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RIGIDITY OF BELTRAMI FIELDS WITH A NON-CONSTANT PROPORTIONALITY FACTOR

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Abstract. We prove that bounded Beltrami fields are symmetric if a proportionality factor depends on 2 variables in the cylindrical coordinate and admits a regular level set diffeomorphic to a cylinder or a torus.

1. Introduction

We consider 3d steady states of ideal incompressible flows

$$(\nabla\times u)\times u+\nabla\pi=0,\quad \nabla\cdot u=0\quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where u is the velocity of fluid and π is the Bernoulli function. Integral curves of the velocity and the vorticity $\nabla\times u$ are called stream lines and vortex lines, respectively. If the Bernoulli function $\pi\not\equiv$ const. is regular, they lie on level sets of π, called Bernoulli surfaces. It is known [2] that Bernoulli surfaces are diffeomorphic to nested cylinders or tori if a domain is bounded and velocity is analytic. The system (1.1) can be written as an elliptic system with constraints, e.g. [23, p.34]. If there exists a current potential η such that $\nabla\times u=\nabla\pi\times\nabla\eta$, u satisfies the elliptic problem,

$$\begin{aligned}&\nabla\times u=\nabla\pi\times\nabla\eta,\quad \nabla\cdot u=0,\\ &u\cdot\nabla\pi=0,\quad u\cdot\nabla\eta=1.\end{aligned}$$

The first line is an elliptic system for given π and η. The second line is constraints to them, called a degenerate hyperbolic system.

The constraints are removed by symmetry, e.g. translation or rotation. In the axisymmetric setting, solutions of (1.1) can be constructed by the Grad-Shafranov equation [24], [29]. Existence of solutions with compactly supported vorticity is well known in the study of vortex rings, e.g. [12]. Moreover, compactly supported solutions are constructed in [20], [11], [13]. Existence of smooth non-symmetric solutions to (1.1) with $\pi\not\equiv$ const. is unknown.

The non-existence of such non-symmetric solutions is a conjecture of Grad [22, p.144], see Constantin et al. [9, p.529]. More precisely, symmetries in this conjecture are 4 types:

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translation, rotation, helix, reflection in a plane. This problem is considered as *rigidity* to (1.1). For the 2d flows, a rigidity result that bounded solutions with no stagnation points are shear flows is proved by Hamel and Nadirashvili [26]. See also rigidity in a strip [25] and in a pipe for axisymmetric flows with swirl [9]. The full 3d rigidity to (1.1) with $\pi \not\equiv$ const. is unknown, cf. [30]. Grad's conjecture is also studied from existence of non-symmetric solutions with piecewise constant Bernoulli function [3], [14] and of smooth non-axisymmetric solutions with small force [10], cf. [5], [4].

Grad [23, p.36] also observed a similar constraint to Beltrami fields,

$$\nabla \times u = fu, \quad \nabla \cdot u = 0 \quad \text{in } \mathbb{R}^3, \tag{1.2}$$

i.e. $\pi \equiv$ const. The function f is called a proportionality factor. If $f \equiv$ const., u is called a strong Beltrami field. Vortex lines of strong Beltrami fields can be chaotic and non-symmetric, e.g. ABC flows [2]. In this sense, the system (1.2) for $f \equiv$ const. is *flexible*, cf. [9]. It is known [15], [16] that strong Beltrami fields describe vortex lines of highly non-trivial topology.

If $f \not\equiv$ const., vortex lines are confined to a level set $f^{-1}(c) = \{x \in \mathbb{R}^3 \mid f(x) = c\}$ for $c \in \mathbb{R}$ since f is a first integral, i.e.

$$u \cdot \nabla f = 0.$$

Topology of the surface $f^{-1}(c)$ is generally unknown since singular points $\{u = 0\}$ may appear on the surface. Under their absence, a closed surface is diffeomorphic to a torus [2]. Existence of solutions to (1.2) is unknown unless $f \equiv$ const or under symmetry, see [7], [31], [1] for axisymmetric solutions.

In contrast to (1.1) for $\pi \not\equiv$ const., rigidity results are known to (1.2). The first rigidity results to (1.2) are Liouville theorems on decay conditions at space infinity [28], [6], e.g. $u = o(|x|^{-1})$ as $|x| \to \infty$. This decay rate is sharp, cf. [15], [16]. The another type Liouville theorem is based on a level set condition for $f \not\equiv$ const.

Theorem 1.1 ([17]). *Suppose that $f \in C^{2+\mu}(\mathbb{R}^3)$ for some $0 < \mu < 1$. If f admits a regular level set diffeomorphic to a sphere, then any solutions to (1.2) is identically zero.*

This Liouville theorem implies non-existence to (1.2) for a broad class of f, e.g. radial or having extrema. On the other hand, it implies a certain relation between existence and symmetry of f since symmetric f does not take extrema.

