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Atsuo Kuniba, Masato Okado

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**TETRAHEDRON EQUATION AND QUANTUM R MATRICES
FOR q -OSCILLATOR REPRESENTATIONS OF
 $U_q(A_{2n}^{(2)})$, $U_q(C_n^{(1)})$ AND $U_q(D_{n+1}^{(2)})$**

ATSUO KUNIBA AND MASATO OKADO

Abstract

The intertwiner of the quantized coordinate ring $A_q(sl_3)$ is known to yield a solution to the tetrahedron equation. By evaluating their n -fold composition with special boundary vectors we generate series of solutions to the Yang-Baxter equation. Finding their origin in conventional quantum group theory is a clue to the link between two and three dimensional integrable systems. We identify them with the quantum R matrices associated with the q -oscillator representations of $U_q(A_{2n}^{(2)})$, $U_q(C_n^{(1)})$ and $U_q(D_{n+1}^{(2)})$.

1. INTRODUCTION

The tetrahedron equation [?] is a generalization of the Yang-Baxter equation [?] and serves as a key to the quantum integrability in three dimension (3d). Among its many formulations the homogeneous version of vertex type has the form

$$\mathcal{R}_{1,2,4}\mathcal{R}_{1,3,5}\mathcal{R}_{2,3,6}\mathcal{R}_{4,5,6} = \mathcal{R}_{4,5,6}\mathcal{R}_{2,3,6}\mathcal{R}_{1,3,5}\mathcal{R}_{1,2,4},$$

where \mathcal{R} is a linear operator on the tensor cube of some vector space F . The equality holds in $\text{End}(F^{\otimes 6})$ where the indices indicate the components on which each \mathcal{R} acts nontrivially. We call a solution to the tetrahedron equation a 3d R .

In the tetrahedron equation one sees that if the spaces 4, 5 and 6 are evaluated away appropriately, it reduces to the Yang-Baxter equation:

$$\mathcal{R}_{1,2}\mathcal{R}_{1,3}\mathcal{R}_{2,3} = \mathcal{R}_{2,3}\mathcal{R}_{1,3}\mathcal{R}_{1,2}.$$

By now, algebraic background of the Yang-Baxter equation has been well understood by the representation theory of quantum groups and their generalizations. Thus the following problem arises. Given a 3d R , find a prescription to reduce it to a solution of the Yang-Baxter equation and clarify its context in the framework of quantum group theory. It is a clue to the connection between integrability in two and three dimensions.

In this paper we present the solution of the problem for the distinguished example of 3d R \mathcal{R} [?, ?]. It acts on the tensor cube of the q -oscillator Fock space F and possesses several remarkable features related to the quantized coordinate ring $A_q(sl_3)$, the PBW bases of the nilpotent subalgebra of $U_q(sl_3)$ and so on. See Section ?? and Appendix ?? for more accounts on the \mathcal{R} .

Our prescription for the reduction is parallel with the earlier work concerning 3d L operator [?]. Namely we take matrix elements of the tetrahedron equation by using certain vectors in the 4, 5 and 6 th components in $F^{\otimes 6}$. These vectors contain a spectral parameter z and serve as special boundary conditions in the context of the 3d lattice model associated with the \mathcal{R} . In fact the tetrahedron equation itself admits a straightforward extension to the n -site situation (see (??)) for which the reduction works equally. In this way the single 3d R \mathcal{R} yields infinite series of solutions of the Yang-Baxter equation labeled by n and the boundary vectors. Up to an overall scalar they are rational functions of q and the (multiplicative) spectral parameter z , leading to integrable 2d vertex models having local states in $F^{\otimes n}$.

Our main result is Theorem ??, which identifies these solutions of the Yang-Baxter equation with the quantum R matrices for the q -oscillator representations of $U_q(D_{n+1}^{(2)})$, $U_q(A_{2n}^{(2)})$ and $U_q(C_n^{(1)})$ on $F^{\otimes n}$ depending on the choice of the boundary vectors. Namely the solutions coincide with the intertwiner of the tensor products up to an overall scalar. Actually Theorem ?? has also guided us to introduce the q -oscillator representation themselves. For type $A_n^{(1)}$ or C_n they were introduced in [?] using a q -analogue of the Weyl algebra. Apart from complementing the latter to $C_n^{(1)}$, the q -oscillator representations for type $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}$ in this paper containing B_n as a classical part seem to be new. An intriguing feature of them is that the quantum parameter q cannot be specialized to be 1. A similar singularity at $q = 1$ has been known for the unitary representations of non-compact real forms of $U_q(sl_2)$ [?, ?]. However our case has another distinctive aspect that the action of some weight generators k_j acquire the factor $i = \sqrt{-1}$ besides a power of q .

We have done the task of determining the spectral decomposition of the associated new quantum R matrices which consists of infinitely many irreducible components. It provides the information complementary with the explicit formula (??).

This work is motivated by several preceding results. A relation between the tetrahedron equation and quantum R matrices goes back, for example, to [?, ?]. In [?], the reduction was made for the same \mathcal{R} by taking the *trace* and the consequent solution to the Yang-Baxter equation was announced to be the direct sum of the quantum R matrices for the symmetric tensor representations of $U_q(A_{n-1}^{(1)})$. We have summarized it in Appendix ?? for comparison. A further result on the trace reduction is available for $n = 2$ [?]. In [?], the reduction based on the same boundary vectors as this paper was studied for the n -product of the 3d L operator [?]. The result was identified with the quantum R matrices for the spin representations of $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ [?] and $U_q(D_{n+1}^{(2)})$. See Remark ?? and [?, Remark 7.2] for the comparison of these quantum affine algebras and those captured in this paper. A notable fact is that the boundary vectors specify the end shape of the Dynkin diagram of the relevant affine Lie algebras. In our previous paper [?], the reduction using the same boundary vectors was applied to the *single* \mathcal{R} and the result was identified with the quantum R matrices for q -oscillator representations of the *rank one* quantum affine algebras $U_q(A_1^{(1)})$ and $U_q(A_2^{(2)})$. The present paper contains these results as the $n = 1$ case by regarding $U_q(D_2^{(2)})$ and $U_q(C_1^{(1)})$ as $U_q(A_1^{(1)})$ appropriately. We note that a more general problem of studying the *mixture* of 3d R and L operators has been formulated and the simplest case has been worked out in [?, Section 5].

The outline of the paper is as follows. In Section ?? we recall the prescription [?, ?] to generate solutions to the Yang-Baxter equation from a solution to the tetrahedron equation using boundary vectors. We then apply it to the 3d R (??) acting on the tensor cube $F^{\otimes 3}$ of the Fock space of q -oscillators. There are two boundary vectors leading to the four families of solutions $S^{s,t}(z)$ ($s, t = 1, 2$) of the Yang-Baxter equation (??)–(??). They correspond to vertex models on planar square lattice whose local states range over $F^{\otimes n}$.

In Section ?? we introduce the q -oscillator representations of the Drinfeld-Jimbo quantum affine algebras $U_q(D_{n+1}^{(2)})$, $U_q(A_{2n}^{(2)})$ and $U_q(C_n^{(1)})$. Their tensor product decomposes into a direct sum of infinitely many irreducible submodules with respect to the classical part $U_q(B_n)$ or $U_q(C_n)$. The spectral decomposition of the R matrices is done in Section ?? and ??, although this part is not used in the rest.

In Section ?? we give our main result Theorem ?. It identifies the solutions $S^{s,t}(z)$ of the Yang-Baxter equation with the quantum R matrices for q -oscillator representations. Depending on the choice of the boundary vectors, $U_q(D_{n+1}^{(2)})$, $U_q(A_{2n}^{(2)})$ and $U_q(C_n^{(1)})$ cases are covered. The correspondence between the boundary vectors and the Dynkin diagrams of the relevant affine Lie algebras in Remark ?? is parallel but not identical to the earlier observation in [?, Remark 7.2] concerning 3d L operators. Our proof of Theorem ?? is done by using the characterization of the quantum R matrices without recourse to their explicit forms. It implies that the commutativity with the q -oscillator representation of U_q is embedded into intertwining relations of the quantized coordinate ring A_q through the evaluation by boundary vectors.

Appendix ?? contains a brief guide to the 3d R \mathcal{R} from the representation theory of the quantized coordinate ring $A_q(sl_3)$ [?]. All the lemmas necessary for the proof of Theorem ?? are prepared.

Appendix ?? is an exposition of the $U_q(A_{n-1}^{(1)})$ case in the setting of this paper. It is relevant to the trace reduction of the tetrahedron equation [?].

Throughout the paper we assume that q is generic and use the following notations:

$$(z; q)_m = \prod_{k=1}^m (1 - zq^{k-1}), \quad (q)_m = (q; q)_m, \quad \binom{m}{k}_q = \frac{(q)_m}{(q)_k (q)_{m-k}},$$

$$[m] = [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]_{q!} = \prod_{k=1}^m [k]_q, \quad \left[\begin{matrix} m \\ k \end{matrix} \right] = \frac{[m]!}{[k]! [m-k]!},$$

where the both q -binomials are to be understood as zero unless $0 \leq k \leq m$.

