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ASYMPTOTIC BEHAVIOR OF LEAST ENERGY SOLUTIONS TO THE FINSLER LANE-EMDEN PROBLEM WITH LARGE EXPONENTS

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ABSTRACT. In this paper we are concerned with the least energy solutions to the Lane-Emden problem driven by an anisotropic operator, so-called the Finsler N-Laplacian, on a bounded domain in $\mathbb{R}^N$. We prove several asymptotic formulae as the nonlinear exponent gets large.

Key words: Finsler Lane-Emden problem, Finsler Laplacian, Least energy solution
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1. INTRODUCTION

Let $N \geq 2$ be an integer. In this paper, we study the following Lane-Emden problem driven by an anisotropic operator Q_N:

$$
(1.1) \quad
\begin{cases}
-Q_N u = u^p & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p > 1$ is any positive number, and Q_N is a quasilinear operator, so-called *the Finsler N-Laplacian*, defined by

$$
Q_N u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(H(\nabla u)^{N-1} H_{\xi_i}(\nabla u) \right).
$$

Here $H \in C^2(\mathbb{R}^N \setminus \{0\})$ is any norm on $\mathbb{R}^N$ and $H_{\xi_i}(\xi) = \frac{\partial H(\xi)}{\partial \xi_i}$. We assume that $H^N \in C^1(\mathbb{R}^N)$ and $\text{Hess}\left(H^N(\xi)\right)$ is positive definite for

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any $\xi \in \mathbb{R}^N$, $\xi \neq 0$. Note that $Q_N u$ can be written as

$$Q_N u = \mathrm{div}\left(\nabla_\xi(\frac{1}{N}H(\xi)^N)\Big|_{\xi=\nabla u}\right) = \sum_{i,j=1}^N a_{ij}(\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j},$$

where $a_{ij}(\nabla u) = \mathrm{Hess}(\frac{1}{N}H^N(\xi))_{i,j}\Big|_{\xi=\nabla u}$. If $H(\xi) = |\xi|$ (the Euclidean norm), then $Q_N u$ coincides with the N-Laplacian $\Delta_N u = \mathrm{div}(|\nabla u|^{N-2}\nabla u)$ of a function u. In this case, the problem (1.1) was treated by Ren and Wei [21] [22] when $N = 2$, and in [23] for general $N \geq 2$. Ren and Wei [23] considered the least energy solution u_p of the following quasilinear problem

$$\begin{cases} -\Delta_N u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$. They studied the asymptotic behavior of u_p as the nonlinear exponent $p \to \infty$, and proved that the least energy solutions remain bounded in L^∞-norm regardless of p. When the dimension $N = 2$, they showed that the least energy solutions must develop one "peak" in the interior of $\Omega \subset \mathbb{R}^2$, that is, the shape of graph of u_p looks like a single spike as $p \to \infty$. Moreover they showed that this peak point must be a critical point of the Robin function of the domain. For other generalizations of this problem to various situations, see for example, [28], [29], [30], [24], [25].

Now, main aim of the paper is to extend the results of Ren and Wei [21], [22], [23] to the anisotropic problem (1.1).

As in [21], [22], [23], we restrict our attention to the least energy solutions to (1.1) constructed as follows:

Consider the constrained minimization problem:

$$(1.2) \quad C_p = \inf\{\int_\Omega H(\nabla u)^N dx : u \in W_0^{1,N}(\Omega), \int_\Omega |u|^{L^{p+1}(\Omega)}dx = 1\}.$$

Since the Sobolev imbedding $W_0^{1,N}(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact for any $p > 1$, we have at least one minimizer $\underline{u}_p$ for the problem (1.2), where $\underline{u}_p \in W_0^{1,N}(\Omega), \|\underline{u}_p\|_{p+1} = 1$. As $|\underline{u}_p| \in W_0^{1,N}(\Omega)$ also achieves C_p, we may assume $\underline{u}_p > 0$. Note that $Q_N(cu) = c^{N-1}Q_N(u)$ for a constant $c > 0$. Thus if we define

$$u_p = C_p^{\frac{1}{p+1-N}}\underline{u}_p,$$

then u_p solves (1.1) and $C_p = \int_\Omega H(\nabla u_p)^N dx/(\int_\Omega |u_p|^{p+1}dx)^{\frac{N}{p+1}}$. Standard regularity argument implies that any weak solution $u \in W_0^{1,N}(\Omega)$

1 satisfies $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. We call u_p the *least energy*
2 *solution* to (1.1).
3   Our first result is the following L^∞-bound of least energy solutions.

4 **Theorem 1.1.** *Let u_p be a least energy solution to* (1.1). *Then there*
5 *exist C_1, C_2 (independent of p), such that*

$$0 < C_1 \le \|u_p\|_{L^\infty(\Omega)} \le C_2 < \infty$$

6 *for p large enough.*
   Furthermore, we have

$$\lim_{p \to \infty} p^{N-1} \int_\Omega H(\nabla u_p)^N dx = \lim_{p \to \infty} p^{N-1} \int_\Omega u_p^{p+1} dx = \left(\frac{Ne\beta_N}{N-1}\right)^{N-1}$$

7 *where $\beta_N = N(N\kappa_N)^{\frac{1}{N-1}}$, $\kappa_N = |\mathcal{W}|$ is the volume (with respect to the*
8 *N-dimensional Hausdorff measure) of the unit Wulff ball associated*
9 *with the dual norm H^0 of H:*

$$\mathcal{W} = \{x \in \mathbb{R}^N : H^0(x) < 1\}.$$

10   On the asymptotic behavior of the L^∞-norm of u_p, we have

11 **Theorem 1.2.** *Let u_p be a least energy solution to* (1.1). *Then it holds*
12 *that*

$$1 \le \limsup_{p \to \infty} \|u_p\|_{L^\infty(\Omega)} \le e^{\frac{N-1}{N}}.$$

13   To state further results, we need some definitions. Set

(1.3)
$$v_p = \frac{u_p}{\left(\int_\Omega u_p^p dx\right)^{\frac{1}{N-1}}}.$$

14 Then v_p is a weak solution of

(1.4)
$$\begin{cases} -Q_N v_p = f_p(x) = \frac{u_p^p}{\int_\Omega u_p^p dx} & \text{in } \Omega, \\ v_p > 0 & \text{in } \Omega, \\ v_p = 0 & \text{on } \partial\Omega. \end{cases}$$

15 For $X, Y \in \mathbb{R}^N, X, Y \ne 0, X \ne Y$, we put

$$d(X,Y) = \frac{\left(H^{N-1}(X)(\nabla_\xi H)(X) - H^{N-1}(Y)(\nabla_\xi H)(Y)\right) \cdot (X - Y)}{H^N(X-Y)},$$

16 and

(1.5) $$d_N = \inf\{d(X,Y) \mid X, Y \in \mathbb{R}^N, X, Y \ne 0, X \ne Y\},$$

1 where $X \cdot Y = \sum_{j=1}^{N} X_j Y_j$ denotes the usual inner product for $X, Y \in$
2 $\mathbb{R}^N$. As in [32] Lemma 5.1, we can obtain the estimate

$$\min\{\frac{\lambda}{\beta^N}, 1\} \leq d_N \leq 1$$

where λ is the least eigenvalue of Hess $\left(\frac{1}{N} H^N(\xi)\right)$, which is positive by
the assumption (2.2) and β is as in (2.3), see §2. Also define

(1.6) $\qquad L_0 = \limsup_{p \to \infty} \frac{p \left(\int_\Omega u_p^p dx\right)^{\frac{1}{N-1}}}{\left(\frac{N}{N-1} e^{\frac{N-1}{N}}\right)}, \quad L_1 = d_N^{-\left(\frac{1}{N-1}\right)} L_0.$

For a sequence v_{p_n} of v_p, we define the *blow-up set S* of $\{v_{p_n}\}$ as usual:

$$S = \{x \in \overline{\Omega} : \exists \text{a subsequence } v_{p'_n}, \exists \{x_n\} \subset \Omega \text{ s.t. } x_n \to x \text{ and } v_{p'_n}(x_n) \to \infty\}.$$

3 In the following, $\sharp A$ denotes the cardinality of a set A and $[\cdot]$ denotes
4 the Gauss symbol.

5 **Theorem 1.3.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Then for*
6 *any sequence v_{p_n} of v_p with $p_n \to \infty$, the blow-up set S of v_{p_n} is non-*
7 *empty. Also there exists a subsequence (still denoted by v_{p_n}) such that*
8 *the estimate*

$$\sharp(S \cap \Omega) \leq \left[\frac{e^{\frac{N-1}{N}}}{d_N}\right]$$

9 *holds true for this subsequence.*
10 *Assume $S \cap \Omega = \{x_1, \cdots, x_k\} \subset \Omega$. Then we have*
  (i)

$$f_n = \frac{u_{p_n}^{p_n}}{\int_\Omega u_{p_n}^{p_n} dx} \xrightarrow{*} \sum_{i=1}^{k} \gamma_i \delta_{x_i}$$

11 *in the sense of Radon measures of Ω, where*

$$\gamma_i \geq \left(\frac{\beta_N}{L_1}\right)^{N-1}$$

12 *and $\sum_{i=1}^{k} \gamma_i \leq 1$.*
13 *(ii) $v_{p_n} \to G$ in $C_{loc}^1(\Omega \setminus (S \cap \Omega))$ for some function G satisfying*

$$\begin{cases} -Q_N G = 0 & in \Omega \setminus (S \cap \Omega), \\ G = +\infty & on S \cap \Omega, \\ G = 0 & on \partial\Omega \setminus (\partial\Omega \cap S). \end{cases}$$

14 *(iii) $\|u_{p_n}\|_{L^\infty(K)} \to 0$ as $n \to \infty$ for any compact set $K \subset \Omega \setminus (S \cap \Omega)$.*

In [23], Ren and Wei obtained an estimate of the number of interior blow-up set

$$\sharp(S \cap \Omega) \leq \left[\frac{1}{d_N} \left(\frac{N}{N-1} \right)^{N-1} \right]$$

when $H(\xi) = |\xi|$ case. Since $e^x < \frac{1}{1-x}$ for $x \in (0,1)$, we check that $e^{\frac{N-1}{N}} < \left(\frac{N}{N-1} \right)^{N-1}$ for all $N \geq 2$. Thus the estimate in Theorem 1.3 is better than that in [23] even when $H(\xi)$ coincides with the Euclidean norm $|\xi|$. Also, Theorem 1.2 seems new even for $H(\xi) = |\xi|$ and $N > 2$ case.