The simplest symmetric f are those depending on 1 variable in the cylindrical coordinate (r, θ, z), i.e. $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = z$. For such f, rigidity to (1.2) is known [19], [8]:

(i) For $f = f(z)$, level sets are planes. Any solutions of (1.2) are harmonic on them and singular points are isolated. In particular, bounded solutions depend only on z.
(ii) For $f = f(r)$, level sets are cylinders. Any solutions of (1.2) are constant on them and axisymmetric.

(iii) For $f = f(\theta)$, level sets are half planes. No solutions exist to (1.2).

These results can be deduced from compatibility of a constrained evolution equation equivalent to (1.2) [17], [8], see Remarks 2.2.

For f depending on 2 variables in (r, θ, z), a variety of surfaces appear as level sets of f such as cylindrical surfaces for $f = f(r, \theta)$, surfaces of revolution for $f = f(r, z)$ and right conoids for $f = f(\theta, z)$. For $f = f(r, \theta)$ and $f = f(r, z)$ having extrema on (x_1, x_2) or (r, z)-planes, their level sets admit a cylinder or a torus, cf. [17]. We prove that for such f, any bounded solutions are symmetric.

Theorem 1.2. *Suppose that $f \in C^{2+\mu}(\mathbb{R}^3)$ for some $0 < \mu < 1$.*
(i) If $f = f(r, \theta)$ admits a regular level set diffeomorphic to a cylinder, then any bounded solutions to (1.2) are translationally symmetric.
(ii) If $f = f(r, z)$ admits a regular level set diffeomorphic to a torus, then any solutions to (1.2) are rotationally symmetric.

In both (i) and (ii), velocity fields on the level sets are expressed by 2 linearly independent tangential vector fields and have no singular points. In particular, stream lines are winding on the level set and linearly independent u of (1.2) are at most 2.

Remarks 1.3. (i) Translationally symmetric solutions of (1.2) are locally expressed as

$$u = \partial_2 \Psi e_1 - \partial_1 \Psi e_2 + u^3(\Psi) e_3, \quad f = \dot{u}^3(\Psi),$$

for some $u^3(\cdot)$ and the stream function $\Psi(x_1, x_2)$ satisfying $-\Delta \Psi = \dot{u}^3(\Psi) u^3(\Psi)$, where e_1, e_2, e_3 is the orthogonal basis in the Cartesian coordinate and $\dot{u}^3(\Psi)$ denotes the derivative in Ψ.
(ii) Rotationally symmetric (axisymmetric) solutions of (1.2) are locally expressed as

$$u = -r^{-1} \partial_z \Psi e_r + r^{-1} \Gamma(\Psi) e_\theta + r^{-1} \partial_r \Psi e_z, \quad f = \dot{\Gamma}(\Psi),$$

for some $\Gamma(\cdot)$ and $\Psi(r, z)$ satisfying $-(\Delta_{z,r} - r^{-1} \partial_r) \Psi = \dot{\Gamma}(\Psi) \Gamma(\Psi)$, where $e_r = {}^t(\cos\theta, \sin\theta, 0)$, $e_\theta = {}^t(-\sin\theta, \cos\theta, 0)$, $e_z = {}^t(0, 0, 1)$ are the orthogonal basis in the cylindrical coordinate. These elliptic problems appear as free boundary problems for translating vortex pairs and vortex rings, see Section 4. Under helical symmetry, (1.1) and (1.2) are reduced to the helical Grad-Shafranov equation [18, p.196].

The proof of Theorem 1.2 is based on the facts that (1.2) can be recasted as a constrained evolution equation on a level set of f [17], [8] and that Beltrami fields are solutions to the elliptic equation $-\Delta u = \nabla f \times u + f^2 u$. Unfortunately, solutions of (1.1) with $\pi \not\equiv$ const. do not possess neither of them and their rigidity is out of reach. Rigidity to (1.1) with $\pi \not\equiv$ const. seems unknown even for π depending on 1 variable in (r, θ, z). A crucial difference is failure of a unique continuation property to (1.1) with $\pi \not\equiv$ const. as compactly supported solutions exist [20], [11], [13].

2. Elliptic equations on surfaces

We derive elliptic equations on surfaces for symmetric $f = f(r,\theta)$ and $f = f(r,z)$ in terms of differential forms.

2.1. **Fourier series.**

A simple proof to Theorem 1.2 (ii) (axisymmetry) is to use the Fourier series in $0 \leq \theta \leq 2\pi$,

$$u = \sum_{n\in\mathbb{Z}} v_n e^{in\theta}, \quad v_n = v^r(r,z)e_r(\theta) + v^\theta(r,z)e_\theta(\theta) + v^z(r,z)e_z.$$

The equations (1.2) are reduced to those for $v = v_n$:

$$\nabla \times v + inv \times \nabla\theta = fv,$$
$$\nabla \cdot v + inv \cdot \nabla\theta = 0.$$

The condition $u \cdot \nabla f = 0$ implies $v \cdot \nabla f = 0$. For $n \neq 0$, taking the divergence to the 1st equation implies that $\nabla \times v \cdot \nabla\theta = 0$. By taking the inner product to the 1st equation with $\nabla\theta$, we have $v \cdot \nabla\theta = 0$. Thus $v = \nabla \times (\Psi\nabla\theta)$ for some $\Psi = \Psi(r,z)$. By taking the inner product to the 1 st equation with $\nabla\Psi$, $\nabla\Psi \equiv 0$ and $v_n \equiv 0$ for $n \neq 0$. Thus $u = v_0$ is axisymmetric.