2. SOLUTIONS OF THE YANG-BAXTER EQUATION FROM 3D R

2.1. General scheme. Let F be a vector space and $\mathcal{R} \in \text{End}(F^{\otimes 3})$. Consider the tetrahedron equation:

$$\mathcal{R}_{1,2,4} \mathcal{R}_{1,3,5} \mathcal{R}_{2,3,6} \mathcal{R}_{4,5,6} = \mathcal{R}_{4,5,6} \mathcal{R}_{2,3,6} \mathcal{R}_{1,3,5} \mathcal{R}_{1,2,4}, \quad (2.1)$$

which is an equality in $\text{End}(F^{\otimes 6})$. Here $\mathcal{R}_{i,j,k}$ acts as \mathcal{R} on the i, j, k th components from the left in the tensor product $F^{\otimes 6}$.

Let us recall the prescription which produces an infinite family of solutions to the Yang-Baxter equation from a solution to the tetrahedron equation based on special boundary vectors [?]. First we regard (??) as a one-site relation, and extend it to the n -site version rather straight-

forwardly. Let $\overset{\alpha_i}{F}, \overset{\beta_i}{F}, \overset{\gamma_i}{F}$ be the copies of F , where α_i, β_i and γ_i ($i = 1, \dots, n$) are just labels and not parameters. Renaming the spaces 1, 2, 3 by them, we have $\mathcal{R}_{\alpha_i, \beta_i, 4} \mathcal{R}_{\alpha_i, \gamma_i, 5} \mathcal{R}_{\beta_i, \gamma_i, 6} \mathcal{R}_{4,5,6} = \mathcal{R}_{4,5,6} \mathcal{R}_{\beta_i, \gamma_i, 6} \mathcal{R}_{\alpha_i, \gamma_i, 5} \mathcal{R}_{\alpha_i, \beta_i, 4}$ for each i . Thus for any i one can carry $\mathcal{R}_{4,5,6}$ through $\mathcal{R}_{\alpha_i, \beta_i, 4} \mathcal{R}_{\alpha_i, \gamma_i, 5} \mathcal{R}_{\beta_i, \gamma_i, 6}$ to the left converting it to the reverse order product $\mathcal{R}_{\beta_i, \gamma_i, 6} \mathcal{R}_{\alpha_i, \gamma_i, 5} \mathcal{R}_{\alpha_i, \beta_i, 4}$. Repeating this n times leads to the relation

$$\begin{aligned} & (\mathcal{R}_{\alpha_1, \beta_1, 4} \mathcal{R}_{\alpha_1, \gamma_1, 5} \mathcal{R}_{\beta_1, \gamma_1, 6}) \cdots (\mathcal{R}_{\alpha_n, \beta_n, 4} \mathcal{R}_{\alpha_n, \gamma_n, 5} \mathcal{R}_{\beta_n, \gamma_n, 6}) \mathcal{R}_{4,5,6} \\ & = \mathcal{R}_{4,5,6} (\mathcal{R}_{\beta_1, \gamma_1, 6} \mathcal{R}_{\alpha_1, \gamma_1, 5} \mathcal{R}_{\alpha_1, \beta_1, 4}) \cdots (\mathcal{R}_{\beta_n, \gamma_n, 6} \mathcal{R}_{\alpha_n, \gamma_n, 5} \mathcal{R}_{\alpha_n, \beta_n, 4}). \end{aligned} \quad (2.2)$$

This is an equality in $\text{End}(\overset{\alpha}{F} \otimes \overset{\beta}{F} \otimes \overset{\gamma}{F} \otimes \overset{4}{F} \otimes \overset{5}{F} \otimes \overset{6}{F})$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is the array of labels and $\overset{\alpha}{F} = \overset{\alpha_1}{F} \otimes \cdots \otimes \overset{\alpha_n}{F}$ ($= F^{\otimes n}$). The notations $\overset{\beta}{F}$ and $\overset{\gamma}{F}$ should be understood similarly. The argument so far is just a 3d analogue of the simple fact in 2d that a single $RLL = LLR$ relation for a local L operator implies a similar relation for the n -site monodromy matrix in the quantum inverse scattering method.

Now we turn to the special boundary vectors. Suppose we have a vector $|\chi_s(x)\rangle \in F$ depending on a variable x such that its tensor product

$$|\chi_s(x, y)\rangle = |\chi_s(x)\rangle \otimes |\chi_s(xy)\rangle \otimes |\chi_s(y)\rangle \in F \otimes F \otimes F \quad (2.3)$$

satisfies the relation

$$\mathcal{R}|\chi_s(x, y)\rangle = |\chi_s(x, y)\rangle. \quad (2.4)$$

The index s is put to distinguish possibly more than one such vectors. Suppose there exist vectors in the dual space

$$\langle \chi_s(x, y) | = \langle \chi_s(x) | \otimes \langle \chi_s(xy) | \otimes \langle \chi_s(y) | \in F^* \otimes F^* \otimes F^*$$

having the similar property

$$\langle \chi_s(x, y) | \mathcal{R} = \langle \chi_s(x, y) |. \quad (2.5)$$

Then evaluating (??) between $\langle \chi_s(x, y) |$ and $|\chi_t(1, 1)\rangle^1$, one encounters the object

$$S_{\alpha, \beta}(z) = \varrho^{s, t}(z) \langle \chi_s(z) | \mathcal{R}_{\alpha_1, \beta_1, 3} \mathcal{R}_{\alpha_2, \beta_2, 3} \cdots \mathcal{R}_{\alpha_n, \beta_n, 3} | \chi_t(1) \rangle \in \text{End}(\overset{\alpha}{F} \otimes \overset{\beta}{F}), \quad (2.6)$$

where the scalar $\varrho^{s, t}(z)$ is inserted to control the normalization. The composition of \mathcal{R} and matrix elements are taken with respect to the space signified by 3. Plainly one may write it as $S(z) \in \text{End}(F^{\otimes n} \otimes F^{\otimes n})$ removing the dummy labels. Remember that $S(z)$ of course depends on s and t although they have been temporarily suppressed in the notation. It follows from (??), (??) and (??) that $S(z)$ satisfies the Yang-Baxter equation:

$$S_{\alpha, \beta}(x) S_{\alpha, \gamma}(xy) S_{\beta, \gamma}(y) = S_{\beta, \gamma}(y) S_{\alpha, \gamma}(xy) S_{\alpha, \beta}(x) \in \text{End}(\overset{\alpha}{F} \otimes \overset{\beta}{F} \otimes \overset{\gamma}{F}). \quad (2.7)$$

This fact holds for each choice of (s, t) . See [?, Section 5] for a further generalization of the procedure to deduce solutions to the Yang-Baxter equation by mixing more than one kind of solutions to the tetrahedron equation.

2.2. 3d R and boundary vectors. Let us proceed to a concrete realization of the above scheme considered in this paper. We will always take F to be an infinite dimensional space $F = \bigoplus_{m \geq 0} \mathbb{Q}(q) |m\rangle$ with a generic parameter q . The dual space will be denoted by $F^* = \bigoplus_{m \geq 0} \mathbb{Q}(q) \langle m|$ with the bilinear pairing

$$\langle m|n\rangle = (q^2)_m \delta_{m, n}. \quad (2.8)$$

The solution \mathcal{R} of the tetrahedron equation we are concerned with is the one obtained as the intertwiner of the quantum coordinate ring $A_q(sl_3)$ [?]², which was also found from a quantum geometry consideration in a different gauge including square roots [?, ?]. They were shown to be essentially the same object and to constitute the solution of the 3d reflection equation in [?]. It can also be identified with the transition matrix of the PBW bases of the nilpotent subalgebra of $U_q(sl_3)$ [?, ?]. Here we simply call it 3d R . It is given by

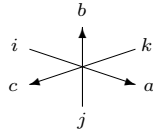
$$\mathcal{R}(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a, b, c \geq 0} \mathcal{R}_{i, j, k}^{a, b, c} |a\rangle \otimes |b\rangle \otimes |c\rangle, \quad (2.9)$$

$$\mathcal{R}_{i, j, k}^{a, b, c} = \delta_{i+j}^{a+b} \delta_{j+k}^{b+c} \sum_{\lambda + \mu = b} (-1)^\lambda q^{i(c-j) + (k+1)\lambda + \mu(\mu-k)} \frac{(q^2)_{c+\mu}}{(q^2)_c} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2}, \quad (2.10)$$

where $\delta_n^m = \delta_{m, n}$ just to save the space. The sum (??) is over $\lambda, \mu \geq 0$ satisfying $\lambda + \mu = b$, which is also bounded by the condition $\mu \leq i$ and $\lambda \leq j$. The formula (??) is taken from [?, eq.(2.20)]. The fact that

$$\mathcal{R}_{i, j, k}^{a, b, c} = 0 \text{ unless } (a + b, b + c) = (i + j, j + k) \quad (2.11)$$

plays an important role and will be referred to as *conservation law*. Further properties of \mathcal{R} have been summarized in Appendix ???. It is natural to depict (??) as follows:



¹This could be chosen as $|\chi_t(x', y')\rangle$ in general. However in all the examples studied in this paper, such a freedom is absorbed into (x, y) .