Finally, we prove that if the blow-up set consists of one point, it must be an interior point of Ω.

Theorem 1.4. *Assume $\sharp S = 1$ and $S = \{x_0\}$, $x_0 \in \overline{\Omega}$. Then $x_0 \in \Omega$ must hold.*

The organization of the paper is as follows: In §2, we recall basic properties of the Finsler norm and collect useful lemmas about the Finsler N-Laplacian. In §3, we obtain asymptotic formula for C_p as $p \to \infty$, and prove the latter half part of Theorem 1.1. In §4, we prove the L^∞-bound of least energy solutions in Theorem 1.1. In §5, we prove Theorem 1.2 using an argument by Adimurthi and Grossi [1]. In §6, we prove Theorem 1.3. We use a notion of (L, δ)-regular, or irregular points, which was originally introduced by Brezis and Merle [6]. Finally in §7, we prove Theorem 1.4 by using a local Pohozaev identity and an idea by Santra and Wei [25].

2. Notations and basic properties

Let H be any *norm* on $\mathbb{R}^N$, i.e., H is convex, $H(\xi) \geq 0$ and $H(\xi) = 0$ if and only if $\xi = 0$, and H satisfies

$$(2.1) \qquad H(t\xi) = |t|H(\xi), \quad \forall \xi \in \mathbb{R}^N, \forall t \in \mathbb{R}.$$

By (2.1), H must be even: $H(-\xi) = H(\xi)$ for all $\xi \in \mathbb{R}^N$. Throughout of the paper, we also assume that $H \in C^2(\mathbb{R}^N \setminus \{0\})$, $H^N \in C^1(\mathbb{R}^N)$, and

$$(2.2) \qquad \text{Hess}\left(H^N(\xi) \right) \text{ is positive definite for any } \xi \in \mathbb{R}^N, \xi \neq 0.$$

Since all norms on $\mathbb{R}^N$ are equivalent to each other, we see the existence of positive constants α and β such that

$$(2.3) \qquad \alpha|\xi| \leq H(\xi) \leq \beta|\xi|, \quad \xi \in \mathbb{R}^N.$$

The dual norm of H is the function $H^0 : \mathbb{R}^N \to \mathbb{R}$ defined by

$$H^0(x) = \sup_{\xi \in \mathbb{R}^N \setminus \{0\}} \frac{\xi \cdot x}{H(\xi)}.$$

It is well-known that H^0 is also a norm on $\mathbb{R}^N$ and satisfies the inequality

$$\frac{1}{\beta}|x| \le H^0(x) \le \frac{1}{\alpha}|x|, \quad \forall x \in \mathbb{R}^N.$$

The set

$$\mathcal{W} = \{x \in \mathbb{R}^N \, : \, H^0(x) < 1\}$$

is called the *Wulff ball*, or the H^0-unit ball, and we denote $\kappa_N = \mathcal{H}^N(\mathcal{W})$, where $\mathcal{H}^N$ denotes the N-dimensional Hausdorff measure on $\mathbb{R}^N$. We also denote $\mathcal{W}_r = \{x \in \mathbb{R}^N \,|\, H^0(x) < r\}$ for any $r > 0$.

For a domain $\Omega \subset \mathbb{R}^N$ and a Borel set $E \subset \mathbb{R}^N$, the *anisotropic H-perimeter* of a set E with respect to Ω is defined as

$$P_H(E, \Omega) = \sup\left\{ \int_{E \cap \Omega} \mathrm{div}\,\sigma\, dx \, : \, \sigma \in C_0^\infty(\Omega, \mathbb{R}^N), H^0(\sigma(x)) \le 1 \right\}.$$

If E is Lipschitz, then it holds $P_H(E, \Omega) = \int_{\Omega \cap \partial^* E} H(\nu) d\mathcal{H}^{N-1}$, where $\partial^* E$ denotes the reduced boundary of the set E and $\nu(x)$ is the measure theoretic outer unit normal of $\partial^* E$ (see [16]). Also we have $P_H(\mathcal{W}, \mathbb{R}^N) = N\kappa_N$. For more explanation about the anisotropic perimeter, see [3] and [5].

Here we just recall some properties of H and H^0. These will be proven by using the homogeneity property of H and H^0, see [4] Lemma 2.1, and Lemma 2.2.

Proposition 2.1. *Let H be a Finsler norm on $\mathbb{R}^N$. Then the following properties hold true:*

(1) $|\nabla_\xi H(\xi)| \le C$ *for any $\xi \ne 0$.*
(2) $\nabla_\xi H(\xi) \cdot \xi = H(\xi)$, $\nabla_x H(x) \cdot x = H(x)$ *for any $\xi \ne 0$, $x \ne 0$.*
(3) $(\nabla_\xi H)(t\xi) = \frac{t}{|t|}(\nabla_\xi H)(\xi)$ *for any $\xi \ne 0$, $t \ne 0$.*
(4) $H(\nabla H^0(x)) = 1$. $H^0(\nabla_\xi H(\xi)) = 1$.
(5) $H^0(x)(\nabla_\xi H)(\nabla_x H^0(x)) = x$.

Finally, given a smooth function u on $\mathbb{R}^N$, *the Finsler Laplace operator* of u (associated with H) is defined by

$$Qu(x) = \mathrm{div}\left(H(\nabla u(x))(\nabla_\xi H)(\nabla u(x)) \right)$$

$$= \sum_{j=1}^N \frac{\partial}{\partial x_j} \left(H(\xi) H_{\xi_j}(\xi) \Big|_{\xi = \nabla u(x)} \right)$$

1 and, more generally, for any $1 < q < \infty$, *the Finsler q-Laplace operator*
2 Q_q by

$$Q_q u(x) = \mathrm{div}\left(H^{q-1}(\nabla u(x))(\nabla_\xi H)(\nabla u(x))\right).$$

3 If we assume that $\mathrm{Hess}(H^q(\xi))$ is positive definite on $\mathbb{R}^N \setminus \{0\}$, Q_q
4 becomes a uniformly elliptic operator locally. The Finsler q-Laplacian
5 has been widely investigated in literature by many authors in different
6 settings, see [2], [5], [8], [9], [10], [12], [13], [14], [17], [20] [34] and the
7 references therein.
8 We collect here several useful facts.

9 **Theorem 2.2.** (Finsler Trudinger-Moser inequality [32]) *Let Ω be a*
10 *bounded domain in $\mathbb{R}^N$, $N \geq 2$. Let $u \in W_0^{1,N}(\Omega)$ satisfy $\int_\Omega H(\nabla u)^N dx \leq$*
11 *1. Then there exists a constant C depending only on the dimension N*
12 *such that*

$$\int_\Omega \exp\left(\beta |u|^{\frac{N}{N-1}}\right) dx \leq C|\Omega|$$

13 *holds for any $\beta \leq \beta_N = N(N\kappa_N)^{\frac{1}{N-1}}$. Furthermore, β_N is opti-*
14 *mal in the sense that there exists a sequence $\{u_n\} \subset W_0^{1,N}(\Omega)$ with*
15 *$\int_\Omega H(\nabla u_n)^N dx \leq 1$, such that $\int_\Omega \exp\left(\beta |u_n|^{\frac{N}{N-1}}\right) dx \to +\infty$ as $n \to \infty$*
16 *for $\beta > \beta_N$.*

17 Next is the unique existence of the Green function for the Finsler
18 p-Laplacian.

19 **Theorem 2.3.** ([32]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain containing the*
20 *origin. Define $\Omega^* = \Omega \setminus \{0\}$ and*

$$\Gamma(x) = \begin{cases} C(p, N)(H^0(x))^{\frac{p-N}{p-1}} & \text{for} \quad 1 < p < N, \\ C(N) \log \frac{1}{H^0(x)} & \text{for} \quad p = N, \end{cases}$$

where $C(p, N) = \frac{p-N}{p-1}(N\kappa_N)^{-\frac{1}{p-1}}$ and $C(N) = (N\kappa_N)^{-\frac{1}{N-1}}$. Then
there exists a unique function $G(\cdot, 0) \in C^{1,\alpha}(\Omega^*)$ with $|\nabla G| \in L^{p-1}(\Omega)$,
$G/\Gamma \in L^\infty(\Omega)$, satisfying

$$\begin{cases} -Q_p G(\cdot, 0) = \delta_0 & \text{in } \Omega, \\ G(\cdot, 0) = 0 & \text{on } \partial\Omega. \end{cases}$$

21 *Moreover, $g = G - \Gamma$ satisfies $g \in C(\Omega)$ and $\lim_{x \to 0} H^0(x)\nabla g(x) = 0$.*

22 We recall here useful regularity estimates which are valid for the
23 Finsler N-Laplacian equations, under the assumption (2.2); see Serrin
24 [26], Tolksdorf [31], DiBenedetto [15] and Lieberman [19].

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Then the following statements are true.*

(1) *Let $u \in W^{1,N}(\Omega)$ be a weak solution of $-Q_N u = f$ in Ω, where $f \in L^q(\Omega)$ for some $q > 1$. Then for any subdomain $\Omega' \subset\subset \Omega$, there exists a constant $C = C(\Omega, \Omega', q, N) > 0$ such that*

$$\|u\|_{L^\infty(\Omega')} \le C \left(\|f\|_{L^q(\Omega)} + \|u\|_{L^N(\Omega)} \right)$$

holds.

(2) *Let $u \in W^{1,N}(\Omega)$ be a weak solution of $-Q_N u = f$ in Ω. Suppose $\|u\|_{L^\infty(\Omega)} \le a$ and $\|f\|_{L^\infty(\Omega)} \le b$ for some $a, b < \infty$. Then $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$ and for any subdomain $\Omega' \subset\subset \Omega$, there exists a constant $C = C(\Omega, \Omega', a, b, \alpha) > 0$ such that*

$$\|u\|_{C^{1,\alpha}(\Omega')} \le C$$

holds. If, in addition, u satisfies the Dirichlet boundary condition $u = \phi$ on $\partial\Omega$ where $\phi \in C^{1,\beta}(\partial\Omega)$, $\beta \in (0,1)$, then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ holds.

(3) *(Harnack inequality) Let $u \in W^{1,N}(\Omega)$ be a nonnegative weak solution of $-Q_N u = f$ in Ω. Suppose $\|f\|_{L^q(\Omega)} \le b$ for some $q > 1$. Then for any subdomain $\Omega' \subset\subset \Omega$, there exists a constant $C = C(\Omega, \Omega', q, b) > 0$ such that*

$$\sup_{x \in \Omega'} u(x) \le C \left(1 + \inf_{x \in \Omega'} u(x) \right)$$

holds.