2.2. **A constrained equation.**

The latter assertion in Theorem 1.2 is based on an elliptic equation on the surface and the 1st de Rham cohomology group, see Remarks 2.1 (i). We derive elliptic equations on surfaces in terms of differential forms and prove not only that u is axisymmetric but also that linearly independent u to (1.2) is at most 2.

We first derive a constrained evolution equation to a dual 1-form [17] for general surfaces. We assume that a level set $f^{-1}(c)$ for $c \in \mathbb{R}$ is regular in the sense that $f^{-1}(c+t)$ is a smooth surface for $0 \leq t \leq t_0$ with some $t_0 > 0$ and $\nabla f(x) \neq 0$ for $x \in f^{-1}(c+t)$. We parametrize the surface $f^{-1}(c)$ by $x = \Phi_0(\xi)$ with $\xi = {}^t(\xi_1, \xi_2)$ and define $\Phi(\xi,t)$ by the flow of $X = \nabla f/|\nabla f|^2$, i.e.

$$\partial_t\Phi = X(\Phi), \quad t > 0,$$
$$\Phi(\xi,0) = \Phi_0(\xi).$$

The flow $\Phi(\xi,t)$ parametrizes the surface $f^{-1}(c+t)$, i.e. $\Phi(\xi,t) \in f^{-1}(c+t)$. Since $f \in C^{2+\mu}$ for some $0 < \mu < 1$, $\Phi(\xi,t)$ is $C^{1+\mu}$. We may assume that $\Phi(\cdot,t)$ is defined for $0 \leq t \leq t_0$. The equations (1.2) for the dual 1-form $\alpha = \sum_{i=1}^3 u^i dx_i$ of $u = (u^i)$ are

$$d_{\mathbb{R}^3}\alpha = f *_{\mathbb{R}^3} \alpha, \quad d_{\mathbb{R}^3} *_{\mathbb{R}^3} \alpha = 0, \tag{2.1}$$

where $d_{\mathbb{R}^3}$ and $*_{\mathbb{R}^3}$ are the exterior derivative and the Hodge star operator in $\mathbb{R}^3$, respectively. By the elliptic equation $-\Delta u = \nabla f \times u + f^2 u$ and $f \in C^{2+\mu}$, u and α are $C^{3+\mu}$. The pullback $\beta = \Phi^*\alpha$ by the map $\Phi : (\xi,t) \longmapsto x = \Phi(\xi,t)$ satisfies

$$d_{\mathbb{R}^3}\beta = (c+t) *_{\mathbb{R}^3} \beta, \quad d_{\mathbb{R}^3} *_{\mathbb{R}^3} \beta = 0, \tag{2.2}$$

With the matrices $F = (\partial_1\Phi, \partial_2\Phi, \partial_t\Phi)$ and $\tilde{F} = |F|F^{-1}$,

$$\begin{aligned} \beta &= (u^1, u^2, u^3)F\, {}^t(d\xi_1, d\xi_2, dt), \\ *_{\mathbb{R}^3}\beta &= (u^1, u^2, u^3)^t\tilde{F}\, {}^t(d\xi_2 \wedge dt, dt \wedge d\xi_1, d\xi_1 \wedge d\xi_2), \end{aligned}$$

where $|F|$ denotes the determinant of F. Since $u \cdot \partial_t\Phi = 0$, the pullback β is a 1-form on a surface and $C^{1+\mu}$,

$$\beta = u(\Phi(\xi,t)) \cdot \partial_1\Phi d\xi_1 + u(\Phi(\xi,t)) \cdot \partial_2\Phi d\xi_2 =: \beta_1(\xi,t)d\xi_1 + \beta_2(\xi,t)d\xi_2.$$

We write the metric tensor by $\mathcal{G} = (\partial_i\Phi \cdot \partial_j\Phi)_{1\le i,j\le 2}$ and $\mathcal{G}^{-1} = (g^{ij})_{1\le i,j\le 2}$. Since

$${}^tFF = \begin{pmatrix} \mathcal{G} & 0 \\ 0 & \chi^2 \end{pmatrix}, \quad \chi = |\nabla f|^{-1},$$

and $(u^1, u^2, u^3) = (\beta_1, \beta_2, 0)F^{-1}$, the Hodge dual in $\mathbb{R}^3$ is

$$*_{\mathbb{R}^3} \beta = \chi|\mathcal{G}|^{1/2}((\beta_1 g^{11} + \beta_2 g^{21})d\xi_2 \wedge dt + (\beta_1 g^{12} + \beta_2 g^{22})dt \wedge d\xi_1).$$