²The formula for it on p194 in [?] contains a misprint unfortunately. Eq. (??) here is a correction of it.

Let us turn to the vectors $|\chi_s(z)\rangle$ and $\langle\chi_s(z)|$ in (??)–(??). We use two such vectors obtained in [?]. In the present notation they read

$$|\chi_1(z)\rangle = \sum_{m \geq 0} \frac{z^m}{(q)_m} |m\rangle, \quad |\chi_2(z)\rangle = \sum_{m \geq 0} \frac{z^m}{(q^4)_m} |2m\rangle, \quad (2.12)$$

$$\langle\chi_1(z)| = \sum_{m \geq 0} \frac{z^m}{(q)_m} \langle m|, \quad \langle\chi_2(z)| = \sum_{m \geq 0} \frac{z^m}{(q^4)_m} \langle 2m|. \quad (2.13)$$

2.3. Solution $S^{s,t}(z)$ to the Yang-Baxter equation. We define the four families of solutions to the Yang-Baxter equation $S(z) = S^{s,t}(z) = S^{s,t}(z, q)$ ($s, t = 1, 2$) by the formula (??) by substituting (??), (??) and (??) into it. Each family consists of the solutions labeled with $n \in \mathbb{Z}_{\geq 1}$. They are the matrices acting on $F^{\otimes n} \otimes F^{\otimes n}$ whose elements are given by [?, Remark 1]

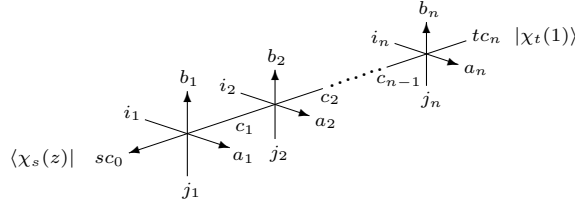
$$S^{s,t}(z)(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle) = \sum_{\mathbf{a}, \mathbf{b}} S^{s,t}(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle, \quad (2.14)$$

$$S^{s,t}(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} = \varrho^{s,t}(z) \sum_{c_0, \dots, c_n \geq 0} \frac{z^{c_0} (q^2)_{sc_0}}{(q^{s^2})_{c_0} (q^{t^2})_{c_n}} \mathcal{R}_{i_1, j_1, c_1}^{a_1, b_1, sc_0} \mathcal{R}_{i_2, j_2, c_2}^{a_2, b_2, c_1} \dots \mathcal{R}_{i_{n-1}, j_{n-1}, c_{n-1}}^{a_{n-1}, b_{n-1}, c_{n-2}} \mathcal{R}_{i_n, j_n, tc_n}^{a_n, b_n, c_{n-1}}, \quad (2.15)$$

where $|\mathbf{a}\rangle = |a_1\rangle \otimes \dots \otimes |a_n\rangle \in F^{\otimes n}$ for $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{Z}_{\geq 0})^n$, etc. The factor $(q^2)_{sc_0}$ originates in (??). By Applying (??) to (??) it is straightforward to show

$$S^{t,s}(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} / \varrho^{t,s}(z) = \left(\prod_{r=1}^n \frac{z^{\frac{1}{t} j_r} (q^2)_{i_r} (q^2)_{j_r}}{z^{\frac{1}{t} b_r} (q^2)_{a_r} (q^2)_{b_r}} \right) S^{s,t}(z^{\frac{s}{t}})_{\bar{\mathbf{a}}, \bar{\mathbf{b}}}^{\bar{\mathbf{i}}, \bar{\mathbf{j}}} / \varrho^{s,t}(z^{\frac{s}{t}}), \quad (2.16)$$

where $\bar{\mathbf{a}} = (a_n, \dots, a_1)$ is the reverse array of $\mathbf{a} = (a_1, \dots, a_n)$ and similarly for $\bar{\mathbf{b}}, \bar{\mathbf{i}}$ and $\bar{\mathbf{j}}$. Henceforth we shall only consider $S^{1,1}(z), S^{1,2}(z)$ and $S^{2,2}(z)$ in the rest of the paper. The matrix element (??) is depicted as follows:



Due to (??), $S^{s,t}(z)$ also obeys the conservation law

$$S^{s,t}(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} = 0 \quad \text{unless} \quad \mathbf{a} + \mathbf{b} = \mathbf{i} + \mathbf{j}. \quad (2.17)$$

Due to the factor δ_{j+k}^{b+c} in (??), the sum (??) is constrained by the n conditions $b_1 + sc_0 = j_1 + c_1, \dots, b_n + c_{n-1} = j_n + tc_n$. Therefore it is actually a *single* sum. For $(s, t) = (2, 2)$, they further enforce a parity constraint

$$S^{2,2}(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} = 0 \quad \text{unless} \quad |\mathbf{a}| \equiv |\mathbf{i}|, \quad |\mathbf{b}| \equiv |\mathbf{j}| \pmod{2}, \quad (2.18)$$

where $|\mathbf{a}| = a_1 + \dots + a_n$, etc. Thus we have a direct sum decomposition

$$S^{2,2}(z) = S^{+,+}(z) \oplus S^{+,-}(z) \oplus S^{-,+}(z) \oplus S^{-,-}(z), \quad (2.19)$$

$$S^{\epsilon_1, \epsilon_2}(z) \in \text{End}((F^{\otimes n})_{\epsilon_1} \otimes (F^{\otimes n})_{\epsilon_2}), \quad (F^{\otimes n})_{\pm} = \bigoplus_{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n, (-1)^{|\mathbf{a}|} = \pm 1} \mathbb{Q}(q)|\mathbf{a}\rangle. \quad (2.20)$$

We dare allow the coexistence of somewhat confusing notations $S^{s,t}(z)$ and $S^{\epsilon_1, \epsilon_2}(z)$ expecting that they can be properly distinguished from the context. (A similar warning applies to $\varrho^{s,t}(z)$ in the sequel.)

We choose the normalization factors as

$$\varrho^{1,1}(z) = \frac{(z; q)_{\infty}}{(-zq; q)_{\infty}}, \quad \varrho^{1,2}(z) = \frac{(z^2; q^2)_{\infty}}{(-z^2q; q^2)_{\infty}}, \quad \varrho^{\epsilon_1, \epsilon_2}(z) = \left(\frac{(z; q^4)_{\infty}}{(zq^2; q^4)_{\infty}} \right)^{\epsilon_1 \epsilon_2}, \quad (2.21)$$

which agrees with [?, eq.(2.22)] for $n = 1$ case. Then the matrix elements of $S^{1,1}(z), S^{1,2}(z)$ and $S^{\epsilon_1, \epsilon_2}(z)$ are rational functions of q and z .

2.4. Examples. Let us demonstrate the calculations of the matrix elements $S^{s,t}(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}}$ (??) on simple examples. We pick a few simple matrix elements derivable from (??) and (??):

$$\begin{aligned} \mathcal{R}_{i,j,k}^{a,0,c} &= q^{ik} \delta_{i+j}^a \delta_{j+k}^c, & \mathcal{R}_{i,0,k}^{a,b,c} &= q^{ac} \frac{(q^2)_i (q^2)_k}{(q^2)_a (q^2)_b (q^2)_c} \delta_i^{a+b} \delta_k^{b+c}, & \mathcal{R}_{1,1,k}^{1,1,k} &= 1 - (1+q^2)q^{2k}, \\ \mathcal{R}_{0,j,k}^{a,b,c} &= (-1)^b q^{b(k+1)} \binom{j}{b}_{q^2} \delta_j^{a+b} \delta_{j+k}^{b+c}, & \mathcal{R}_{i,j,k}^{0,b,c} &= (-1)^j q^{j(c+1)} \frac{(q^2)_k}{(q^2)_c} \delta_{i+j}^b \delta_{j+k}^{b+c}. \end{aligned}$$