Next is the result from [33] (Theorem 1.1 and Theorem 1.2).

Theorem 2.5. *(Finsler Brezis-Merle type inequality [33]) Let Ω be a bounded domain in $\mathbb{R}^N$, $N \ge 2$.*

(1) *Suppose u is a weak solution to*

$$\begin{cases} -Q_N u = f(x) & in \ \Omega, \\ u = 0 & on \ \partial\Omega, \end{cases}$$

where $f \in L^1(\Omega)$. Then for any $\varepsilon \in (0, \beta_N)$ where $\beta_N = N(N\kappa_N)^{\frac{1}{N-1}}$, it holds that

$$\int_\Omega \exp\left(\frac{(\beta_N - \varepsilon)|u(x)|}{\|f\|_{L^1(\Omega)}^{\frac{1}{N-1}}} \right) dx \le \frac{\beta_N}{\varepsilon} |\Omega|.$$

1   *(2) Suppose u and v are weak solutions to*

$$-Q_N u = f(x) > 0 \quad in \ \Omega$$

2   *and*

$$-Q_N v = 0 \quad in \ \Omega \quad v = u \quad on \ \partial\Omega,$$

3   *respectively. Then for any $\varepsilon \in (0, \beta_N)$, we have*

$$\int_\Omega \exp\left(\frac{(\beta_N - \varepsilon)d_N^{\frac{1}{N-1}}|u(x) - v(x)|}{\|f\|_{L^1(\Omega)}^{\frac{1}{N-1}}} \right) dx \leq \frac{|\Omega|}{\varepsilon},$$

4   *where d_N is defined in (1.5).*

5   Next is the Pohozaev identity for the Finsler q-Laplacian problem
6 without the boundary condition. This is a special case of much more
7 general identity proved in [11]. The identity below is known to hold
8 for solutions in $C^1(\overline{\Omega}) \cap C^2(\Omega)$. The important point is that we can
9 remove the condition $u \in C^2(\Omega)$ with the cost of the convexity and the
10 $C^1(\mathbb{R}^N)$-regularity of the map $\mathbb{R}^N \ni \xi \mapsto H^q(\xi)$. This improvement is
11 crucial for the application to the Finsler Laplacian problem, since the
12 best possible regularity result of solutions is $C^{1,\alpha}$, not C^2, see Theorem
13 2.4.

Theorem 2.6. ([11]) *Let $1 < q < \infty$. Let $u \in C^1(\overline{\Omega})$ be a weak solution
of $-Q_q u = f(u)$ in Ω, where $\Omega \subset \mathbb{R}^N$ is a domain with the boundary
of class C^1, and $f \in C(\mathbb{R}, \mathbb{R})$. Assume the map $\mathbb{R}^N \ni \xi \mapsto H^q(\xi)$ is
convex and belongs to $C^1(\mathbb{R}^N)$. Then the identity*

$$N \int_\Omega F(u)dx - \left(\frac{N-q}{q}\right) \int_\Omega H^q(\nabla u)dx$$

$$= \int_{\partial\Omega} F(u)(x-y) \cdot \nu(x)ds_x$$

$$- \frac{1}{q} \int_{\partial\Omega} H^q(\nabla u)(x-y) \cdot \nu(x)ds_x$$

$$+ \int_{\partial\Omega} \left(H^{q-1}(\nabla u)(\nabla_\xi H)(\nabla u) \cdot \nu(x)\right)((x-y) \cdot \nu(x))ds_x$$

14 *holds true for any $y \in \mathbb{R}^N$. Here ν is the outer unit normal of $\partial\Omega$ and*
15 $F(s) = \int_0^s f(t)dt$.

Proof. Indeed, since $\mathcal{L}(x, s, \xi) = \frac{1}{q}H^q(\xi) - F(s)$ is of the "splitting"
form, $F \in C^1(\mathbb{R})$, and $\xi \mapsto H^q(\xi)$ is convex and in $C^1(\mathbb{R}^N)$, Lemma 5,
thus the equation (3) in [11] holds as it is. Also, if we do not impose

the boundary condition $u = 0$ on $\partial\Omega$ (and put $f = 0$ there) in Lemma 2 in [11], we obtain the identity

$$\int_{\partial\Omega} \mathcal{L}(x, u, \nabla u)(h \cdot \nu)ds_x - \sum_{i,j=1}^{N} \int_{\partial\Omega} h_j D_{\xi_i}\mathcal{L}(x, u, \nabla u)D_{x_j}u\nu_i ds_x$$

$$= \int_{\Omega} (\mathrm{div} h)\mathcal{L}(x, u, \nabla u)dx - \sum_{i,j=1}^{N} \int_{\partial\Omega} D_i h_j D_{\xi_i}\mathcal{L}(x, u, \nabla u)D_{x_j}u dx$$

for every $h \in C^1(\overline{\Omega}, \mathbb{R}^N)$. Inserting $h(x) = x$ leads to the claim.   □

Finally, we prove the following simple lemma.

Lemma 2.7. *Let $u \in W_0^{1,N}(\Omega)$ be a weak solution to $-Q_N u = f(u)$ in $\Omega \subset \mathbb{R}^N$, where $f : \mathbb{R} \to \mathbb{R}$ is continuous. Let $a, c > 0$, $d \in \mathbb{R}$ and $b \in \mathbb{R}^N$. Then $v(x) = cu(ax + b) + d$, $x \in \Omega_{a,b} = \frac{\Omega - b}{a}$ is a weak solution to*

$$-Q_N v = a^N c^{N-1} f\left(\frac{v - d}{c}\right) \quad \text{in } \Omega_{a,b}, \quad v = 0 \quad \text{on } \partial\Omega_{a,b}.$$

Proof. For $x \in \Omega_{a,b}$, put $y = ax + b \in \Omega$. Then for any $\phi \in C_0^\infty(\Omega_{a,b})$, $\tilde{\phi}(y) = \phi(x)$ belongs to $C_0^\infty(\Omega)$. Therefore we have

$$\int_{\Omega_{a,b}} H^{N-1}(\nabla v(x))(\nabla_\xi H)(\nabla v(x)) \cdot \nabla\phi(x)dx$$

$$= \int_{\Omega_{a,b}} H^{N-1}(ca(\nabla u)(ax + b))(\nabla_\xi H)(ca(\nabla u)(ax + b)) \cdot \nabla\phi(x)dx$$

$$= \int_{\Omega} c^{N-1}a^{N-1}H^{N-1}(\nabla u(y))(\nabla_\xi H)((\nabla u(y)) \cdot a\nabla\tilde{\phi}(y)a^{-N}dy$$

$$= c^{N-1} \int_{\Omega} H^{N-1}(\nabla u(y))(\nabla_\xi H)(\nabla u(y)) \cdot \nabla\tilde{\phi}(y)dy$$

$$= c^{N-1} \int_{\Omega} f(u(y))\tilde{\phi}(y)dy$$

$$= c^{N-1} \int_{\Omega_{a,b}} f\left(\frac{v(x) - d}{c}\right)\phi(x)a^N dx,$$

where we have used (2.1) and Proposition 2.1 (3). Thus we see

$$\int_{\Omega_{a,b}} H^{N-1}(\nabla v(x))(\nabla_\xi H)(\nabla v(x)) \cdot \nabla\phi(x)dx = a^N c^{N-1} \int_{\Omega_{a,b}} f\left(\frac{v(x) - d}{c}\right)\phi(x)dx.$$

This holds true for any $\phi \in C_0^\infty(\Omega_{a,b})$, which implies Lemma.   □

1 ### 3. Asymptotic estimate for C_p

2 In this section, first by using the Finsler Trudinger-Moser inequality
3 Theorem 2.2, we establish the refined Sobolev embedding.

4 **Lemma 3.1.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For any $t \geq 2$, there*
5 *exists $D_t > 0$ such that for any $u \in W_0^{1,N}(\Omega)$,*

$$\|u\|_{L^t(\Omega)} \leq D_t t^{\frac{N-1}{N}} \|H(\nabla u)\|_{L^N(\Omega)}$$

6 *holds true. Furthermore, we have*

$$\lim_{t \to \infty} D_t = \left(\frac{1}{N\kappa_N^{1/N}}\right) \left(\frac{N-1}{Ne}\right)^{\frac{N-1}{N}}.$$

Proof. Let $u \in W_0^{1,N}(\Omega)$. By the elementary inequality $\frac{x^s}{\Gamma(s+1)} \leq e^x$ for
$x \geq 0$ and $s \geq 0$, where $\Gamma(s)$ is the Gamma function, and the Finsler
Trudinger-Moser inequality, we have

$$\frac{1}{\Gamma(\frac{N-1}{N}t + 1)} \int_\Omega |u|^t dx$$

$$= \frac{1}{\Gamma(\frac{N-1}{N}t + 1)} \int_\Omega \left(\beta_N \left(\frac{|u|}{\|H(\nabla u)\|_{L^N(\Omega)}}\right)^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}t} dx \beta_N^{-\frac{N-1}{N}t} \|H(\nabla u)\|_{L^N(\Omega)}^t$$

$$\leq \int_\Omega \exp\left(\left(\beta_N \frac{|u(x)|}{\|\nabla u\|_{L^N(\Omega)}}\right)^{\frac{N}{N-1}}\right) dx \beta_N^{-\frac{N-1}{N}t} \|H(\nabla u)\|_{L^N(\Omega)}^t$$

$$\leq C|\Omega| \beta_N^{-\frac{N-1}{N}t} \|H(\nabla u)\|_{L^N(\Omega)}^t.$$

7 Put

$$D_t = \Gamma\left(\frac{N-1}{N}t + 1\right)^{1/t} C^{1/t} |\Omega|^{1/t} \beta_N^{-\frac{N-1}{N}} t^{-\frac{N-1}{N}}.$$

8 Then we have

$$\|u\|_{L^t(\Omega)} \leq D_t t^{\frac{N-1}{N}} \|H(\nabla u)\|_{L^N(\Omega)}.$$

9 Stirling's formula implies that

$$\left(\Gamma\left(\frac{(N-1)t}{N} + 1\right)\right)^{\frac{1}{t}} \sim \left(\frac{N-1}{Ne}\right)^{\frac{N-1}{N}} t^{\frac{N-1}{N}}$$

10 as $t \to \infty$. So we have

$$\lim_{t \to \infty} D_t = \beta_N^{-\frac{N-1}{N}} \left(\frac{N-1}{Ne}\right)^{\frac{N-1}{N}} = \left(\frac{1}{N\kappa_N^{1/N}}\right) \left(\frac{N-1}{Ne}\right)^{\frac{N-1}{N}},$$

11 which is a desired result. $\square$

Recall that C_p is defined in (1.2). Using the above Lemma and energy comparison, we get the following.