Then the equations (2.2) imply

$$\begin{aligned} &\partial_1\beta_2 - \partial_2\beta_1 = 0, \\ &\partial_t\beta_1 = (c+t)\chi|\mathcal{G}|^{1/2}(\beta_1 g^{12} + \beta_2 g^{22}), \\ &\partial_t\beta_2 = -(c+t)\chi|\mathcal{G}|^{1/2}(\beta_1 g^{11} + \beta_2 g^{21}), \\ &\partial_1(\chi|\mathcal{G}|^{1/2}(\beta_1 g^{11} + \beta_2 g^{21})) + \partial_2(\chi|\mathcal{G}|^{1/2}(\beta_1 g^{12} + \beta_2 g^{22})) = 0. \end{aligned}$$

The last equation follows from the first 3 equations. They can be written as

$$v_t = Av, \quad \nabla^\perp \cdot v = 0, \tag{2.3}$$

for $v = {}^t(v^1, v^2)$, $v^i = \beta_i$ with the matrix

$$A = (c+t)\chi|\mathcal{G}|^{1/2} \begin{pmatrix} g^{12} & g^{22} \\ -g^{11} & -g^{21} \end{pmatrix},$$

where $\nabla = {}^t(\partial_{\xi_1}, \partial_{\xi_2})$ and $\nabla^\perp = {}^t(\partial_{\xi_2}, -\partial_{\xi_1})$. Taking the rotation implies that v satisfies the elliptic equation $\nabla^\perp \cdot (Av) = 0$ and $\nabla^\perp \cdot v = 0$. With the Hodge star operator on the surface,

$$*_t\beta = (\beta_1, \beta_2)|\mathcal{G}|^{1/2}\mathcal{G}^{-1}\,{}^t(d\xi_2, -d\xi_1),$$

the equations (2.3) are expressed as

$$\beta_t = -(c+t)\chi *_t \beta, \quad d\beta = 0, \tag{2.4}$$

where d is the exterior derivative on the surface.

Remarks 2.1. (i) Differentiating the first equation of (2.4) by d yields the elliptic equation

$$d(\chi *_t \beta) = 0, \quad d\beta = 0. \tag{2.5}$$

If $\chi \equiv$ const., β is a harmonic differential form. For a closed surface with genus $g \geq 0$, the space of harmonic differential forms has dimension $2g$.
(ii) Theorem 1.1 is the case $g = 0$. If the surface $f^{-1}(c+t)$ is diffeomorphic to a sphere, β is an exact form, i.e. $\beta = d\psi$. Thus ψ satisfies an elliptic equation of the divergence form

$$d(\chi *_t d\psi) = 0.$$

By integration by parts on the surface, ψ is constant. Thus u vanishes in a neighborhood of the regular level set and in $\mathbb{R}^3$ by unique continuation [17]. Theorem 1.2 (ii) is the case $g = 1$, see Proposition 3.2.
(iii) The system (2.4) is overdetermined in the sense that the irrotational condition $d\beta = 0$ is generally *not* compatible with the evolution equation $\beta_t = -(c+t)\chi *_t \beta$. Regarding (1.2) as a constrained evolution equation originates from [15] in which Cauchy-Kowalevski theorem is used to construct strong Beltrami fields for given initial surface and tangential data.
(iv) Clelland and Klotz [8] derived a similar evolution equation as (2.4) by using a moving frame and studied (1.2) in terms of an integral manifold to an equivalent exterior differential system by using the Cartan's method. Among other results, they showed that associated integral manifolds are at most 3-dimensional if level sets of f have no umbilic points.

2.3. **Elliptic equations for symmetric f.** We study symmetry of solutions to (2.5) for symmetric f depending on 2 variables in (r, θ, z).

(i) $f = f(r, \theta)$. We parametrize a curve in a plane by $(r(\xi_1, t), \theta(\xi_1, t))$, i.e. $x_1 = r\cos\theta$ $x_2 = r\sin\theta$. Then, the map

$$\Phi(\xi, t) = re_r(\theta) + ze_z \tag{2.6}$$

for $(r, \theta, z) = (r(\xi_1, t), \theta(\xi_1, t), \xi_2)$, parametrizes a cylindrical surface. The matrix A forms

$$A = (c+t)\chi \begin{pmatrix} 0 & \nu \\ -1/\nu & 0 \end{pmatrix} \tag{2.7}$$

for

$$\chi = \sqrt{|\partial_t r|^2 + r^2|\partial_t \theta|^2}, \quad \nu = \sqrt{|\partial_1 r|^2 + r^2|\partial_1 \theta|^2}.$$

(ii) $f = f(r, z)$. We parametrize a curve in the (r, z)-plane by $(r(\xi_1, t), z(\xi_1, t))$. Then the map (2.6) for $(r, \theta, z) = (r(\xi_1, t), \xi_2, z(\xi_1, t))$, parametrizes a surface of revolution. The matrix A is the same form as (2.7) with different coefficients

$$\chi = \sqrt{|\partial_t r|^2 + |\partial_t z|^2}, \quad \nu = \sqrt{|\partial_1 r|^2 + |\partial_1 z|^2}/r.$$

