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Ken Abe

Citation	Communications in Mathematical Physics. 391(2); 873–899
Issue Date	2022-04
Published	2022-02-16
Type	Journal Article
Textversion	Author
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DOI	10.1007/s00220-022-04331-y

Abe, K. Existence of Vortex Rings in Beltrami Flows. *Communications in Mathematical Physics*. 391, 873–899 (2022). https://doi.org/10.1007/s00220-022-04331-y

EXISTENCE OF VORTEX RINGS IN BELTRAMI FLOWS

KEN ABE

Abstract. Axisymmetric vortex rings are traveling wave solutions to the 3d Euler equations, first constructed by Fraenkel for the case without swirl via the variational principle. In this paper, we consider axisymmetric vortex rings with swirl consisting of Beltrami fields with a non-constant proportionality factor. They provide first examples to C^1-traveling wave solutions, axisymmetric with swirl.

1. Introduction

We consider the 3d Euler equations

$$v_t + v\cdot\nabla v + \nabla q = 0,\quad \nabla\cdot v = 0,\quad x\in\mathbb{R}^3,\ t>0, \tag{1.1}$$

where the vector field $v(x,t)$ denotes the velocity of a fluid and the scalar field $q(x,t)$ denotes the pressure. The subscript v_t denotes the partial derivative for t and $\nabla = \nabla_x$ denotes the gradient for x. We study traveling wave solutions to (1.1) among classical and axisymmetric solutions.

Traveling waves of the Euler equations are related with translating vortex motion dating back to a pioneering work of Helmholtz [33]. They form

$$v(x,t) = u(x+u_\infty t) - u_\infty,\quad q(x,t) = p(x+u_\infty t),$$

where v vanishes at space infinity with a constant u_∞. By substituting this into the Euler equation, one sees that the profile (u,p) solves the steady equations

$$\begin{aligned}(\nabla\times u)\times u + \nabla\Pi = 0,\quad \nabla\cdot u = 0,\quad &\text{in } \mathbb{R}^3,\\ u\to u_\infty\quad &\text{as } |x|\to\infty,\end{aligned} \tag{1.2}$$

with the Bernoulli function $\Pi = p + |u|^2/2$. This equation can describe a broad class of translating vortices. Because of the 3d nature of the domain, a vortex can be supported on a knotted and linked solid torus. Existence of such a vortex is a conjecture of Kelvin [44, p.264].

Besides a support of a vortex, integral curves of the velocity (stream lines) and the vorticity (vortex lines) can be also knotted and linked. If the Bernoulli function Π is not constant, it

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Date: January 18, 2022.
2010 *Mathematics Subject Classification.* 35Q31, 35Q35.
Key words and phrases. Vortex rings, Beltrami fields, Grad-Shafranov equation.

acts as a first integral of stream lines and vortex lines. They lie on level sets of Π (Bernoulli surfaces). The Bernoulli surfaces are known to be nested tori or surfaces diffeomorphic to cylinders by the structure theorem of Arnold [6, p.69], though his theorem is stated for steady Euler flows in a bounded domain with analytic velocity. All stream lines on each torus are *closed* or *quasi-periodic*, i.e. dense on the torus. Such a torus is called an *invariant torus* in terms of mechanics [5, p.272].

If the Bernoulli function is constant, the velocity and the vorticity are collinear, i.e. $u \parallel \nabla \times u$. Such a vector field is called a *Beltrami field*. With a proportionality factor f, the steady Euler equations (1.2) are reduced to

$$\begin{aligned} \nabla \times u = fu, \quad \nabla \cdot u = 0, &\quad \text{in } \mathbb{R}^3, \\ u \to u_\infty &\quad \text{as } |x| \to \infty. \end{aligned} \tag{1.3}$$

If the factor f is not constant, it is a first integral of vortex lines (stream lines) and plays an alternative role of the Bernoulli function [6, p.71].

An exceptional case is when the factor f is constant. A Beltrami field with a constant factor is called a *strong* Beltrami field. Due to absence of the first integrals, vortex lines of a strong Beltrami field have topological freedom and can be chaotic, e.g. ABC flows [6]. It is known [19] that for any linked curves, there exists a strong Beltrami field (with $u_\infty = 0$) having a vortex line diffeomorphic to the curve. Kelvin's conjecture is revisited by Enciso and Peralta-Salas [20] by constructing a strong Beltrami field with a *thin* knotted and linked *vortex tube*. A vortex tube is a union of vortex lines embedded to a solid torus. The strong Beltrami field [20] has knotted and linked vortex lines. The complexity of vortex lines of strong Beltrami fields is recently studied in [22] in terms of random Beltrami fields.

A vortex of a strong Beltrami field can not be compactly supported since a strong Beltrami field is an eigenfunction of the Laplace operator $-\Delta u = f^2 u$, which is trivial if $u \in L^2(\mathbb{R}^3)$ by the Fourier transform. The strong Beltrami fields [19], [20] decay by the sharp order $u = O(|x|^{-1})$ as $|x| \to \infty$. It is known [38], [11] that Beltrami fields must be trivial if $u \in L^q(\mathbb{R}^3)$ for $q \in [2,3]$ or $u = o(|x|^{-1})$ as $|x| \to \infty$.

In the axisymmetric setting, a vortex of a steady Euler flow can be supported on an unknotted and unlinked solid torus, usually referred to as a *vortex ring*. A vortex ring is a vortex tube with compactly supported vorticity. Due to the axisymmetry, circulation of the velocity acts as an additional first integral of stream lines. We denote by $2\pi\Gamma$ circulation of the velocity along a circle around the symmetric axis. There are 3 particular types of vortex rings:

Type I: $\Gamma \equiv 0$. This is an axisymmetric flow without swirl. The velocity and the vorticity are perpendicular as 2d, i.e. $u \perp \nabla \times u$. Vortex lines are circles, unknotted and unlinked. Existence of this type vortex ring is proved by Fraenkel [23], [24].

Type II: $\Pi \equiv$ const. This is a Beltrami flow. The factor f is related with the circulation. Vortex lines (stream lines) are *knotted* and *linked*. An explicit solution of this type vortex ring with a non-constant factor is found by Chandrasekhar [12].

Type III: $u_\infty = 0$. This is a steady flow rest at infinity. The velocity can be also compactly supported besides the vorticity. An explicit solution is found by Prendergast [41]. This type vortex ring is recently constructed by Gavrilov [31], Constantin et al. [16] and Domínguez-Vázquez et al. [18].

Hicks [34] and Moffatt [37] found an explicit solution to the steady Euler equations (1.2), called the *Hicks-Moffatt vortex*, which is a 3 parameter family of solutions with a vortex supported on a *ball* including the above 3 types as particular cases, see Section 2. One of them agrees with the Beltrami field of Chandrasekhar [12]. Besides it, a Beltrami field with a non-constant factor is constructed by Turkington [48, p.68] by a variational principle.

In contrast to many strong Beltrami fields, existence of a Beltrami field with a non-constant factor seems unknown besides [12], [48]. Enciso and Peralta-Salas [21] proved non-existence of Beltrami fields if $f \in C^{2+\mu}(\mathbb{R}^3)$ for some $\mu \in (0,1)$ and a level set $f^{-1}(k)$ is diffeomorphic to a *sphere* for some $k \in \mathbb{R}$, e.g. f is radial or has local extrema. The non-existence result might imply that realization of Kelvin's conjecture by Beltrami fields with a non-constant factor is subtle. The Beltrami fields [12], [48] have discontinuous factors f whose level set $f^{-1}(k)$ is a *ball* or a *solid torus* for some k and its complement or an empty set for other k, see Section 2. These Beltrami fields have discontinuous vortex and are *not* C^1-solutions.

The aim of this paper is to prove existence of Beltrami fields with a continuous factor f whose level sets $f^{-1}(k)$ are *nested invariant tori*. The topology of constructed Beltrami fields is consistent with the structure theorem of Arnold. They make a vortex ring that can be supported on a *thick* solid torus.

Theorem 1.1. *For any $u_\infty \neq 0$, there exist axisymmetric solutions $u \in C^1(\mathbb{R}^3)$ to (1.3) such that $\nabla \times u$ is compactly supported on a solid torus with some $f \in C(\mathbb{R}^3)$ such that $f^{-1}(k)$ are nested invariant tori for $0 < k < k_0$ with some k_0 and empty sets for other $k \neq 0, k_0$. In particular, solutions with regular f exist, e.g. $f \in C^{2+\mu}(\mathbb{R}^3)$ for $\mu \in (0,1)$. The nested tori are symmetric in the direction of u_∞. On each torus, all vortex lines are closed or quasi-periodic. The torus degenerates to a circle for $k = k_0$.*

It is contrastive to the non-existence results [38], [11] that the solution in Theorem 1.1 has the asymptotics

$$u = u_\infty + O(|x|^{-3}) \quad \text{as } |x| \to \infty, \tag{1.4}$$

due to the compactly supported vortex, see Remark 5.10. The proportionality factor f depends on a stream function of u and can be smooth, see Section 2. The axisymmetric Beltrami fields in Theorem 1.1 have constant swirl component ru^θ on the circle $k = k_0$ with vanishing poloidal components $u^r = u^z = 0$, i.e. $u^r e_r + u^z e_z$ admits a stagnation point in the cross section. Existence of such solutions are contrastive to a rigidity result [15, Theorem 1.6] that axisymmetric Beltrami fields with no stagnation points are shear flow type.

Our approach constructing a vortex ring is based on a variational principle for the Grad-Shafranov equation, cf. [31], [16], [18]. This equation describes not only Beltrami flows but also steady Euler flows (1.2) with non-constant Π having stream lines and vortex lines not collinear. On the other hand, knotted or linked vortex tubes can not be constructed by this approach due to the axisymmetry.

Vortex rings in Theorem 1.1 are first examples of C^1-traveling wave solutions to the 3d Euler equations, axisymmetric with swirl. A question on global existence and finite time blow-up of solutions to the 3d Euler equations may be related with stability and instability of these traveling waves. In the study of shallow water equations, stability of a traveling wave is considered in terms of shape of a wave. Such stability is called *orbital stability*. Orbital stability of a traveling wave of the KdV equation is a well-known property [7]. This property also holds for a traveling wave of the Camassa-Holm equation [14], [13] even though it exhibits wave breaking. These traveling waves have variational characterization as a minimizer of an energy which deduces their orbital stability by conservation laws.

The Euler equations also admit a variational principle constructing a stationary solution and proving its stability, called a vorticity method, initiated by Arnold [6, p.88]. This method is mainly applied to 2d problems. Benjamin [8], [30] developed it to vortex rings without swirl and suggested their orbital stability, cf [10], [1]. A certain class of vortex rings with swirl is constructed by Turkington [48] by a vorticity method. This method involves some quantity *not* conserved by the Euler equations, see Section 2.

We prove existence of vortex rings by an elementary approach separately from the stability problem by using another variational principle, called a stream function method. This method is a minimax method constructing a vortex ring as a critical point of some functional with given physical constants, e.g. speed and flux of rings. We construct axisymmetric Beltrami fields via the Grad-Shafranov equation as a least energy critical point by a minimization on the Nehari manifold, see Section 3 for a variational formulation.

2. The Grad-Shafranov equation

2.1. **Invariant tori.** We derive an equation of vortex rings for the steady Euler equations (1.2). The equation of vortex rings is called Hicks equation, Bragg-Hawthorne equation or Squire-Long equation, e.g. [26], [45]. It is identical to the Grad-Shafranov equation [32], [46] in magnetohydrodynamics.

An axisymmetric vector field is written as

$$u(x) = u^r(z,r)e_r(\theta) + u^\theta(z,r)e_\theta(\theta) + u^z(z,r)e_z,$$

with the cylindrical coordinate (r,θ,z) and $e_r(\theta) = {}^t(\cos\theta, \sin\theta, 0)$, $e_\theta(\theta) = {}^t(-\sin\theta, \cos\theta, 0)$, $e_z = {}^t(0,0,1)$, i.e. $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = z$ for $x = {}^t(x_1, x_2, x_3)$. Each component of u is a function defined in the cross section $\mathbb{R}^2_+ = \{{}^t(z,r) \mid z \in \mathbb{R},\ r > 0\}$. The divergence-free condition for the axisymmetric vector field is $\partial_z(ru^z) + \partial_r(ru^r) = 0$. Thus there exists a stream function Ψ such that $ru^z = \partial_r\Psi$, $ru^r = -\partial_z\Psi$ and Ψ is constant on $\partial\mathbb{R}^2_+$. We assume that $u \to u_\infty$ as $|x| \to \infty$ for $u_\infty = {}^t(0,0,-W)$, $W > 0$ and $\Psi(z,0) = -\gamma$ for $\gamma \geq 0$. The circulation of u along the circle $C(z,r) = \{x = re_r(\theta) + ze_z,\ 0 \leq \theta \leq 2\pi\}$ is denoted by $2\pi\Gamma(z,r)$. By computation,

$$\Gamma(z,r) = \frac{1}{2\pi}\int_{C(z,r)} u \cdot \mathrm{d}l(x) = ru^{\theta}(z,r).$$

Thus an axisymmetric divergence-free vector field is represented by Ψ and Γ with the Clebsch representation, cf. [9, p.56],

$$u = \nabla \times (\Psi \nabla \theta) + \Gamma \nabla \theta. \tag{2.1}$$

The vorticity is written as $\nabla \times u = -L\Psi\nabla\theta + \nabla\Gamma \times \nabla\theta$ for $L = \partial_z^2 + \partial_r^2 - r^{-1}\partial_r$.