Using them we find

$$S^{1,1}(z)(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle) = S^{1,2}(z)(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle) = S^{+,+}(z)(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle) = |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle, \quad (2.22)$$

$$S^{+,-}(z)_{\mathbf{0}, \mathbf{e}_1}^{\mathbf{0}, \mathbf{e}_1} = \frac{-q}{1-z}, \quad S^{-,+}(z)_{\mathbf{e}_1, \mathbf{0}}^{\mathbf{e}_1, \mathbf{0}} = \frac{1}{1-z}, \quad S^{-,-}(z)_{\mathbf{e}_1, \mathbf{e}_1}^{\mathbf{e}_1, \mathbf{e}_1} = \frac{z-q^2}{1-zq^2}, \quad (2.23)$$

where $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{e}_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) \in \mathbb{Z}^n$. In fact for any $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{Z}_{\geq 0})^n$, the formulas

$$\begin{aligned} S^{1,t}(z)_{\mathbf{a}, \mathbf{0}}^{\mathbf{a}, \mathbf{0}} &= (-q)^{-|\mathbf{a}|} S^{1,t}(z)_{\mathbf{0}, \mathbf{a}}^{\mathbf{0}, \mathbf{a}} = \frac{(z^t; q^t)_{|\mathbf{a}|}}{(-z^t q; q^t)_{|\mathbf{a}|}} \quad (t = 1, 2), \\ S^{+,+}(z)_{\mathbf{a}, \mathbf{0}}^{\mathbf{a}, \mathbf{0}} &= (-q)^{-|\mathbf{a}|} S^{+,+}(z)_{\mathbf{0}, \mathbf{a}}^{\mathbf{0}, \mathbf{a}} = \frac{(z; q^4)_{|\mathbf{a}|/2}}{(zq^2; q^4)_{|\mathbf{a}|/2}} \quad (|\mathbf{a}| \in 2\mathbb{Z}) \end{aligned}$$

are valid. We also have

$$\begin{aligned} S^{1,1}(z)_{\mathbf{e}_1, \mathbf{e}_1}^{2\mathbf{e}_1, \mathbf{0}} &= (-q)^{-1} S^{1,1}(z)_{\mathbf{e}_1, \mathbf{e}_1}^{\mathbf{0}, 2\mathbf{e}_1} = \frac{(1+q)(1-z)}{(1+zq)(1+zq^2)}, \\ S^{-,-}(z)_{\mathbf{e}_n, \mathbf{e}_n}^{\mathbf{e}_n, \mathbf{e}_n} &= S^{-,-}(z)_{\mathbf{e}_1, \mathbf{e}_1}^{\mathbf{e}_1, \mathbf{e}_1}, \quad S^{-,-}(z)_{\mathbf{e}_1, \mathbf{e}_n}^{\mathbf{e}_n, \mathbf{e}_1} = z^{-1} S^{-,-}(z)_{\mathbf{e}_n, \mathbf{e}_1}^{\mathbf{e}_1, \mathbf{e}_n} = \frac{1-q^2}{1-zq^2}, \\ S^{-,-}(z)_{\mathbf{e}_1, \mathbf{e}_n}^{\mathbf{e}_1, \mathbf{e}_n} &= S^{-,-}(z)_{\mathbf{e}_n, \mathbf{e}_1}^{\mathbf{e}_n, \mathbf{e}_1} = -\frac{q(1-z)}{1-zq^2}. \end{aligned}$$

For instance to derive the last result in (??), one looks at the corresponding sum (??):

$$\sum_{c_0, \dots, c_n \geq 0} \frac{z^{c_0} (q^2)_{2c_0}}{(q^4)_{c_0} (q^4)_{c_n}} \mathcal{R}_{1,1,c_1}^{1,1,2c_0} \mathcal{R}_{0,0,c_2}^{0,0,c_1} \dots \mathcal{R}_{0,0,c_{n-1}}^{0,0,c_{n-2}} \mathcal{R}_{0,0,2c_n}^{0,0,c_{n-1}}.$$

Due to (??) this is a single sum over $k = c_0 = c_n = c_1/2 = \dots = c_{n-1}/2$. Moreover the product of \mathcal{R} 's is equal to $\mathcal{R}_{1,1,2k}^{1,1,2k} = 1 - (1+q^2)q^{4k}$. Thus it is calculated as

$$\begin{aligned} \sum_{k \geq 0} \frac{z^k (q^2)_{2k}}{(q^4)_k^2} (1 - (1+q^2)q^{4k}) &= \sum_{k \geq 0} \frac{z^k (q^2; q^4)_k}{(q^4; q^4)_k} (1 - (1+q^2)q^{4k}) \\ &= \frac{(zq^2; q^4)_\infty}{(z; q^4)_\infty} - (1+q^2) \frac{(zq^6; q^4)_\infty}{(zq^4; q^4)_\infty} = \varrho^{-,-}(z)^{-1} \frac{z-q^2}{1-zq^2} \end{aligned}$$

by means of the identity [?, eq.(1.3.12)]

$$\sum_{k \geq 0} \frac{(x; p)_k}{(p; p)_k} z^k = \frac{(zx; p)_\infty}{(z; p)_\infty}.$$

General matrix elements for $n = 1$ has been obtained in [?, Proposition 2].

3. QUANTUM R MATRICES FOR q -OSCILLATOR REPRESENTATIONS

3.1. Quantum affine algebras. The Drinfeld-Jimbo quantum affine algebras (without derivation operator) $U_q = U_q(A_{2n}^{(2)})$, $U_q(C_n^{(1)})$ and $U_q(D_{n+1}^{(2)})$ are the Hopf algebras generated by $e_i, f_i, k_i^{\pm 1}$ ($0 \leq i \leq n$) satisfying the relations

$$\begin{aligned} k_i k_i^{-1} &= k_i^{-1} k_i = 1, \quad [k_i, k_j] = 0, \\ k_i e_j k_i^{-1} &= q_i^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j, \quad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu e_i^{(1-a_{ij}-\nu)} e_j e_i^{(\nu)} &= 0, \quad \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu f_i^{(1-a_{ij}-\nu)} f_j f_i^{(\nu)} = 0 \quad (i \neq j), \end{aligned} \quad (3.1)$$

where $e_i^{(\nu)} = e_i^\nu / [\nu]_{q_i}!$, $f_i^{(\nu)} = f_i^\nu / [\nu]_{q_i}!$. The data q_i will be specified for the algebras under consideration in the sequel. The Cartan matrix $(a_{ij})_{0 \leq i, j \leq n}$ [?] is given by

$$a_{i,j} = \begin{cases} 2 & i = j, \\ -\max((\log q_j)/(\log q_i), 1) & |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

We use the coproduct Δ of the form

$$\Delta k_i^{\pm 1} = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta e_i = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta f_i = f_i \otimes 1 + k_i^{-1} \otimes f_i.$$

3.2. q -oscillator representations. We introduce representations of $U_q(D_{n+1}^{(2)})$, $U_q(A_{2n}^{(2)})$ and $U_q(C_n^{(1)})$ on the tensor product of the Fock space $\hat{F}^{\otimes n}$ or $F^{\otimes n}$. Here $\hat{F} = \bigoplus_{m \geq 0} \mathbb{C}(q^{\frac{1}{2}}) |m\rangle$ is a slight extension of the coefficient field of $F = \bigoplus_{m \geq 0} \mathbb{Q}(q) |m\rangle \subset \hat{F}$. For $U_q(A_{2n}^{(2)})$ and $U_q(C_n^{(1)})$, they are essentially the affinization of the q -oscillator representation of the classical part $U_q(C_n)$ [?] which factors through the algebra homomorphism from U_q to the q -Weyl algebra. A similar feature is expected also for $U_q(D_{n+1}^{(2)})$. As in the previous section we write the elements of $\hat{F}^{\otimes n}$ as

$$|\mathbf{m}\rangle = |m_1\rangle \otimes \cdots \otimes |m_n\rangle \in \hat{F}^{\otimes n} \quad \text{for } \mathbf{m} = (m_1, \dots, m_n) \in (\mathbb{Z}_{\geq 0})^n \quad (3.3)$$

and describe the changes in \mathbf{m} by the vectors $\mathbf{e}_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) \in \mathbb{Z}^n$.

Consider $U_q(D_{n+1}^{(2)})$. The Dynkin diagram of $D_{n+1}^{(2)}$ looks as

$$\begin{array}{ccccccc} q^{\frac{1}{2}} & q & q & & q & q^{\frac{1}{2}} \\ \circ & \leftarrow \circ & \circ & \cdots & \circ & \rightarrow \circ \\ 0 & 1 & 2 & & n-1 & n \end{array}$$

Here the vertices are numbered with $\{0, \dots, n\}$ as indicated. The q_i associated to the vertex i is specified above it, so $q_0 = q_n = q^{\frac{1}{2}}$ and $q_j = q$ for $0 < j < n$. The Cartan matrix is given according to (??) as $a_{0,1} = a_{n,n-1} = -2$, $a_{1,0} = -1$, etc. Similar conventions will also be adopted for the other algebras under consideration. Somewhat unusual convention to include $q^{\frac{1}{2}}$ is to make the presentation and proof of our main Theorem ?? simple and uniform. In Proposition ??, ?? and ??, the symbol $[m]$ denotes $[m]_q$.