(iii) $f = f(\theta, z)$. We parametrize a curve in the (θ, z)-plane by $(\theta(\xi_1, t), z(\xi_1, t))$. The map (2.6) for $(r, \theta, z) = (\xi_2, \theta(\xi_1, t), z(\xi_1, t))$ parametrizes a right conoid. The matrix A is the same form as (2.7) with different coefficients

$$\chi = \sqrt{r^2|\partial_t \theta|^2 + |\partial_t z|^2}, \quad \nu = \sqrt{r^2|\partial_1 \theta|^2 + |\partial_1 z|^2}.$$

In all the cases (i)-(iii), the elliptic problem for v is expressed as

$$\nabla \cdot (Bv) = 0, \quad \nabla^{\perp} \cdot v = 0 \tag{2.8}$$

with the matrix

$$B = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \quad p = \chi/\nu,\ q = \chi\nu.$$

In the cases (i) and (ii), B is independent of ξ_2. Hence $\partial_2 v = \nabla v^2$ and

$$\nabla \cdot (B\nabla v^2) = 0. \tag{2.9}$$

In terms of the differential form, (2.9) is expressed as

$$d(\chi *_t d\beta_2) = 0. \tag{2.10}$$

Remarks 2.2. (i) For $f = f(z)$, (1.2) is expressed as

$$\partial_z v = -f v^{\perp}, \quad \nabla \cdot v = 0, \quad \nabla^{\perp} \cdot v = 0,$$

for $u = {}^t(v, 0)$ and $v = {}^t(v^1, v^2)$, where $v^{\perp} = {}^t(-v^2, v^1)$ and $\nabla^{\perp} = (\partial_2, -\partial_1)$. The evolution equation for z is *compatible* with the elliptic constraints and for a given harmonic vector field $v(r, \theta, 0)$ on $\{z = 0\}$ and $f = f(z)$, there exists a (non-symmetric) solution to (1.2). If u is bounded, v is constant in $\mathbb{R}^2$. Hence any bounded solutions depend only on z, i.e. $u = u(z)$.
(ii) For $f = f(r)$, we take $r = r(t)$ satisfying $df(r(t))/dt = 1$ and $\theta = \xi_1$, $z = \xi_2$. Then, the function Φ in (2.6) parametrizes a cylinder and v satisfies (2.8) for $B = B(t)$. If $p \equiv 1$, differentiating $\nabla \cdot (Bv) = 0$ by t implies

$$0 = \nabla \cdot (\partial_t B v + B \partial_t v) = \nabla \cdot (\partial_t B v) = \partial_t q \partial_2 v^2.$$

By $\nabla \cdot (Bv) = 0$, $\partial_i v^i = 0$ for $i = 1, 2$. By differentiating each components of $\partial_t v = Av$ by ξ_1 and ξ_2, $v = v(t)$ follows. If $p \not\equiv 1$, applying the same argument to $\nabla \cdot (p^{-1} B v) = 0$ imples $v = v(t)$. Thus, $u = u^{\theta}(r) e_{\theta} + u^{z}(r) e_z$ is constant and axisymmetric on cylinders.
(iii) For $f = f(\theta)$, we take $\theta = \theta(t)$ satisfying $df(\theta(t))/dt = 1$ and $r = \xi_2$, $z = \xi_1$. Then, the function Φ in (2.8) parametrizes a half plane. The first equation of (2.8) is $\nabla \cdot (\xi_2 v) = 0$. By differentiating this by t, $\nabla \cdot (\xi_2^2 v^{\perp}) = 0$ and $v = 0$ follow. Thus, no solutions exist to (1.2).

3. The Liouville theorem

The elliptic equation (2.10) implies local symmetry of u. The global symmetry follows from unique continuation.

3.1. **Local symmetry.** For $f = f(r, \theta)$, we use the parameter $\xi = {}^t(\xi_1, \xi_2)$ for $0 \leq \xi_1 \leq 2\pi$ and $\xi_2 \in \mathbb{R}$ to denote the surface diffeomorfic to a cylinder $\Phi(\xi, t) = r(\xi_1, 1) e_r(\theta(\xi_1, t)) + \xi_2 e_z$. For $f = f(r, z)$, we use the parameter $\xi = {}^t(\xi_1, \xi_2)$ for $0 \leq \xi_1 \leq 2\pi$ and $0 \leq \xi_2 \leq 2\pi$ to denote the surface diffeomorfic to a torus $\Phi(\xi, t) = r(\xi_1, t) e_r(\xi_2) + z(\xi_1, t) e_z$.

Proposition 3.1.

$$\beta_1 = \beta_1(\xi_1, t),\ \beta_2 = \beta_2(t), \quad 0 \leq \xi_1 \leq 2\pi,\ 0 \leq t \leq t_0. \tag{3.1}$$

For each $0 \leq t \leq t_0$, $\beta_1(\cdot, t) \equiv 0$ *or* $\beta_1(\xi_1, t) \neq 0$ *for all* $0 \leq \xi_1 \leq 2\pi$.