A stream line of u is a solution to the autonomous system

$$\dot{x}(\tau) = u(x(\tau)),$$

where $\dot{x}(\tau)$ denotes the derivative in τ. For 2d incompressible flows, this equation is a (finite-dimensional) Hamiltonian system with 1 degree of freedom. Indeed, the velocity is represented by a stream function $\tilde{\Psi}$ as $\nabla_y^{\perp}\tilde{\Psi}$ for $\nabla_y^{\perp} = {}^t(\partial_{y_2}, -\partial_{y_1})$ and $y = {}^t(y_1, y_2)$. Therefore a stream line $y(\tau)$ of 2d flows is a solution to

$$\dot{y}(\tau) = \nabla_y^{\perp}\tilde{\Psi}(y(\tau)).$$

This is a Hamiltonian system with 1 degree of freedom. Hence a stream line is understood as a level set of the Hamiltonian $\tilde{\Psi}$.

A stream line of axisymmetric Euler flows can be written also by a Hamiltonian system though general 3d flows do not admit this property. We reduce the autonomous system to a Hamiltonian system with 1 degree of freedom, similarly as a Hamiltonian system with 2 degree of freedom with a central field potential, e.g. [5, p.33]. A stream line $x(\tau)$ of an axisymmetric divergence-free vector field is represented as $x(\tau) = r(\tau)e_r(\theta(\tau)) + z(\tau)e_z$ with the cylindrical coordinate $(r(\tau), \theta(\tau), z(\tau))$. The 2d component $y(\tau) = {}^t(z(\tau), r(\tau)^2/2)$ is a solution to the Hamiltonian system with the stream function $\tilde{\Psi}(y) = \Psi(z,r)$ in the cross section. If u is a solution to the Euler equation (1.2), the circulation $\tilde{\Gamma}(y) = \Gamma(z,r)$ acts as a first integral of the stream line, i.e.

$$\frac{d}{d\tau}\tilde{\Gamma}(y(\tau)) = \dot{z}(\tau)\partial_z\Gamma + \dot{r}(\tau)\partial_r\Gamma = u \cdot \nabla_x\Gamma = 0.$$

Since $y(\tau)$ is a level set of $\tilde{\Psi}$, $\tilde{\Gamma}$ is regarded as a function of $\tilde{\Psi}$, i.e. $\Gamma = \Gamma(\Psi)$. Thus for a given constant l, $y(\tau) = {}^t(z(\tau), r(\tau)^2/2)$ is identified with $\tilde{\Psi}^{-1}(l)$ and $\theta(\tau)$ is computed by integrating $\dot{\theta}(\tau) = \Gamma(l)/r(\tau)^2$.

If $\tilde{\Psi}^{-1}(l)$ is a closed curve in $\mathbb{R}^2_+$ for some l, the stream line $x(\tau) = r(\tau)e_r(\theta(\tau)) + z(\tau)e_z$ lies on a torus $\Sigma(l)$ in $\mathbb{R}^3$. We denote by $\Theta(l)$ increment of $\theta(\tau)$ while $y(\tau)$ goes around the closed curve $\tilde{\Psi}^{-1}(l)$. If $\Theta(l)$ is *commensurable* with 2π, i.e. $\Theta(l)/(2\pi)$ is a rational number, the stream line $x(\tau)$ is closed on the torus. If $\Theta(l)$ is not commensurable with 2π, $x(\tau)$ is quasi-periodic. We call $\Sigma(l)$ an invariant torus [5].

A vortex line can be computed by the stream line $x(\tau)$. Since the Bernoulli function $\tilde{\Pi}(y) = \Pi(z, r)$ is also a first integral of u,

$$\frac{d}{d\tau}\tilde{\Pi}(y(\tau)) = \dot{z}(\tau)\partial_z\Pi + \dot{r}(\tau)\partial_r\Pi = u \cdot \nabla_x\Pi = 0.$$

Thus, $\Pi = \Pi(\Psi)$ and the each component of $\omega = \nabla \times u$ is represented as

$$\omega^r = \dot{\Gamma}(\Psi)u^r, \quad \omega^\theta = -r\dot{\Pi}(\Psi) + \dot{\Gamma}(\Psi)u^\theta, \quad \omega^z = \dot{\Gamma}(\Psi)u^z. \tag{2.2}$$

From the θ-component of ω and the boundary conditions for Ψ, one sees that Ψ satisfies the Grad-Shafranov equation:

$$\begin{aligned} -\frac{1}{r^2}L\Psi &= -\dot{\Pi}(\Psi) + \frac{1}{r^2}\dot{\Gamma}(\Psi)\Gamma(\Psi) && \text{in } \mathbb{R}^2_+, \\ \Psi &= -\gamma && \text{on } \partial\mathbb{R}^2_+, \\ \frac{1}{r}\partial_z\Psi \to 0,& \quad \frac{1}{r}\partial_r\Psi \to -W && \text{as } z^2 + r^2 \to \infty. \end{aligned} \tag{2.3}$$

This equation is a semi-linear elliptic equation for prescribed $\Pi(t)$ and $\Gamma(t)$. It includes axisymmetric flows without swirl $\Gamma \equiv 0$ and axisymmetric Beltrami flows $\Pi \equiv$ const with the proportionality factor $f = \dot{\Gamma}(\Psi)$ as particular cases. Strong Beltrami fields can be also described if Γ is proportional to Ψ. The constants $W > 0$ and $\gamma \geq 0$ are speed of a ring and flux measuring distance from the z-axis to the ring.

2.2. **The Hicks-Moffatt vortex.** The Grad-Shafranov equation (2.3) admits an explicit solution, called the Hicks-Moffatt vortex, a solution for $\gamma = 0$ with

$$-\dot{\Pi}_{HM}(t) = \lambda_1 1_{(0,\infty)}(t),\ \lambda_1 \in \mathbb{R}, \quad \dot{\Gamma}_{HM}(t) = \lambda_2^{1/2} 1_{(0,\infty)}(t),\ \lambda_2 \geq 0,$$

for the indicator function $1_{(0,\infty)}(t)$. The explicit form of Ψ [25, p.20] is

$$\Psi_{HM}(z, r) = \begin{cases} -\frac{3}{2}Wr^2\left(B(\kappa) - C(\kappa)\frac{J_{3/2}(\lambda_2^{1/2}\sigma)}{(\lambda_2^{1/2}\sigma)^{3/2}}\right), & \sigma < a, \\ -\frac{1}{2}Wr^2\left(1 - \frac{a^3}{\sigma^3}\right), & \sigma \geq a, \end{cases}$$

for $\sigma = \sqrt{z^2 + r^2}$, with the constants $a = \kappa/\lambda_2^{1/2}$,

$$B(\kappa) = \frac{J_{3/2}(\kappa)}{\kappa J_{5/2}(\kappa)}, \quad C(\kappa) = \frac{\kappa^{1/2}}{J_{5/2}(\kappa)},$$

and $\kappa \in (0, c_{5/2})$ satisfying

$$\lambda_1 = \frac{3}{2} W B(\kappa) \lambda_2.$$

Here, J_m denotes the m-th order Bessel function of the first kind and c_m is the first zero point of J_m, i.e. $c_{3/2} = 4.4934\cdots$, $c_{5/2} = 5.7634\cdots$. For given $\lambda_1 \in \mathbb{R}$, $\lambda_2 > 0$ and $W > 0$, the constant $\kappa \in (0, c_{5/2})$ uniquely exists since $B(\kappa)$ is decreasing on $(0, c_{5/2})$, i.e. $\kappa = B^{-1}(2\lambda_1/(3W\lambda_2))$.

The vorticity of the Hicks-Moffatt vortex is supported on a ball $\{x \in \mathbb{R}^3 \mid |x| \leq a\}$ since $\{{}^t(z,r) \in \mathbb{R}^2_+ \mid \Psi_{HM} > 0\} = \{\sigma < a\}$ and $\operatorname{spt} \omega^r, \operatorname{spt} \omega^\theta, \operatorname{spt} \omega^z = \{\sigma \leq a\}$. Here, $\operatorname{spt} \omega^r$ denotes the support of ω^r in the cross section $\mathbb{R}^2_+ = \{{}^t(z,r) \mid z \in \mathbb{R},\ r > 0\}$.

The Hick-Moffatt vortex is a 3-parameter family of solutions for $\lambda_1 \in \mathbb{R}$, $\lambda_2 > 0$, $W > 0$ with the following particular cases:

Type I: $\lambda_2 = 0$. This is a limiting case. Sending $\lambda_2 \to 0$ implies that $\kappa = 0$ and

$$\Psi_H(z,r) = \begin{cases} \dfrac{3}{4} W r^2 \left(1 - \dfrac{\sigma^2}{a^2}\right), & \sigma < a, \\ -\dfrac{1}{2} W r^2 \left(1 - \dfrac{a^3}{\sigma^3}\right), & \sigma \geq a, \end{cases}$$

for $\lambda_1 > 0$ and $W > 0$ with $a = (15W/2\lambda_1)^{1/2}$. This is Hill's spherical vortex ring [35].

Type II: $\lambda_1 = 0$. This implies that $\kappa = c_{3/2}$ and

$$\Psi_B(z,r) = \begin{cases} \dfrac{3}{2} W r^2 C(c_{3/2}) \dfrac{J_{3/2}(\lambda_2^{1/2}\sigma)}{(\lambda_2^{1/2}\sigma)^{3/2}}, & \sigma < a, \\ -\dfrac{1}{2} W r^2 \left(1 - \dfrac{a^3}{\sigma^3}\right), & \sigma \geq a, \end{cases}$$

for $\lambda_2 > 0$ and $W > 0$ with $a = c_{3/2}/\lambda_2^{1/2}$. The associated velocity (2.1) is a Beltrami field (1.3) with the discontinuous factor $f = \lambda_2^{1/2} 1_{(0,\infty)}(\Psi_B)$ [12].

Type III: $W = 0$. This is another limiting case. Sending $W \to 0$ implies that $\kappa = c_{5/2}$ and

$$\Psi_{\mathrm{com}}(z,r) = \begin{cases} -\dfrac{\lambda_1}{\lambda_2} r^2 \left(1 - \dfrac{c_{5/2}^{3/2}}{J_{3/2}(c_{5/2})} \dfrac{J_{3/2}(\lambda_2^{1/2}\sigma)}{(\lambda_2^{1/2}\sigma)^{3/2}}\right), & \sigma < a, \\ 0, & \sigma \geq a, \end{cases}$$

for $\lambda_1 < 0$ and $\lambda_2 > 0$ with $a = c_{5/2}/\lambda_2^{1/2}$. This solution is compactly supported in a half disk $\{{}^t(z,r) \in \mathbb{R}^2_+ \mid \sigma \leq a\}$ [41].

The Hicks-Moffatt vortex is viewed as a family of solutions for $0 < \kappa < c_{5/2}$. The parameter $\lambda_1 \in \mathbb{R}$ changes the sign at $\kappa = c_{3/2}$. If $\lambda_1 \geq 0$, the Hicks-Moffatt vortex is a unique solution to (2.3) with $\Pi_{HM}(t)$ and $\Gamma_{HM}(t)$ [25].

2.3. **A free-boundary problem.** A vortex can be supported on a solid torus for $\gamma > 0$. The problem (2.3) is a free-boundary problem for Ψ with a priori *unknown* vortex core

$$\overline{\Omega} = \operatorname{spt} \omega^{\theta}.$$

Once the core is found, one can find Ψ by solving two problems:

$$\begin{aligned}
&-\frac{1}{r^2}L\Psi = -\dot{\Pi}(\Psi) + \frac{1}{r^2}\dot{\Gamma}(\Psi)\Gamma(\Psi) \quad \text{in } \Omega, \quad \Psi = 0 \quad \text{on } \partial\Omega, \\
&-\frac{1}{r^2}L\Psi = 0 \quad \text{in } \mathbb{R}^2_+\backslash\Omega, \quad \Psi = -\gamma \quad \text{on } \partial\mathbb{R}^2_+, \quad \frac{1}{r}\partial_z\Psi \to 0, \ \frac{1}{r}\partial_r\Psi \to -W \ \text{ as } r^2+z^2 \to \infty.
\end{aligned}$$

On the other hand, the core is characterized as $\Omega = \{{}^t(z,r) \mid \Psi > 0\}$ by a maximum principle if $-L\Psi$ is non-negative. A typical choice of $-\dot{\Pi}(t)$ is a non-negative and non-decreasing function, e.g.

$$-\dot{\Pi}(t) = \lambda t_+^{\alpha}, \quad 0 \leq \alpha < \infty, \ \lambda > 0,$$

for $t_+ = \max\{t, 0\}$.