Proposition 3.2. *We have*

$$\lim_{p\to\infty} p^{N-1}C_p = \left(\frac{Ne}{N-1}\beta_N\right)^{N-1}.$$

where $\beta_N = N(N\kappa_N)^{\frac{1}{N-1}}$.

Proof. Lower bound $\liminf_{p\to\infty}(p+1)^{N-1}C_p \geq \left(\frac{Ne}{N-1}\beta_N\right)^{N-1}$ is a direct consequence of Lemma 3.1 and the fact

$$(3.1) \qquad\qquad C_p = \frac{\|H(\nabla u_p)\|_{L^N(\Omega)}^N}{\|u_p\|_{L^{p+1}(\Omega)}^N}$$

for least energy solutions u_p.

Therefore we must prove only the upper bound. We will do this by constructing a suitable test function for the value C_p.

We may assume that $0 \in \Omega$ and $\mathcal{W}_L \subset \Omega$ where $\mathcal{W}_L = \{x \in \mathbb{R}^N : H^0(x) < L\}$. For $0 < l < L$, consider the Finsler Moser function

$$m_l(x) = \frac{1}{(N\kappa_N)^{1/N}}\begin{cases} \left(\log\frac{L}{l}\right)^{\frac{N-1}{N}}, & 0 \leq H^0(x) \leq l, \\[2mm] \frac{\log\frac{L}{H^0(x)}}{(\log\frac{L}{l})^{\frac{1}{N}}}, & l \leq H^0(x) \leq L, \\[2mm] 0, & L \leq H^0(x). \end{cases}$$

We check that the Moser function $m_l \in W_0^{1,N}(\Omega)$ and $\|H(\nabla m_l)\|_{L^N(\Omega)} = 1$. Also it is easily checked that

$$\left(\int_\Omega m_l^{p+1}dx\right)^{\frac{1}{p+1}} \geq \left(\int_{\mathcal{W}_l} m_l^{p+1}dx\right)^{\frac{1}{p+1}} \geq \frac{1}{(N\kappa_N)^{1/N}}\left(\log\frac{L}{l}\right)^{\frac{N-1}{N}}\left(l^N\kappa_N\right)^{\frac{1}{p+1}}.$$

Choosing $l = L\exp\left(-(\frac{N-1}{N^2})(p+1)\right)$, we have

$$\|m_l\|_{L^{p+1}(\Omega)} \geq \frac{1}{(N\kappa_N)^{1/N}}\left(\frac{N-1}{N^2}\right)^{\frac{N-1}{N}}e^{-\frac{N-1}{N}}(p+1)^{\frac{N-1}{N}}\left(L^N\kappa_N\right)^{\frac{1}{p+1}}.$$

and

$$C_p \leq \frac{\|H(\nabla m_l)\|_{L^N(\Omega)}^N}{\|m_l\|_{L^{p+1}(\Omega)}^N} \leq N\kappa_N\left(\frac{N^2e}{N-1}\right)^{N-1}(p+1)^{-(N-1)}\left(L^N\kappa_N\right)^{-\frac{N}{p+1}},$$

which implies $\limsup_{p\to\infty}(p+1)^{N-1}C_p \leq \left(\frac{Ne}{N-1}\beta_N\right)^{N-1}$. □

Since
$$\int_\Omega H(\nabla u_p)^N dx = \int_\Omega u_p^{p+1} dx$$

and (3.1), we have the following lemma.

Lemma 3.3.
$$\lim_{p\to\infty} p^{N-1} \int_\Omega H(\nabla u_p)^N dx = \lim_{p\to\infty} p^{N-1} \int_\Omega u_p^{p+1} dx = \left(\frac{Ne}{N-1}\beta_N\right)^{N-1}.$$

4. Proof of Theorem 1.1

To obtain a lower bound for $\|u_p\|_{L^\infty(\Omega)}$, define the first eigenvalue of the Finsler N-Laplacian Q_N:

$$\lambda_1(\Omega) = \inf\left\{\int_\Omega H(\nabla u)^N dx : u \in W_0^{1,N}(\Omega), \int_\Omega |u|^N dx = 1\right\}.$$

It is known that $0 < \lambda_1(\Omega) < \infty$ and
$$\int_\Omega u_p^{p+1} dx = \int_\Omega H(\nabla u_p)^N dx \geq \lambda_1(\Omega) \int_\Omega u_p^N dx.$$

Thus
$$\int_\Omega (u_p^{p+1} - \lambda_1(\Omega)u_p^N) dx \geq 0,$$

which implies

(4.1)
$$\|u_p\|_{L^\infty(\Omega)}^{p+1-N} \geq \lambda_1(\Omega).$$

To obtain a uniform upper bound of $\|u_p\|_{L^\infty(\Omega)}$, we use an argument with the coarea formula and the Finsler isoperimetric inequality in $\mathbb{R}^N$. Set

$$\gamma_p = \max_{x\in\Omega} u_p(x),$$
$$\Omega_t = \{x \in \Omega : u_p(x) > t\},$$
$$\mathcal{A} = \{x \in \Omega : u_p(x) > \frac{\gamma_p}{2}\}.$$

By Lemma 3.1 with $t = \frac{Np}{N-1}$ and by Lemma 3.3, we have

$$\left(\int_\Omega u_p^{\frac{Np}{N-1}} dx\right)^{\frac{N-1}{Np}} \leq D_{\frac{Np}{N-1}} \left(\frac{Np}{N-1}\right)^{\frac{N-1}{N}} \|H(\nabla u_p)\|_{L^N(\Omega)} \leq M$$

where M is independent of p if p large. From this and Chebyshev's inequality, we have

(4.2)
$$\left(\frac{\gamma_p}{2}\right)^{\frac{Np}{N-1}} |\mathcal{A}| \leq M^{\frac{Np}{N-1}}.$$

On the other hand, by approximating the constant 1 on Ω_t by C_0^∞-functions, we have

$$-\int_{\Omega_t} \operatorname{div}\left(H(\nabla u_p)^{N-1}(\nabla_\xi H)(\nabla u_p) \right) dx = \int_{\Omega_t} u_p^p dx.$$

Thus integration by parts leads to

$$\begin{aligned}
\int_{\Omega_t} u_p^p dx &= -\int_{\partial\Omega_t} H(\nabla u_p)^{N-1}(\nabla_\xi H)(\nabla u_p) \cdot \nu\, ds \\
(4.3) \qquad &= \int_{\partial\Omega_t} \frac{H(\nabla u_p)^{N-1}(\nabla_\xi H)(\nabla u_p) \cdot \nabla u_p}{|\nabla u_p|} ds \\
&= \int_{\partial\Omega_t} \frac{H(\nabla u_p)^N}{|\nabla u_p|} ds,
\end{aligned}$$

since the outer unit normal ν to $\partial\Omega_t$ is $\nu = -\frac{\nabla u_p}{|\nabla u_p|}$. Here we used Proposition 2.1 (3) in the last equality. Coarea formula implies

$$|\Omega_t| = \int_{\Omega_t} 1\, dx = \int_t^\infty \int_{\{u_p = s\}} \frac{ds}{|\nabla u_p|}.$$

Thus

$$(4.4) \qquad -\frac{d}{dt}|\Omega_t| = \int_{\partial\Omega_t} \frac{ds}{|\nabla u_p|}.$$

By (4.3), (4.4), and the Schwartz inequality, we have

$$\begin{aligned}
\left(-\frac{d}{dt}|\Omega_t| \right)^{N-1} \int_{\Omega_t} u_p^p dx &= \left(\int_{\partial\Omega_t} \frac{1}{|\nabla u_p|} ds \right)^{N-1} \left(\int_{\partial\Omega_t} \frac{H^N(\nabla u_p)}{|\nabla u_p|} ds \right) \\
(4.5) \qquad &\geq \left(\int_{\partial\Omega_t} \frac{H(\nabla u_p)}{|\nabla u_p|} ds \right)^N \\
&= \left(\int_{\partial\Omega_t} H(\nu) ds \right)^N \\
&= P_H(\Omega_t, \mathbb{R}^N)^N \geq N^N \kappa_N |\Omega_t|^{N-1}.
\end{aligned}$$

In the last inequality of (4.5), we used the Finsler isoperimetric inequality in $\mathbb{R}^N$ [3], [27], [18]:

$$(4.6) \qquad P_H(E, \mathbb{R}^N) \geq N\kappa_N^{\frac{1}{N}} |E|^{\frac{N-1}{N}}$$

for any set of finite perimeter $E \subset \mathbb{R}^N$ with respect to H.

Now, define $r(t) > 0$ such that

$$|\Omega_t| = \kappa_N r^N(t).$$

Then

$$\frac{d}{dt}|\Omega_t| = N\kappa_N r^{N-1}(t)r'(t).$$

Note that $r'(t) < 0$. Putting this in (4.5), we have

$$\left(-N\kappa_N r^{N-1}(t)\frac{dr}{dt}(t)\right)^{N-1}\int_{\Omega_t} u_p^p dx \geq N^N \kappa_N |\Omega_t|^{N-1},$$

$$\left(-\frac{dr}{dt}\right)^{N-1}\int_{\Omega_t} u_p^p dx \geq (N\kappa_N)r^{N-1},$$

$$-\frac{dt}{dr} \leq \left(\int_{\Omega_t} u_p^p dx\right)^{\frac{1}{N-1}}(N\kappa_N)^{-\frac{1}{N-1}}r^{-1}$$

$$\leq Cr^{-1}\gamma_p^{\frac{p}{N-1}}|\Omega_t|^{\frac{1}{N-1}} = C\gamma_p^{\frac{p}{N-1}}r^{\frac{1}{N-1}},$$

where C is a constant dependent only on N and varies from line to line. Integrating the last inequality from $r = 0$ to $r = r_0$, we have

$$t(0) - t(r_0) \leq C\gamma_p^{\frac{p}{N-1}}r_0^{\frac{N}{N-1}}.$$

Choose r_0 such that $t(r_0) = \frac{\gamma_p}{2}$. Then the above inequality implies

$$\gamma_p \leq C\gamma_p^{\frac{p}{N-1}}r_0^{\frac{N}{N-1}}, \quad \text{i.e.,} \quad \gamma_p \leq C\gamma_p^{\frac{p}{N-1}}|\mathcal{A}|^{\frac{1}{N-1}}.$$

Combining this with (4.2), we have

$$\gamma_p \leq C\gamma_p^{\frac{p}{N-1}}\left(\frac{M^{\frac{Np}{N-1}}}{\left(\frac{\gamma_p}{2}\right)^{\frac{Np}{N-1}}}\right)^{\frac{1}{N-1}} = C\gamma_p^{-\frac{p}{(N-1)^2}}M^{\frac{Np}{(N-1)^2}},$$

$$\gamma_p^{1+\frac{p}{(N-1)^2}} \leq CM^{\frac{Np}{(N-1)^2}},$$

$$\gamma_p \leq C^{\frac{(N-1)^2}{(N-1)^2+p}}M^{\frac{Np}{(N-1)^2+p}}.$$

From this, we conclude that there exists $C > 0$ (independent of p) such that $\gamma_p \leq C$ for p large.