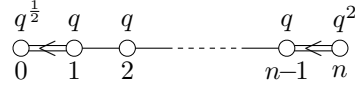
Proposition 1. *The following defines an irreducible $U_q(D_{n+1}^{(2)})$ module structure on $\hat{F}^{\otimes n}$.*

$$\begin{aligned}
e_0|\mathbf{m}\rangle &= x|\mathbf{m} + \mathbf{e}_1\rangle, \\
f_0|\mathbf{m}\rangle &= i\kappa[m_1]x^{-1}|\mathbf{m} - \mathbf{e}_1\rangle, \\
k_0|\mathbf{m}\rangle &= -iq^{m_1 + \frac{1}{2}}|\mathbf{m}\rangle, \\
e_j|\mathbf{m}\rangle &= [m_j]|\mathbf{m} - \mathbf{e}_j + \mathbf{e}_{j+1}\rangle \quad (1 \leq j \leq n-1), \\
f_j|\mathbf{m}\rangle &= [m_{j+1}]|\mathbf{m} + \mathbf{e}_j - \mathbf{e}_{j+1}\rangle \quad (1 \leq j \leq n-1), \\
k_j|\mathbf{m}\rangle &= q^{-m_j + m_{j+1}}|\mathbf{m}\rangle \quad (1 \leq j \leq n-1), \\
e_n|\mathbf{m}\rangle &= i\kappa[m_n]|\mathbf{m} - \mathbf{e}_n\rangle, \\
f_n|\mathbf{m}\rangle &= |\mathbf{m} + \mathbf{e}_n\rangle, \\
k_n|\mathbf{m}\rangle &= iq^{-m_n - \frac{1}{2}}|\mathbf{m}\rangle,
\end{aligned}$$

where x is a nonzero parameter and

$$\kappa = \frac{q+1}{q-1}. \quad (3.4)$$

Consider $U_q(A_{2n}^{(2)})$. The Dynkin diagram of $A_{2n}^{(2)}$ looks as



Proposition 2. *The following defines an irreducible $U_q(A_{2n}^{(2)})$ module structure on $\hat{F}^{\otimes n}$.*

$$\begin{aligned}
e_0|\mathbf{m}\rangle &= x|\mathbf{m} + \mathbf{e}_1\rangle, \\
f_0|\mathbf{m}\rangle &= i\kappa[m_1]x^{-1}|\mathbf{m} - \mathbf{e}_1\rangle, \\
k_0|\mathbf{m}\rangle &= -iq^{m_1 + \frac{1}{2}}|\mathbf{m}\rangle, \\
e_j|\mathbf{m}\rangle &= [m_j]|\mathbf{m} - \mathbf{e}_j + \mathbf{e}_{j+1}\rangle \quad (1 \leq j \leq n-1), \\
f_j|\mathbf{m}\rangle &= [m_{j+1}]|\mathbf{m} + \mathbf{e}_j - \mathbf{e}_{j+1}\rangle \quad (1 \leq j \leq n-1), \\
k_j|\mathbf{m}\rangle &= q^{-m_j + m_{j+1}}|\mathbf{m}\rangle \quad (1 \leq j \leq n-1), \\
e_n|\mathbf{m}\rangle &= \frac{[m_n][m_n - 1]}{[2]^2}|\mathbf{m} - 2\mathbf{e}_n\rangle, \\
f_n|\mathbf{m}\rangle &= |\mathbf{m} + 2\mathbf{e}_n\rangle, \\
k_n|\mathbf{m}\rangle &= -q^{-2m_n - 1}|\mathbf{m}\rangle,
\end{aligned}$$

where x is a nonzero parameter and κ is defined by (??).

Consider $U_q(C_n^{(1)})$. The Dynkin diagram of $C_n^{(1)}$ looks as



Proposition 3. *The following defines an irreducible $U_q(C_n^{(1)})$ module structure on $(F^{\otimes n})_+$ and $(F^{\otimes n})_-$ defined in (??).*

$$\begin{aligned}
e_0|\mathbf{m}\rangle &= x|\mathbf{m} + 2\mathbf{e}_1\rangle, \\
f_0|\mathbf{m}\rangle &= \frac{[m_1][m_1 - 1]}{[2]^2}x^{-1}|\mathbf{m} - 2\mathbf{e}_1\rangle, \\
k_0|\mathbf{m}\rangle &= -q^{2m_1+1}|\mathbf{m}\rangle, \\
e_j|\mathbf{m}\rangle &= [m_j]|\mathbf{m} - \mathbf{e}_j + \mathbf{e}_{j+1}\rangle \quad (1 \leq j \leq n-1), \\
f_j|\mathbf{m}\rangle &= [m_{j+1}]|\mathbf{m} + \mathbf{e}_j - \mathbf{e}_{j+1}\rangle \quad (1 \leq j \leq n-1), \\
k_j|\mathbf{m}\rangle &= q^{-m_j+m_{j+1}}|\mathbf{m}\rangle \quad (1 \leq j \leq n-1), \\
e_n|\mathbf{m}\rangle &= \frac{[m_n][m_n - 1]}{[2]^2}|\mathbf{m} - 2\mathbf{e}_n\rangle, \\
f_n|\mathbf{m}\rangle &= |\mathbf{m} + 2\mathbf{e}_n\rangle, \\
k_n|\mathbf{m}\rangle &= -q^{-2m_n-1}|\mathbf{m}\rangle,
\end{aligned}$$

where x is a nonzero parameter.

Direct calculation verifies these are representations of U_q . To see the irreducibility decompose $\hat{F}^{\otimes n}$ as a $U_q(A_{n-1})$ module forgetting the action of generators indexed by 0 and n . By $\hat{F}^{\otimes n} \simeq \bigoplus_{l=0}^{\infty} U_q(A_{n-1})|l\mathbf{e}_1\rangle$ and considering the action of e_0 and f_0 the irreducibility follows. We call the irreducible representations given there the q -oscillator representations of U_q . We remark that for the twisted case $U_q(D_{n+1}^{(2)})$ and $U_q(A_{2n}^{(2)})$, they are *singular* at $q = 1$ because of the factor κ (??).

3.3. Quantum R matrices. Let $V = \hat{F}^{\otimes n}$ for $U_q(D_{n+1}^{(2)}), U_q(A_{2n}^{(2)})$ and $V = F^{\otimes n}$ for $U_q(C_n^{(1)})$. First we consider $U_q(D_{n+1}^{(2)})$ and $U_q(A_{2n}^{(2)})$. Let $V_x = \hat{F}^{\otimes n}[x, x^{-1}]$ be the representation space of U_q in Propositions ?? and ??. By the existence of the universal R matrix [?] there exists an element $R \in \text{End}(V_x \otimes V_y)$ such that

$$\Delta'(g)R = R\Delta(g) \quad \forall g \in U_q \quad (3.5)$$

up to an overall scalar. Here Δ' is the opposite coproduct defined by $\Delta' = P \circ \Delta$, where $P(u \otimes v) = v \otimes u$ is the exchange of the components. Another useful form of (??) is

$$(\pi_y \otimes \pi_x)\Delta(g)PR = PR(\pi_x \otimes \pi_y)\Delta(g) \quad \forall g \in U_q, \quad (3.6)$$

where $\pi_x : U_q \rightarrow \text{End } V_x$ denotes the representation.

A little inspection of our representations shows that R depends on x and y only through the ratio $z = x/y$. Moreover, by the irreducibility of $V_x \otimes V_y$ (Proposition ??) R is determined only by postulating (??) for $g = k_r, e_r$ and f_r with $0 \leq r \leq n$. Thus denoting the R by $R(z)$, we may claim [?] that it is determined by the conditions

$$(k_r \otimes k_r)R(z) = R(z)(k_r \otimes k_r), \quad (3.7)$$

$$(e_r \otimes 1 + k_r \otimes e_r)R(z) = R(z)(1 \otimes e_r + e_r \otimes k_r), \quad (3.8)$$

$$(1 \otimes f_r + f_r \otimes k_r^{-1})R(z) = R(z)(f_r \otimes 1 + k_r^{-1} \otimes f_r) \quad (3.9)$$

for $0 \leq r \leq n$ up to an overall scalar. We fix the normalization of $R(z)$ by

$$R(z)(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle) = |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle, \quad (3.10)$$

where $|\mathbf{0}\rangle \in \hat{F}^{\otimes n}$ is defined after (??). We call the intertwiner $R(z)$ the *quantum R matrix* for q -oscillator representation. It satisfies the Yang-Baxter equation

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x). \quad (3.11)$$

Next we consider $U_q(C_n^{(1)})$. Denote by $V_x^\pm = (F^{\otimes n})_\pm[x, x^{-1}]$ the representation spaces in Proposition ?? and set $V_x = V_x^+ \oplus V_x^- = F^{\otimes n}[x, x^{-1}]$. See (??) for the definition of $(F^{\otimes n})_\pm$. We define the quantum R matrix $R(z)$ to be the direct sum

$$R(z) = R^{+,+}(z) \oplus R^{+,-}(z) \oplus R^{-,+}(z) \oplus R^{-,-}(z), \quad (3.12)$$

where each $R^{\epsilon_1, \epsilon_2}(z) \in \text{End}(V_x^{\epsilon_1} \otimes V_y^{\epsilon_2})$ is the quantum R matrix with the normalization condition

$$\begin{aligned} R^{+,+}(z)(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle) &= |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle, & R^{+,-}(z)(|\mathbf{0}\rangle \otimes |\mathbf{e}_1\rangle) &= \frac{-iq^{1/2}}{1-z} |\mathbf{0}\rangle \otimes |\mathbf{e}_1\rangle, \\ R^{-,+}(z)(|\mathbf{e}_1\rangle \otimes |\mathbf{0}\rangle) &= \frac{-iq^{1/2}}{1-z} |\mathbf{e}_1\rangle \otimes |\mathbf{0}\rangle, & R^{-,-}(z)(|\mathbf{e}_1\rangle \otimes |\mathbf{e}_1\rangle) &= \frac{z-q^2}{1-zq^2} |\mathbf{e}_1\rangle \otimes |\mathbf{e}_1\rangle. \end{aligned} \quad (3.13)$$