Existence of vortex rings goes back to Fraenkel [23], [24] who constructed solutions to (2.3) with $\Gamma \equiv 0$ by an implicit function theorem for Ψ, called a stream function method. Later, a variational principle is employed to construct solutions for given $W > 0$ and $\gamma \geq 0$ with the Lagrange multiplier $\lambda > 0$ [27]. It is known [2], [39] that solutions can be constructed with given constants λ, W, γ.

The another variational principle is a vorticity method of Friedman-Turkington [29], [28] which provides solutions to (2.3) with $\Gamma \equiv 0$ by maximizing an energy subject to constraints on impulse and mass for ω^{θ}. The constants $W > 0$, $\gamma \geq 0$ are obtained as Lagrange multipliers. The constant γ vanishes if impulse is small or mass is large.

Turkington [48] constructed solutions to (2.3) with $\Pi_{HM}(t)$ and $\Gamma_{HM}(t)$ for $\lambda_1 \geq 0$ and $\lambda_2 > 0$ with unknown $W > 0$, $\gamma \geq 0$ by a vorticity method maximizing

$$\text{the kinetic energy} - \frac{1}{2\lambda_2}\int_{\mathbb{R}^3} r^2 \left(\frac{\omega^{\theta}}{r} - \lambda_1\right)_+^2 \mathrm{d}x,$$

subject to constraints on impulse and mass for ω^{θ}. This result includes the Beltrami flow $\lambda_1 = 0$. Unlike the case without swirl, the second term is not conserved by the evolution of the Euler equations.

2.4. **Existence of Beltrami fields.** In this paper, we confine ourselves to the problem of Beltrami fields (1.3). The Grad-Shafranov equation (2.3) with $\Pi \equiv \text{const}$ is equivalent to (1.3) in the axisymmetric setting with the proportionality factor

$$f = \dot{\Gamma}(\Psi).$$

A level set of f can be identified with that of Ψ if $\dot{\Gamma}(t)$ is invertible. We consider a non-negative and non-decreasing function

$$\dot{\Gamma}(t) = (\lambda q)^{1/2} t_+^{q-1}, \quad \lambda > 0,\ 1 < q < \infty. \tag{2.4}$$

The function $\dot{\Gamma}(t)$ is continuous at $t = 0$ and increasing for $t > 0$. The level sets of f are axisymmetric and determined by those of Ψ, i.e.

$$\text{the cross section of } f^{-1}(k) = \begin{cases} \left\{ {}^t(z,r) \in \overline{\mathbb{R}^2_+} \,\middle|\, \Psi = \left(\frac{k}{\sqrt{\lambda q}} \right)^{1/(q-1)} \right\} & k > 0, \\ \left\{ {}^t(z,r) \in \overline{\mathbb{R}^2_+} \,\middle|\, \Psi \le 0 \right\} & k = 0, \\ \emptyset & k < 0. \end{cases}$$

If the level set $\Psi^{-1}(l)$ for $l = (k/\sqrt{\lambda q})^{1/(q-1)}$ is a closed curve in $\mathbb{R}^2_+$, the level set $f^{-1}(k)$ is a torus in $\mathbb{R}^3$. By changing the unknown function from Ψ to ψ by setting $\Psi = \psi - Wr^2/2 - \gamma$, (2.3)-(2.4) with $\Pi \equiv$ const is transformed into the homogeneous problem

$$\begin{aligned} -L\psi &= \lambda \left(\psi - \frac{W}{2} r^2 - \gamma \right)_+^{2q-1} && \text{in } \mathbb{R}^2_+, \\ \psi &= 0 && \text{on } \partial\mathbb{R}^2_+, \\ \frac{1}{r} \nabla_{z,r} \psi &\to 0 && \text{as } z^2 + r^2 \to \infty. \end{aligned} \tag{2.5}$$

The key result of this paper is existence of solutions to this problem.

Theorem 2.1. *Let* $1 < q < \infty$ *and* $0 < \lambda, W, \gamma < \infty$. *There exists a solution* $\psi \in C^{2+\nu}(\overline{\mathbb{R}^2_+})$ *of (2.5) for* $\nu \in (0,1)$ *such that*

$$\psi(z,r) = \psi(-z,r), \quad \partial_z \psi(z,r) < 0,\ z, r > 0,$$
$$\Omega = \left\{ {}^t(z,r) \in \mathbb{R}^2_+ \,\middle|\, \psi(z,r) - \frac{1}{2} W r^2 - \gamma > 0 \right\}.$$

The vortex core Ω *is bounded, connected and simply-connected with boundary of class* $C^{2+\nu}$. *The level sets* $\{{}^t(z,r) \in \mathbb{R}^2_+ \mid (\psi - Wr^2/2 - \gamma)_+ = l\}$ *are nested closed curves for* $0 < l < l_0$ *with some* $l_0 > 0$, *empty sets for other* $l \neq 0, l_0$, *and points for* $l = l_0$.

At the limit $q = 1$, the function (2.4) is that of the Hicks-Moffatt vortex, i.e. $\dot{\Gamma}_{HM}(t) = \lambda^{1/2} 1_{(0,\infty)}(t)$. This function is discontinuous at $t = 0$ and constant for $t > 0$. A level set of $f = \dot{\Gamma}(\Psi)$ is identified with the vortex core, i.e.

$$\text{the cross section of } f^{-1}(k) = \begin{cases} \left\{ {}^t(z,r) \in \overline{\mathbb{R}^2_+} \,\middle|\, \Psi > 0 \right\} & k = \lambda^{1/2}, \\ \left\{ {}^t(z,r) \in \overline{\mathbb{R}^2_+} \,\middle|\, \Psi \le 0 \right\} & k = 0, \\ \emptyset & k \neq \lambda^{1/2}, 0. \end{cases}$$

The vortex core of the Hicks-Moffatt vortex (with $\Pi \equiv \text{const}$) is a half disk in $\mathbb{R}^2_+$. Therefore the level set $f^{-1}(k)$ is a ball for $k = \lambda^{1/2}$ in $\mathbb{R}^3$. Turkington's solution has a vortex core supported away from the boundary $\partial\mathbb{R}^2_+$ if $\gamma > 0$. Hence $f^{-1}(k)$ is a solid torus for $k = \lambda^{1/2}$ in $\mathbb{R}^3$. The choice of $\dot{\Gamma}(t)$ has more freedom than (2.4), see Remark 3.3.

3. A VARIATIONAL PRINCIPLE

3.1. **A weighted Sobolev inequality.** In the sequel, we prove Theorem 2.1. Theorem 1.1 is deduced from Theorem 2.1 at the end of this paper.

Without loss of generality, we may assume that $W = 2$ by dividing ψ by $W/2$. We construct a solution to (2.5) by a variational principle. Let $H(\mathbb{R}^2_+; r^{-1})$ denote the weighted L^2-Sobolev space of trace zero functions on $\partial\mathbb{R}^2_+$ with the weight $2\pi^2 r^{-1}$ [40, p.243]. This space is a Hilbert space equipped with the inner product

$$(\psi, \phi)_{H(\mathbb{R}^2_+; r^{-1})} = 2\pi^2 \int_{\mathbb{R}^2_+} \nabla_{z,r}\psi \cdot \nabla_{z,r}\phi \frac{1}{r} \mathrm{d}z\mathrm{d}r,$$

where $\nabla_{z,r}$ denotes the gradient for the (z, r)-variable. The equation (2.5) can be written as a critical point of the functional

$$I[\psi] = \frac{1}{2}\|\psi\|^2_{H(\mathbb{R}^2_+; r^{-1})} - J[\psi], \quad J[\psi] = \frac{\pi^2\lambda}{q} \int_{\mathbb{R}^2_+} \left(\psi - r^2 - \gamma\right)^{2q}_+ \frac{1}{r} \mathrm{d}z\mathrm{d}r. \tag{3.1}$$

This functional is deduced by regarding (2.5) as the 5d problem

$$-\Delta_y \varphi = \frac{\lambda}{r^2}\left(r^2\varphi - r^2 - \gamma\right)^{2q-1}_+ \quad \text{in } \mathbb{R}^5,$$

by the transform

$$\psi(z, r) \longmapsto \varphi(y) = \frac{\psi(z, r)}{r^2}, \quad y = {}^t(y_1, y') \in \mathbb{R}^5,\ y_1 = z,\ |y'| = r.$$

The functional associated with the 5d problem is

$$\frac{1}{2}\int_{\mathbb{R}^5} |\nabla_y \varphi|^2 \mathrm{d}y - \frac{\lambda}{2q}\int_{\mathbb{R}^5} \frac{1}{r^4}(r^2\varphi - r^2 - \gamma)^{2q}_+ \mathrm{d}y.$$

Since $H(\mathbb{R}^2_+; r^{-1})$ is isometrically isomorphic to a subspace of axisymmetric functions in $\dot{H}^1(\mathbb{R}^5)$ by the above transform [4, Lemma 2.2], i.e. $\|\psi\|_{H(\mathbb{R}^2_+; r^{-1})} = \|\nabla_y\varphi\|_{L^2(\mathbb{R}^5)}$ for $\psi = r^2\varphi$, the functional (3.1) is deduced.

The main tool to work with the functional I is the weighted Sobolev inequality [25, Theorem 2],

$$\left(\int_{\mathbb{R}^2_+} |\psi|^p \frac{1}{r^{2+p/2}} \mathrm{d}z\mathrm{d}r\right)^{1/p} \leq C \left(\int_{\mathbb{R}^2_+} |\nabla_{z,r}\psi|^2 \frac{1}{r} \mathrm{d}z\mathrm{d}r\right)^{1/2}, \quad 2 \leq p < \infty. \tag{3.2}$$

This inequality is found in a more general form by Koch [36, Theorem 4.2.2] and de Valeriola-Van Schaftingen [17, Lemma 3]. The main mathematical contribution of the present work is a minimax method based on (3.2). Though vortex rings without swirl can be constructed without (3.2), e.g. [2], [39], this inequality is crucial to apply a minimax method to (2.5), see the next subsection.

The inequality (3.2) implies the continuous embedding from $H(\mathbb{R}^2_+; r^{-1})$ to the weighted Lebesgue space

$$H(\mathbb{R}^2_+; r^{-1}) \subset L^p(\mathbb{R}^2_+; r^{-2-p/2}), \quad 2 \leq p < \infty.$$

The functional $I : H(\mathbb{R}^2_+; r^{-1}) \longrightarrow \mathbb{R}$ is bounded since

$$\frac{q}{\pi^2\lambda} J[\psi] \leq \int_{\{\psi > r^2\}} \psi^{2q} \frac{1}{r} \mathrm{d}z\mathrm{d}r \leq \int_{\mathbb{R}^2_+} \psi^p \frac{1}{r^{2+p/2}} \mathrm{d}z\mathrm{d}r \lesssim \|\psi\|^p_{H(\mathbb{R}^2_+; r^{-1})}, \quad p = \frac{2}{3}(4q+1).$$

It is also continuous by $|s_+^{2q} - t_+^{2q}| \leq 2q(s_+^{2q-1} + t_+^{2q-1})((s-t)_+ + (t-s)_+)$ and

$$|J[\psi] - J[\phi]| \lesssim \left(\|\psi\|_{H(\mathbb{R}^2_+; r^{-1})} + \|\phi\|_{H(\mathbb{R}^2_+; r^{-1})}\right)^{p(1-1/(2q))} \|\psi - \phi\|^{p/(2q)}_{H(\mathbb{R}^2_+; r^{-1})}.$$

A similar argument implies that I is Fréchet differentiable, i.e. $I \in C^1(H(\mathbb{R}^2_+; r^{-1}); \mathbb{R})$, and

$$\begin{aligned} I'[\psi]\phi &= (\psi, \phi)_{H(\mathbb{R}^2_+; r^{-1})} - J'[\psi]\phi, \\ J'[\psi]\phi &= 2\pi^2\lambda \int_{\mathbb{R}^2_+} (\psi - r^2 - \gamma)_+^{2q-1} \phi \frac{1}{r} \mathrm{d}z\mathrm{d}r, \quad \phi \in H(\mathbb{R}^2_+; r^{-1}). \end{aligned} \tag{3.3}$$

A function $\psi \in H(\mathbb{R}^2_+; r^{-1})$ is a critical point $I'[\psi] = 0$ if and only if

$$(\psi, \phi)_{H(\mathbb{R}^2_+; r^{-1})} = 2\pi^2\lambda \int_{\mathbb{R}^2_+} (\psi - r^2 - \gamma)_+^{2q-1} \phi \frac{1}{r} \mathrm{d}z\mathrm{d}r, \quad \phi \in H(\mathbb{R}^2_+; r^{-1}).$$

We find a critical point by a minimization on the Nehari manifold

$$c = \inf_{N(\mathbb{R}^2_+; r^{-1})} I, \quad N(\mathbb{R}^2_+; r^{-1}) = \left\{\psi \in H(\mathbb{R}^2_+; r^{-1}) \;\middle|\; I'[\psi]\psi = 0,\ \psi \neq 0\right\}. \tag{3.4}$$

The Nehari manifold includes all critical points of I. A minimizer on the Nehari manifold is called a *least energy critical point* or a *ground state* [49, p.71].