Proof. For $f = f(r, z)$, (2.10) and an integration by parts on the surface imply (3.1). For $f = f(r, \theta)$, we consider the periodic extension of v^2 for the ξ_1-variable to $\mathbb{R}$. Since the level set $f^{-1}(c)$ is regular, $\nabla f \neq 0$ and $\partial_1 \Phi \neq 0$ for $0 \leq t \leq t_0$ and $\xi_1 \in \mathbb{R}$. Thus, $\chi = |\nabla f|^{-1}$ and $\nu = |\partial_1 \Phi|$ are bounded from above and below by positive constants. We take some $\lambda(t)$ and $\Lambda(t)$ such that

$$0 < \lambda(t) \leq p(\xi_1, t), q(\xi_1, t) \leq \Lambda(t), \quad \xi_1 \in \mathbb{R},\ 0 \leq t \leq t_0.$$

Since the diagonal matrix B satisfies the elliptic condition, applying the Liouville theorem

[21, Corollally 3.12, Theorem 8.20] for a bounded weak solution $v^2 \in C^{1+\mu}(\mathbb{R}^2)$ to (2.9) implies that v^2 is constant. By $\nabla^\perp \cdot v = 0$, v^1 is independent of ξ_2 and (3.1) follows.

For each $0 \le t \le t_0$, $\beta_1(\xi_1, t)p(\xi_1, t) = C(t)$ for some $C(t)$ by (2.8). Since $p(\xi_1, t) \neq 0$, $\beta_1(\cdot, t) \equiv 0$ or $\beta_1(\xi_1, t) \neq 0$ for all $0 \le \xi_1 \le 2\pi$. □

Proposition 3.2. *The space of solutions to (2.5) has dimension 2 in the sense that any solutions are expressed by linear combinations of $\beta_1(\xi_1, t)d\xi_1$ and $\beta_2(t)d\xi_2$ such that $\beta_1(\xi_1, t) \neq 0$ for all $0 \le \xi_1 \le 2\pi$. In particular, linearly independent u of (1.2) is at most 2.*

Proof. Let $\beta(\xi, t) = \beta_1(\xi_1, t)\mathrm{d}\xi_1 + \beta_2(t)\mathrm{d}\xi_2$ be a solution of (2.5) such that $\beta_1(\xi_1, t) \neq 0$ for all $0 \le \xi_1 \le 2\pi$. Let $\tilde{\beta}(\xi, t) = \tilde{\beta}_1(\xi_1, t)\mathrm{d}\xi_1 + \tilde{\beta}_2(t)\mathrm{d}\xi_2$ be another solution to (2.5). We take $\mu(t) \in \mathbb{R}$ such that $\tilde{\beta}_1(0, t) = \mu(t)\beta_1(0, t)$. Since $\tilde{\beta} - \mu\beta$ is also a solution to (2.5), $\tilde{\beta}_1(\xi_1, t) = \mu(t)\beta_1(\xi_1, t)$ for all $0 \le \xi_1 \le 2\pi$ by Proposition 3.1. Thus $\tilde{\beta}$ is expressed by the linear combinations of $\beta_1\mathrm{d}\xi_2$ and $\beta_2\mathrm{d}\xi_2$. □

Lemma 3.3. *The solution u is translationally or rotationally symmetric in some symmetric open set $U \subset \mathbb{R}^3$.*

Proof. In both cases (i) and (ii) of Theorem 1.2, $\partial_1\Phi$ and $\partial_2\Phi$ are orthogonal. Thus $u(\Phi) = u(\Phi(\xi, t))$ satisfies

$$\begin{aligned} u(\Phi) &= (u(\Phi)\cdot\partial_1\Phi)\frac{\partial_1\Phi}{|\partial_1\Phi|^2} + (u(\Phi)\cdot\partial_2\Phi)\frac{\partial_2\Phi}{|\partial_2\Phi|^2} \\ &= \beta_1(\xi_1, t)\frac{\partial_1\Phi}{|\partial_1\Phi|^2} + \beta_2(t)\frac{\partial_2\Phi}{|\partial_2\Phi|^2}. \end{aligned}$$

In the case (i), by differentiating $\Phi(\xi, t) = r(\xi_1, t)e_r(\theta(\xi_1, t)) + \xi_2 e_z$ by ξ_1 and ξ_2,

$$u(\Phi) = \frac{\beta_1(\xi_1, t)}{|\partial_1 r|^2 + r^2|\partial_1\theta|^2}(\partial_1 r e_r + r\partial_1\theta e_\theta) + \beta_2(t)e_z.$$

The right-hand side is independent of $\xi_2 = z$. Thus u is translationally symmetric on the level set $f^{-1}(c + t)$ for $0 \le t \le t_0$. In particular, u is translationally symmetric in $U = D \times \mathbb{R}$ for some open set D in a plane.