The R matrix $R(z)$ satisfies the Yang-Baxter equation (??). In fact it is decomposed into the finer equalities ($\epsilon_1, \epsilon_2, \epsilon_3 = \pm$)

$$R_{12}^{\epsilon_1, \epsilon_2}(x) R_{13}^{\epsilon_1, \epsilon_3}(xy) R_{23}^{\epsilon_2, \epsilon_3}(y) = R_{23}^{\epsilon_2, \epsilon_3}(y) R_{13}^{\epsilon_1, \epsilon_3}(xy) R_{12}^{\epsilon_1, \epsilon_2}(x).$$

3.4. Singular vectors. In this subsection we find all singular vectors in $V^{\otimes 2}$, namely those $v \in V^{\otimes 2}$ killed by e_1, \dots, e_n , as a $U_q(B_n)$ module or $U_q(C_n)$ module. Since $V^{\otimes 2}$ is not finite-dimensional, we cannot say at this stage that they are actually the highest weight vectors of irreducible modules, but we will see it later in the next subsection.

Proposition 4. *As a $U_q(B_n)$ module any singular vector in $V^{\otimes 2}$ is given by*

$$v_l = \sum_{k=0}^l i^k q^{lk-k^2/2} \begin{bmatrix} l \\ k \end{bmatrix} |k\mathbf{e}_n\rangle \otimes |(l-k)\mathbf{e}_n\rangle$$

for some $l \in \mathbb{Z}_{\geq 0}$.

Proof. One can assume that a singular vector v is a weight vector of weight $\mathbf{l} = \sum_{j=1}^n l_j \mathbf{e}_j \in (\mathbb{Z}_{\geq 0})^n$. Hence v can be written as

$$v = \sum_{\mathbf{m}} c_{\mathbf{m}} |\mathbf{m}\rangle \otimes |\mathbf{l} - \mathbf{m}\rangle. \quad (3.14)$$

The conditions $e_j v = 0$ ($1 \leq j \leq n-1$) and $e_n v = 0$ read respectively as

$$\begin{aligned} \sum c_{\mathbf{m}} ([l_j - m_j] |\mathbf{m}\rangle \otimes |\mathbf{l} - \mathbf{m} - \mathbf{e}_j + \mathbf{e}_{j+1}\rangle \\ + q^{-l_j + l_{j+1} + m_j - m_{j+1}} [m_j] |\mathbf{m} - \mathbf{e}_j + \mathbf{e}_{j+1}\rangle \otimes |\mathbf{l} - \mathbf{m}\rangle) = 0, \end{aligned} \quad (3.15)$$

$$\begin{aligned} \sum c_{\mathbf{m}} ([l_n - m_n] |\mathbf{m}\rangle \otimes |\mathbf{l} - \mathbf{m} - \mathbf{e}_n\rangle \\ + iq^{-l_n + m_n - 1/2} [m_n] |\mathbf{m} - \mathbf{e}_n\rangle \otimes |\mathbf{l} - \mathbf{m}\rangle) = 0. \end{aligned} \quad (3.16)$$

We first show there is no singular vector of weight \mathbf{l} such that $l_j > 0, l_{j+1} = 0$ for some $j < n$. Suppose $l_{j+1} = 0$. Looking at the coefficient of $|\dots, m_j, 0, \dots\rangle \otimes |\dots, l_j - m_j - 1, 1, \dots\rangle$ in (??) one sees $c_{\mathbf{m}} = 0$ if $m_j < l_j$. Similarly, the coefficient of $|\dots, m_j - 1, 1, \dots\rangle \otimes |\dots, l_j - m_j, 0, \dots\rangle$ in (??) gives $c_{\mathbf{m}} = 0$ if $m_j > 0$. Hence $c_{\mathbf{m}} = 0$ for all \mathbf{m} unless $l_j = 0$.

We next show there is no singular vector of weight \mathbf{l} such that $l_{n-1} > 0$. Looking at the coefficient of $|\dots, m_{n-1}, 0\rangle \otimes |\dots, l_{n-1} - m_{n-1} - 1, l_n + 1\rangle$ in (??) _{$j=n-1$} one sees $c_{\mathbf{m}} = 0$ if $m_{n-1} < l_{n-1}, m_n = 0$. Together with (??) we get $c_{\mathbf{m}} = 0$ if $m_{n-1} < l_{n-1}$. Similarly, the coefficient of $|\dots, m_{n-1} - 1, l_n + 1\rangle \otimes |\dots, l_{n-1} - m_{n-1}, 0\rangle$ in (??) _{$j=n-1$} gives $c_{\mathbf{m}} = 0$ if $m_{n-1} > 0, m_n = l_n$. Together with (??) we get $c_{\mathbf{m}} = 0$ for $m_{n-1} > 0$. Hence $c_{\mathbf{m}} = 0$ for all \mathbf{m} unless $l_{n-1} = 0$. The coefficient $c_{\mathbf{m}}$ for $\mathbf{l} = l\mathbf{e}_n$ can easily be determined by solving (??). \square

Proposition 5. *As a $U_q(C_n)$ module any singular vector in $V^{\otimes 2}$ is given by*

$$v_l^\epsilon = \sum_{\substack{0 \leq k \leq l \\ k \equiv p(\epsilon) \pmod{2}}} q^{k(2l-k-1)/2} \begin{bmatrix} l \\ k \end{bmatrix} |k\mathbf{e}_n\rangle \otimes |(l-k)\mathbf{e}_n\rangle \in V^\epsilon \otimes V^{\epsilon\eta(l)} \quad (l \geq p(\epsilon)),$$

or

$$v_0^- = |\mathbf{e}_{n-1}\rangle \otimes |\mathbf{e}_n\rangle - q|\mathbf{e}_n\rangle \otimes |\mathbf{e}_{n-1}\rangle \in V^- \otimes V^-$$

for some $l \in \mathbb{Z}_{\geq 0}$ and $\epsilon = +$ or $-$, where $p(\epsilon) = 0$ ($\epsilon = +$), $= 1$ ($\epsilon = -$) and $\eta(l) = +$ (l :even), $= -$ (l :odd).

Proof. The proof is similar to Proposition ???. A singular vector v can be written as (??). The condition $e_j v = 0$ reads (??) and

$$\begin{aligned} & \sum c_{\mathbf{m}}([l_n - m_n][l_n - m_n - 1]|\mathbf{m}\rangle \otimes |\mathbf{l} - \mathbf{m} - 2\mathbf{e}_n\rangle \\ & - q^{-2l_n + 2m_n - 1}[m_n][m_n - 1]|\mathbf{m} - 2\mathbf{e}_n\rangle \otimes |\mathbf{l} - \mathbf{m}\rangle) = 0. \end{aligned} \quad (3.17)$$

The fact that there is no singular vector of weight \mathbf{l} such that $l_j > 0, l_{j+1} = 0$ for some $j < n$ is the same. Similarly to the next argument in the proof of Proposition ??? with (??), we see that $c_{\mathbf{m}} = 0$ if $m_{n-1} \leq l_{n-1} - 1$ & $m_n \equiv 0$, $m_{n-1} \leq l_{n-1} - 2$ & $m_n \equiv 1$, $m_{n-1} \geq 1$ & $m_n \equiv l_n$, or $m_{n-1} \geq 2$ & $m_n \equiv l_n - 1$, where the congruence \equiv is modulo 2. Hence we can conclude that if nontrivial $c_{\mathbf{m}}$ exists, then $l_{n-1} \leq 1$ if l_n is odd and $l_{n-1} = 0$ if l_n is even.