The latter half part of Theorem 1.1 is already proven in Lemma 3.3. Thus we have completed the proof of Theorem 1.1. $\square$

From Theorem 1.1, we have the following consequence.

Corollary 4.1. *There exist $C, C' > 0$ independent of p large such that*

$$C \leq p^{N-1}\int_\Omega u_p^p dx \leq C'$$

holds true.

Proof. By Theorem 1.1, we have

$$\frac{1}{C_2}p^{N-1}\int_\Omega u_p^{p+1}dx \le \frac{\|u_p\|_{L^\infty(\Omega)}}{C_2}p^{N-1}\int_\Omega u_p^p dx \le p^{N-1}\int_\Omega u_p^p dx$$

where C_2 is as in Theorem 1.1. The left-hand side of the above inequality is bounded from below by a positive constant by Lemma 3.3. On the other hand, Hölder's inequality implies

$$p^{N-1}\int_\Omega u_p^p dx \le \left(p^{N-1}\int_\Omega u_p^{p+1}dx\right)^{\frac{p}{p+1}} p^{\frac{1}{p+1}}|\Omega|^{\frac{1}{p+1}}$$

and the right-hand side of the above inequality is bounded from above by Lemma 3.3. This proves the conclusion. $\square$

5. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2. Since $\limsup_{p\to\infty}\|u_p\|_{L^\infty(\Omega)} \ge 1$ immediately follows from (4.1) (this is true for any solution sequence, not necessary least energy solutions), we just need to prove $\limsup_{p\to\infty}\|u_p\|_{L^\infty(\Omega)} \le e^{\frac{N-1}{N}}$. For this purpose, we follow the argument by Adimurthi and Grossi [1].

Let $x_p \in \Omega$ be a point so that the least energy solution to (1.1) takes its maximum: $u_p(x_p) = \|u_p\|_{L^\infty(\Omega)}$. As in [1], We make a change of variable

$$(5.1) \quad z_p(x) = \frac{p}{u_p(x_p)}\left(u_p(\varepsilon_p x + x_p) - u_p(x_p)\right), \quad x \in \Omega_p = \frac{\Omega - x_p}{\varepsilon_p},$$

where $\varepsilon_p > 0$ is defined so that

$$(5.2) \qquad\qquad \varepsilon_p^N p^{N-1} u_p(x_p)^{p+1-N} \equiv 1.$$

By Theorem 1.1, we see $\varepsilon_p \to 0$ as $p \to \infty$. Since u_p is a weak solution to (1.1), z_p is a weak solution to

$$(5.3) \qquad \begin{cases} -Q_N z_p = \left(1 + \frac{z_p}{p}\right)^p & \text{in } \Omega_p, \\ z_p|_{\partial\Omega_p} = -p, \\ \max_{x\in\overline\Omega_n} z_n(x) = z_n(0) = 0, \\ -p < z_p \le 0 & \text{in } \Omega_p \end{cases}$$

by Lemma 2.7. We want to pass to the limit as $p \to \infty$ in (5.3). For this purpose, take any ball $B_R(0) \subset \Omega_p$ centered at the origin and radius R. Consider

$$\begin{cases} -Q_N w_p = \left(1 + \frac{z_p}{p}\right)^p & \text{in } B_R(0), \\ w_p|_{\partial B_R(0)} = 0. \end{cases}$$

1 Comparison principle for $-Q_N$ (see for example, [33] Theorem 3.1) and
2 Serrin's elliptic estimate Theorem 2.4 yield that $0 \le w_p \le C$ on $B_R(0)$
3 where C is a constant independent of p. Set $\psi_p(x) = w_p(x) - z_p(x), x \in$
4 $B_R(0)$. Then ψ_p is a nonnegative in $B_R(0)$ and $\psi_p(0) = w_p(0) - z_p(0) =$
5 $w_p(0) \le C$ uniformly in p. Moreover, we have

$$0 = -(Q_N w_p - Q_N z_p) = -\tilde{Q}_N(w_p - z_p) = -\tilde{Q}_N \psi_p$$

where

$$\tilde{Q}_N(w_p - z_p)$$
$$= \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left[\int_0^1 \frac{1}{N}\frac{\partial^2 H^N}{\partial \xi_i \partial \xi_j}(t\nabla w_p + (1-t)\nabla z_p)dt \frac{\partial}{\partial x_j}(w_p(x) - z_p(x))\right].$$

6 Thanks to the assumption that $\mathrm{Hess}H^N(\xi)$ is positive definite, $\tilde{Q}_N$ is
7 a quasilinear elliptic differential operator. Thus we can apply Serrin's
8 Harnack inequality (Theorem 2.4 (3)) to ψ_p, which implies that there
9 exists $C = C(R, r) > 0$ for any $0 < r < R$ such that

$$\sup_{B_r(0)} \psi_p(x) \le C\left(1 + \inf_{x \in B_r(0)} \psi_p(x)\right) \le C(1+\psi_p(0)) = C(1+w_p(0)) \le C.$$

10 Thus we have

$$0 \ge z_p(x) = w_p(x) - \psi_p(x) \ge -C$$

11 for $x \in B_r(0)$. Since $0 < r < R$ is arbitrary, we have $\{|z_p|\} \subset$
12 $L^\infty_{loc}(B_R(0)))$ is uniformly bounded in p. Again Serrin's regularity es-
13 timate implies that $\{z_p\}$ is bounded in $C^{1,\alpha}_{loc}(B_R(0)))$ for any $R > 0$
14 uniformly in p.
15   Now, we consider two cases:
16   **Case (i)**: $\frac{\mathrm{dist}(x_p, \partial\Omega_p)}{\varepsilon_p} \to +\infty$
17   **Case (ii)**: $\frac{\mathrm{dist}(x_p, \partial\Omega_p)}{\varepsilon_p}$ is bounded and

$$\Omega_p \to \mathbb{R}^N_+(s_0) = \{x = (x', x_N) \in \mathbb{R}^N : x_N > s_0\} \quad (p \to \infty)$$

18 for some s_0.
19   In the case (i), note that $\Omega_p \to \mathbb{R}^N$ as $p \to \infty$. Hence by the Ascoli-
20 Arzelá theorem, we know that (up to a subsequence), $\{z_p\}$ converges
21 to some function $z \in C^1(\mathbb{R}^N)$ and z satisfies

$$-Q_N z = e^z \quad \text{in } \mathbb{R}^N.$$

22   Now we claim that $\int_{\mathbb{R}^N} e^z dx < +\infty$. In fact, since $z_p \to z$ in
23 $C^1_{loc}(\mathbb{R}^N)$, we obtain

$$1_{\Omega_p}(x)\left(1 + \frac{z_p(x)}{p}\right)^p \to e^{z(x)}$$

pointwisely for $x \in \mathbb{R}^N$, where 1_{Ω_p} is the characteristic function of Ω_p. By using Fatou's lemma and Hölder's inequality, we deduce

$$
\begin{aligned}
\int_{\mathbb{R}^N} e^z dx &\leq \liminf_{p \to \infty} \int_{\Omega_p} \left(1 + \frac{z_p(x)}{p} \right)^p dx \\
&\leq \lim_{p \to \infty} \frac{p^{N-1}}{(u_p(x_p))^{N-1}} \int_\Omega (u_p(y))^p dy \\
&\leq \lim_{p \to \infty} \frac{p^{N-1}}{(u_p(x_p))^{N-1}} \left(\int_\Omega (u_p(y))^{p+1} dy \right)^{p/(p+1)} |\Omega|^{1/(p+1)} \\
&\leq C < \infty
\end{aligned}
$$

where we have used the facts that $\int_\Omega u_p^{p+1} dy = \frac{O(1)}{p^{N-1}}$ by Lemma 3.3 and $u_p(x_p) \geq C_1 > 0$ by Theorem 1.1. Hence, we check that the limit function satisfies

(5.4)
$$
\begin{cases}
-Q_N z = e^z & \text{in } \mathbb{R}^N, \\
z \leq 0, & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} e^z dx < \infty.
\end{cases}
$$

In the case (ii), almost the same proof works, and we see that the limit function z is a solution of

(5.5)
$$
\begin{cases}
-Q_N z = e^z & \text{in } \mathbb{R}^N_+(s_0), \\
z \leq 0, & \text{in } \mathbb{R}^N_+(s_0), \\
z = -\infty, & \text{on } \partial \mathbb{R}^N_+(s_0), \\
\int_{\mathbb{R}^N_+(s_0)} e^z dx < \infty.
\end{cases}
$$

Now we prove the following lemma. The case $N = 2$ was treated by Ding (see [7]) when $H(\xi) = |\xi|$, and by Wang and Xia [32] for general $H(\xi)$.