3.2. **Compactness of an embedding.** We construct a ground state in a half space by that in a half disk $D = \{{}^t(z,r) \in \mathbb{R}^2_+ \mid \sqrt{z^2+r^2} < R\}$ for $R > 0$:

$$
\begin{aligned}
-L\psi &= \lambda \left(\psi - r^2 - \gamma\right)_+^{2q-1} && \text{in } D,\\
\psi &= 0 && \text{on } \partial D.
\end{aligned}
\tag{3.5}
$$

A ground state in a half disk is constructed by the same variational principle on the Nehari manifold $N(D;r^{-1})$ and the Hilbert space $H(D;r^{-1})$ defined by the same manner as $\mathbb{R}^2_+$. By the zero extension, elements of $H(D;r^{-1})$ are regarded as those of $H(\mathbb{R}^2_+;r^{-1})$, i.e. $H(D;r^{-1}) \subset H(\mathbb{R}^2_+;r^{-1})$. The functional I is also regarded as that on $H(D;r^{-1})$, i.e. $I \in C^1(H(D;r^{-1});\mathbb{R})$.

The space $H(D;r^{-1})$ is isometrically isomorphic to a homogeneous L^2-Sobolev space on a ball $B \subset \mathbb{R}^5$ centered at the origin with radius R [4, Lemma 2.2]. We denote by $F(B)$ the space of all axisymmetric functions in $H^1_0(B)$, a homogeneous L^2-Sobolev space of trace zero functions on ∂B equipped with the inner product $(\varphi,\eta)_{H^1_0(B)} = \int_B \nabla_y\varphi \cdot \nabla_y\eta \mathrm{d}y$. The transform $\psi(z,r) \longmapsto \varphi(y) = \psi(z,r)/r^2$ is a unitary operator from $H(D;r^{-1})$ to $F(B)$, i.e.

$$
(\psi,\phi)_{H(D;r^{-1})} = (\varphi,\eta)_{H^1_0(B)}, \quad \varphi(y) = \frac{\psi(z,r)}{r^2}, \quad \eta(y) = \frac{\phi(z,r)}{r^2}. \tag{3.6}
$$

Thus $H(D;r^{-1})$ is isometrically isomorphic to $F(B)$, i.e.

$$
H(D;r^{-1}) \cong F(B).
$$

The equation for vortex rings of a Beltrami field (3.5) has more *singular* operator and force than those of vortex pairs $-\Delta_{z,r}\psi = \lambda(\psi - r - \gamma)_+^{2q-1}$ [50] and vortex rings without swirl $-L\psi = \lambda r^2(\psi - r^2 - \gamma)_+^{2q-1}$ [2], [39]. Associated functionals to these equations admit a ground state by the *compactness* of the embedding $H(D;r^{-1}) \subset H^1_0(D) \subset\subset L^p(D)$ for $p \in [1,\infty)$. Due to the singular force, this compactness is not sufficient to prove existence of a ground state to (3.5).

A key fact to construct a ground state to (3.5) is the following compactness property thanks to (3.2) from $H(D;r^{-1})$ to the weighted Lebesgue space, cf. [40].

Lemma 3.1 (Compact embedding)**.**

$$
H(D;r^{-1}) \subset\subset L^p(D;r^{-1}), \quad 1 \le p < \infty. \tag{3.7}
$$

Proof. We prove (3.7) for $p = 1$. By the Rellich-Kondrachov theorem,

$$
F(B) \subset\subset L^s(B), \quad 2 < s < \frac{10}{3}.
$$

The conjugate exponent $10/7 < s' < 2$ satisfies $(-1 + 3/s)s' < 1$. Applying the Hölder's inequality implies that $\varphi = \psi/r^2$ satisfies

$$\int_D |\psi|\frac{1}{r}\mathrm{d}z\mathrm{d}r \le \left(\int_D |\psi|^s \frac{1}{r^{2s-3}}\mathrm{d}z\mathrm{d}r\right)^{1/s}\left(\int_D \frac{1}{r^{(-1+3/s)s'}}\mathrm{d}z\mathrm{d}r\right)^{1/s'} \lesssim \|\varphi\|_{L^s(B)}$$
$$\lesssim \|\nabla_y\varphi\|_{L^2(B)} = \|\psi\|_{H(D;r^{-1})}.$$

Thus, (3.7) holds for $p = 1$.

The continuous embedding $H(D; r^{-1}) \subset L^p(D; r^{-1})$ holds for $2 \le p < \infty$ since $1 < R/r$ for $(z, r) \in D$ and the zero extension of $\psi \in H(D; r^{-1})$ to $\mathbb{R}^2_+$ satisfies (3.2) and

$$\int_D |\psi|^p \frac{1}{r}\mathrm{d}z\mathrm{d}r \le R^{1+p/2}\int_D |\psi|^p \frac{1}{r^{2+p/2}}\mathrm{d}z\mathrm{d}r \lesssim R^{1+p/2}\|\psi\|^p_{H(D;r^{-1})}.$$

The compact embedding (3.7) for $1 \le p < \infty$ follows from the Hölder's inequality $\|\psi\|_{L^p(D;r^{-1})} \le \|\psi\|^{\theta}_{L^q(D;r^{-1})}\|\psi\|^{1-\theta}_{L^1(D;r^{-1})}$ for $1/p = \theta/q + 1 - \theta$ and $2 \le q < \infty$. □

3.3. **Regularity of a critical point.** We use a short-hand notation $H = H(D; r^{-1})$. We show that all critical points are classical solutions to (3.5).

Lemma 3.2. *If $\psi \in H$ satisfies $I'[\psi] = 0$, then $\varphi = \psi/r^2 \in C^{2+\nu}(\overline{B})$ for any $\nu \in (0, 1)$, and*

$$\begin{aligned} -\Delta_y\varphi &= \frac{\lambda}{r^2}(r^2\varphi - r^2 - \gamma)^{2q-1}_+ && \text{in } B, \\ \varphi &= 0 && \text{on } \partial B. \end{aligned} \tag{3.8}$$

In particular, $\psi \in C^{2+\nu}(\overline{D})$ is a classical solution to (3.5).

Proof. The force term of the 5d problem for φ belongs to $L^3(B)$ by (3.2) and

$$\int_B \left|\frac{\lambda}{r^2}(r^2\varphi - r^2 - \gamma)^{2q-1}_+\right|^3 \mathrm{d}y = 2\pi^2\lambda^3\int_D \frac{1}{r^3}(\psi - r^2 - \gamma)^{3(2q-1)}_+\mathrm{d}z\mathrm{d}r$$
$$\le 2\pi^2\lambda^3\int_D |\psi|^p\frac{1}{r^{2+p/2}}\mathrm{d}z\mathrm{d}r \lesssim \|\psi\|^p_H, \quad p = 8q - \frac{14}{3}.$$

Thus an axisymmetric solution of the Poisson equation $-\Delta_y\tilde{\varphi} = \lambda r^{-2}(r^2\varphi - r^2 - \gamma)^{2q-1}_+$ in B and $\tilde{\varphi} = 0$ on ∂B belongs to $W^{2,3}(B)$. Here, $W^{k,s}(B)$ denotes the Sobolev space of order k with exponent s. Since $I'[\psi] = 0$ and $\tilde{\psi} = r^2\tilde{\varphi}$ satisfies

$$(\tilde{\psi}, \phi)_H = 2\pi^2\lambda\int_D (\psi - r^2 - \gamma)^{2q-1}_+\phi\frac{1}{r}\mathrm{d}z\mathrm{d}r, \quad \phi \in H,$$

the function $\tilde{\psi}$ agrees with ψ. Thus $\varphi = \tilde{\varphi} \in W^{2,3}(B)$ is a solution to (3.8). By the Sobolev inequality, $\varphi \in L^\infty(B)$ and $-\Delta_y\varphi \in L^\infty(B)$. Thus $\varphi \in W^{2,s}(B)$, $1 \le s < \infty$, by elliptic regularity. Since $\gamma > 0$,

$$-\Delta_y\varphi = \lambda r^{4(q-1)}\left(\varphi - 1 - \frac{\gamma}{r^2}\right)_+^{2q-1} \in W^{1,\infty}(B).$$

Thus $\varphi \in W^{3,s}(B)$ and $\varphi \in C^{2+\nu}(\overline{B})$ for any $\nu \in (0,1)$. □

Remark 3.3. The problem (2.5) is written as

$$\begin{aligned} -L\psi &= \lambda g(\psi, r) && \text{in } \mathbb{R}^2_+, \\ \psi &= 0 && \text{on } \partial\mathbb{R}^2_+, \\ \frac{1}{r}\nabla_{z,r}\psi &\to 0 && \text{as } z^2 + r^2 \to \infty, \end{aligned} \tag{3.9}$$

for $g(\psi, r) = h(\psi - Wr^2/2 - \gamma)$. The associated functional is

$$I[\psi] = \frac{1}{2}\|\psi\|^2_{H(\mathbb{R}^2_+;r^{-1})} - J[\psi], \quad J[\psi] = \int_{\mathbb{R}^2_+} G(\psi, r)\frac{2\pi^2}{r}\mathrm{d}z\mathrm{d}r,$$

for $G(s, r) = \int_0^s g(\tau, r)\mathrm{d}\tau$. The choice $h(s) = s_+^{2q-1}$ for $q > 1$ is the problem (2.5). A more general choice is non-decreasing $h \in C(\mathbb{R})$ growing in a polynomial order as $s \to \infty$ such that $h(s) = 0$ for $s \leq 0$, $h(s) = o(s)$ as $s \to 0$, and there exists $\theta \in [0, 1/2)$ such that for $s \geq 0$,

$$\int_0^s h(\tau)\mathrm{d}\tau \leq \theta s h(s).$$

The last condition means that h is *superlinear* as $s \to \infty$. For such h, solutions of (3.9) can be constructed by the same minimization as (3.4), see [2, p.27], [39, p.211] for the case without swirl and [43, p.9], [49, p.72] for the superlinear growth condition. On the other hand, the case $q = 1$ does not satisfy the above conditions at $s = 0$ and $s \to \infty$ and is excluded in this paper. For the case without swirl $g(\psi, r) = r^2 h(\psi - Wr^2/2 - \gamma)$, mountain pass solutions are constructed in [3] for bounded and discontinuous h.

4. Ground states in a half disk

4.1. **A deformation theorem.** In this section, we prove existence of a ground state and its properties in a half disk. We consider the minimization

$$c = \inf_N I, \quad N = \left\{\psi \in H \;\middle|\; I'[\psi]\psi = 0,\ \psi \neq 0\right\}. \tag{4.1}$$

By the compact embedding $H = H(D; r^{-1}) \subset\subset L^p(D; r^{-1})$ for $p \in [1, \infty)$, the functional I satisfies the Palais-Smale condition and admits a deformation theorem.

Proposition 4.1.

$$\frac{1}{2}\left(1-\frac{1}{q}\right)\|\psi\|_H^2 \leq I[\psi]-\frac{1}{2q}I'[\psi]\psi, \quad \psi \in H. \tag{4.2}$$

Proof. This follows from (3.1) and (3.3). □

Proposition 4.2 (Palais-Smale condition)**.** *Any sequence $\{\psi_n\} \subset H$ satisfying*

$$\sup_n I[\psi_n] < \infty, \quad \|I'[\psi_n]\|_{H^*} \to 0,$$

has a convergent subsequence in H.

Proof. A sequence satisfying the above condition is bounded in H by (4.2) and has a convergent subsequence weakly in $H = H(D; r^{-1})$ and strongly in $L^p(D; r^{-1})$ for $1 \leq p < \infty$ by (3.7). Thus by choosing a subsequence (still denoted by $\{\psi_n\}$), there exists some ψ such that

$$\begin{aligned} \psi_n &\rightharpoonup \psi \quad \text{in } H(D; r^{-1}), \\ \psi_n &\to \psi \quad \text{in } L^p(D; r^{-1}). \end{aligned}$$

This implies that $J'[\psi_n]\psi_n$ and $J'[\psi_n]\psi$ converge to $J'[\psi]\psi$. By (3.3),

$$\begin{aligned} I'[\psi_n]\psi &= (\psi_n, \psi)_H - J'[\psi_n]\psi, \\ I'[\psi_n]\psi_n &= (\psi_n, \psi_n)_H - J'[\psi_n]\psi_n. \end{aligned}$$

Since the left-hand sides vanish as $n \to \infty$, $\lim_{n\to\infty} \|\psi_n\|_H = \|\psi\|_H$. Hence $\psi_n \to \psi$ in H. □

We set a filtration and a set of critical points with a critical value $c \in \mathbb{R}$ by

$$\begin{aligned} A_c &= \{\psi \in H \mid I[\psi] \leq c\ \}, \\ K_c &= \{\psi \in H \mid I[\psi] = c,\ I'[\psi] = 0\ \}. \end{aligned}$$

Lemma 4.3 (Deformation Theorem)**.** *There exists $\varepsilon_0 > 0$ such that for $c \in \mathbb{R}$ and a neighborhood $\mathcal{U}$ of K_c, there exists $\varepsilon_1 \in (0, \varepsilon_0)$ and a homeomorphism $\iota : H \to H$ such that*

$$\begin{aligned} &\iota(\psi) = \psi, \quad \psi \notin I^{-1}[c-\varepsilon_0, c+\varepsilon_0], \\ &I[\iota(\psi)] \leq I[\psi], \quad \psi \in H, \\ &\iota(A_{c+\varepsilon_1} \backslash \mathcal{U}) \subset A_{c-\varepsilon_1}. \end{aligned}$$

Proof. See [43, p.82, Theorem A.4]. □

4.2. **Characterization of a critical value.** We characterize a critical value (4.1) as a minimax value by using convexity of $I[t\psi]$ for $t > 0$. Then the deformation theorem implies that a ground state is a critical point, cf. [49, p.74, Theorem 4.3].