We wish to show that nontrivial $c_{\mathbf{m}}$ exists only when $\mathbf{l} = l\mathbf{e}_n$ for some nonnegative integer l or $\mathbf{l} = \mathbf{e}_{n-1} + \mathbf{e}_n$. Thus the remaining thing to show is that if $l_{n-1} = 1$ and l_n is odd, then (i) $l_{n-2} = 0$ and (ii) $l_n = 1$. To show (i) by contradiction suppose $l_{n-2} > 0$. From the result of the previous paragraph we know $c_{\mathbf{m}} = 0$ if $m_{n-1} = 0$ & $m_n \equiv 0$ or $m_{n-1} = 1$ & $m_n \equiv 1$. Hence it suffices to show $c_{\mathbf{m}} = 0$ when (a) $m_{n-2} > 0$ & $m_{n-1} = 0$ & $m_n \equiv 1$ (b) $m_{n-2} = m_{n-1} = 0$ & $m_n \equiv 1$ (c) $m_{n-2} < l_{n-2}$ & $m_{n-1} = 1$ & $m_n \equiv 0$ (d) $m_{n-2} = l_{n-2}$ & $m_{n-1} = 1$ & $m_n \equiv 0$. Case (a) (resp. (c)) is shown from (??) _{$j=n-2$} and $c_{(\dots, m_{n-2}-1, 1, m_n)} = 0$ if $m_n \equiv 1$ (resp. $c_{(\dots, m_{n-2}+1, 0, m_n)} = 0$ if $m_n \equiv 0$). Case (b) (resp. (d)) is shown by comparing the coefficient of $|\dots, 0, 0, m_n\rangle \otimes |\dots, l_{n-2} - 1, 2, l_n - m_n\rangle$ (resp. $|\dots, l_{n-2} - 1, 2, m_n\rangle \otimes |\dots, 0, 0, l_n - m_n\rangle$) in (??) _{$j=n-2$} . To show (ii) by contradiction suppose $l_n \geq 3$. One can compute $c_{\mathbf{e}_{n-1}+2\mathbf{e}_n}/c_{\mathbf{e}_n}$ either by $(c_{3\mathbf{e}_n}/c_{\mathbf{e}_n})(c_{\mathbf{e}_{n-1}+2\mathbf{e}_n}/c_{3\mathbf{e}_n})$ or $(c_{\mathbf{e}_{n-1}}/c_{\mathbf{e}_n})(c_{\mathbf{e}_{n-1}+2\mathbf{e}_n}/c_{\mathbf{e}_{n-1}})$. The former gives $-q^{l_n-3}[l_n - 1][l_n - 2]/([3][2])$ while the latter $-q^{l_n-3}[l_n][l_n - 1]/([2][1])$, which is a contradiction.

For the survived cases it is easy to obtain nontrivial $c_{\mathbf{m}}$. □

3.5. Spectral decomposition. We calculate the spectral decomposition of $PR(z)$. In this subsection we denote the subspace generated by f_1, \dots, f_n from the singular vector v_l (resp. v_l^ϵ) by V_l (resp. V_l^ϵ). The complete reducibility of $V^{\otimes 2}$ as a $U_q(B_n)$ or $U_q(C_n)$ module is valid since $PR(z)$ has a different eigenvalue on each subspace V_l or V_l^ϵ as we will see below, and the singular vectors obtained in the previous subsection are actually highest weight vectors of each irreducible component.

We prepare a lemma, which is obtained by direct calculation.

Lemma 6. For $U_q(D_{n+1}^{(2)})$ we have

$$(\pi_x \otimes \pi_y)\Delta(e_{n-1} \cdots e_1 e_0)v_l = \frac{1}{1 - q^{2l+1}} \{(q^{l+1}x + y)v_{l+1} + q^l(x + q^l y)f_n^2 v_{l-1}\} \quad (l \geq 1), \quad (3.18)$$

$$(\pi_x \otimes \pi_y)\Delta(e_{n-1} \cdots e_1 e_0)v_0 = \frac{1}{1 - q} \{(qx + y)v_1 - iq^{1/2}(x + y)f_n v_0\}, \quad (3.19)$$

$$(\pi_x \otimes \pi_y)\Delta(f_0 f_1 \cdots f_{n-1})v_l = -\kappa[l]q^{-1/2}(q^l x^{-1} + y^{-1})v_{l-1} \quad (l \geq 1). \quad (3.20)$$

Proposition 7. For $U_q(D_{n+1}^{(2)})$, $PR(z)$ has the following spectral decomposition.

$$PR(z) = \sum_{l=0}^{\infty} \prod_{j=1}^l \frac{z + q^j}{1 + q^j z} P_l,$$

where P_l is the projector on V_l .

Proof. First we note that by $PR(z)$ v_l is mapped to a scalar multiple of v_l since $PR(z)$ commutes with $U_q(B_n)$. Suppose $PR(z)v_l = \rho_l(z)v_l$. Setting $g = f_0 f_1 \cdots f_{n-1}$ in (??), applying both sides to v_l and using (??) we get

$$(q^l y^{-1} + x^{-1})\rho_l(z)v_{l-1} = \rho_{l-1}(z)(q^l x^{-1} + y^{-1})v_{l-1}.$$

Due to $v_0 = |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle$ the normalization condition (??) is satisfied by choosing $\rho_0(z) = 1$. Thus we obtain

$$\rho_l(z) = \prod_{j=1}^l \frac{z + q^j}{1 + q^j z}.$$

□

Lemma 8. For $U_q(A_{2n}^{(2)})$ we have

$$\begin{aligned} (\pi_x \otimes \pi_y)\Delta(e_{n-1} \cdots e_1 e_0)(v_l^+ \pm v_l^-) &= \frac{[l]}{[2l]} \{q^{-l}(y \mp iq^{l+1/2}x)(v_{l+1}^+ \pm v_{l+1}^-) \\ &\quad \mp iq^{-1/2}(x \mp iq^{l-1/2}y)f_n(v_{l-1}^+ \pm \chi(l \neq 1)v_{l-1}^-)\} \quad (l \geq 1), \\ (\pi_x \otimes \pi_y)\Delta(e_{n-1} \cdots e_1 e_0)v_0^+ &= yv_1^+ - iq^{1/2}xv_1^-, \\ (\pi_x \otimes \pi_y)\Delta(e_{n-2} \cdots e_0 e_{n-1} \cdots e_0)v_0^+ &= \frac{1}{[2]^2} \{(q^{-1}y^2 - x^2)f_{n-1}v_2^+ - i[2]xyf_{n-1}v_2^- \\ &\quad - (x^2 + y^2)f_{n-1}f_n v_0^+ + i[2](q^{1/2} - q^{-1/2})xyv_0^-\}, \\ (\pi_x \otimes \pi_y)\Delta(e_{n-1}e_n e_{n-1} \cdots e_0)v_0^- &= \frac{1}{[2]}(-iq^{1/2}xv_1^+ + yv_1^-), \\ (\pi_x \otimes \pi_y)\Delta(f_0 f_1 \cdots f_{n-1})(v_l^+ \pm v_l^-) &= -\kappa[l]q^{-1/2}(\mp iq^{l-1/2}x^{-1} + y^{-1})(v_{l-1}^+ \pm \chi(l \neq 1)v_{l-1}^-) \quad (l \geq 1). \end{aligned}$$

Here $\chi(\theta) = 1$ if θ is true, $= 0$ otherwise.

Proposition 9. For $U_q(A_{2n}^{(2)})$, $PR(z)$ has the following spectral decomposition.

$$PR(z) = P_0 + \sum_{l=1}^{\infty} \left(\prod_{j=1}^l \frac{z - iq^{j-1/2}}{1 - iq^{j-1/2}z} P_l^+ + \prod_{j=1}^l \frac{z + iq^{j-1/2}}{1 + iq^{j-1/2}z} P_l^- \right),$$

where P_0 is the projector on $V_0^+ \oplus V_0^-$ and P_l^\pm with $l \geq 1$ is the one on the $U_q(C_n)$ invariant subspace containing $v_l^+ \pm v_l^-$.

Proof. Set

$$PR(z) = \sum_{l=0}^{\infty} (\rho_l^+(z)P_l^+ + \rho_l^-(z)P_l^-)$$

with $\rho_0^\pm(z) = 1$. P_0^ϵ is the projector on V_0^ϵ . The determination of $\rho_l^\epsilon(z)$ is similar to Proposition ???. The normalization condition (??) is satisfied since $v_0^\pm = |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle$. □

Lemma 10. For $U_q(C_n^{(1)})$ we have

$$\begin{aligned} (\pi_x \otimes \pi_y) \Delta((e_{n-1} \cdots e_1)^2 e_0) v_l^\epsilon &= \frac{[2]}{\langle l+1 \rangle \langle l \rangle \langle l-1 \rangle} \{q^{-2l-1} \langle l-1 \rangle (y - q^{2l+2} x) v_{l+2}^\epsilon \\ &\quad - [2] \langle l \rangle (x+y) f_n v_l^\epsilon - q^{-1} \langle l+1 \rangle (x - q^{2l-2} y) f_n^2 v_{l-2}^\epsilon\} \quad (l \geq 2, (\epsilon, l) \neq (-, 2)), \\ (\pi_x \otimes \pi_y) \Delta((e_{n-1} \cdots e_1)^2 e_0) v_1^\epsilon &= \frac{1}{\langle 2 \rangle} \{q^{-3} (y - q^4 x) v_3^\epsilon - ([2 - \epsilon] x + [2 + \epsilon] y) f_n v_1^\epsilon\}, \\ (\pi_x \otimes \pi_y) \Delta(e_{n-1}^2 e_n e_{n-1} e_{n-2}^2 \cdots e_1^2 e_0) v_0^- &= q^{-1} [2]^{n-2} (y - q^2 x) v_2^-, \\ (\pi_x \otimes \pi_y) \Delta(f_0 (f_1 \cdots f_{n-1})^2) v_l^\epsilon &= q^{-1} \frac{[l][l-1]}{[2]} (q^{2l-2} x^{-1} - y^{-1}) v_{l-2}^\epsilon \quad (l \geq 2, (\epsilon, l) \neq (-, 2)), \\ (\pi_x \otimes \pi_y) \Delta(f_0 f_1^2 \cdots f_{n-2}^2 f_{n-1} f_n f_{n-1}^2) v_2^- &= [2]^{n-2} \{(y^{-1} + x^{-1}) f_{n-1} v_2^- + q[2] (y^{-1} - q^2 x^{-1}) v_0^-\}. \end{aligned}$$

Here $\langle m \rangle = q^m + q^{-m}$.