Lemma 5.1. *If z is a C^1 weak solution of (5.4), then we have*

$$
\int_{\mathbb{R}^N} e^z dx \geq \left(\frac{N}{N-1} \right)^{N-1} N^N \kappa_N.
$$

If z is a C^1 weak solution of (5.5), then we have

$$
\int_{\mathbb{R}^N_+(s_0)} e^z dx \geq \left(\frac{N}{N-1} \right)^{N-1} N^N \kappa_N.
$$

Proof. As in the proof of Theorem 1.1, we use a level set argument. First, we assume z is a solution of (5.4). Put

$$
\Omega_t = \{x \in \mathbb{R}^N \; : \; z(x) > t\}, \quad \mu(t) = |\Omega_t|.
$$

Integration by parts on Ω_t leads to

$$\int_{\Omega_t} e^z dx = -\int_{\Omega_t} Q_N z dx = \int_{\partial\Omega_t} H^{N-1}(\nabla z)(\nabla_\xi H)(\nabla z) \cdot \frac{\nabla z}{|\nabla z|} ds_x$$

$$= \int_{\partial\Omega_t} \frac{H^N(\nabla z)}{|\nabla z|} ds_x.$$

By the Finsler isoperimetric inequality (4.6) and Hölder's inequality, we see

$$N\kappa_N^{1/N} |\Omega_t|^{\frac{N-1}{N}} \leq P_H(\Omega_t, \mathbb{R}^N) = \int_{\partial\Omega_t} \frac{H(\nabla z)}{|\nabla z|} ds_x$$

$$\leq \left(\int_{\partial\Omega_t} \frac{H^N(\nabla z)}{|\nabla z|} ds_x\right)^{\frac{1}{N}} \left(\int_{\partial\Omega_t} \frac{ds_x}{|\nabla z|}\right)^{\frac{N-1}{N}}$$

$$= \left(\int_{\Omega_t} e^z dx\right)^{\frac{1}{N}} (-\mu'(t))^{\frac{N-1}{N}},$$

here we have used coarea formula

$$\mu(t) = \int_t^\infty \int_{\{x:z(x)=s\}} \frac{ds_x}{|\nabla z|} ds.$$

Thus we have

$$\mu(t) \leq \left\{\frac{1}{N\kappa_N^{1/N}} \left(\int_{\Omega_t} e^z dx\right)^{\frac{1}{N}} (-\mu'(t))^{\frac{N-1}{N}}\right\}^{\frac{N}{N-1}}.$$

Therefore, we obtain

$$\int_{\mathbb{R}^N} e^z dx = \int_{-\infty}^{\max z} e^t \mu(t) dt$$

$$\leq \left(\frac{1}{N\kappa_N^{1/N}}\right)^{\frac{N}{N-1}} \int_{-\infty}^{\max z} e^t \left(\int_{\Omega_t} e^z dx\right)^{\frac{1}{N-1}} (-\mu'(t)) dt$$

$$= \left(\frac{1}{N\kappa_N^{1/N}}\right)^{\frac{N}{N-1}} \left(\frac{N-1}{N}\right) \int_{-\infty}^{\max z} \frac{d}{dt} \left(\int_{\Omega_t} e^z dx\right)^{\frac{N}{N-1}} dt$$

$$= \left(\frac{1}{N\kappa_N^{1/N}}\right)^{\frac{N}{N-1}} \left(\frac{N-1}{N}\right) \left(\int_{\mathbb{R}^N} e^z dx\right)^{\frac{N}{N-1}},$$

which implies

$$\left(\frac{N}{N-1}\right)^{N-1} N^N \kappa_N \leq \int_{\mathbb{R}^N} e^z dx.$$

The proof when z is a solution to (5.5) is similar, since the boundary condition $z = -\infty$ on $\partial\mathbb{R}_+^N(s_0)$ assures that all level sets of z are confined in $\mathbb{R}_+^N(s_0)$. $\square$

By the change of variables, we have

$$(5.6) \qquad p^{N-1}\int_\Omega u_p^{p+1}(y)dy = u_p^N(x_p)\int_{\Omega_p}\left(1 + \frac{z_p(x)}{p}\right)^{p+1}dx.$$

Let us take $\limsup_{p\to\infty}$ of both sides of (5.6). Then we see

$$\limsup_{n\to\infty}\text{LHS of }(5.6) = \left(\frac{Ne}{N-1}\right)^{N-1}N^N\kappa_N$$

by Lemma 3.3. On the other hand, Fatou's lemma and Lemma 5.1 implies

$$\limsup_{p\to\infty}\text{RHS of }(5.6) \geq \left(\limsup_{p\to\infty}u_p(x_p)\right)^N \times \begin{cases} \int_{\mathbb{R}^N}e^z dx & \text{when case (i)} \\ \int_{\mathbb{R}_+^N(s_0)}e^z dx & \text{when case (ii)} \end{cases}$$

$$\geq \left(\limsup_{p\to\infty}u_p(x_p)\right)^N\left(\frac{N}{N-1}\right)^{N-1}N^N\kappa_N.$$

Hence, we have

$$e^{N-1} \geq \left(\limsup_{p\to\infty}\|u_p\|_{L^\infty(\Omega)}\right)^N.$$

which implies Theorem 1.2 $\square$

6. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. Given any sequence p_n of p with $p_n \to \infty$, let us recall (1.3) and (1.6) for $p = p_n$, $u_n = u_{p_n}$.

$$v_n = \frac{u_n}{\lambda_n} = \frac{u_n}{\left(\int_\Omega u_n^{p_n}dx\right)^{\frac{1}{N-1}}}, \quad \lambda_n = \left(\int_\Omega u_n^{p_n}dx\right)^{\frac{1}{N-1}},$$

$$f_n(x) = \frac{u_n^{p_n}}{\int_\Omega u_n^{p_n}dx},$$

$$L_0 = \limsup_{n\to\infty}\frac{p_n\left(\int_\Omega u_n^{p_n}dx\right)^{\frac{1}{N-1}}}{\left(\frac{N}{N-1}e^{\frac{N-1}{N}}\right)}, \quad L_1 = d_N^{-\left(\frac{1}{N-1}\right)}L_0.$$

Then v_n is a weak solution of (1.4) for $p = p_n$. By Hölder's inequality and Theorem 1.1, we see

$$p_n^{N-1}\int_\Omega u_n^{p_n}dx \leq p_n^{N-1}\left(\int_\Omega u_n^{p_n+1}dx\right)^{\frac{p_n}{p_n+1}}|\Omega|^{\frac{1}{p_n+1}} \to \left(\frac{Ne}{N-1}\beta_N\right)^{N-1}$$

as $n \to \infty$. This shows that

$$L_0 \leq e^{\frac{1}{N}} \beta_N, \quad L_1 \leq e^{\frac{1}{N}} \beta_N d_N^{-(\frac{1}{N-1})}.$$

First, we prove $S \neq \phi$ for any sequence $v_n = v_{p_n}$ of v_p with $p_n \to \infty$. Indeed, by Theorem 1.1, we have $\|u_n\|_{L^\infty(\Omega)} \geq C_1 > 0$ for any $n \in \mathbb{N}$. Let $x_n \in \Omega$ be a point such that $u_n(x_n) = \|u_n\|_{L^\infty(\Omega)}$, then

$$v_n(x_n) = \frac{u_n(x_n)}{(\int_\Omega u_n^{p_n} dx)^{\frac{1}{N-1}}} \geq \frac{C_1}{(\int_\Omega u_n^{p_n} dx)^{\frac{1}{N-1}}} = \frac{C_1}{O(\frac{1}{p_n})} \to +\infty$$

by Lemma 3.3. This implies that any accumulation point of $\{x_n\}$ is contained in S and hence $S \neq \phi$.

Next, as in [6], [21], [22], we define (L, δ)-*regular set* and (L, δ)-*irregular set* of a sequence $\{u_n\}$. Since

$$f_n = \frac{u_n^{p_n}}{\int_\Omega u_n^{p_n} dx} \in L^1(\Omega), \quad f_n \geq 0, \quad \int_\Omega f_n dx = 1,$$

there exists a subsequence (still denoted by n) such that

$$f_n \overset{*}{\rightharpoonup} \mu, \quad \mu(\Omega) \leq 1$$

in the sense of Radon measures of Ω, where μ is a nonnegative Radon measure.

Given $L > 0$ and $\delta > 0$, we call a point $x_0 \in \Omega$ a (L, δ)-*regular point of* $\{u_n\}$ if there exists $\varphi \in C_0(\Omega), 0 \leq \varphi \leq 1$ with $\varphi \equiv 1$ near x_0 such that

$$\int_\Omega \varphi d\mu < \left(\frac{\beta_N}{L + 3\delta}\right)^{N-1}$$

where $\beta_N = N(N\kappa_N)^{\frac{1}{N-1}}$ is as in Theorem 2.2. We put

$$R_L(\delta) = \{x_0 \in \Omega : x_0 \text{ is a } (L, \delta)\text{-regular point}\},$$
$$\Sigma_L(\delta) = \Omega \setminus R_L(\delta).$$

We call a point in $\Sigma_L(\delta)$ an (L, δ)-*irregular point* of the sequence $\{u_n\}$. Note that (L, δ)-regular, or (L, δ)-irregular points are automatically interior points of Ω. Also note that if $x_0 \in \Sigma_L(\delta)$, then we have

(6.1)
$$\mu(\{x_0\}) \geq \left(\frac{\beta_N}{L + 3\delta}\right)^{N-1}.$$

Since

$$1 \geq \mu(\Omega) \geq \left(\frac{\beta_N}{L + 3\delta}\right)^{N-1} \sharp \Sigma_L(\delta)$$

by (6.1), we see that $\Sigma_L(\delta)$ is a finite set for any $L > 0$ and $\delta > 0$.

Next Lemma is the key to analyze the interior blow-up set $S \cap \Omega$.

