local processes is specified by (3), (4) and (5). For $n = 1$ and the homogeneous choice $\mu_1 = \cdots = \mu_L$, it reduces to the model introduced in [20].

Example 1. Consider $L = 2, n = 2$ and the sector $m = (2,1)$, which is the six dimensional space. We denote the basis in terms of multiset of particles as $|12,1\rangle, |\emptyset,112\rangle$, etc, instead of the corresponding multiplicity arrays $|(1,1),(1,0)\rangle, |(0,0),(2,1)\rangle$, etc. Then $T(\lambda|\mu_1,\mu_2)|1,12\rangle$ is equal to

$$\begin{aligned}
&-\frac{q\mu_1\mu_2(\lambda-1)^2(\lambda-\mu_2)|2,11\rangle}{(\mu_1-1)(\mu_2-1)\lambda^3(q\mu_2-1)} + \frac{\mu_1(\lambda-1)(\lambda-\mu_2)(\lambda-q\mu_2)|\emptyset,112\rangle}{(\mu_1-1)(\mu_2-1)\lambda^3(q\mu_2-1)}\\
&+\frac{\mu_2(\lambda-1)(\lambda-\mu_1)(\lambda-\mu_2)|11,2\rangle}{(\mu_1-1)(\mu_2-1)\lambda^3(q\mu_2-1)} - \frac{\mu_2^2(\lambda-1)(q\lambda-1)(\lambda-\mu_1)|112,\emptyset\rangle}{(\mu_1-1)(\mu_2-1)\lambda^3(q\mu_2-1)}\\
&+\frac{\mu_2(\lambda-1)\left(q\mu_1\mu_2\lambda^2 - q\mu_1\lambda - q\mu_2\lambda - q\mu_1\mu_2\lambda + q\mu_1\mu_2 + q\lambda^2 - \mu_1\mu_2\lambda + \mu_1\mu_2\right)|12,1\rangle}{(\mu_1-1)(\mu_2-1)\lambda^3(q\mu_2-1)}\\
&-\frac{(\lambda-\mu_2)\left(-q\mu_2\lambda + q\mu_1\mu_2 + \mu_1\mu_2\lambda^2 - \mu_1\lambda - 2\mu_1\mu_2\lambda + \mu_1\mu_2 + \lambda^2\right)|1,12\rangle}{(\mu_1-1)(\mu_2-1)\lambda^3(q\mu_2-1)}.
\end{aligned}$$

2.3. **Continuous time $U_q(A_n^{(1)})$-ZRP.** From the homogeneous case $\mu_1 = \cdots = \mu_L = \mu$ of the Markov transfer matrix $T(\lambda|\mu) = T(\lambda|\mu,\ldots,\mu)$, we extract the two "Hamiltonians" by the so called Baxter formula (cf. [2, Chap. 10.14]):

$$H^{(1)} = -\epsilon\mu^{-1}\frac{\partial \log T(\lambda|\mu)}{\partial\lambda}\bigg|_{\lambda=1}, \qquad H^{(2)} = \epsilon\mu\,\frac{\partial \log T(\lambda|\mu)}{\partial\lambda}\bigg|_{\lambda=\mu}. \tag{20}$$

From (16) the commutativity $[H^{(1)}, H^{(2)}] = 0$ follows. Moreover they both satisfy

(i)' Non-negativity; all the off-diagonal elements are nonnegative,
(ii)' Sum-to-zero property; the sum of elements in any column is zero.

Thus an infinitesimal version of (19) of the form

$$\frac{d}{dt}|P(t)\rangle = H|P(t)\rangle, \qquad H = aH^{(1)} + bH^{(2)} \quad (a, b \in \mathbb{R}_{\geq 0}) \tag{21}$$

defines a continuous time integrable Markov process for any $a, b \in \mathbb{R}_{\geq 0}$. The Markov matrices consist of the pairwise interaction terms as $H^{(r)} = \sum_{i\in\mathbb{Z}_L} h^{(r)}_{i,i+1}$, where $h^{(r)}_{i,i+1}$ acts on the $(i, i+1)$th components from the left in $|P(t)\rangle$ as $h^{(r)} \in \mathrm{End}(W \otimes W)$ and as the identity elsewhere. The local Markov matrices are given by

$$h^{(1)}|\alpha, \beta\rangle = -\epsilon\mu^{-1} \sum_{\gamma\in\mathbb{Z}^n_{\geq 0}} \Phi'_q(\gamma|\alpha; 1, \mu)|\alpha - \gamma, \beta + \gamma\rangle, \tag{22}$$

$$h^{(2)}|\alpha, \beta\rangle = \epsilon\mu \sum_{\gamma\in\mathbb{Z}^n_{\geq 0}} \Phi'_q(\beta - \gamma|\beta; \mu, \mu)|\alpha + \gamma, \beta - \gamma\rangle, \tag{23}$$

where $'$ denotes $\frac{\partial}{\partial\lambda}$ and the local transition rate is explicitly given by

$$\begin{aligned} -\epsilon\mu^{-1}\Phi'_q(\gamma|\alpha; 1, \mu) &= \epsilon\frac{q^{\varphi(\alpha-\gamma,\gamma)}\mu^{|\gamma|-1}(q)_{|\gamma|-1}}{(\mu q^{|\alpha|-|\gamma|})_{|\gamma|}} \prod_{i=1}^{n} \binom{\alpha_i}{\gamma_i}_q \quad (|\gamma| \geq 1), \\ &= -\epsilon \sum_{i=0}^{|\beta|-1} \frac{q^i}{1-\mu q^i} \quad (|\gamma| = 0), \end{aligned} \tag{24}$$

$$\begin{aligned} \epsilon\mu\Phi'_q(\beta - \gamma|\beta; \mu, \mu) &= \epsilon\frac{q^{\varphi(\gamma,\beta-\gamma)}(q)_{|\gamma|-1}}{(\mu q^{|\beta|-|\gamma|})_{|\gamma|}} \prod_{i=1}^{n} \binom{\beta_i}{\gamma_i}_q \quad (|\gamma| \geq 1), \\ &= -\epsilon \sum_{i=0}^{|\beta|-1} \frac{1}{1-\mu q^i} \quad (|\gamma| = 0). \end{aligned} \tag{25}$$

From (22) and (23), $H^{(1)}$ and $H^{(2)}$ individually defines an n-species totally asymmetric zero range processes (n-TAZRP) in which particles hop to the to right and to the left neighbor sites with the rate (24) and (25), respectively. In the both cases $\gamma = (\gamma_1, \ldots, \gamma_n)$ gives the number of particles of species $1, \ldots, n$ jumping out the departure site. The opposite directional move originates in the different behavior of $T(\lambda|\mu)$ at the two "Hamiltonian points" $\lambda = 1$ and $\lambda = \mu$. The Markov matrix H (21) is a mixture of them yielding an n-species asymmetric zero range process. In [15], Bethe eigenvalues of $T(\lambda|\mu_1, \ldots, \mu_L)$ and H have been obtained.

The above integrable Markov processes cover several models considered earlier. When $(\epsilon, \mu) = (1, 0)$ in $H^{(1)}$, the nontrivial local transitions in (24) are limited to the case $|\gamma| = 1$. So if $\gamma_a = 1$ and the other components of γ are 0, the rate (24) becomes $q^{\alpha_1+\cdots+\alpha_{a-1}}\frac{1-q^{\alpha_a}}{1-q}$. This reproduces the n-species q-boson process in [25] whose $n = 1$ case further goes back to [22]. When $n = 1$, the system (19) was also studied in [20, 5, 4]. The transition rate (24) for $n = 1$ and general μ reproduces the one in [24, p2] by a suitable adjustment. When $(\epsilon, \mu, q) = (1, 0, 0)$ in $H^{(2)}$, a kinematic constraint $\varphi(\gamma, \beta - \gamma) = \sum_{1\leq i<j\leq n} \gamma_i(\beta_j - \gamma_j) = 0$ arises from (25). In fact, in order that $\gamma_a > 0$ happens, one must have $\gamma_{a+1} = \beta_{a+1}, \gamma_{a+2} = \beta_{a+2}, \ldots, \gamma_n = \beta_n$. It means that larger species particles have the priority to jump out, which precisely reproduces the n-species TAZRP explored in [17, 18] after reversing the labeling of the species $1, 2, \ldots, n$ of the particles.