Proposition 4.4. *For $\psi \in H \backslash \{0\}$, set*

$$g(t) = I[t\psi], \quad t \geq 0.$$

There exists some $t(\psi) > 0$ such that $\dot{g}(t(\psi)) = 0$, i.e. $t(\psi)\psi \in N$. The function $g(t)$ is increasing for $t < t(\psi)$ and decreasing for $t > t(\psi)$. Moreover,

$$g(t(\psi)) = \frac{\pi^2\lambda}{q}\int_D (t(\psi)\psi - r^2 - \gamma)_+^{2q-1}\left(r^2 + \gamma + (q-1)t(\psi)\psi\right)\frac{1}{r}dzdr, \tag{4.3}$$

and $t(\cdot) : H\backslash\{0\} \to (0,\infty)$ is continuous.

Proof. By

$$\begin{aligned}
\frac{\dot{g}(t)}{t} &= \|\psi\|_H^2 - \frac{2\pi^2\lambda}{t}\int_D (t\psi - r^2 - \gamma)_+^{2q-1}\psi\frac{1}{r}\mathrm{d}z\mathrm{d}r, \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{g}(t)}{t}\right) &= -\frac{2\pi^2\lambda}{t^2}\int_D (t\psi - r^2 - \gamma)_+^{2q-2}\left(2(q-1)t\psi + r^2 + \gamma\right)\psi\frac{1}{r}\mathrm{d}z\mathrm{d}r < 0,
\end{aligned}$$

$\lim_{t\to 0}\dot{g}(t)/t = \|\psi\|_H^2 > 0$ and $\dot{g}(t)/t$ is decreasing. Hence there exists a unique $t(\psi) > 0$ such that $\dot{g}(t(\psi)) = 0$ and $t(\psi)\psi \in N$. The identity (4.3) follows from

$$g(t(\psi)) = I[t(\psi)\psi] = \frac{1}{2}\|t(\psi)\psi\|_H^2 - J[t(\psi)\psi] = \frac{1}{2}J'[t(\psi)\psi](t(\psi)\psi) - J[t(\psi)\psi].$$

To prove continuity of $t(\cdot)$, we take a sequence $\{\psi_n\} \subset H\backslash\{0\}$ such that $\psi_n \to \psi$ in $H\backslash\{0\}$. Since $g_n(t) = I[t\psi_n]$ satisfies $\dot{g}_n(t(\psi_n)) = 0$,

$$0 = \frac{\dot{g}_n(t(\psi_n))}{t(\psi_n)} = \|\psi_n\|_H^2 - 2\pi^2\lambda t(\psi_n)^{2(q-1)}\int_D\left(\psi_n - \frac{r^2+\gamma}{t(\psi_n)}\right)_+^{2q-1}\psi_n\frac{1}{r}\mathrm{d}z\mathrm{d}r.$$

Since $\lim_{n\to\infty}\|\psi_n\|_H = \|\psi\|_H \neq 0$, the sequence $\{t(\psi_n)\}$ is bounded.

Suppose that $\{t(\psi_n)\}$ does not converge to $t(\psi)$. Then, there exists a subsequence (still denoted by $\{t(\psi_n)\}$) such that $t(\psi_n) \to t_0$ for some $t_0 \geq 0$. Since $\psi \neq 0$, $t_0 > 0$. Sending $n \to \infty$ to the above equality implies that $\dot{g}(t_0) = 0$ for $g(t) = I[t\psi]$. Thus $t_0 = t(\psi)$. This is a contradiction and we conclude that $t(\psi_n) \to t(\psi)$. □

Proposition 4.5.

$$c=\inf_{\psi\in N} I[\psi]=\inf_{\psi\in H\backslash\{0\}}\sup_{t\geq 0} I[t\psi]=\inf_{p\in\Lambda}\sup_{0\leq t\leq 1} I[p(t)], \tag{4.4}$$

for $\Lambda=\{p\in C([0,1];H)\mid p(0)=0,\ I[p(1)]<0\ \}$.

Proof. Since $\sup_{t\geq 0} I[t\psi]\leq I[\psi]$ for $\psi\in N$ by Proposition 4.4,

$$\inf_{\psi\in H\backslash\{0\}}\sup_{t\geq 0} I[t\psi]\leq \inf_{\psi\in N} I[\psi]=c.$$

Since

$$\inf_{p\in\Lambda}\sup_{0\leq t\leq 1} I[p(t)]\leq \inf_{\psi\in H\backslash\{0\}}\sup_{t\geq 0} I[t\psi],$$

it suffices to show that the left-hand side is larger than c. We set $h(t)=I'[p(t)]p(t)$ for $p\in\Lambda$. By (3.3) and (3.7),

$$I'[\psi]\psi=\|\psi\|_H^2-2\pi^2\lambda\int_D(\psi-r^2-\gamma)_+^{2q-1}\psi\frac{1}{r}\mathrm{d}z\mathrm{d}r\geq \|\psi\|_H^2-C\|\psi\|_H^{2q},\quad \psi\in H,$$

for some $C>0$. This implies that $\lim_{t\to 0} h(t)/\|p(t)\|_H^2=1$. Hence $h(t)$ is positive near $t=0$. Since $h(1)<0$ by (4.2), by the intermediate value theorem, $h(t_1)=0$ for some $t_1\in(0,1)$. Thus $p(t_1)\in N$ and

$$c=\inf_{\psi\in N} I[\psi]\leq I[p(t_1)]\leq \sup_{0\leq t\leq 1} I[p(t)].$$

Since $p\in\Lambda$ is arbitrary, (4.4) follows. □

Lemma 4.6. *If* $\psi\in N$ *satisfies* $c=I[\psi]$*, then* $I'[\psi]=0$*.*

Proof. Suppose on the contrary that there exists a ground state $\psi\in N$ such that $I'[\psi]\neq 0$. By the continuity of $I':H\to H^*$, there exists $\delta>0$ such that $I'[\phi]\neq 0$ for all $\phi\in\mathcal{B}$, where $\mathcal{B}=\{\phi\in H\mid \|\psi-\phi\|_H\leq\delta\}$. Thus the set of critical points with the critical value c is not included in $\mathcal{B}$, i.e. $K_c\cap\mathcal{B}=\emptyset$. Thus $\mathcal{U}=\mathcal{B}^c$ is a neighborhood of K_c. We apply Lemma 4.3 and take $\varepsilon_1>0$ and a homeomorphism $\iota:H\to H$ such that

$$\iota(A_{c+\varepsilon_1}\cap\mathcal{B})\subset A_{c-\varepsilon_1}.$$

We take $p\in\Lambda$ such that $p(t)\in A_{c-\varepsilon_1}\cup(A_{c+\varepsilon_1}\cap\mathcal{B})$ for all $0\leq t\leq 1$ and set $\hat{p}=\iota(p)\in\Lambda$. Then, $\hat{p}(t)\in A_{c-\varepsilon_1}$ for all $0\leq t\leq 1$. Thus by (4.4),

$$c\leq \sup_{0\leq t\leq 1} I[\tilde{p}(t)]\leq c-\varepsilon_1.$$

This is a contradiction. We conclude that $I'[\psi] = 0$. □

4.3. **Existence of a ground state.** We prove existence of a symmetric ground state for the z-variable. We say that ψ^* is the Steiner symmetrization of ψ if $\psi^*(z,r) = \psi^*(-z,r)$, ${}^t(z,r) \in D$, ψ^* is non-increasing for $|z|$, and ψ^* is equi-measurable, i.e.

$$|\{z \in \mathbb{R} \mid \psi(z,r) \geq t,\ {}^t(z,r) \in D\}| = |\{z \in \mathbb{R} \mid \psi^*(z,r) \geq t,\ {}^t(z,r) \in D\}|, \quad t \geq 0,\ r \geq 0.$$

The Steiner symmetrization exists for any $\psi \in H$ and does not increase the Dirichlet energy [27, Appendix I], i.e. $\|\psi^*\|_H \leq \|\psi\|_H$. We show that a ground state can be replaced with the Steiner symmetrization.

Proposition 4.7. *If $\psi \in N$ satisfies $c = I[\psi]$, then $\psi^* \in N$ and $c = I[\psi^*]$.*

Proof. Since ψ^* is equi-measurable,

$$\begin{aligned} I'[\psi^*]\psi^* &= \|\psi^*\|_H^2 - 2\pi^2\lambda \int_D (\psi^* - r^2 - \gamma)_+^{2q-1}\psi^* \frac{1}{r}\mathrm{d}z\mathrm{d}r \\ &\leq \|\psi\|_H^2 - 2\pi^2\lambda \int_D (\psi - r^2 - \gamma)_+^{2q-1}\psi \frac{1}{r}\mathrm{d}z\mathrm{d}r = I'[\psi]\psi = 0. \end{aligned}$$

Thus $g(t) = I[t\psi^*]$ satisfies $\dot{g}(1) \leq 0$. Since $g(t)$ is decreasing for $t > t(\psi^*)$ by Proposition 4.4, we have $t(\psi^*) \leq 1$. By (4.3), $g(t(\psi^*)) \leq g(1)$ and

$$c \leq I[t(\psi^*)\psi^*] = g(t(\psi^*)) \leq g(1) = I[\psi^*] \leq I[\psi] = c.$$

Thus $t(\psi^*) = 1$, $\psi^* \in N$ and $c = I[\psi^*]$. □

Proposition 4.8.

$$\inf_{\psi \in N} \|\psi\|_H > 0. \tag{4.5}$$

Proof. We take arbitrary $\psi \in N$. By (4.1) and (3.3), $\psi \neq 0$ and

$$0 = I'[\psi]\psi = \|\psi\|_H^2 - 2\pi^2\lambda \int_D (\psi - r^2 - \gamma)_+^{2q-1}\psi \frac{1}{r}\mathrm{d}z\mathrm{d}r.$$

Since $q > 1$, $|s|^{2(q-1)} \to 0$ as $|s| \to 0$. Thus for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that $|s|^{2(q-1)} \leq \varepsilon$ for $|s| \leq \delta$. Since $(\psi - r^2 - \gamma)_+ = 0$ if $\psi \leq 0$,

$$\int_D (\psi - r^2 - \gamma)_+^{2q-1} \psi \frac{1}{r} \mathrm{d}z\mathrm{d}r = \int_{D\cap\{0<\psi<\delta\}} (\psi - r^2 - \gamma)_+^{2q-1} \psi \frac{1}{r} \mathrm{d}z\mathrm{d}r + \int_{D\cap\{\psi\geq\delta\}} (\psi - r^2 - \gamma)_+^{2q-1} \psi \frac{1}{r} \mathrm{d}z\mathrm{d}r.$$

We estimate

$$\int_{D\cap\{0<\psi<\delta\}} (\psi - r^2 - \gamma)_+^{2q-1} \psi \frac{1}{r} \mathrm{d}z\mathrm{d}r \leq \int_{D\cap\{0<\psi<\delta\}} \psi^{2q} \frac{1}{r} \mathrm{d}z\mathrm{d}r \leq \varepsilon \int_D \psi^2 \frac{1}{r} \mathrm{d}z\mathrm{d}r,$$
$$\int_{D\cap\{\psi\geq\delta\}} (\psi - r^2 - \gamma)_+^{2q-1} \psi \frac{1}{r} \mathrm{d}z\mathrm{d}r \leq \int_{D\cap\{\psi\geq\delta\}} \psi^{2q} \frac{1}{r} \mathrm{d}z\mathrm{d}r \leq \int_D \psi^{2q} \frac{1}{r} \mathrm{d}z\mathrm{d}r.$$

By Lemma 3.1, $||\psi||_{L^p(D;r^{-1})} \leq C||\psi||_H$ for $1 \leq p < \infty$, with some constant C, independent of ψ. Thus there exist positive constants C_1 and C_2 independent of ψ and $\varepsilon > 0$ such that

$$||\psi||_H^2 \leq C_1\varepsilon||\psi||_H^2 + C_2||\psi||_H^{2q}.$$

Thus for small $\varepsilon > 0$ such that $\varepsilon C_1 < 1$, we have $0 < (1 - C_1\varepsilon) \leq C_2||\psi||_H^{2(q-1)}$. Taking the infimum for $\psi \in N$ implies (4.5). □

Lemma 4.9 (Existence of a ground state)**.** *There exists $\psi \in N$ such that $c = I[\psi]$, $I'[\psi] = 0$ and $\psi = \psi^*$.*

Proof. We take a minimizing sequence $\{\psi_n\} \subset N$ of (4.1). By (4.2) and (3.7), there exists a subsequence (still denoted by $\{\psi_n\}$) and some ψ such that

$$\begin{aligned} \psi_n &\rightharpoonup \psi \quad \text{in } H(D; r^{-1}), \\ \psi_n &\to \psi \quad \text{in } L^p(D; r^{-1}), \quad 1 \leq p < \infty. \end{aligned}$$

The limit ψ is non-trivial by (4.5). This convergence implies $J'[\psi_n]\psi_n \to J'[\psi]\psi$ and

$$||\psi||_H^2 - J'[\psi]\psi \leq \liminf_{n\to\infty} \left(||\psi_n||_H^2 - J'[\psi_n]\psi_n\right) = \lim_{n\to\infty} I'[\psi_n]\psi_n = 0.$$

Hence $g(t) = I[t\psi]$ satisfies $\dot{g}(1) \leq 0$. Since $g(t)$ is decreasing for $t > t(\psi)$ for some $t(\psi) > 0$ by Proposition 4.4, we have $t(\psi) \leq 1$. By (4.3), $g(t(\psi)) \leq g(1)$ and

$$\inf_N I \leq g(t(\psi)) \leq g(1) \leq \lim_{n\to\infty} I[\psi_n] = \inf_N I.$$

Hence, $t(\psi) = 1$, $\psi \in N$ and $c = I[\psi]$. Thus ψ is a ground state. By Proposition 4.7, we replace it by the Steiner symmetrization. □

4.4. **Shape of vortex cores.** We set $\Omega = \{{}^t(z,r) \in D \mid \psi - r^2 - \gamma > 0\}$ for the ground state ψ. By (2.2) and (2.4), Ω is the vortex core, i.e. $\overline{\Omega} = \operatorname{spt} \omega^\theta$. The vortex core Ω is symmetric in the z-direction since ψ is. We show that ψ is decreasing for $|z|$ and Ω consists of simply-connected components with regular boundaries, cf. [27, Theorem 3D].