Lemma 6.1. (smallness of μ implies boundedness) *Let x_0 be a (L_1, δ)-regular point of a sequence $\{u_n\}$ where L_1 is defined in (1.6). Then $\{v_n\}$ is bounded in $L^\infty(B_{R_0}(x_0))$ for some $R_0 > 0$.*

Proof. Key point in the proof is to get the following pointwise estimate

$$(6.2) \qquad f_n(x) < \exp\left((L_1 + \delta/2) d_N^{\frac{1}{N-1}} v_n(x) \right), \quad x \in \Omega.$$

In checking (6.2), we use the elementary inequality

$$(6.3) \qquad \frac{\log x}{x} \leq \frac{\log y}{y} \quad \text{for any } 0 < x \leq y \leq e.$$

Let

$$\alpha_n = \frac{\|u_n\|_{L^\infty(\Omega)}}{\left(\int_\Omega u_n^{p_n} dx\right)^{\frac{1}{p_n}}} = \frac{\|u_n\|_{L^\infty(\Omega)}}{\lambda_n^{\frac{N-1}{p_n}}},$$

and recall that $\lambda_n = O\left(\frac{1}{p_n}\right)$ by Corollary 4.1, so $\lambda_n^{\frac{N-1}{p_n}} = O\left(\frac{1}{p_n}\right)^{\frac{N-1}{p_n}} \to 1$ as $n \to \infty$. Thus we have

$$\limsup_{n\to\infty} \alpha_n = \limsup_{n\to\infty} \|u_n\|_{L^\infty(\Omega)} \leq e^{\frac{N-1}{N}}$$

by Theorem 1.2. From this, we see that for any small $\varepsilon' > 0$,

$$\frac{u_n(x)}{\lambda_n^{\frac{N-1}{p_n}}} \leq \alpha_n \leq e^{\frac{N-1}{N}} + \varepsilon' < e$$

holds for any $x \in \Omega$ and for large n. Therefore by (6.3), we have for fixed small $\varepsilon > 0$

$$\frac{\log\left(\frac{u_n(x)}{\lambda_n^{\frac{N-1}{p_n}}}\right)}{\frac{u_n(x)}{\lambda_n^{\frac{N-1}{p_n}}}} \leq \frac{\log \alpha_n}{\alpha_n} \leq \left(\frac{N-1}{N}\right) \frac{1}{e^{\frac{N-1}{N}}} + \varepsilon$$

for large n. Hence

$$\begin{aligned}
\log f_n(x) = p_n \log \frac{u_n(x)}{\lambda_n^{\frac{N-1}{p_n}}} &\leq p_n \left(\frac{u_n(x)}{\lambda_n^{\frac{N-1}{p_n}}}\right)\left(\frac{N-1}{N}\frac{1}{e^{\frac{N-1}{N}}} + \varepsilon\right) \\
&= p_n \lambda_n \left(\frac{N-1}{N} e^{-\frac{N-1}{N}} + \varepsilon\right) \frac{v_n(x)}{\lambda_n^{\frac{N-1}{p_n}}} \\
&\leq \left(\frac{N}{N-1} e^{\frac{N-1}{N}} L_1 d_N^{\frac{1}{N-1}} + \varepsilon\right)\left(\frac{N-1}{N} e^{-\frac{N-1}{N}} + 2\varepsilon\right) v_n(x),
\end{aligned}$$

here we have used $\lim_{n\to\infty} \lambda_n^{\frac{N-1}{p_n}} = 1$ and

$$p_n\lambda_n \leq \frac{N}{N-1}e^{\frac{N-1}{N}}L_1 d_N^{\frac{1}{N-1}} + \varepsilon$$

for large n by the definition of L_1. Therefore

$$\log f_n(x) \leq \left((L_1 + \delta/2)d_N^{\frac{1}{N-1}} \right) v_n(x)$$

holds if we choose $\varepsilon > 0$ small enough. This proves the pointwise estimate (6.2).

Next, by the use of Brezis-Merle theory for the Finsler N-Laplacian, we obtain the integral estimate

$$(6.4) \qquad \int_{B_{R_1/2}(x_0)} \exp\left((L_1 + \delta)d_N^{\frac{1}{N-1}}v_n(x) \right) dx \leq C$$

for some $R_1 > 0$ small and $C > 0$ independent of n, here x_0 is a (L_1, δ)-regular point.

Indeed, by the definition of (L_1, δ)-regular point, we can find $R_1 > 0$ such that

$$\int_{B_{R_1}(x_0)} f_n dx \leq \left(\frac{\beta_N}{L_1 + 2\delta} \right)^{N-1}.$$

Also by Theorem 2.5 (i) and the fact that $\|f_n\|_{L^1(\Omega)} = 1$, we have

$$\int_\Omega \exp\left((\beta_N - \varepsilon)v_n(x) \right) dx \leq \frac{\beta_N}{\varepsilon}|\Omega|$$

for any $\varepsilon \in (0, \beta_N)$. From this, we obtain

$$(6.5) \qquad\qquad\qquad \|v_n\|_{L^N(\Omega)} \leq C$$

where $C > 0$ is independent of n. Next, let ϕ_n be a weak solution of

$$-Q_N\phi_n = 0 \quad \text{in } B_{R_1}(x_0) \quad \phi_n = v_n \quad \text{on } \partial B_{R_1}(x_0).$$

Then by Theorem 2.5 (2) and the fact that $\|f_n\|_{L^1(B_{R_1}(x_0))}^{\frac{1}{N-1}} < \frac{\beta_N}{L_1+2\delta}$, we have

$$(6.6) \qquad \int_\Omega \exp\left((L_1 + \delta)d_N^{\frac{1}{N-1}}|v_n(x) - \phi_n(x)| \right) dx \leq C$$

if we choose $\varepsilon \in (0, \beta_N)$ sufficiently small. By the comparison principle for the Finsler N-Laplacian (see [33] Theorem 3.2) and Serrin's estimates Theorem 2.4 (i), we have

$$\|\phi_n\|_{L^\infty(B_{R_1/2}(x_0))} \leq \|v_n\|_{L^\infty(B_{R_1/2}(x_0))} \leq C\|v_n\|_{L^N(B_{R_1}(x_0))} \leq C$$

where we have used (6.5). Combining this with (6.6), we obtain the desired integral estimate (6.4).

Comparing (6.4) and (6.2), we see that f_n is bounded uniformly in n in $L^q(B_{R_1/2}(x_0))$ where $q = \frac{L_1+\delta}{L_1+\delta/2} > 1$. Therefore, Serrin's regularity estimate Theorem 2.4 (i) again implies that

$$\|v_n\|_{L^\infty(B_{R_1/4}(x_0))} \leq C$$

independent of n. Taking $R_0 = R_1/4$ ends the proof of Lemma 6.1. $\square$

We know that $\Sigma_{L_1}(\delta)$ is a set of finite points, all of those are interior of Ω. From Lemma 6.1, we obtain $S \cap \Omega = \Sigma_{L_1}(\delta)$ for any $\delta > 0$ and

$$1 \geq \mu(\Omega) \geq \left(\frac{\beta_N}{L_1+3\delta}\right)^{N-1} \sharp(\Sigma_{L_1}(\delta)) = \left(\frac{\beta_N}{L_1+3\delta}\right)^{N-1} \sharp(S \cap \Omega).$$

Hence

$$\sharp(S \cap \Omega) \leq \left(\frac{L_1+3\delta}{\beta_N}\right)^{N-1} \leq \left(\frac{e^{\frac{1}{N}}\beta_N d_N^{-\left(\frac{1}{N-1}\right)} + 3\delta}{\beta_N}\right)^{N-1}.$$

Taking a limit $\delta \to 0$, we have

$$\sharp(S \cap \Omega) \leq e^{\frac{N-1}{N}} d_N^{-1}$$

This proves the first part of Theorem 1.3.

If $x_0 \in S \cap \Omega = \Sigma_{L_1}(\delta)$, then for any $R > 0$, we have

$$(6.7) \qquad \lim_{n\to\infty} \|v_n\|_{L^\infty(B_R(x_0))} = +\infty.$$

Indeed, if for some $R > 0$, assume there exists $C > 0$ independent of n such that $\|v_n\|_{L^\infty(B_R(x_0))} \leq C$ for all large n. Then

$$f_n = \frac{v_n^{p_n}}{\lambda_n^{N-1-p_n}} \leq C^{p_n} O\left(\frac{1}{p_n}\right)^{p_n-(N-1)} \to 0 \quad (n \to \infty)$$

uniformly on $B_R(x_0)$. This implies x_0 is a (L_1, δ)-regular point, which is absurd. The same kind of argument leads to that the limit measure μ is atomic and of the form

$$\mu = \sum_{i=1}^{k} \gamma_i \delta_{x_i}$$

where $S \cap \Omega = \{x_1, \cdots, x_k\}$. Since $\mu(\Omega) \leq 1$, we have $\sum_{i=1}^{k} \gamma_i \leq 1$ and

$$\gamma_i \geq \left(\frac{\beta_N}{L_1}\right)^{N-1}$$

for all $i = 1, \cdots, k$ by letting $\delta \to 0$ in (6.1) with $L = L_1$. This proves Theorem 1.3 (i).

On any compact sets in $\Omega \setminus (S \cap \Omega)$, $\{v_n\}$ is uniformly bounded. Then by Serrin's and Tolksdorf's regularity estimate, $\{v_n\}$ is also bounded in $C_{loc}^{1,\alpha}(\Omega \setminus (S \cap \Omega))$ for some $\alpha \in (0,1)$. By Ascoli-Arzelá theorem, we have a subsequence and a function G such that $v_n \to G$ in $C_{loc}^1(\Omega \setminus (S \cap \Omega))$. That this G satisfies Theorem 1.3 (ii) is clear.

Finally, since $\lambda_n = O(\frac{1}{p_n})$ as $n \to \infty$ and $v_n(x) = \frac{u_n(x)}{\lambda_n}$ is uniformly bounded in $L_{loc}^\infty(\Omega \setminus (S \cap \Omega))$, we easily see that Theorem 1.3 (iii) holds.

Thus all the proof of Theorem 1.3 has been completed.

7. Proof of Theorem 1.4

In this section, we prove Theorem 1.4.