Remark 2. Denote the Markov matrix H in (21) by $H(a, b, \epsilon, q, \mu)$ exhibiting the dependence on the parameters. Then a duality relation $H(a, b, -\epsilon, q^{-1}, \mu^{-1}) = \mathcal{P}H(\mu b, \mu a, \epsilon, q, \mu)\mathcal{P}^{-1}$ holds, where $\mathcal{P} = \mathcal{P}^{-1} \in \mathrm{End}(W^{\otimes L})$ is the "parity" operator reversing the sites as $\mathcal{P}|\sigma_1, \ldots, \sigma_L\rangle = |\sigma_L, \ldots, \sigma_1\rangle$ [15, Remark 9]. Under the duality, the condition $0 < \mu^\epsilon, q^\epsilon < 1$ on the parameters is preserved.

Example 3. With the same convention as Example 1 we have

$$H^{(1)}|1, 12\rangle = -\frac{(2 + q - 3q\mu)|1, 12\rangle}{(1-\mu)(1-q\mu)} + \frac{q|12, 1\rangle}{1-q\mu} + \frac{|11, 2\rangle}{1-q\mu} + \frac{(1-q)\mu|112, \emptyset\rangle}{(1-\mu)(1-q\mu)} + \frac{|\emptyset, 112\rangle}{1-\mu},$$

$$H^{(2)}|1, 12\rangle = -\frac{(3 - \mu - 2q\mu)|1, 12\rangle}{(1-\mu)(1-q\mu)} + \frac{|12, 1\rangle}{1-q\mu} + \frac{q|11, 2\rangle}{1-q\mu} + \frac{(1-q)|112, \emptyset\rangle}{(1-\mu)(1-q\mu)} + \frac{|\emptyset, 112\rangle}{1-\mu}.$$

3. Steady states

3.1. **General remarks and examples.** By definition a steady state of the discrete time $U_q(A_n^{(1)})$-ZRP (19) is a vector $|\overline{P}\rangle \in W^{\otimes L}$ such that

$$|\overline{P}\rangle = T(\lambda|\mu_1,\dots,\mu_L)|\overline{P}\rangle. \tag{26}$$

The steady state is unique within each sector m. Apart from m, it depends on q and the inhomogeneity parameters $\mu_1,\dots,\mu_L$ but *not* on λ thanks to the commutativity (16). Sectors $m=(m_1,\dots,m_n)$ such that $\forall m_a \geq 1$ are called *basic*. Non-basic sectors are equivalent to a basic sector of some $n'<n$ models with a suitable relabeling of the species. Henceforth we concentrate on the basic sectors. The coefficient appearing in the expansion

$$|\overline{P}(m)\rangle = \sum_{(\sigma_1,\dots,\sigma_L)\in S(m)} \mathbb{P}(\sigma_1,\dots,\sigma_L)|\sigma_1,\dots,\sigma_L\rangle$$

is the steady state probability if it is properly normalized as $\sum_{(\sigma_1,\dots,\sigma_L)\in S(m)}\mathbb{P}(\sigma_1,\dots,\sigma_L)=1$. In this paper unnormalized ones will also be refereed to as steady state probabilities by abuse of terminology.

If the dependence on the inhomogeneity parameters are exhibited as $\mathbb{P}(\sigma_1,\dots,\sigma_L;\mu_1,\dots,\mu_L)$, we have the cyclic symmetry $\mathbb{P}(\sigma_1,\dots,\sigma_L;\mu_1,\dots,\mu_L)=\mathbb{P}(\sigma_L,\sigma_1,\dots,\sigma_{L-1};\mu_L,\mu_1,\dots,\mu_{L-1})$ by the definition.

Example 4. For $L=2, n=2$ and the sector $m=(1,1)$, we have

$$\begin{aligned}|\overline{P}(1,1)\rangle &= \mu_1^2\,(1-\mu_2)\,(1-q\mu_2)\,(\mu_1+\mu_2-2\mu_2\mu_1)\,|\emptyset,12\rangle \\ &\quad + \mu_1\mu_2\,(1-\mu_1)\,(1-\mu_2)\,(\mu_1+q\mu_2-\mu_1\mu_2-q\mu_1\mu_2)\,|1,2\rangle + \text{cyclic}.\end{aligned} \tag{27}$$

For $L=3, n=2$ and the sector $m=(1,1)$, we have

$$\begin{aligned}|\overline{P}(1,1)\rangle &= \mu_1^2\mu_2^2\,(1-\mu_3)\,(1-q\mu_3)\,(\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3-3\mu_1\mu_3\mu_2)\,|\emptyset,\emptyset,12) \\ &\quad + \mu_1^2\mu_2\mu_3\,(1-\mu_2)\,(1-\mu_3)\,(q\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3-2\mu_1\mu_2\mu_3-q\mu_1\mu_2\mu_3)\,|\emptyset,2,1\rangle \\ &\quad + \mu_1^2\mu_2\mu_3\,(1-\mu_2)\,(1-\mu_3)\,(\mu_1\mu_2+q\mu_1\mu_3+q\mu_2\mu_3-\mu_1\mu_2\mu_3-2q\mu_1\mu_2\mu_3)\,|\emptyset,1,2\rangle + \text{cyclic}.\end{aligned}$$

Here cyclic means the sum of terms obtained by the replacement $\mu_j \to \mu_{j+i}$ and $|\sigma_1,\dots,\sigma_L\rangle \to |\sigma_{1+i},\dots,\sigma_{L+i}\rangle$ over $i\in\mathbb{Z}_L$ with $i\neq 0$.

Let us proceed to the steady states of the continuous time $U_q(A_n^{(1)})$-ZRP in Section 2.3. By the construction (20) and the commutativity (16), the Markov matrices $H^{(1)}, H^{(2)}$ and H share the common steady state in each sector. It is given by specializing the discrete time result $\mathbb{P}(\sigma_1,\dots,\sigma_L;\mu_1,\dots,\mu_L)$ to the homogeneous case $\mu_1=\cdots=\mu_L=\mu$. For instance, the latter $|\overline{P}(1,1)\rangle$ in Example 4 reproduces $|\bar{P}_3\rangle$ in [15, Ex.13] via the specialization $\mu_1=\mu_2=\mu_3=\mu$.

Example 5. For $L=2, n=2$ and the sector $m=(2,1)$, we have

$$\begin{aligned}|\overline{P}(2,1)\rangle &= (1-q^2\mu)(3+q-\mu-3q\mu)|\emptyset,112\rangle + (1-\mu)(1+q+2q^2-2q\mu-q^2\mu-q^3\mu)|2,11\rangle \\ &\quad + (1+q)(1-\mu)(2+q+q^2-\mu-q\mu-2q^2\mu)|1,12\rangle + \text{cyclic}.\end{aligned}$$

For $L=3, n=2$ and the sector $m=(2,1)$, we have

$$\begin{aligned}|\overline{P}(2,1)\rangle &= 3(1-q\mu)(1-q^2\mu)(2+q-(1+2q)\mu)|\emptyset,\emptyset,112\rangle \\ &\quad + (1-\mu)(1-q\mu)(3+3q+3q^2-(1+5q+2q^2+q^3)\mu)|\emptyset,2,11\rangle \\ &\quad + (1+q)(1-\mu)(1-q\mu)(3+3q+3q^2-(2+2q+5q^2)\mu)|\emptyset,1,12\rangle \\ &\quad + (1+q)(1-\mu)(1-q\mu)(5+2q+2q^2-(3+3q+3q^2)\mu)|\emptyset,12,1\rangle \\ &\quad + (1-\mu)(1-q\mu)(1+2q+5q^2+q^3-(3q+3q^2+3q^3)\mu)|\emptyset,11,2\rangle \\ &\quad + (1+q)(1+q+q^2)(1-\mu)^2(2+q-(1+2q)\mu)|1,1,2\rangle + \text{cyclic}.\end{aligned}$$

The cyclic here means the sum of terms obtained by the replacement $|\sigma_1,\dots,\sigma_L\rangle \to |\sigma_{1+i},\dots,\sigma_{L+i}\rangle$ over $i\in\mathbb{Z}_L$ with $i\neq 0$.