In the case (ii), by differentiating $\Phi(\xi, t) = r(\xi_1, t)e_r(\xi_2) + z(\xi_1, t)e_z$ by ξ_1 and ξ_2,

$$u(\Phi) = \frac{\beta_1(\xi_1, t)}{|\partial_1 r|^2 + |\partial_1 z|^2}(\partial_1 r e_r + \partial_1 z e_z) + \frac{\beta_2(t)}{r}e_\theta.$$

Each components in the cylindrical coordinate are independent of $\xi_2 = \theta$. Thus u is rotationally symmetric on the level set $f^{-1}(c + t)$ for $0 \le t \le t_0$. In particular, u is rotationally symmetric in a region U, rotation of some open set in the (r, z)-plane around the z-axis. □

3.2. **Unique continuation.** We use a classical unique continuation result under the boundedness of $|\Delta w|/|w|$, e.g. [32].

Proposition 3.4. *Let $w \in C^2(\mathbb{R}^3)$ satisfy*

$$|\Delta w| \leq C_R |w| \quad \text{in } \{|x| < R\},$$

for each $R > 0$ with some $C_R > 0$. Assume that w vanishes in some open set in $\{|x| < R\}$. Then, $w \equiv 0$.

Proof of Theorem 1.2. For translationally symmetric u in U, set $w(x) = u(x) - u(x + \tau e_z)$ for $\tau \in \mathbb{R}$. Then, w is a Beltrami field with f and vanishes in U. Since $-\Delta w = \nabla f \times w + f^2 w$ in $\mathbb{R}^3$, by unique continuation, $w \equiv 0$ in $\mathbb{R}^3$. Thus u is translationally symmetric in $\mathbb{R}^3$, i.e. $u = u(x_1, x_2)$.

Similarly, for rotationally symmetric u in U, set $w(x) = u(x) - {}^t R_\tau u(R_\tau x)$ with $R_\tau = (e_r(\tau), e_\theta(\tau), e_z)$ for $\tau \in [0, 2\pi]$. Then, applying unique continuation to w implies that u is rotationally symmetric in $\mathbb{R}^3$, i.e. $u = u^r(r,z)e_r(\theta) + u^\theta(r,z)e_\theta(\theta) + u^z(r,z)e_z$.

By Proposition 3.2, velocity fields on the surface are expressed by combinations of 2 linearly independent tangential vector fields and have no singular points. The proof is complete. ◻

Remarks 3.5. (i) A crucial step in the proof of Theorem 1.2 is to prove that the space of solutions to (2.5) has dimension 2 (Proposition 3.2). Once we know this, we are able to prove Theorem 1.2 without differential forms. Indeed, for a solution u to (1.2) with $f = f(r,z)$, $\partial_\theta u$ is also a solution to (1.2). The fields u and $\nabla\theta$ are linearly independent 2 tangential vector fields on each torus. Therefore by Proposition 3.2, $\partial_\theta u$ is expressed by their linear combinations as

$$\partial_\theta u = A_1(f)u + A_2(f)\nabla\theta,$$

with some constants $A_1(f)$ and $A_2(f)$. By differentiating both sides by θ, $\partial_\theta(\partial_\theta u) = A_1(f)\partial_\theta u$. Since $\partial_\theta u$ is periodic in θ, $A_1(f) = 0$. By taking the rotation to both sides of $\partial_\theta u = A_2(f)\nabla\theta$,

$$\begin{aligned} \nabla \times (\partial_\theta u) &= f\partial_\theta u = fA_2(f)\nabla\theta, \\ \nabla \times (A_2(f)\nabla\theta) &= \dot{A}_2(f)\nabla f \times \nabla\theta. \end{aligned}$$

By taking the inner product to $fA_2(f)\nabla\theta = \dot{A}_2(f)\nabla f \times \nabla\theta$ with $\nabla\theta$, $A_2(f) = 0$ and $\partial_\theta u = 0$. Thus u is locally axisymmetric and, by unique continuation, globally axisymmetric in $\mathbb{R}^3$.
(ii) Similarly by Proposition 3.2, we are able to express a solution u to (1.2) with $f = f(r,\theta)$ on a cylinder as

$$\partial_z u = A_1(f)u + A_2(f)\nabla z.$$

By integrating $\partial_z(\partial_z u) = A_1(f)(\partial_z u)$ in z, $\partial_z u(\cdot, z) = e^{A_1(f)z}\partial_z u(\cdot, 0)$. If $A_1(f) = 0$, by taking the rotation to both sides of $\partial_z u = A_2(f)\nabla z$, similarly as the rotationally symmetric case, $A_2(f) = 0$ and $\partial_z u = 0$. If $A_2(f) \neq 0$, by integrating $\partial_z u(\cdot, z) = e^{A_1(f)z}\partial_z u(\cdot, 0)$,

$$u(\cdot, z) = u(\cdot, 0) + \frac{1}{A_2(f)} e^{A_2(f)z}\partial_z u(\cdot, 0), \quad z \in \mathbb{R}.$$