Proof. Assume the contrary that $x_0 \in \partial\Omega$, where x_0 is the unique blow-up point of a sequence $v_n = v_{p_n}$ with $p_n \to +\infty$ as $n \to \infty$. For $R > 0$ small, we may use the Pohozaev identity Theorem 2.6 on $\Omega \cap B_R(x_0)$, with the aid of Theorem 2.4:

(7.1)
$$\frac{N}{p_n+1} \int_{\Omega \cap B_R(x_0)} u_n^{p_n+1} dx = \int_{\partial(\Omega \cap B_R(x_0))} \frac{u_n^{p_n+1}}{p_n+1} (x-y) \cdot \nu(x) ds_x$$
$$- \frac{1}{N} \int_{\partial(\Omega \cap B_R(x_0))} H^N(\nabla u_n)(x-y) \cdot \nu(x) ds_x$$
$$+ \int_{\partial(\Omega \cap B_R(x_0))} \left(H^{N-1}(\nabla u_n)(\nabla_\xi H)(\nabla u_n) \cdot \nu(x) \right)(x-y) \cdot \nu(x) ds_x.$$

In order to remove the integral terms involving $\partial\Omega$, we use a trick in [25]. Define

$$\rho_n = \frac{\int_{\partial\Omega \cap B_R(x_0)} H^N(\nabla u_n)(x-x_0) \cdot \nu(x) ds_x}{\int_{\partial\Omega \cap B_R(x_0)} H^N(\nabla u_n)\nu(x_0) \cdot \nu(x) ds_x}$$

and put $y_n = x_0 + \rho_n \nu(x_0)$. We assume $R > 0$ so small such that $1/2 \le \nu(x_0) \cdot \nu(x) \le 1$ for $x \in \partial\Omega \cap B_R(x_0)$. Then we have that $\rho_n \le 2R$. By the definition of y_n and ρ_n, we see that

$$\int_{\partial\Omega \cap B_R(x_0)} H^N(\nabla u_n)(x-y_n) \cdot \nu(x) ds_x \equiv 0$$

for all $n \in \mathbb{N}$. Also since $u_n = 0$ on $\partial\Omega$ and $u_n > 0$ in Ω, we see $\nu(x) = -\frac{\nabla u_n(x)}{|\nabla u_n(x)|}$. By using these, we see (7.1) with $y = y_n$ becomes

(7.2)
$$\frac{N}{p_n + 1} \int_{\Omega \cap B_R(x_0)} u_n^{p_n+1} dx = \frac{1}{p_n + 1} \int_{\Omega \cap \partial B_R(x_0)} u_n^{p_n+1}(x - y_n) \cdot \nu(x) ds_x$$
$$- \frac{1}{N} \int_{\Omega \cap \partial B_R(x_0)} H^N(\nabla u_n)(x - y_n) \cdot \nu(x) ds_x$$
$$+ \int_{\Omega \cap \partial B_R(x_0))} \left(H^{N-1}(\nabla u_n)(\nabla_\xi H)(\nabla u_n) \cdot \nu(x) \right) (x - y_n) \cdot \nabla u_n(x) ds_x.$$

Multiplying $\left(\frac{1}{\lambda_n}\right)^N$ to both sides of (7.2) and recalling $v_n = \frac{u_n}{\lambda_n}$, we have

(7.3)
$$\frac{N}{p_n + 1} \left(\frac{1}{\lambda_n} \right)^N \int_{\Omega \cap B_R(x_0)} u_n^{p_n+1} dx$$
$$= \frac{1}{p_n + 1} \left(\frac{1}{\lambda_n} \right)^N \int_{\Omega \cap \partial B_R(x_0)} u_n^{p_n+1}(x - y_n) \cdot \nu(x) ds_x$$
$$- \frac{1}{N} \int_{\Omega \cap \partial B_R(x_0)} H^N(\nabla v_n)(x - y_n) \cdot \nu(x) ds_x$$
$$+ \int_{\Omega \cap \partial B_R(x_0))} \left(H^{N-1}(\nabla v_n)(\nabla_\xi H)(\nabla v_n) \cdot \nu(x) \right) (x - y_n) \cdot \nabla v_n(x) ds_x$$
$$= I + II + III.$$

We estimate the terms I, II, III on the right-hand side of (7.3) as follows:

$$|I| = \frac{1}{p_n + 1} \left(\frac{1}{\lambda_n} \right)^N \left| \int_{\Omega \cap \partial B_R(x_0)} u_n^{p_n+1}(x - y_n) \cdot \nu(x) ds_x \right|$$
$$\leq \frac{O(p_n^N)}{p_n^{N-1}(p_n + 1)} \| p_n^{N-1} u_n^{p_n+1} \|_{L^\infty(\Omega \cap \partial B_R(x_0))} \int_{\Omega \cap \partial B_R(x_0)} |(x - y_n) \cdot \nu(x)| ds_x$$
$$= \frac{O(p_n^N)}{p_n^{N-1}(p_n + 1)} \| p_n^{N-1} u_n^{p_n+1} \|_{L^\infty(\Omega \cap \partial B_R(x_0))} O(R^{N-1}).$$

We note that since $S \cap \Omega = \phi$ by assumption,
$$f_n = \frac{u_n^{p_n}}{\lambda_n^{N-1}} \to 0$$

uniformly on compact sets in Ω and
$$p_n^{N-1} u_n^{p_n+1}(x) \leq \|u_n\|_{L^\infty(\Omega)} p_n^{N-1} u_n^{p_n}(x) \leq C \frac{u_n^{p_n}(x)}{\lambda_n^{N-1}} \leq C f_n(x)$$

1   by Theorem 1.1 and the fact that $\lambda_n = O(\frac{1}{p_n})$ as $n \to \infty$. Thus we

2   have

$$\|p_n^{N-1} u_n^{p_n+1}\|_{L^\infty(\Omega \cap \partial B_R(x_0))} \to 0 \quad \text{as} \quad n \to \infty$$

3   and thus

$$\lim_{R \to 0} \lim_{n \to \infty} |I| = 0.$$

Also, by Theorem 1.3 (ii), we have $v_n \to G$ in $C^{1,\alpha}_{loc}(\Omega \setminus (S \cap \Omega))$. Thus we have $H^N(\nabla v_n) = O(1)$ on $\Omega \cap \partial B_R(x_0)$, which implies

$$|II| = \frac{1}{N} \left| \int_{\Omega \cap \partial B_R(x_0)} H^N(\nabla v_n)(x - y_n) \cdot \nu(x) ds_x \right|$$

$$\leq O(1) \int_{\Omega \cap \partial B_R(x_0)} |(x - y_n) \cdot \nu(x)| ds_x \leq O(1) O(R^{N-1}),$$

$$|III| = \left| \int_{\Omega \cap \partial B_R(x_0))} \left(H^{N-1}(\nabla v_n)(\nabla_\xi H)(\nabla v_n) \cdot \nu(x) \right) (x - y_n) \cdot \nabla v_n(x) ds_x \right|$$

$$\leq O(1) \int_{\Omega \cap \partial B_R(x_0)} |(x - y_n) \cdot \nu(x)| ds_x \leq O(1) O(R^{N-1}).$$

4   Therefore we have

$$\lim_{R \to 0} \lim_{n \to \infty} |II| = \lim_{R \to 0} \lim_{n \to \infty} |III| = 0.$$

5   From these, we obtain

(7.4) $$\lim_{R \to 0} \lim_{n \to \infty} \left(\text{RHS of } (7.3) \right) = 0.$$

On the other hand, recall

$$z_n(x) = \frac{p_n}{u_n(x_n)} \left(u_n(\varepsilon_n x + x_n) - u_n(x_n) \right),$$

$$x \in \Omega_{R,n} = \frac{(\Omega \cap B_R(x_0)) - x_n}{\varepsilon_n},$$

where $\varepsilon_n^N p_n^{N-1} u_n(x_n)^{p_n+1-N} \equiv 1$. Then we see from Fatou's lemma, Theorem 1.1, and Lemma 5.1, that

$$\lim_{R \to 0} \lim_{n \to \infty} \int_{\Omega \cap B_R(x_0))} p_n^{N-1} u_n^{p_n+1}(y) dy$$

$$= \lim_{R \to 0} \lim_{n \to \infty} u_n(x_n)^N \int_{\Omega_{R,n}} \left(1 + \frac{z_n(x)}{p_n} \right)^{p_n+1} dx$$

$$\geq C_1^N \int_U e^z dx \geq C_1^N \left(\frac{N}{N-1} \right)^{N-1} N^N \kappa_N$$

6   where $u = \mathbb{R}^N$ or $\mathbb{R}^N_+(s_0)$ for some $s_0 > 0$ according to the cases

7    $\frac{\text{dist}(x_n, \partial \Omega_{R,n})}{\varepsilon_n} \to +\infty$ or $\frac{\text{dist}(x_n, \partial \Omega_{R,n})}{\varepsilon_n} \to s_0$. Note that our assumption

$\sharp S = 1$ assures that we can choose x_n as a maximum points of u_n. From this and the fact that $\lambda_n = O(\frac{1}{p_n})$ as $n \to \infty$, we have

$$(7.5) \qquad \lim_{R \to 0} \lim_{n \to \infty} (\text{LHS of } (7.3)) \geq C > 0$$

for some positive constant $C > 0$ independent of n.

Clearly (7.5) contradicts to (7.4), and we conclude that $x_0 \notin \partial\Omega$. $\square$

Finally, as a corollary, we prove the following.

Corollary 7.1. *Let $R > 0$ and let $\{u_p\}$ be a sequence of least energy solutions to*

$$(7.6) \qquad \begin{cases} -Q_N u_p = u_p^p & \text{in } \mathcal{W}_R, \\ u_p > 0 & \text{in } \mathcal{W}_R, \\ u_p = 0 & \text{on } \partial\mathcal{W}_R \end{cases}$$

where $\mathcal{W}_R = \{x \in \mathbb{R}^N : H^0(x) < R\}$. Then the blow-up set S of v_p satisfies $S \cap \mathcal{W}_R = \{0\}$, and

$$u_p \to G(\cdot, 0) \quad \text{in } C^1_{loc}(\mathcal{W}_R \setminus \{0\})$$

where G is the unique Green function on $\mathcal{W}_R$ obtained in Theorem 2.3, and

$$f_p = \frac{u_p^p}{\int_{\mathcal{W}_R} u_p^p dx} \xrightarrow{*} \delta_0$$

in the sense of Radon measures on $\mathcal{W}_R$, along the full sequence.

Proof. The usual method of moving plane to prove the symmetry of solutions is not applicable in the anisotropic situation. However, we can use Theorem 4.1 in [5] under the convexity and C^1-assumption of the map $\xi \mapsto H^N(\xi)$. (Note that the key point of the proof of Theorem 4.1 in [5] is the Pohozaev identity Theorem 4.2 in [5] for $C^1(\overline{\Omega})$-weak solutions, which is valid by the above assumptions). Thus we assure that any positive solution u_p to (7.6) is *Finsler-radial*, that is, all level sets of u_p are homothetic to $\mathcal{W}_R$ for any $p > 1$. Let S be the blow-up set of v_p. Then we see that $S \cap \mathcal{W}_R = \{0\}$. Indeed, if there were a point $x_0 \in S \cap \mathcal{W}_R$, then all points on the level set of u_p passing through x_0 must be blow-up points of v_p, which contradicts to the fact that $\sharp(S \cap \mathcal{W}_R)$ is finite. Thus by Theorem 1.3, we see

$$v_p \to G(\cdot, 0) \quad \text{in } C^1_{loc}(\mathcal{W}_R \setminus \{0\})$$

for some function G along a subsequence. The limit function must be the unique Green function constructed in Theorem 2.3, and by the uniqueness, the convergence is true for the full sequence. $\square$

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