Examples 4 and 5 indicate that $\mathbb{P}(\sigma_1,\dots,\sigma_L;\mu_1,\dots,\mu_L)\in\mathbb{Z}_{\geq 0}[-\mu_1,\dots,-\mu_L,q]$ holds in an appropriate normalization.

3.2. **Matrix product construction.** Let us consider the discrete time $U_q(A_n^{(1)})$-ZRP in Section 2.2 whose master equation is (19). We seek the steady state probability in the matrix product form

$$\mathbb{P}(\sigma_1,\ldots,\sigma_L)=\mathrm{Tr}(X_{\sigma_1}(\mu_1)\cdots X_{\sigma_L}(\mu_L)) \tag{28}$$

in terms of some operator $X_\alpha(\mu)$ with $\alpha\in\mathbb{Z}^n_{\ge 0}$. Our strategy is to invoke the following result.

Proposition 6. *Suppose the operators $X_\alpha(\mu)$ ($\alpha\in\mathbb{Z}^n_{\ge 0}$) obey the relation*

$$X_\alpha(\mu)X_\beta(\lambda)=\sum_{\gamma,\delta\in\mathbb{Z}^n_{\ge 0}}\mathcal{S}(\lambda,\mu)^{\beta,\alpha}_{\gamma,\delta}X_\gamma(\lambda)X_\delta(\mu). \tag{29}$$

Suppose further that $X_0(\lambda)$ is invertible and the following auxiliary condition is satisfied:

$$X_\beta(\mu)X_0(\lambda)^{-1}X_\gamma(\lambda)=q^{\varphi(\beta,\gamma)}\frac{g_\beta(\mu)g_\gamma(\lambda)}{g_{\beta+\gamma}(\mu)}X_{\beta+\gamma}(\mu). \tag{30}$$

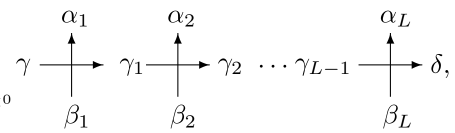
Then (28) gives the steady state probability of the system (19) if the trace is convergent and not identically zero.

Proof. In view of (26) we are to show

$$\mathrm{Tr}(X_{\alpha_1}(\mu_1)\cdots X_{\alpha_L}(\mu_L))=\sum_{\beta_1,\ldots,\beta_L\in\mathbb{Z}^n_{\ge 0}}T(\lambda|\mu_1,\ldots,\mu_L)^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L}\mathrm{Tr}(X_{\beta_1}(\mu_1)\cdots X_{\beta_L}(\mu_L)). \tag{31}$$

Introduce the elements $M^{\alpha_1,\ldots,\alpha_L;\delta}_{\gamma;\beta_1,\ldots,\beta_L}=M(\lambda|\mu_1,\ldots,\mu_L)^{\alpha_1,\ldots,\alpha_L;\delta}_{\gamma;\beta_1,\ldots,\beta_L}$ of the monodromy matrix by the diagram similar to (17) as

$$M^{\alpha_1,\ldots,\alpha_L;\delta}_{\gamma;\beta_1,\ldots,\beta_L}=\sum_{\gamma_1,\ldots,\gamma_{L-1}\in\mathbb{Z}^n_{\ge 0}}$$



where the i th vertex from the left denotes the element (5) of $\mathcal{S}(\lambda,\mu_i)$. By the definition we have $M(\lambda|\mu_1,\ldots,\mu_L)^{\alpha_1,\ldots,\alpha_L;\delta}_{\gamma;\beta_1,\ldots,\beta_L}=0$ unless $\delta+\sum_{i=1}^L\alpha_i=\gamma+\sum_{i=1}^L\beta_i$. Elements of the Markov transfer matrix is given by $T(\lambda|\mu_1,\ldots,\mu_L)^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L}=\sum_{\gamma\in\mathbb{Z}^n_{\ge 0}}M(\lambda|\mu_1,\ldots,\mu_L)^{\alpha_1,\ldots,\alpha_L;\gamma}_{\gamma;\beta_1,\ldots,\beta_L}$, where the sum is bounded by $\gamma\le\alpha_1,\beta_L$. See the remarks after (4). Now we have

$$\begin{aligned}
\text{LHS of (31)} &= \mathrm{Tr}(X_{\alpha_1}(\mu_1)\cdots X_{\alpha_L}(\mu_L)X_0(\lambda)X_0(\lambda)^{-1})\\
&\overset{(29)}{=}\sum_{\gamma,\beta_1,\ldots,\beta_L}M^{\alpha_1,\ldots,\alpha_L;0}_{\gamma;\beta_1,\ldots,\beta_L}\,\mathrm{Tr}(X_{\beta_1}(\mu_1)\cdots X_{\beta_{L-1}}(\mu_{L-1})X_{\beta_L}(\mu_L)X_0(\lambda)^{-1}X_\gamma(\lambda))\\
&\overset{(30)}{=}\sum_{\gamma,\beta_1,\ldots,\beta_L}M^{\alpha_1,\ldots,\alpha_L;0}_{\gamma;\beta_1,\ldots,\beta_L}\,\mathrm{Tr}(X_{\beta_1}(\mu_1)\cdots X_{\beta_{L-1}}(\mu_{L-1})X_{\beta_L+\gamma}(\mu_L))q^{\varphi(\beta_L,\gamma)}\frac{g_{\beta_L}(\mu_L)g_\gamma(\lambda)}{g_{\beta_L+\gamma}(\mu_L)}\\
&\overset{(12)}{=}\sum_{\gamma,\beta_1,\ldots,\beta_L}M^{\alpha_1,\ldots,\alpha_L;\gamma}_{\gamma;\beta_1,\ldots,\beta_{L-1},\beta_L+\gamma}\,\mathrm{Tr}(X_{\beta_1}(\mu_1)\cdots X_{\beta_{L-1}}(\mu_{L-1})X_{\beta_L+\gamma}(\mu_L)).
\end{aligned}$$

By replacing $\beta_L+\gamma$ with β_L and summing over γ, the coefficient of the trace in the last expression becomes $T(\lambda|\mu_1,\ldots,\mu_L)^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L}$. □

Observe a perfect fit of the auxiliary condition (30) and the property of the stochastic R matrix (12).

As explained before Example 5, a matrix product formula for the continuous time $U_q(A_n^{(1)})$-ZRP in Section 2.3 follows from (28) just by the specialization $\mu_1=\cdots=\mu_L=\mu$.

The relation of the form (29) possessing a solution to the Yang-Baxter equation as the structure function is often called the *Zamolodchikov-Faddeev* (ZF) *algebra*. In our case its associativity is assured by the transposed Yang-Baxter equation (9) rather than (6). Proposition 6 implies that the RHS of (28) satisfies the Knizhnik-Zamolodchikov type equation (cf. [11]) as a function of $\mu_1,\ldots,\mu_L$.