Proposition 4.10. *The ground state in Lemma 4.9 satisfies $\psi \in C^{2+\nu}(\overline{D})$ for $\nu \in (0,1)$ and*

$$\frac{\partial \psi}{\partial z}(z,r) < 0, \quad z > 0,\ {}^t(z,r) \in D. \tag{4.6}$$

The level sets $\{\psi - r^2 - \gamma = l\}$ are nested closed curves of class $C^{2+\nu}$ for $0 \leq l < l_0$ with $l_0 = \|(\psi - r^2 - \gamma)_+\|_\infty$ and points for $l = l_0$.

Proof. The regularity of the ground state follows from Lemma 3.2. Since ψ is Steiner symmetric,

$$\frac{\partial \psi}{\partial z}(z,r) \leq 0, \quad z > 0,\ {}^t(z,r) \in D.$$

We set $\varphi(y) = \psi(z,r)/r^2$ for $y = {}^t(y_1, y')$, $y_1 = z$, $|y'| = r$. By Lemma 3.2, $\varphi \in C^{2+\nu}(\overline{B})$ for $\nu \in (0,1)$ and φ is a solution to the 5d Dirichlet problem (3.8) in B. For arbitrary $\tau \in (0,R)$, we set a hyperplane and a cap by

$$T_\tau = \{{}^t(y_1,y') \in B \mid y_1 = \tau\}, \quad \Sigma_\tau = \{{}^t(y_1,y') \in B \mid y_1 > \tau\}.$$

We denote a reflection point of $y = {}^t(y_1,y') \in \Sigma_\tau$ with respect to T_τ by $y_\tau = {}^t(2\tau - y_1, y')$ and set $\tilde{\varphi}(y) = \varphi(y) - \varphi(y_\tau)$ for $y \in \Sigma_\tau$. Since $\varphi(y_\kappa)$ solves (3.8) for $y \in \Sigma_\kappa$ and $\varphi(y)$ is non-increasing for $y_1 > 0$, $\tilde{\varphi}(y) \leq 0$ in Σ_τ and

$$\begin{aligned}
-\Delta_y \tilde{\varphi}(y) &= \frac{\lambda}{r^2}(r^2\varphi(y) - r^2 - \gamma)_+^{2q-1} - \frac{\lambda}{r^2}(r^2\varphi(y_\kappa) - r^2 - \gamma)_+^{2q-1} \leq 0, \quad y \in \Sigma_\kappa, \\
\tilde{\varphi}(y) &= 0, \quad y \in T_\kappa.
\end{aligned}$$

Hence by Hopf's lemma [42, p.65, Theorem 7],

$$\frac{\partial \tilde{\varphi}}{\partial y_1}(y) < 0, \quad y \in T_\tau.$$

Thus (4.6) holds. By the implicit function theorem, a level set of $\psi - r^2 - \gamma = l$ for $0 \leq l < l_0$ is written as a graph of a $C^{2+\nu}$-function. For $l = l_0$, the level set is points lying on the r-axis. □

The connectedness of Ω follows from the least energy property of a ground state, cf. [2, Theorem 4]. Our proof is based on that of de Valeriola-Van Schaftingen [17, Lemma 11] using energy identities [27, Lemma 5A].

Proposition 4.11. *The identities*

$$\int_\Omega |\nabla_{z,r}\Psi|^2 = \lambda \int_\Omega \Psi^{2q}, \tag{4.7}$$

$$\int_\Omega |\nabla_{z,r}\Psi|^2 = \int_\Omega |\nabla_{z,r}\psi|^2 - \int_\Omega |\nabla_{z,r}(r^2+\gamma)|^2, \tag{4.8}$$

hold for $\Psi = \psi - r^2 - \gamma$ *and the ground state* ψ *in Lemma 4.9, where the measure* $2\pi^2 r^{-1} dzdr$ *is suppressed.*

Proof. Since Ψ satisfies $-L\Psi = \lambda\Psi^{2q-1}$ in Ω and $\Psi = 0$ on $\partial\Omega$, $\Phi = \Psi/r^2$ satisfies the 5d problem

$$-\Delta_y \Phi = \lambda r^{4(q-1)}\Phi^{2q-1} \quad \text{in } U, \qquad \Phi = 0 \quad \text{on } \partial U,$$

for $U = \{y = {}^t(y_1, y') \in B \mid y_1 = z,\ |y'| = r,\ {}^t(z,r) \in \Omega\}$. By multiplying Φ by the equation and integration by parts,

$$\int_U |\nabla_y \Phi|^2 \mathrm{d}y = \lambda \int_U r^{4(q-1)}\Phi^{2q}\mathrm{d}y.$$

Since $H(D; r^{-1}) \cong F(B)$ by the transform $\Psi \longmapsto \Phi = \Psi/r^2$, (4.7) follows.

The identity (4.8) follows from $|\nabla_{z,r}\Psi|^2 = |\nabla_{z,r}\psi|^2 - |\nabla_{z,r}(r^2+\gamma)|^2 - 2\nabla_{z,r}\Psi \cdot \nabla_{z,r}(r^2+\gamma)$, and

$$\int_\Omega \nabla_{z,r}\Psi \cdot \nabla_{z,r}(r^2+\gamma) = \int_U \nabla_y \Phi \cdot \nabla_y \left(1 + \frac{\gamma}{r^2}\right)\mathrm{d}y = 0.$$

□

Lemma 4.12. *The vortex core* Ω *of the ground state in Lemma 4.9 is connected.*

Proof. Let $\Omega_0 \subset \Omega$ be a connected component of $\Omega = \Omega_0 \cup \Omega_1$. We set

$$\Psi_0 = \begin{cases} \Psi & \text{in } \Omega_0, \\ 0 & \text{in } D \backslash \overline{\Omega_0}, \end{cases}$$

and set $\psi_0 = \Psi_0 + \chi$ with $\chi = \min\{\psi, r^2+\gamma\}$. Then,

$$\psi_0 = \begin{cases} \psi & \text{in } D \backslash \overline{\Omega_1}, \\ r^2+\gamma & \text{in } \Omega_1. \end{cases}$$

By $I'[\psi]\psi = 0$,

$$\begin{aligned}I'[\psi_0]\psi_0 &= \|\psi_0\|_H^2 - \lambda\int_D(\psi_0 - r^2-\gamma)_+^{2q-1}\psi_0\\ &= -\int_\Omega|\nabla_{z,r}\psi|^2 + \int_\Omega|\nabla_{z,r}\chi|^2 + \int_{\Omega_0}|\nabla_{z,r}\psi|^2 - \int_{\Omega_0}|\nabla_{z,r}\chi|^2 + \lambda\int_{\Omega_1}\Psi_+^{2q-1}\psi.\end{aligned}$$

Since the identities (4.7) and (4.8) hold also on Ω_0,

$$I'[\psi_0]\psi_0 = \lambda\int_{\Omega_1}\Psi_+^{2q-1}\chi \geq 0.$$

Thus $I'[t\Psi_0+\chi](t\Psi_0+\chi)$ is non-negative at $t=1$. By (3.3), $I'[t\Psi_0+\chi](t\Psi_0+\chi)$ is negative for large $t>1$. By the intermediate value theorem, there exists $t_0\geq 1$ such that $t_0\Psi_0+\chi\in N$. By (4.8),

$$\int_D|\nabla_{z,r}\chi|^2 = \int_{D\backslash\Omega}|\nabla_{z,r}\psi|^2 + \int_\Omega|\nabla_{z,r}(r^2+\gamma)|^2 = \int_D|\nabla_{z,r}\psi|^2 - \int_\Omega|\nabla_{z,r}\Psi|^2.$$

Applying (4.7) yields

$$\begin{aligned}I[\psi] \leq I[t_0\Psi_0+\chi] &= \frac{1}{2}\|t_0\Psi_0+\chi\|_H^2 - \frac{\lambda}{2q}\int_D(t_0\Psi_0+\chi-r^2-\gamma)_+^{2q}\\ &= I[\psi] + \frac{t_0^2}{2}\left(1-\frac{t_0^{2(q-1)}}{q}\right)\int_{\Omega_0}|\nabla_{z,r}\Psi|^2 - \frac{1}{2}\left(1-\frac{1}{q}\right)\int_\Omega|\nabla_{z,r}\Psi|^2\\ &\leq I[\psi] - \frac{1}{2}\left(1-\frac{1}{q}\right)\int_{\Omega_1}|\nabla_{z,r}\Psi|^2 \leq I[\psi].\end{aligned}$$

We conclude that $\Omega_0=\Omega$.

□

5. Ground states in a half plane

5.1. A pointwise estimate of a ground state. We construct a ground state in $\mathbb{R}^2_+$ by sending $R\to\infty$ to that in a half disk $D=D(R)$. To take a limit, we estimate a ground state in L^∞ uniformly for R by using the Green function of the Dirichlet problem

$$\begin{aligned}-L\psi &= r^2\zeta \quad \text{in } \mathbb{R}^2_+,\\ \psi &= 0 \qquad \text{on } \partial\mathbb{R}^2_+.\end{aligned}$$

The Grad-Shafranov equation (2.5) is the problem for $\zeta=\lambda r^{-2}(\psi-r^2-\gamma)_+^{2q-1}$. The operator $-r^{-2}L$ is viewed as the 5d Laplace operator by the transform $\psi\longmapsto\varphi=\psi/r^2$, i.e. $-\Delta_y\varphi=\zeta$ in $\mathbb{R}^5$. Solutions to this problem are represented by

$$\psi(z,r) = \int_{\mathbb{R}^2_+}G(z,r,z',r')\zeta(z',r')r'\mathrm{d}z'\mathrm{d}r', \tag{5.1}$$

with the Green function

$$G(z,r,z',r') = \frac{rr'}{2\pi}\int_0^{\pi} \frac{\cos\theta \mathrm{d}\theta}{\sqrt{|z-z'|^2+r^2+r'^2-2rr'\cos\theta}}.$$

The formula (5.1) is the axisymmetric Biot-Savart law and is available for decaying ψ/r as $z^2+r^2 \to \infty$. Since $-\Delta(\psi\nabla\theta) = r^2\xi\nabla\theta$ in $\mathbb{R}^3$, (5.1) is deduced from a representation using the Newton potential, e.g. [29, p.4], [28, p.472], [47, 19.1].

The Green function can be expressed in terms of the complete elliptic integrals of the first and second kind. By their asymptotic expansions with $\xi^2 = (|z-z'|^2+|r-r'|^2)/4rr'$, e.g. [28, p.482],

$$\frac{G(z,r,z',r')}{\sqrt{rr'}} \lesssim \begin{cases} |\log\xi| & \xi \le 1, \\ \xi^{-3} & \xi \ge 1. \end{cases}$$

Thus the Green function satisfies the pointwise estimate

$$G(z,r,z',r') \le C\frac{(rr')^{1/2+\tau}}{(|z-z'|^2+|r-r'|^2)^{\tau}}, \quad 0<\tau\le\frac{3}{2}, \quad {}^t(z,r),\ {}^t(z',r') \in \mathbb{R}^2_+. \tag{5.2}$$

In terms of $\zeta = \omega^{\theta}/r$, impulse and circulation (mass) can be written as

$$\frac{1}{2}\int_{\mathbb{R}^3} x\times\omega \mathrm{d}x = \pi\left(\int_{\mathbb{R}^2_+} r^3\zeta \mathrm{d}z\mathrm{d}r\right)e_z, \quad \int_{\{r=0\}} u\cdot \mathrm{d}l(x) = \int_{\mathbb{R}^2_+} r\zeta \mathrm{d}z\mathrm{d}r.$$

Besides them, we use a weighted $L^{1+\beta}$-norm to estimate the stream function.