Since u is bounded, $\partial_z u(\cdot, 0) = 0$ and u is translationally symmetric.
(iii) Under the translational symmetry, (1.2) is reduced to

$$\partial_2 u^3 = fu^1, \quad -\partial_1 u^3 = fu^2, \quad \partial_1 u^2 - \partial_2 u^1 = fu^3, \quad \partial_1 u^1 + \partial_2 u^2 = 0.$$

With a stream function Ψ, ${}^t(u^1, u^2) = \nabla^{\perp}\Psi$ and ${}^t(u^1, u^2)\cdot\nabla u^3 = 0$. Hence $u^3 = u^3(\Psi)$, $f = \dot{u}^3(\Psi)$ and Ψ is a solution to $-\Delta\Psi = \dot{u}^3(\Psi)u^3(\Psi)$.
(iv) Under the rotational symmetry, (1.2) is reduced to

$$-\partial_z u^\theta = fu^r, \quad \partial_z u^r - \partial_r u^z = fu^\theta, \quad \partial_r u^\theta + u^\theta/r = fu^z, \quad \partial_r u^r + u^r/r + \partial_z u^z = 0.$$

With a stream function Ψ, $ru^z = \partial_r\Psi$, $ru^r = -\partial_z\Psi$ and ${}^t(ru^z, ru^r)\cdot\nabla_{z,r}\Gamma = 0$ for $\Gamma = ru^\theta$. Hence $\Gamma = \Gamma(\Psi)$, $f = \dot{\Gamma}(\Psi)$ and Ψ is a solution to $-(\Delta_{z,r} - r^{-1}\partial_r)\Psi = \dot{\Gamma}(\Psi)\Gamma(\Psi)$.

4. Examples of symmetric solutions

We review existence of translationally and rotationally symmetric solutions to (1.2).

4.1. **Vortex pairs.** Translationally symmetric solutions can be constructed by the elliptic problem for given $u^3(\cdot)$,

$$-\Delta\Psi = \dot{u}^3(\Psi)u^3(\Psi) \quad \text{in } \mathbb{R}^2.$$

The simplest solutions are rotationally symmetric solutions, i.e. $\Psi = \Psi(r)$. For such Ψ, level sets of f are cylinders, i.e. $f = f(r)$. If $\dot{u}^3(\Psi)u^3(\Psi)$ is compactly supported, the Biot-Savart law implies the decay ${}^t(u^1, u^2) = O(r^{-1})$ as $r \to \infty$, cf. [28], [6]. Besides rotationally symmetric solutions, there exist periodic solutions for $\dot{u}^3(t)u^3(t) = t$ or e^t. For such solutions, level sets of f are deformed cylinders in $\mathbb{R}^3$ and ${}^t(u^1, u^2)$ is merely bounded, e.g. [27, 2.2.2].

Variational solutions also exist. A vortex pair is a pair of translating 2 vortices with opposite signs in $\mathbb{R}^2$. They are symmetric for the x_2-variable and constructed via the half plane problem:

$$\begin{aligned} -\Delta\Psi &= \dot{u}^3(\Psi)u^3(\Psi) && \text{in } \mathbb{R}^2_+, \\ \Psi &= -\gamma && \text{on } \partial\mathbb{R}^2_+, \\ \partial_1\Psi \to 0, \quad \partial_2\Psi &\to -W && \text{as } x_1^2 + x_2^2 \to \infty. \end{aligned}$$

The constant $W > 0$ is a speed of a vortex and $\gamma \geq 0$ is a flux measuring a distance from a vortex to the boundary $x_2 = 0$. A typical choice is $u^3(t) = t_+^l$ for $l > 1$ and $t_+ = \max\{t, 0\}$. For such u^3, variational solutions exist and their vortex is compactly supported in $\mathbb{R}^2$ [33].

Level sets of f are symmetric and deformed cylinders in $\mathbb{R}^3$ and the decay is ${}^t(u^1, u^2) =$ const. $+ O(r^{-1})$ as $r \to \infty$.

4.2. **Vortex rings.** Rotationally symmetric solutions can be constructed via the elliptic problem for given $\Gamma(\cdot)$:

$$\begin{aligned} -(\Delta_{z,r} - r^{-1}\partial_r)\Psi &= \dot{\Gamma}(\Psi)\Gamma(\Psi) && \text{in } \mathbb{R}^2_+, \\ \Psi &= -\gamma && \text{on } \partial\mathbb{R}^2_+, \\ r^{-1}\partial_z\Psi \to 0, \quad r^{-1}\partial_r\Psi &\to -W && \text{as } z^2 + r^2 \to \infty. \end{aligned}$$

For the choice $\Gamma(s) = s_+^l$ and $l > 1$, variational solutions exist and their vortex is compactly supported in $\mathbb{R}^3$ [1]. Level sets of f are tori in $\mathbb{R}^3$ and the decay is $u =$ const. $+ O(|x|^{-3})$ as $|x| \to \infty$.

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