From (3) the ZF algebra (29) reads more explicitly as

$$X_\alpha(\mu)X_\beta(\lambda)=\sum_{\gamma\le\alpha}\Phi_q(\beta|\alpha+\beta-\gamma;\lambda,\mu)X_\gamma(\lambda)X_{\alpha+\beta-\gamma}(\mu).$$

We find it convenient to work with $Z_\alpha(\mu)$ defined by

$$X_\alpha(\mu)=g_\alpha(\mu)Z_\alpha(\mu), \tag{32}$$

where $g_\alpha(\mu)$ was defined in (11). Then by using the identity (14), the ZF algebra and the auxiliary condition are cast into

$$Z_\alpha(\mu)Z_\beta(\lambda) = \sum_{\gamma\le\alpha} q^{\varphi(\alpha-\gamma,\beta-\gamma)}\Phi_q(\gamma|\alpha;\lambda,\mu)Z_\gamma(\lambda)Z_{\alpha+\beta-\gamma}(\mu), \tag{33}$$

$$Z_\beta(\mu)Z_0(\lambda)^{-1}Z_\gamma(\lambda) = q^{\varphi(\beta,\gamma)}Z_{\beta+\gamma}(\mu). \tag{34}$$

The expression (34) or equivalently $Y_\beta(\mu)Y_\gamma(\lambda) = q^{\varphi(\beta,\gamma)}Y_{\beta+\gamma}(\mu)$ in terms of the operator $Y_\alpha(\mu) := Z_0(0)^{-1}Z_\alpha(\mu)$ is the simplest presentation of the auxiliary condition.

Proposition 7. *The ZF algebra (29) (or (33)) and the auxiliary condition (30) (or (34)) admit a "trivial representation" in terms of an operator K_α satisfying $K_0 = 1$ and $K_\alpha K_\beta = q^{\varphi(\alpha,\beta)}K_{\alpha+\beta}$ as*

$$X_\alpha(\mu) \mapsto g_\alpha(\mu)K_\alpha, \quad Z_\alpha(\mu) \mapsto K_\alpha.$$

Proof. The auxiliary condition is easily checked. The relation (33) is rewritten as

$$1 = \sum_{\gamma\le\alpha} q^{\varphi(\alpha-\gamma,\beta-\gamma)-\varphi(\alpha,\beta)+\varphi(\gamma,\alpha+\beta-\gamma)}\Phi_q(\gamma|\alpha;\lambda,\mu).$$

Let $\alpha' = (\alpha_n,\dots,\alpha_1)$ be the reverse ordered array of α as defined after (11). Then the obvious identity $\varphi(\alpha,\beta) = \varphi(\beta',\alpha')$ leads to $\Phi_q(\gamma|\alpha;\lambda,\mu) = q^{\varphi(\alpha,\gamma)-\varphi(\gamma,\alpha)}\Phi_q(\gamma'|\alpha';\lambda,\mu)$, hence the RHS of the above relation is $\sum_{\gamma\le\alpha}\Phi_q(\gamma'|\alpha';\lambda,\mu)$. This equals 1 thanks to (13). □

When $n = 1$, $\varphi(\alpha,\beta) = 0$ holds by (4). Therefore we may set $Z_\alpha(\mu) = K_\alpha = 1$. Then (28) gives the result

$$\mathbb{P}(\sigma_1,\dots,\sigma_L) = \prod_{i=1}^{L} g_{\sigma_i}(\mu_i) = \prod_{i=1}^{L}\frac{\mu_i^{-\sigma_i}(\mu_i)_{\sigma_i}}{(q)_{\sigma_i}}, \tag{35}$$

where the i th site variable is a single integer $\sigma_i \in \mathbb{Z}_{\ge0}$. In the homogeneous case $\mu_1 = \cdots = \mu_L = \mu$, one can remove the common overall factor within a given sector. It leads to $\mathbb{P}(\sigma_1,\dots,\sigma_L) = \prod_{i=1}^{L}\frac{(\mu)_{\sigma_i}}{(q)_{\sigma_i}}$ up to an overall normalization. This reproduces the product measure in [20, eqs. (3), (7)]. See also [9].

It is easy to construct K_α obeying $K_\alpha K_\beta = q^{\varphi(\alpha,\beta)}K_{\alpha+\beta}$ for general n by using q-commuting operators. However, doing so naively runs into the trouble $\mathrm{Tr}(K_{\alpha_1}\cdots K_{\alpha_L}) = 0$. The content of the next section grew out of an effort to overcome it for $n = 2$. See also the remark after (40).

4. $U_q(A_2^{(1)})$ case

The operator $Z_\alpha(\mu)$ in (32) captures the multispecies effect beyond the product measure (35) for $n = 1$. From now on we concentrate on the next nontrivial case $n = 2$. We consider the regime $\epsilon = +1, 0 < \forall\mu_i,\mu,q < 1$ without losing generality thanks to Remark 2.

4.1. ***q*-boson realization.** Consider the Fock space $F = \bigoplus_{m\ge0}\mathbb{C}|m\rangle$, its dual $F^* = \bigoplus_{m\ge0}\mathbb{C}\langle m|$ and the operators $\mathbf{b}_+,\mathbf{b}_-,\mathbf{k}$ acting on them as

$$\begin{aligned}
&\mathbf{b}_+|m\rangle = |m+1\rangle, &&\mathbf{b}_-|m\rangle = (1-q^m)|m-1\rangle, &&\mathbf{k}|m\rangle = q^m|m\rangle,\\
&\langle m|\mathbf{b}_- = \langle m+1|, &&\langle m|\mathbf{b}_+ = \langle m-1|(1-q^m), &&\langle m|\mathbf{k} = \langle m|q^m,
\end{aligned} \tag{36}$$

where $|-1\rangle = \langle -1| = 0$. They satisfy

$$\mathbf{k}\mathbf{b}_\pm = q^{\pm1}\mathbf{b}_\pm\mathbf{k}, \qquad \mathbf{b}_+\mathbf{b}_- = 1-\mathbf{k}, \qquad \mathbf{b}_-\mathbf{b}_+ = 1-q\mathbf{k}. \tag{37}$$

We specify the bilinear pairing of F^* and F as $\langle m|m'\rangle = \theta(m = m')(q)_m$. Then $\langle m|(X|m'\rangle) = (\langle m|X)|m'\rangle$ holds and the trace is given by $\mathrm{Tr}(X) = \sum_{m\ge0}\frac{\langle m|X|m\rangle}{(q)_m}$.

Let $\mathcal{B}$ denote the q-boson algebra generated by $1,\mathbf{b}_\pm,\mathbf{k}$ obeying the relations (37). As a vector space, it has the direct sum decomposition $\mathcal{B} = \mathbb{C}1\oplus\mathcal{B}_{\mathrm{fin}}$, where $\mathcal{B}_{\mathrm{fin}} = \oplus_{r\ge1}(\mathcal{B}_+^r\oplus\mathcal{B}_-^r\oplus\mathcal{B}_0^r)$ with $\mathcal{B}_\pm^r = \oplus_{s\ge0}\mathbb{C}\mathbf{k}^s\mathbf{b}_\pm^r$ and $\mathcal{B}_0^r = \mathbb{C}\mathbf{k}^r$. The trace $\mathrm{Tr}(X)$ is convergent if $X\in\mathcal{B}_{\mathrm{fin}}$. It vanishes unless $X\in\oplus_{r\ge1}\mathcal{B}_0^r$ when it is evaluated by $\mathrm{Tr}(\mathbf{k}^r) = (1-q^r)^{-1}$. It is an easy exercise to verify the following generalization:

$$\mathrm{Tr}(\mathbf{k}^{m_2}\mathbf{b}_-^{m_1}\mathbf{b}_+^{m_1}) = \frac{(q)_{m_1}(q)_{m_2-1}}{(q)_{m_1+m_2}} \qquad (m_1\ge0,\ m_2\ge1). \tag{38}$$

The trace is invariant under the replacement $\mathbf{b}_\pm \mapsto c^{\pm1}\mathbf{b}_\pm$ for any nonzero constant c since it is an automorphism $\mathcal{B}$. The q-boson algebra $\mathcal{B}$ here is slightly different from those in [16, Sec.3.1] and [18, eq.(3.2)].

The following result is the main source of our matrix product formula.

Theorem 8. *For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_{\geq 0}$, the operator*

$$X_\alpha(\mu) = g_\alpha(\mu) Z_\alpha(\mu), \quad Z_\alpha(\mu) = \frac{(\mathbf{b}_+)_\infty}{(\mu^{-1}\mathbf{b}_+)_\infty} \mathbf{k}^{\alpha_2} \mathbf{b}_-^{\alpha_1} \tag{39}$$

satisfies the ZF algebra (29) and the auxiliary condition (30).