Proposition 5.1. *Let* $0<\beta\le 1$ *and* $0<\delta<1$. *The estimate*

$$|\psi(z,r)| \le C\min\left\{r,\ \frac{1}{r^{1-\delta}}\right\}\left(\|r^3\zeta\|_1 + \|r\zeta\|_1 + \|r^{1+2\beta}\zeta^{1+\beta}\|_1^{1/(1+\beta)}\right),\ {}^t(z,r)\in\mathbb{R}^2_+, \tag{5.3}$$

holds for ψ *in (5.1) with some constant* C.

Proof. We set $s = \sqrt{|z-z'|^2+|r-r'|^2}$ and

$$|\psi(z,r)| \le \int_{\mathbb{R}^2_+} G(z,r,z',r')|\zeta(z',r')|r'\mathrm{d}z'\mathrm{d}r' = \int_{s<r/2} + \int_{s\ge r/2} =: I + II.$$

Since $r'^2/s^3 \le C/r$ for $s\ge r/2$, by the estimate of the Green function (5.2) for $\tau = 3/2$,

$$
\begin{aligned}
II &\lesssim \int_{s\geq r/2} \frac{(rr')^2}{s^3}|\zeta(z',r')|r'\mathrm{d}z'\mathrm{d}r' \lesssim r\|r\zeta\|_1,\\
II &\lesssim \frac{1}{r}\int_{s\geq r/2} r'^2|\zeta(z',r')|r'\mathrm{d}z'\mathrm{d}r' \lesssim \frac{1}{r}\|r^3\zeta\|_1.
\end{aligned}
$$

Thus, $II \lesssim \min\{r, r^{-1}\}(\|r\zeta\|_1 + \|r^3\zeta\|_1)$. By (5.3) for $0 < \tau \leq 3/2$ and the Hölder's inequality, for $0 < \alpha \leq \beta \leq 1$,

$$
\begin{aligned}
I &\lesssim r^{1/2+\tau}\int_{s<r/2} \frac{(r')^{1/2+\tau}}{s^{2\tau}} r'^2\zeta(z',r')\frac{1}{r'}\mathrm{d}z'\mathrm{d}r'\\
&\leq r^{1/2+\tau}\left(\int_{s<r/2} \frac{(r')^{(1/2+\tau)\sigma}}{s^{2\tau\sigma}}\frac{1}{r'}\mathrm{d}z'\mathrm{d}r'\right)^{1/\sigma}\left(\int_{s<r/2} (r'^2\zeta(z',r'))^{1+\alpha}\frac{1}{r'}\mathrm{d}z'\mathrm{d}r'\right)^{1/(1+\alpha)}\\
&= Cr^{2-1/(1+\alpha)}\|r^2\zeta\|_{L^{1+\alpha}(\{s<r/2\};r^{-1})},
\end{aligned}
$$

with some constant C, independent of r. The constant σ is the Hölder conjugate to $1+\alpha$. We chose $\tau < \alpha/(1+\alpha)$ so that the integral is finite. By the Hölder's inequality,

$$
\|r^2\zeta\|_{L^{1+\alpha}(\{s<r/2\};r^{-1})} \leq \|r^2\zeta\|^{\theta}_{L^{1+\beta}(\{s<r/2\};r^{-1})}\|r^2\zeta\|^{1-\theta}_{L^{1}(\{s<r/2\};r^{-1})}, \qquad \frac{1}{1+\alpha} = \frac{\theta}{1+\beta} + 1 - \theta.
$$

Since $\|r^2\zeta\|_{L^{1+\beta}(\mathbb{R}^2_+;r^{-1})} = \|r^{1+2\beta}\zeta^{1+\beta}\|_1^{1/(1+\beta)}$ and $\|r^2\zeta\|_{L^{1}(\{s<r/2\};r^{-1})} \lesssim \min\left\{1, r^{-2}\right\}(\|r\zeta\|_1 + \|r^3\zeta\|_1)$, we obtain

$$
I \lesssim \min\left\{r^{2-1/(1+\alpha)}, \frac{1}{r^{1/(1+\alpha)-2\theta}}\right\}\left(\|r^3\zeta\|_1 + \|r\zeta\|_1 + \|r^{1+2\beta}\zeta^{1+\beta}\|_1^{1/(1+\beta)}\right).
$$

The right-hand side is $O(r^{2-1/(1+\beta)})$ as $r \to 0$ for $\alpha = \beta$ and $O(r^{-1+\delta})$ as $r \to \infty$ for sufficiently small α. Thus

$$
I \lesssim \min\left\{r^{2-1/(1+\beta)}, \frac{1}{r^{1-\delta}}\right\}\left(\|r^3\zeta\|_1 + \|r\zeta\|_1 + \|r^{1+2\beta}\zeta^{1+\beta}\|_1^{1/(1+\beta)}\right).
$$

By combining the estimates for I and II, the desired estimate follows. □

Proposition 5.2. *The estimate (5.3) holds for solutions to the Dirichlet problem*

$$
\begin{aligned}
-L\psi &= r^2\zeta \quad \text{in } D(R),\\
\psi &= 0 \qquad \text{on } \partial D(R),
\end{aligned}
$$

with some constant independent of R.

Proof. The function $\varphi=\psi/r^2$ is a solution to $-\Delta_y\varphi=\zeta$ in B and $\varphi=0$ on ∂B. We may assume that ζ is non-negative. By zero extension of ζ, we set $\tilde{\psi}$ by (5.1). Then $\tilde{\varphi}=\tilde{\psi}/r^2$ is positive and satisfies $-\Delta_y\tilde{\varphi}=\zeta$ in B. Since $\varphi-\tilde{\varphi}$ is harmonic in B and negative on ∂B, $\varphi<\tilde{\varphi}$ by the maximum principle. Thus the result follows from Proposition 5.1. ◻

We apply the estimate (5.3) to the ground state constructed in Lemma 4.9. The following energy identity is essentially due to Friedman-Turkington [29, Lemma 3.2.].

Proposition 5.3. *The identity*

$$\frac{1}{2\pi^2}\|\psi\|_H^2=\frac{1}{\lambda^\beta}\|r^{1+2\beta}\zeta^{1+\beta}\|_1+\|r^3\zeta\|_1+\gamma\|r\zeta\|_1 \tag{5.4}$$

holds for solutions to (3.5) with $\zeta=\lambda r^{-2}(\psi-r^2-\gamma)_+^{2q-1}$ *and* $\beta=(2q-1)^{-1}\in(0,1)$.

Proof. By multiplying $\Psi=\psi-r^2-\gamma$ by $r\zeta$ and integrating it on D, we have

$$\int_D\psi r\zeta \mathrm{d}z\mathrm{d}r=\int_D\Psi r\zeta\mathrm{d}z\mathrm{d}r+\|r^3\zeta\|_1+\gamma\|r\zeta\|_1.$$

Since $\zeta=\lambda r^{-2}\Psi_+^{2q-1}$, $\Psi_+=(\lambda^{-1}r^2\zeta)^\beta$ for $\beta=(2q-1)^{-1}$ and

$$\int_D\Psi r\zeta\mathrm{d}z\mathrm{d}r=\frac{1}{\lambda^\beta}\int_D r^{1+2\beta}\zeta^{1+\beta}\mathrm{d}z\mathrm{d}r=\frac{1}{\lambda^\beta}\|r^{1+2\beta}\zeta^{1+\beta}\|_1.$$

Since $\varphi=\psi/r^2$ satisfies $-\Delta_y\varphi=\zeta$ in B, $\varphi=0$ on ∂B and $H(D;r^{-1})\cong F(B)$,

$$\int_D\psi r\zeta\mathrm{d}z\mathrm{d}r=-\int_D\varphi\Delta_y\varphi r^3\mathrm{d}z\mathrm{d}r=-\frac{1}{2\pi^2}\int_B\varphi\Delta_y\varphi\mathrm{d}y=\frac{1}{2\pi^2}\int_B|\nabla_y\varphi|^2\mathrm{d}y=\frac{1}{2\pi^2}\|\psi\|_H^2.$$

We obtained (5.4). ◻

Lemma 5.4. *The ground sate in Lemma 4.9 satisfies*

$$|\psi(z,r)|\leq C\min\left\{r,\ \frac{1}{r^{1-\delta}}\right\},\quad {}^t(z,r)\in D(R), \tag{5.5}$$

for $0<\delta<1$. *The constant* C *depends only on* $\|\psi\|_H$, λ *and* γ.

Proof. The result follows from Propositions 5.2 and 5.3. ◻

5.2. **Uniform boundedness of the vortex core.** We take an increasing sequence $\{R_n\}$ and set $H(D_n; r^{-1})$, $N(D_n; r^{-1})$ and $c_n = \inf_{N(D_n;r^{-1})} I$ with $D_n = D(R_n)$. By the zero extension,

$$\begin{aligned} &H(D_n; r^{-1}) \subset H(D_{n+1}; r^{-1}) \subset \cdots \subset H(\mathbb{R}^2_+; r^{-1}), \\ &N(D_n; r^{-1}) \subset N(D_{n+1}; r^{-1}) \subset \cdots \subset N(\mathbb{R}^2_+; r^{-1}), \\ &c_n \geq c_{n+1} \geq \cdots \geq c. \end{aligned}$$

Proposition 5.5.

$$\lim_{n\to\infty} c_n = c. \tag{5.6}$$

Proof. We prove that for arbitrary $\phi \in N(\mathbb{R}^2_+; r^{-1})$ there exists $\phi_n \in N(D_n; r^{-1})$ such that $\phi_n \to \phi$ in $H(\mathbb{R}^2_+; r^{-1})$. This implies that

$$c \leq \lim_{m\to\infty} c_n = \lim_{n\to\infty} \inf_{N(D_n;r^{-1})} I \leq \lim_{n\to\infty} I[\phi_n] = I[\phi].$$

By taking infimum for ϕ, (5.6) follows.

We construct $\tilde{\phi}_n \in H(\mathbb{R}^2_+; r^{-1})$ supported in D_n such that $\tilde{\phi}_n \to \phi$ in $H(\mathbb{R}^2_+; r^{-1})$ by a cut-off function argument. By Proposition 4.4, for $g_n(t) = I[t\tilde{\phi}_n]$ there exists $t(\tilde{\phi}_n) > 0$ such that

$$0 = \frac{\dot{g}_n(t(\tilde{\phi}_n))}{t(\tilde{\phi}_n)} = \|\tilde{\phi}_n\|^2_{H(D_n;r^{-1})} - 2\pi^2\lambda t(\tilde{\phi}_n)^{2(q-1)} \int_{D_n} \left(\tilde{\phi}_n - \frac{r^2+\gamma}{t(\tilde{\phi}_n)}\right)^{2q-1}_+ \tilde{\phi}_n \frac{1}{r} \mathrm{d}z\mathrm{d}r.$$

Since $\lim_{n\to\infty} \|\tilde{\phi}_n\|_{H(D_n;r^{-1})} = \|\phi\|_{H(\mathbb{R}^2_+;r^{-1})} \neq 0$, the sequence $\{t(\tilde{\phi}_n)\}$ is bounded.

Suppose that $\{t(\tilde{\phi}_n)\}$ does not converge to 1. Then, there exists a subsequence such that $t(\tilde{\phi}_n) \to t_0$ for some $t_0 > 0$. Sending $n \to \infty$ implies that $\dot{g}(t_0) = 0$ for $g(t) = I[t\phi]$. Since Proposition 4.4 holds also for $\mathbb{R}^2_+$ and $\phi \in N(\mathbb{R}^2_+; r^{-1})$, $t_0 = t(\phi) = 1$. We thus conclude that $t(\tilde{\phi}_n) \to 1$.

The desired sequence is obtained by setting $\phi_n = t(\tilde{\phi}_n)\tilde{\phi}_n \in N(D_n; r^{-1})$. □

The ground state ψ_n of $c_n = \inf_{N(D_n;r^{-1})} I$ is uniformly bounded and equi-continuous in $\overline{\mathbb{R}^2_+}$ with a bounded vortex core.