The proof will be presented in Section 4.3. The ratio of the infinite products are defined in terms of the series expansion:

$$\frac{(zw)_\infty}{(z)_\infty} = \sum_{j\geq 0} \frac{(w)_j}{(q)_j} z^j. \tag{40}$$

In (39), the factor $K_\alpha = \mathbf{k}^{\alpha_2}\mathbf{b}_-^{\alpha_1}$ realizes the trivial representation in Proposition 7. However, taking it only as $Z_\alpha(\mu) = \mathbf{k}^{\alpha_2}\mathbf{b}_-^{\alpha_1}$ leads to the vanishing trace $\mathrm{Tr}(Z_{\sigma_1}(\mu_1)\cdots Z_{\sigma_L}(\mu_L)) = 0$. In this sense the representation (39) of the ZF algebra is a perturbation series from the trivial representation with respect to $\mathbf{b}_+$ such that the trace acquires nonzero contribution.

4.2. **Steady state probability.** Let $(\sigma_1, \ldots, \sigma_L) \in S(m)$ be a configuration in a basic sector $m = (m_1, m_2) \in \mathbb{Z}^2_{\geq 1}$, where each local state is the two component array $\sigma_i = (\sigma_{i,1}, \sigma_{i,2}) \in \mathbb{Z}^2_{\geq 0}$. Then the formula (28) becomes

$$\begin{aligned}\mathbb{P}(\sigma_1, \ldots, \sigma_L) &= \left(\prod_{i=1}^L g_{\sigma_i}(\mu_i)\right) \mathrm{Tr}(Z_{\sigma_1}(\mu_1)\cdots Z_{\sigma_L}(\mu_L)) \\ &= \left(\prod_{i=1}^L \frac{\mu_i^{-|\sigma_i|}(\mu_i)_{|\sigma_i|}}{(q)_{\sigma_{i,1}}(q)_{\sigma_{i,2}}}\right) \mathrm{Tr}\left(\frac{(\mathbf{b}_+)_\infty}{(\mu_1^{-1}\mathbf{b}_+)_\infty}\mathbf{k}^{\sigma_{1,2}}\mathbf{b}_-^{\sigma_{1,1}} \cdots \frac{(\mathbf{b}_+)_\infty}{(\mu_L^{-1}\mathbf{b}_+)_\infty}\mathbf{k}^{\sigma_{L,2}}\mathbf{b}_-^{\sigma_{L,1}}\right).\end{aligned} \tag{41}$$

The element in the trace belongs to (a completion of) $\mathcal{B}_{\mathrm{fin}}$ thanks to $\sigma_{1,2} + \cdots + \sigma_{L,2} = m_2 \geq 1$. Thus the trace is convergent and (41) provides a matrix product formula of the steady state probability.

Example 9. Consider $L = 2$ and the sector $m = (1, 1)$. We calculate (41) as

$$\begin{aligned}\mathbb{P}(\emptyset, 12) &= g_{0,0}(\mu_1)g_{1,1}(\mu_2)\mathrm{Tr}(Z_{0,0}(\mu_1)Z_{1,1}(\mu_2)) = \frac{\mu_2^{-2}(\mu_2)_2}{(q)_1^2}\mathrm{Tr}\left(\frac{(\mathbf{b}_+)_\infty}{(\mu_1^{-1}\mathbf{b}_+)_\infty}\frac{(\mathbf{b}_+)_\infty}{(\mu_2^{-1}\mathbf{b}_+)_\infty}\mathbf{k}\mathbf{b}_-\right) \\ &= \frac{\mu_2^{-2}(\mu_2)_2}{(q)_1^2}\left(\frac{\mu_1^{-1}(\mu_1)_1}{(q)_1} + \frac{\mu_2^{-1}(\mu_2)_1}{(q)_1}\right)\mathrm{Tr}(\mathbf{b}_+\mathbf{k}\mathbf{b}_-), \\ \mathbb{P}(1, 2) &= g_{1,0}(\mu_1)g_{0,1}(\mu_2)\mathrm{Tr}(Z_{1,0}(\mu_1)Z_{0,1}(\mu_2)) = \frac{(\mu_1\mu_2)^{-1}(\mu_1)_1(\mu_2)_1}{(q)_1^2}\mathrm{Tr}\left(\frac{(\mathbf{b}_+)_\infty}{(\mu_1^{-1}\mathbf{b}_+)_\infty}\mathbf{b}_-\frac{(\mathbf{b}_+)_\infty}{(\mu_2^{-1}\mathbf{b}_+)_\infty}\mathbf{k}\right) \\ &= \frac{(\mu_1\mu_2)^{-1}(\mu_1)_1(\mu_2)_1}{(q)_1^2}\left(\frac{\mu_1^{-1}(\mu_1)_1}{(q)_1}\mathrm{Tr}(\mathbf{b}_+\mathbf{b}_-\mathbf{k}) + \frac{\mu_2^{-1}(\mu_2)_1}{(q)_1}\mathrm{Tr}(\mathbf{b}_-\mathbf{b}_+\mathbf{k})\right)\end{aligned}$$

by means of (40). In view of $\mathrm{Tr}(\mathbf{b}_+\mathbf{k}\mathbf{b}_-) = q^{-1}\mathrm{Tr}(\mathbf{b}_+\mathbf{b}_-\mathbf{k}) = \mathrm{Tr}(\mathbf{b}_-\mathbf{b}_+\mathbf{k}) = (1-q^2)^{-1}$, these results coincide with (27) if they are commonly multiplied by $(\mu_1\mu_2)^3(q)_2(1-q)^2$ which is symmetric in μ_1 and μ_2.

One way to generally characterize our normalization of $\mathbb{P}(\sigma_1, \ldots, \sigma_L)$ implied by the formula (41) is the following calculation extending the first case of Example 9.

Example 10. For general system size L and a general basic sector $m = (m_1, m_2) \in \mathbb{Z}^2_{\geq 1}$, consider the most condensed configuration in which all the particles are contained in a particular site i. Namely $\sigma_j = m$ if $j = i$ and $\emptyset$ otherwise. Then we have

$$\begin{aligned}\mathbb{P}(\emptyset, \ldots, \overset{i}{m}, \ldots, \emptyset) &= \frac{\mu_i^{-m_1-m_2}(\mu_i)_{m_1+m_2}}{(q)_{m_1}(q)_{m_2}}\mathrm{Tr}\left(\mathbf{k}^{m_2}\mathbf{b}_-^{m_1}\frac{(\mathbf{b}_+)_\infty}{(\mu_1^{-1}\mathbf{b}_+)_\infty}\cdots\frac{(\mathbf{b}_+)_\infty}{(\mu_L^{-1}\mathbf{b}_+)_\infty}\right) \\ &= \frac{\mu_i^{-m_1-m_2}(\mu_i)_{m_1+m_2}}{(q)_{m_1+m_2}(1-q^{m_2})}\sum_{r_1+\cdots+r_L=m_1}\frac{(\mu_1)_{r_1}\cdots(\mu_L)_{r_L}}{(q)_{r_1}\cdots(q)_{r_L}}\mu_1^{-r_1}\cdots\mu_L^{-r_L},\end{aligned}$$

where (38) has been used and the sum extends over $r_1, \ldots, r_L \in \mathbb{Z}_{\geq 0}$ under the specified condition.

We note that (41) is convergent also at $m_1 = \sigma_{1,1} + \cdots + \sigma_{L,1} = 0$ which is outside the basic sector, and reproduces the $n = 1$ result (29) up to an overall normalization as long as $m_2 \geq 1$.

So far we have treated the inhomogeneous discrete time $U_q(A_2^{(1)})$-ZRP in Section 2.2. A matrix product formula for continuous time $U_q(A_2^{(1)})$-ZRP in Section 2.3 is obtained from (41) by the specialization $\mu_1 = \cdots = \mu_L = \mu$. Making the replacement $\mathbf{b}_\pm \to \mu^{\pm 1}\mathbf{b}_\pm$ (see the remark after (38)) and removing a further common factor within a sector, we arrive at

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \left(\prod_{i=1}^{L} \frac{(\mu)_{\sigma_{i,1}+\sigma_{i,2}}}{(q)_{\sigma_{i,1}}(q)_{\sigma_{i,2}}}\right) \mathrm{Tr}\left(\frac{(\mu\mathbf{b}_+)_\infty}{(\mathbf{b}_+)_\infty}\mathbf{k}^{\sigma_{1,2}}\mathbf{b}_-^{\sigma_{1,1}} \cdots \frac{(\mu\mathbf{b}_+)_\infty}{(\mathbf{b}_+)_\infty}\mathbf{k}^{\sigma_{L,2}}\mathbf{b}_-^{\sigma_{L,1}}\right). \tag{42}$$

Example 11. Consider the homogeneous ZRP $\mu_1 = \cdots = \mu_L = \mu$. Suppose $m = (m_1, m_2) \geq l = (l_1, l_2) \in \mathbb{Z}_{\geq 0}^2$ and consider the state $(m - l, \emptyset, \ldots, \overset{j}{l}, \ldots, \emptyset)$ for some $j \in [2, L]$. It is a less condensed state than Example 10 where $|l| = l_1 + l_2$ particles have been separated from the 1 st to the j th site. One of them is assumed to be the 1 st site without losing generality by the $\mathbb{Z}_L$-cyclic symmetry. Based on computer calculation of (42) we conjecture

$$\mathbb{P}(m, \emptyset, \ldots, \emptyset)^{-1} \sum_{|l|=r,\, l \leq m} \mathbb{P}(m - l, \emptyset, \ldots, \overset{j}{l}, \ldots, \emptyset) = \frac{f_{|m|-r} f_r}{f_{|m|}}, \qquad f_s = \frac{(\mu)_s}{(q)_s} \tag{43}$$

for any site $j \in [2, L]$ and $0 \leq r \leq |m|$. The independence on L, j and $m_1 - m_2$ is curious. The conjecture has been proven for $r = 1$. See Section 5 for more comments.

At $q = \mu = 0$ (and $a = 0$ in (21)), the present model reduces to the $n = 2$ case of the n-TAZRP studied in [17, 18]. The formula (42) simplifies to

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \mathrm{Tr}(\tilde{Z}_{\sigma_1} \cdots \tilde{Z}_{\sigma_L}), \qquad \tilde{Z}_{\sigma_i} = \tilde{Z}_{\sigma_{i,1},\sigma_{i,2}} = \Big(\sum_{j\geq 1} \mathbf{b}_+^j\Big)\mathbf{k}^{\sigma_{i,2}}\mathbf{b}_-^{\sigma_{i,1}}. \tag{44}$$

This result agrees with [17, 18]. In fact $X_{\alpha_1,\alpha_2}(1)$ in [18, Example 2.1] coincides with $\tilde{Z}_{\alpha_2,\alpha_1}$ here.

The matrix product formulas (41) and (42) for the steady state probabilities, which are corollaries of Proposition 6 and Theorem 8, are our main results on the $U_q(A_2^{(1)})$-ZRP in this paper.

4.3. **Proof of Theorem 8.** From $X_0(\mu) = \frac{(\mathbf{b}_+)_\infty}{(\mu^{-1}\mathbf{b}_+)_\infty}$, its inverse $X_0(\mu)^{-1} = \frac{(\mu^{-1}\mathbf{b}_+)_\infty}{(\mathbf{b}_+)_\infty}$ certainly exists. The auxiliary condition (30) is also straightforward to check. In what follows we shall focus on the proof of the relation (33) among the $Z_\alpha(\mu)$ specified in (39). We use a subsidiary variable $\nu = \mu\lambda^{-1}$ throughout. Substituting (39) into (33) and using the relations

$$\mathbf{k}\frac{(\eta\mathbf{b}_+)_\infty}{(\zeta\mathbf{b}_+)_\infty} = \frac{(q\eta\mathbf{b}_+)_\infty}{(q\zeta\mathbf{b}_+)_\infty}\mathbf{k}, \qquad \left[\mathbf{b}_-, \frac{(\eta\mathbf{b}_+)_\infty}{(\zeta\mathbf{b}_+)_\infty}\right] = (\zeta - \eta)\frac{(q\eta\mathbf{b}_+)_\infty}{(\zeta\mathbf{b}_+)_\infty}\mathbf{k},$$

one can remove the ratio of infinite products. The result reads

$$\begin{aligned}
&(-1)^{\alpha_1} q^{\frac{1}{2}\alpha_1(\alpha_1-1)} (\lambda^{-1}\mathbf{b}_+)_{\alpha_2} (q^{1-\alpha_1}W)_{\alpha_1} \\
&= \sum_{\gamma\leq\alpha} (-1)^{\gamma_1} q^{(\gamma_1-\alpha_1)\alpha_2 + \frac{1}{2}\gamma_1(\gamma_1-1)} \nu^{|\gamma|} \frac{(\lambda)_{|\gamma|}(\nu)_{|\alpha|-|\gamma|}}{(\mu)_{|\alpha|}} \binom{\alpha_1}{\gamma_1}_q \binom{\alpha_2}{\gamma_2}_q \\
&\qquad \times (\mu^{-1}\mathbf{b}_+)_{\gamma_2} (q^{|\gamma|}\mathbf{b}_+)_{|\alpha|-|\gamma|} (q^{1-\gamma_1}X_{\gamma_2})_{\gamma_1} \mathbf{b}_-^{\alpha_1-\gamma_1},
\end{aligned} \tag{45}$$

where $W = q^{-\alpha_2}\mathbf{b}_- + \lambda^{-1}\mathbf{k}$ and $X_{\gamma_2} = q^{-\gamma_2}\mathbf{b}_- + \mu^{-1}\mathbf{k}$. Curiously μ is contained in the RHS only. In what follows we prove (45) by induction on α_1 utilizing the following remark.

Remark 12. Suppose a relation $F(\mathbf{b}_+, \mathbf{b}_-, \mathbf{k}) = 0$ holds in the q-boson algebra $\mathcal{B}$. Then $F(c\mathbf{b}_+, c^{-1}\mathbf{b}_-, \mathbf{k}) = 0$ also holds for any $c \neq 0$ since $\mathbf{b}_\pm \mapsto c^{\pm 1}\mathbf{b}_\pm$ is an automorphism of $\mathcal{B}$.

Lemma 13. *The relation (45) is valid at $\alpha_1 = 0$, namely the following holds:*

$$(\lambda^{-1}\mathbf{b}_+)_{\alpha_2} = \sum_{0\leq\gamma_2\leq\alpha_2} \nu^{\gamma_2} \frac{(\lambda)_{\gamma_2}(\nu)_{\alpha_2-\gamma_2}}{(\mu)_{\alpha_2}} \binom{\alpha_2}{\gamma_2}_q (\mu^{-1}\mathbf{b}_+)_{\gamma_2} (q^{\gamma_2}\mathbf{b}_+)_{\alpha_2-\gamma_2}. \tag{46}$$

Proof. One can prove it as an identity of polynomials of order α_2 in $\mathbf{b}_+$. At $\mathbf{b}_+ = 0$ it is equivalent to

$$\frac{(\mu)_{\alpha_2}}{(q)_{\alpha_2}} = \sum_{i+j=\alpha_2} \frac{\nu^j (\lambda)_j (\nu)_i}{(q)_j (q)_i}.$$

Due to (40) this indeed holds since the RHS is the coefficient of z^{α_2} in $\frac{(\nu z)_\infty(\mu z)_\infty}{(z)_\infty(\nu z)_\infty} = \frac{(\mu z)_\infty}{(z)_\infty}$. It suffices to check (46) further at $\mathbf{b}_+ = \lambda q^{-k}$ with $k = 0, 1, \ldots, \alpha_2 - 1$. By the identity $(\lambda)_{\gamma_2}(\lambda q^{\gamma_2 - k})_{\alpha_2-\gamma_2} = (\lambda q^{-k})_{\alpha_2}(\lambda q^{\gamma_2-k})_k/(\lambda q^{-k})_k$, the relation to show takes the form

$$0 = \sum_{0\le j\le\alpha_2} \nu^j\,(\nu)_{\alpha_2-j}\left(\nu^{-1}q^{-k}\right)_j \binom{\alpha_2}{j}_q (\lambda q^{j-k})_k \quad (0 \le k < \alpha_2).$$