Proposition 5.6. *The ground state $\psi_n \in N(D_n; r^{-1})$ in Lemma 4.9 satisfies*

$$\sup_n \|\psi_n\|_{H(D_n;r^{-1})} < \infty. \tag{5.7}$$

Proof. By (4.2),

$$\frac{1}{2}\left(1 - \frac{1}{q}\right)\|\psi_n\|^2_{H(D_n;r^{-1})} \leq I[\psi_n] = c_n \leq c_1.$$

□

Proposition 5.7. *There exists some $\nu \in (0,1)$ such that $\varphi_n = \psi_n/r^2$ satisfies*

$$\sup_n \|\varphi_n\|_{C^\nu(\mathbb{R}^5)} < \infty. \tag{5.8}$$

Proof. By (5.5) and (5.7),

$$|\psi_n(z,r)| \le C \min\left\{r,\ \frac{1}{r^{1-\delta}}\right\}, \quad {}^t(z,r) \in D_n,$$

for $0 < \delta < 1$, with some constant C, independent of n. Thus $\zeta_n = \lambda r^{-2}(\psi_n - r^2 - \gamma)_+^{2q-1}$ satisfies $\zeta_n \le C(1 + r^{-1})$, $(z,r) \in D_n$. Hence,

$$\sup_n \|\zeta_n\|_{L^s_{\mathrm{ul}}(\mathbb{R}^5)} < \infty, \quad \frac{5}{2} < s < 4,$$

where $L^s_{\mathrm{ul}}(\mathbb{R}^5)$ denotes the uniformly local L^s space on $\mathbb{R}^5$. Since $-\Delta\varphi_n = \zeta_n$ in B_n and $\varphi_n = 0$ on ∂B_n, by the elliptic regularity, uniformly local L^s-norm of φ_n is uniformly bounded up to second orders. Thus (5.8) holds for $\nu = 1 - 5/s^*$ and $1/s^* = 1/s - 1/5$ by the Sobolev embedding. □

The equi-continuity of $\{\varphi_n\}$ implies that the vortex core is uniformly bounded [3, Lemma 4.1].

Proposition 5.8. *There exists $R > 0$ such that $\Omega_n = \{{}^t(z,r) \in \mathbb{R}^2_+ \mid \psi_n - r^2 - \gamma > 0\} \subset D(R)$ for all $n \ge 1$.*

Proof. For notational simplicity, we write $\varphi_n(z,r) = \varphi_n(y)$. By the Sobolev inequality $F(\mathbb{R}^5) \subset L^{10/3}(\mathbb{R}^5)$,

$$\begin{aligned}
|\{z \in \mathbb{R} \mid \varphi_n(z,r_0) \ge 1/2\}| &\lesssim \int_{-\infty}^{\infty} \varphi_n^{8/3}(z,r_0)\mathrm{d}z \\
&= -\int_{r_0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}r} \int_{-\infty}^{\infty} \varphi_n^{8/3}\mathrm{d}z\mathrm{d}r \\
&\lesssim \frac{1}{r_0^3} \int_0^{\infty}\int_{-\infty}^{\infty} |\nabla\varphi_n|\varphi_n^{5/3} r^3 \mathrm{d}z\mathrm{d}r \lesssim \frac{1}{r_0^3}\|\varphi_n\|^{8/3}_{F(B_n)}.
\end{aligned}$$

Thus by (5.7) and $H(D_n; r^{-1}) \cong F(B_n)$,

$$|\{z \in \mathbb{R} \mid \varphi_n(z,r_0) \ge 1/2\}| \le \frac{C}{r_0^3}, \quad r_0 > 0,$$

with some constant C, independent of n. We take $r_n > 0$ such that $r_n = \sup\{r \mid {}^t(z,r) \in \Omega_n\}$. The maximum point lies on the r-axis and $\partial\Omega_n$ since Ω_n is symmetric for z. Thus $\varphi_n(0,r_n) =$

$1 + r_n^{-2}\gamma \geq 1$. Since $\{\varphi_n\}$ is equi-continuous around ${}^t(0, r_n)$ by (5.8), there exists $\delta_1 > 0$ such that

$$\varphi_n(z, r_n) \geq \frac{1}{2}, \quad |z| \leq \delta_1.$$

Thus $r_n \lesssim \delta_1^{-1/3}$ and Ω_n is bounded in the r-direction.

In a similar way, we take $z_n > 0$ such that $z_n = \sup\{z \mid {}^t(z, r) \in \Omega_n\}$ and the maximum point ${}^t(z_n, r_n)$. Since $\varphi_n(z_n, r_n) \geq 1$, by equi-continuity of φ_n, there exists $\delta_2 > 0$ such that

$$\varphi_n(z_n, r_n + \delta_2) \geq \frac{1}{2}.$$

Since φ_n is decreasing for z, $z_n \lesssim \delta_2^{-3}$ and Ω_n is bounded in the z-direction. □

5.3. **Convergence to a ground state in $\mathbb{R}^2_+$.**

Lemma 5.9. *The sequence $\{\varphi_n\}$ subsequently converges to a limit φ locally uniformly in $\mathbb{R}^5$. The limit $\psi = r^2\varphi \in N(\mathbb{R}^2_+; r^{-1})$ is a ground state to (3.4) such that $I'[\psi] = 0$ and $\psi = \psi^*$ with a bounded vortex core $\Omega = \{{}^t(z, r) \in \mathbb{R}^2_+ \mid \psi - r^2 - \gamma > 0\}$.*

Proof. The sequence $\{\varphi_n\}$ is uniformly bounded and equi-continuous by the uniform estimate (5.8). By choosing a subsequence, φ_n converges to a limit φ locally uniformly in $\mathbb{R}^5$ by Ascoli-Arzelá theorem. By Proposition 5.8, $U_n = \{y = {}^t(y_1, y') \in \mathbb{R}^5 \mid y_1 = z,\ |y'| = r,\ {}^t(z, r) \in \Omega_n\}$ is uniformly bounded. Thus, $\zeta_n = \lambda r^{4(q-1)}(\varphi_n - 1 - r^{-2}\gamma)_+^{2q-1}$ converges to $\zeta = \lambda r^{4(q-1)}(\varphi - 1 - r^{-2}\gamma)_+^{2q-1} = \lambda r^{-2}(\psi - r^2 - \gamma)_+^{2q-1}$ uniformly in $\mathbb{R}^5$ and $\Omega = \{\psi - r^2 - \gamma > 0\}$ is bounded. By $\operatorname{spt}\omega^\theta = \operatorname{spt}\zeta = \overline{\Omega}$, Ω is the vortex core. By the uniform estimate of the Dirichlet energy (5.7), $\psi \in H(\mathbb{R}^2_+; r^{-1})$. The properties $I'[\psi] = 0$ and $\psi = \psi^*$ follow from those of ψ_n.

We show that the limit φ is non-trivial. Suppose that $\varphi \equiv 0$. Then, $0 \leq \varphi_n \leq 1$ in B_n for large n. By Lemma 3.2, φ_n solves $-\Delta_y\varphi_n = \lambda r^{4(q-1)}(\varphi_n - 1 - r^{-2}\gamma)_+^{2q-1} = 0$ in B_n and $\varphi_n = 0$ on ∂B_n. Thus $\varphi_n \equiv 0$. This contradicts $\psi_n \not\equiv 0$. Thus the limit is non-trivial. In particular, $\psi \in N(\mathbb{R}^2_+; r^{-1})$.

It remains to show that ψ is a ground state to (3.4). By the uniform boundedness of the vortex core $\Omega_n = \{\psi_n - r^2 - \gamma > 0\} \subset D(R)$,

$$\begin{aligned} J'[\psi_n]\psi_n &= 2\pi^2\lambda \int_{D(R)} (\psi_n - r^2 - \gamma)_+^{2q-1}\psi_n \frac{1}{r}\mathrm{d}z\mathrm{d}r \\ &\to 2\pi^2\lambda \int_{D(R)} (\psi - r^2 - \gamma)_+^{2q-1}\psi \frac{1}{r}\mathrm{d}z\mathrm{d}r = J'[\psi]\psi \quad \text{as } n \to \infty. \end{aligned}$$

By $0 = I'[\psi_n]\psi_n = \|\psi_n\|^2_{H(\mathbb{R}^2_+; r^{-1})} + J'[\psi_n]\psi_n$ and $0 = I'[\psi]\psi = \|\psi\|^2_{H(\mathbb{R}^2_+; r^{-1})} + J'[\psi]\psi$, we have $\lim_{n\to\infty}\|\psi_n\|_{H(\mathbb{R}^2_+; r^{-1})} = \|\psi\|_{H(\mathbb{R}^2_+; r^{-1})}$. By choosing a subsequence, $\psi_n \to \psi$ in $H(\mathbb{R}^2_+; r^{-1})$. By continuity of I and (5.6),

$$\inf_{N(\mathbb{R}^2_+;r^{-1})} I = c = \lim_{n\to\infty} c_n = \lim_{n\to\infty} I[\psi_n] = I[\psi].$$

Thus ψ is a ground state to (3.4). □

Proof of Theorem 2.1. The critical point $\psi \in H(\mathbb{R}^2_+; r^{-1})$ is a classical solution $\psi \in C^{2+\nu}(\overline{\mathbb{R}^2_+})$ of (2.5) for $\nu \in (0,1)$ and ψ is decreasing for $|z|$ by a maximum principle as we proved Lemma 3.2 and Proposition 4.10 for a ground state in a half disk. The level sets $\{(\psi - r^2 - \gamma)_+ = l\}$ are closed curves of class $C^{2+\nu}$ for $0 \leq l < l_0$ and $l_0 = \|(\psi - r^2 - \gamma)_+\|_\infty$ and points for $l = l_0$. The vortex core Ω is bounded, connected and simply-connected as we proved Lemma 4.12 for a ground state in D. Since $r^{-1}\nabla_{z,r}G(z,r,z',r') \to 0$ as $z^2 + r^2 \to \infty$ for each ${}^t(z',r') \in \mathbb{R}^2_+$ and $\zeta = \lambda r^{-2}(\psi - r^2 - \gamma)_+^{2q-1}$ is supported in $\overline{\Omega}$, $r^{-1}\nabla_{z,r}\psi \to 0$ follows from (5.1). □

Proof of Theorem 1.1. By rotational invariance of (1.3), we may assume that $u_\infty = {}^t(0,0,-W)$ for $W > 0$. We take $1 < q < \infty$ and $0 < \lambda, \gamma < \infty$. Then by Theorem 2.1, there exists a solution $\psi \in C^{2+\nu}(\overline{\mathbb{R}^2_+})$ of (2.5) for $\nu \in (0,1)$. We set $f = \dot{\Gamma}(\Psi)$ by $\Psi = \psi - Wr^2/2 - \gamma$ and $\dot{\Gamma}(t) = (\lambda q)^{1/2} t_+^{q-1}$. Then, $f \in C(\overline{\mathbb{R}^2_+})$. By regarding it to an axisymmetric function in $\mathbb{R}^3$, $f \in C(\mathbb{R}^3)$. If $q > 3$, $f \in C^{2+\mu}(\mathbb{R}^3)$ for some $\mu \in (0,1)$.

The velocity u defined by (2.1) satisfies $u \in C^1(\mathbb{R}^3 \backslash \{r = 0\})$ and $\nabla \times u = fu$ by (2.2). Since f is supported on a solid torus rotating $\overline{\Omega}$ around the z-axis, u is harmonic near the z-axis. By (2.1) and (2.5),

$$\begin{aligned} u &= -\frac{1}{r}\partial_z \Psi e_r(\theta) + \frac{1}{r}\Gamma(\Psi) e_\theta(\theta) + \frac{1}{r}\partial_r \Psi e_z \\ &= -\frac{1}{r}\partial_z \psi e_r(\theta) + \frac{1}{r}\Gamma(\Psi) e_\theta(\theta) + \frac{1}{r}\partial_r \psi e_z + u_\infty \to u_\infty \quad \text{as } |x| \to \infty. \end{aligned}$$

Thus $u \in C^1(\mathbb{R}^3)$ is an axisymmetric solution to (1.3). Since the level sets $\{(\psi - Wr^2/2 - \gamma)_+ = l\}$ for $0 < l < l_0$ are symmetric nested closed curves in $\mathbb{R}^2_+$ (resp. points for $l = l_0$), the level sets $f^{-1}(k)$ are symmetric nested tori for $0 < k < k_0$ with $k_0 = \sqrt{\lambda q} l_0^{q-1}$ (resp. circles for $k = k_0$). The torus $f^{-1}(k) = \Sigma(l)$ includes vortex lines (stream lines) associated with $l = (k/\sqrt{\lambda q})^{1/(q-1)}$ for $0 < k < k_0$. All vortex lines are closed or quasi-periodic if $\Theta(l)$ is commensurable with 2π or not. Thus $f^{-1}(k)$ is an invariant torus. The proof is now complete. □

Remark 5.10. The asymptotics (1.4) follows from the decay $r^{-1}\nabla_{z,r}\psi = O(|x|^{-3})$ as $|x| \to \infty$. Since $\varphi(y) = \psi(z,r)/r^2$ for $y = {}^t(y_1, y')$, $z = y_1$, $r = |y'|$, is an axisymmetric solution to $-\Delta_y \varphi = \zeta$ in $\mathbb{R}^5$ for compactly supported ζ, this decay follows from $\varphi = C|y|^{-3} *_{\mathbb{R}^5} \zeta$.

Acknowledgements

The property (1.4) and the references [31], [16], [18] were informed by Professor Daniel Peralta-Salas. The reference [30] was informed by Professor Yasuhide Fukumoto. The introduction and the proof of Proposition 4.8 were improved by suggestions of the referee. The author is grateful to Professors Daniel Peralta-Salas, Yasuhide Fukumoto and referees for their helpful comments and suggestions. This work is partially supported by JSPS through the Grant-in-aid for Young Scientist 20K14347, Scientific Research (B) 17H02853 and MEXT Promotion of Distinctive Joint Research Center Program Grant Number JPMXP0619217849.

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(K. ABE) Department of Mathematics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku Osaka, 558-8585, Japan

E-mail address: kabe@osaka-cu.ac.jp