By expanding the last factor into powers of λ, the RHS is expressed as a sum $\sum_{0\le s\le k}\lambda^s A_{k,s}f_{\alpha_2,k,s}(\nu)$ with

$$f_{a,k,s}(\nu) = \sum_{0\le j\le a}(q^s\nu)^j(\nu)_{a-j}(\nu^{-1}q^{-k})_j\binom{a}{j}_q \qquad (0\le s\le k<a),$$

$$A_{k,s} = (-1)^s q^{\frac{1}{2}s(s-1)-ks}\binom{k}{s}_q.$$

Consider the identity

$$(q\nu z)_s(q^{-k+s+1}z)_{k-s} = \frac{(q^{-k+s+1}z)_\infty(q\nu z)_\infty}{(q^{s+1}\nu z)_\infty(qz)_\infty} = \sum_{a\ge 0}\frac{(qz)^a f_{a,k,s}(\nu)}{(q)_a},$$

where the last equality is due to (40). Since the LHS is an order k polynomial in z, it follows that $f_{a,k,s}(\nu) = 0$ for $k < a$. □

Lemma 14. *Set $Y = \mathbf{b}_- + \mathbf{k}$. Then the following equality is valid for any $m \in \mathbb{Z}_{\ge 0}$.*

$$(\mathbf{b}_+)_m\mathbf{b}_-^m = (-1)^m q^{\frac{1}{2}m(m-1)}\sum_{s=0}^m \mu^{m-s}\binom{m}{s}_q(\mu)_s(\mu^{-1}q^{1-m}Y)_{m-s}.$$

Proof. It is equivalent to

$$(\mathbf{b}_+ - 1)(\mathbf{b}_+ - q^{-1})\cdots(\mathbf{b}_+ - q^{1-m})\mathbf{b}_-^m = (q)_m\sum_{s+t=m}\frac{(\mu)_s}{(q)_s}\frac{(\mu^{-1}q^{1-m}Y)_t}{(q)_t}\mu^t,$$

where the sum extends over $s, t \in \mathbb{Z}_{\ge 0}$ under the specified constraint. Applying the identity

$$(\mathbf{b}_+ - q^{-n+1})\mathbf{b}_-^n = (1 - \mathbf{k} - q^{-n+1}\mathbf{b}_-)\mathbf{b}_-^{n-1} = \mathbf{b}_-^{n-1}(1 - q^{-n+1}Y)$$

successively, one finds that the LHS is equal to $(q^{1-m}Y)_m$. On the other hand from (40), the RHS is equal to the coefficient of z^m in the power series

$$(q)_m\frac{(z\mu)_\infty}{(z)_\infty}\frac{(zq^{1-m}Y)_\infty}{(z\mu)_\infty} = (q)_m\frac{(zq^{1-m}Y)_\infty}{(z)_\infty} = (q)_m\sum_{j\ge 0}\frac{(q^{1-m}Y)_j}{(q)_j}z^j.$$

□

Proof of (45) for general α_1. Let us write (45) as $\mathcal{L}(\mathbf{b}_\pm, \alpha_1, \alpha_2; \lambda) = \mathcal{R}(\mathbf{b}_\pm, \alpha_1, \alpha_2; \lambda, \mu)$. We are going to prove it by induction on α_1. The case $\alpha_1 = 0$ was shown in Lemma 13. In Lemma 14 with $m = \gamma_1$, replace $\mathbf{b}_\pm$ by $(\mu^{-1}q^{\gamma_2})^{\pm 1}\mathbf{b}_\pm$. Then $(\mu^{-1}q^{1-m}Y)_{m-s}$ becomes $(q^{1-\gamma_1}X_{\gamma_2})_{\gamma_1-s}$. Solving it for the $s = 0$ term we get

$$(q^{1-\gamma_1}X_{\gamma_2})_{\gamma_1} = (-\mu)^{-\gamma_1}q^{-\frac{1}{2}\gamma_1(\gamma_1-1)}(\mu^{-1}q^{\gamma_2}\mathbf{b}_+)_{\gamma_1}(\mu q^{-\gamma_2}\mathbf{b}_-)^{\gamma_1} - \sum_{s=1}^{\gamma_1}\mu^{-s}\binom{\gamma_1}{s}_q(\mu)_s(q^{1-\gamma_1}X_{\gamma_2})_{\gamma_1-s}.$$

Substituting this into the RHS of (45) we have the decomposition $\mathcal{R}(\mathbf{b}_\pm, \alpha_1, \alpha_2; \lambda, \mu) = \mathcal{R}_0(\mathbf{b}_\pm, \alpha_1, \alpha_2; \lambda, \mu) - \sum_{s=1}^{\gamma_1}\mathcal{R}_s(\mathbf{b}_\pm, \alpha_1, \alpha_2; \lambda, \mu)$, where

$$\mathcal{R}_0(\mathbf{b}_\pm, \alpha_1, \alpha_2; \lambda, \mu) = \sum_{\gamma\le\alpha} q^{(\gamma_1-\alpha_1)\alpha_2-\gamma_1\gamma_2}\nu^{|\gamma|}\frac{(\lambda)_{|\gamma|}(\nu)_{|\alpha|-|\gamma|}}{(\mu)_{|\alpha|}}\binom{\alpha_1}{\gamma_1}_q\binom{\alpha_2}{\gamma_2}_q(\mu^{-1}\mathbf{b}_+)_{|\gamma|}(q^{|\gamma|}\mathbf{b}_+)_{|\alpha|-|\gamma|}\mathbf{b}_-^{\alpha_1},$$

$$\mathcal{R}_s(\mathbf{b}_\pm, \alpha_1, \alpha_2; \lambda, \mu) = \sum_{\gamma\le\alpha} q^{(\gamma_1-\alpha_1)\alpha_2}\nu^{|\gamma|}\frac{(\lambda)_{|\gamma|}(\nu)_{|\alpha|-|\gamma|}}{(\mu)_{|\alpha|}}\binom{\alpha_1}{\gamma_1}_q\binom{\alpha_2}{\gamma_2}_q(\mu^{-1}\mathbf{b}_+)_{\gamma_2}(q^{|\gamma|}\mathbf{b}_+)_{|\alpha|-|\gamma|}$$
$$\times(-1)^{\gamma_1}q^{\frac{1}{2}\gamma_1(\gamma_1-1)}\mu^{-s}\binom{\gamma_1}{s}_q(\mu)_s(q^{1-\gamma_1}X_{\gamma_2})_{\gamma_1-s}\mathbf{b}_-^{\alpha_1-\gamma_1}.$$

In $\mathcal{R}_0(\mathbf{b}_\pm,\alpha_1,\alpha_2;\lambda,\mu)$, replace γ_2 by $\gamma_2-\gamma_1$. Then the sum over γ_1 can be taken by means of $\binom{|\alpha|}{\gamma_2}_q = \sum_{0\le\gamma_1\le\min(\gamma_2,\alpha_1)} q^{\gamma_1(\alpha_2-\gamma_2+\gamma_1)}\binom{\alpha_1}{\gamma_1}_q\binom{\alpha_2}{\gamma_2-\gamma_1}_q$, yielding

$$\begin{aligned}\mathcal{R}_0(\mathbf{b}_\pm,\alpha_1,\alpha_2;\lambda,\mu) &= q^{-\alpha_1\alpha_2}\sum_{\gamma_2\le\alpha_2}\nu^{\gamma_2}\frac{(\lambda)_{\gamma_2}(\nu)_{|\alpha|-\gamma_2}}{(\mu)_{|\alpha|}}\binom{|\alpha|}{\gamma_2}_q(\mu^{-1}\mathbf{b}_+)_{\gamma_2}(q^{\gamma_2}\mathbf{b}_+)_{|\alpha|-\gamma_2}\mathbf{b}_-^{\alpha_1}\\ &= q^{-\alpha_1\alpha_2}(\lambda^{-1}\mathbf{b}_+)_{|\alpha|}\mathbf{b}_-^{\alpha_1}\\ &= (-1)^{\alpha_1}q^{\frac{1}{2}\alpha_1(\alpha_1-1)}(\lambda^{-1}\mathbf{b}_+)_{\alpha_2}\sum_{t=0}^{\alpha_1}\lambda^{-t}(\lambda)_t\binom{\alpha_1}{t}_q(q^{1-\alpha_1}W)_{\alpha_1-t},\end{aligned} \tag{47}$$

where the second equality is due to Lemma 13. The third equality is obtained by applying Lemma 14 with m,μ and $\mathbf{b}_\pm$ replaced by α_1,λ and $(q^{\alpha_2}\lambda^{-1})^{\pm1}\mathbf{b}_\pm$, respectively.

To evaluate $\mathcal{R}_s(\mathbf{b}_\pm,\alpha_1,\alpha_2;\lambda,\mu)$ with $s\ge1$, rewrite $\binom{\alpha_1}{\gamma_1}_q\binom{\gamma_1}{s}_q$ as $\binom{\alpha_1-s}{\gamma_1-s}_q\binom{\alpha_1}{s}_q$ and then change γ_1 into γ_1+s. By this procedure, the formula for $\mathcal{R}(\mathbf{b}_\pm,\alpha_1,\alpha_2;\lambda,\mu)$ gets replaced by

$$\mathcal{R}(\mathbf{b}_\pm,\alpha_1,\alpha_2;\lambda,\mu) = \mathcal{R}_0(\mathbf{b}_\pm,\alpha_1,\alpha_2;\lambda,\mu) - \sum_{s=1}^{\alpha_1}\mathcal{R}_s(\mathbf{b}_\pm,\alpha_1,\alpha_2;\lambda,\mu). \tag{48}$$

As for the summand, comparing the resulting expression with $\mathcal{R}(q^{\pm s}\mathbf{b}_\pm,\alpha_1-s,\alpha_2;q^s\lambda,q^s\mu)$, we find

$$\begin{aligned}\mathcal{R}_s(\mathbf{b}_\pm,\alpha_1,\alpha_2;\lambda,\mu) &= (-\lambda^{-1})^s(\lambda)_s q^{s\alpha_1-\frac{1}{2}s(s+1)}\binom{\alpha_1}{s}_q\mathcal{R}(q^{\pm s}\mathbf{b}_\pm,\alpha_1-s,\alpha_2;q^s\lambda,q^s\mu)\\ &= (-\lambda^{-1})^s(\lambda)_s q^{s\alpha_1-\frac{1}{2}s(s+1)}\binom{\alpha_1}{s}_q\mathcal{L}(q^{\pm s}\mathbf{b}_\pm,\alpha_1-s,\alpha_2;q^s\lambda)\\ &= (-1)^{\alpha_1}q^{\frac{1}{2}\alpha_1(\alpha_1-1)}(\lambda^{-1}\mathbf{b}_+)_{\alpha_2}\lambda^{-s}(\lambda)_s\binom{\alpha_1}{s}_q(q^{1-\alpha_1}W)_{\alpha_1-s},\end{aligned} \tag{49}$$

where the second equality is due to the induction assumption. Now from (47) and (49) we see that the difference (48) leaves the $t=0$ term of (47) only, which exactly coincides with $\mathcal{L}(\mathbf{b}_\pm,\alpha_1,\alpha_2,\lambda)$, i.e., the LHS of (45). □

5. Summary and discussion

We have studied the steady state probabilities of the $U_q(A_n^{(1)})$-ZRPs [15]. The main results are the attribution to the ZF algebra and the auxiliary condition for general n (Proposition 6), a concrete realization of them for $n=2$ (Theorem 8) and the resulting matrix product formulae in (41) and (42).

They serve as a starting point for studying physical properties of the system. For instance, the RHS of (43) is viewed as a naive measure of the *condensation* (cf. [8, 12]). Apart from the statistical factors L and $L(L-1)/2$ for the relevant configurations, the crude estimation

$$\frac{f_{|m|-r}f_r}{f_{|m|}} \simeq \frac{1-\mu}{1-q}\quad (q\searrow 0),\qquad \frac{f_{|m|-r}f_r}{f_{|m|}} \simeq \exp\left(\left(\log\frac{1-q}{1-\mu}\right)\frac{(1-q^r)(1-q^{|m|-r})}{\log q}\right)\quad (q\nearrow 1)$$

for $0<r<|m|$ indicates that particles are more likely to condense in the region $\mu>q$ than $\mu<q$.

Another commonly undertaken approach is to switch to the grand canonical picture and investigate the generating function involving "fugacity" x,y:

$$\mathcal{Z}(w,x,y) = \sum_{\sigma_1,\dots,\sigma_L\in\mathbb{Z}^2_{\ge0}}\mathbb{P}_w(\sigma_1,\dots,\sigma_L)x^{\sigma_{1,1}+\cdots+\sigma_{L,1}}y^{\sigma_{1,2}+\cdots+\sigma_{L,2}}.$$

Here $\mathbb{P}_w(\sigma_1,\dots,\sigma_L)$ is a regularization of (41) avoiding the divergence at the non-basic sector $\sigma_{1,2}=\cdots=\sigma_{L,2}=0$. An example of such a prescription is to insert a boson-counter $\mathbf{h}=\log_q\mathbf{k}$ (see (36)) into the trace as $\mathrm{Tr}\left(w^{\mathbf{h}}(\cdots)\right)$. Then (41) allows one to express it as

$$\begin{aligned}\mathcal{Z}(w,x,y) &= \mathrm{Tr}\left(w^{\mathbf{h}}V(\mu_1,x,y)\cdots V(\mu_L,x,y)\right),\\ V(\mu,x,y) &= \frac{(\mathbf{b}_+)_\infty}{(\mu^{-1}\mathbf{b}_+)_\infty}\sum_{l,m\ge0}\frac{x^m y^l\mu^{-l-m}(\mu)_{l+m}}{(q)_l(q)_m}\mathbf{k}^l\mathbf{b}_-^m = \frac{(\mathbf{b}_+)_\infty}{(\mu^{-1}\mathbf{b}_+)_\infty}\sum_{m\ge0}\frac{(x\mu^{-1})^m(\mu)_m}{(q)_m}\frac{(q^m y\mathbf{k})_\infty}{(y\mu^{-1}\mathbf{k})_\infty}\mathbf{b}_-^m\\ &= \frac{(\mathbf{b}_+)_\infty}{(\mu^{-1}\mathbf{b}_+)_\infty}\Gamma(x\mu^{-1},y\mu^{-1})^{-1}\Gamma(x,y),\qquad \Gamma(x,y)=(x\mathbf{b}_-)_\infty(y\mathbf{k})_\infty.\end{aligned}$$

Similarly in the homogeneous case $\mu_1 = \cdots = \mu_L = \mu$, the result (42) corresponding to another normalization of $\mathbb{P}_w(\sigma_1, \ldots, \sigma_L)$ leads to the alternative form

$$\mathcal{Z}(w,x,y) = \mathrm{Tr}\left(w^{\mathbf{h}}\tilde{V}(\mu,x,y)^L\right), \quad \tilde{V}(\mu,x,y) = \frac{(\mu\mathbf{b}_+)_\infty}{(\mathbf{b}_+)_\infty}\Gamma(x,y)^{-1}\Gamma(x\mu,y\mu).$$

This formula remains valid at $\mu = 0$ and may be useful to extract the large L asymptotics in the corresponding $(n=2)$-species q-boson model [25].

As noted in (44), the formula (42) with $q = \mu = 0$ agrees with the earlier result on the $(n=2)$-TAZRP [17, 18] based on the combinatorial R and the tetrahedron equation. It is an interesting question how the approaches in the present paper and [17, 18] are related for general n. We plan to address it in a future publication.

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