Proposition 11. For $U_q(C_n^{(1)})$, $PR(z)$ has the following spectral decomposition.

$$PR^{\epsilon, \epsilon}(z) = \sum_{l=0}^{\infty} \prod_{j=1}^l \frac{z - q^{4j-2}}{1 - q^{4j-2} z} P_{2l}^\epsilon, \quad PR^{\epsilon, -\epsilon}(z) = \frac{-iq^{1/2}}{1-z} \sum_{l=0}^{\infty} \prod_{j=1}^l \frac{z - q^{4j}}{1 - q^{4j} z} P_{2l+1}^\epsilon,$$

where P_{2l}^ϵ is the projector on V_{2l}^ϵ , and P_{2l+1}^ϵ is the $U_q(C_n)$ linear map sending v_{2l+1}^ϵ to $v_{2l+1}^{-\epsilon}$ and other singular vectors to 0.

Proof. Set

$$PR^{\epsilon, \epsilon}(z) = \sum_{l=0}^{\infty} \rho_{2l}^\epsilon(z) P_{2l}^\epsilon, \quad PR^{\epsilon, -\epsilon}(z) = \sum_{l=0}^{\infty} \rho_{2l+1}^\epsilon(z) P_{2l+1}^\epsilon$$

with $\rho_0^\epsilon(z) = 1$ and $\rho_1^\epsilon(z) = \frac{-iq^{1/2}}{1-z}$. The necessary data to derive the recursion relations of $\rho_l^\epsilon(z)$ are given in the lemma. The four vectors $|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle$, $|\mathbf{0}\rangle \otimes |\mathbf{e}_1\rangle$, $|\mathbf{e}_1\rangle \otimes |\mathbf{0}\rangle$ and $|\mathbf{e}_1\rangle \otimes |\mathbf{e}_1\rangle$ in (??) are contained in the irreducible components generated from v_0^+ , v_1^+ , v_1^- and v_2^- , respectively. Thus the condition (??) agrees with the above normalization of the eigenvalues. \square

Finally we prove

Proposition 12. As a $U_q(D_{n+1}^{(2)})$ or $U_q(A_{2n}^{(2)})$ module $V_x \otimes V_y$ is irreducible. As a $U_q(C_n^{(1)})$ module each $V_x^{\epsilon_1} \otimes V_y^{\epsilon_2}$ ($\epsilon_1, \epsilon_2 = \pm$) is irreducible.

Proof. We prove the $U_q(D_{n+1}^{(2)})$ case only. Suppose a submodule contains a nonzero weight vector. One can assume it is a singular vector. Hence it is v_l for some $l \in \mathbb{Z}_{\geq 0}$. By (??) the submodule contains v_0 . Then by (??) it contains a linear combination of v_1 and $f_n v_0$. However, since eigenvalues of $PR(z)$ for v_1 and $f_n v_0$ are different by Proposition ??, the submodule contains v_1 . Arguing similarly using (??), it contains v_l for any $l \geq \mathbb{Z}_{\geq 0}$, and the submodule is nothing but $V_x \otimes V_y$. \square

4. $S^{s,t}(z)$ AS QUANTUM R MATRIX

4.1. Main theorem. Define the operator K acting on $\hat{F}^{\otimes n}$ by

$$K|\mathbf{m}\rangle = (-iq^{\frac{1}{2}})^{m_1 + \cdots + m_n} |\mathbf{m}\rangle.$$

See (??) for the notation. Introduce the gauge transformed quantum R matrix by

$$\tilde{R}(z) = (K^{-1} \otimes 1)R(z)(1 \otimes K). \quad (4.1)$$

It is easy to see that $\tilde{R}(z)$ also satisfies the Yang-Baxter equation (??).

In Section ?? we have constructed the solutions $S^{s,t}(z)$ of the Yang-Baxter equation from the 3d R in (??), (??) and (??). In Section ?? the quantum R matrices for q -oscillator representations of $U_q(D_{n+1}^{(2)})$, $U_q(A_{2n}^{(2)})$ and $U_q(C_n^{(1)})$ have been obtained. The next theorem, which is the main result of the paper, states the precise relation between them.

Theorem 13. Denote by $\tilde{R}_{\mathfrak{g}}(z)$ the gauge transformed quantum R matrix (??) for $U_q(\mathfrak{g})$. Then the following equalities hold:

$$\begin{aligned} S^{1,1}(z) &= \tilde{R}_{D_{n+1}^{(2)}}(z), \\ S^{1,2}(z) &= \tilde{R}_{A_{2n}^{(2)}}(z), \\ S^{2,2}(z) &= \tilde{R}_{C_n^{(1)}}(z), \end{aligned}$$

where the last one means $S^{\epsilon_1, \epsilon_2}(z) = \tilde{R}^{\epsilon_1, \epsilon_2}(z)$ between (??) and (??) with the gauge transformation (??).

For $S^{2,1}(z)$, see (??).

Remark 14. Theorem ?? suggests the following correspondence between the boundary vectors (??) and (??) with the end shape of the Dynkin diagrams:

$$\begin{array}{ccc} \langle \chi_1(z) | & \begin{array}{c} 0 \\ \circ \longleftarrow \end{array} & \begin{array}{c} n \\ \longrightarrow \circ \end{array} | \chi_1(1) \rangle \\ \langle \chi_2(z) | & \begin{array}{c} 0 \\ \circ \longrightarrow \end{array} & \begin{array}{c} n \\ \longleftarrow \circ \end{array} | \chi_2(1) \rangle \end{array}$$

From this viewpoint it may be natural to interpret $S^{2,1}(z)$, which is reducible to $S^{1,2}(z^{1/2})$ by (??), in terms of another $U_q(A_{2n}^{(2)})$ realized as the affinization of the classical part $U_q(B_n)$. (Proposition ?? corresponds to taking the classical part to be $U_q(C_n)$.) As far as $\langle \chi_1(z) |$ and $| \chi_1(1) \rangle$ are concerned, the above correspondence agrees with the observation made in [?, Remark 7.2] on the similar result concerning a 3d L operator. With regard to $\langle \chi_2(z) |$ and $| \chi_2(1) \rangle$, the relevant affine Lie algebras $A_{2n}^{(2)}$ and $C_n^{(1)}$ in this paper are the subalgebras of $B_{n+1}^{(1)}$ and $D_{n+2}^{(1)}$ in [?, Theorem 7.1] obtained by folding their Dynkin diagrams.

4.2. Proof. Let us present an expository proof of Theorem ?. Comparing (??), (??) and (??), (??) with the gauge transformation (??) taken into account, one finds that $S^{s,t}(z)$ and $\tilde{R}(z)$ satisfy the same normalization condition. Moreover the conservation law (??) and the commutativity (??) are equivalent conditions on the matrices acting on $\hat{F}^{\otimes n} \otimes \hat{F}^{\otimes n}$. Thus it remains to show that $S^{s,t}(z)$ satisfies the same equation as the gauge transformed version of (??) and (??) for $\tilde{R}(z)$:

$$(\tilde{e}_r \otimes 1 + k_r \otimes e_r) S^{s,t}(z) = S^{s,t}(z) (1 \otimes \tilde{e}_r + e_r \otimes k_r), \quad (4.2)$$

$$(1 \otimes f_r + \tilde{f}_r \otimes k_r^{-1}) S^{s,t}(z) = S^{s,t}(z) (f_r \otimes 1 + k_r^{-1} \otimes \tilde{f}_r) \quad (4.3)$$

$0 \leq r \leq n$. Here $\tilde{e}_r = K^{-1} e_r K$, $\tilde{f}_r = K^{-1} f_r K$ are the gauge transformed Chevalley generators. We first treat (??). The actions of k_r^{-1} , f_r and \tilde{f}_r are to be taken from Proposition ??, ?? and ?? according to $(s, t) = (1, 1)$, $(1, 2)$ and $(2, 2)$, respectively.

Consider the actions of the both sides of (??) on a base vector $| \mathbf{i} \rangle \otimes | \mathbf{j} \rangle \in V_x \otimes V_y$:

$$\varrho^{s,t}(z)^{-1} y^{\delta_{r,0}} (1 \otimes f_r + \tilde{f}_r \otimes k_r^{-1}) S^{s,t}(z) (| \mathbf{i} \rangle \otimes | \mathbf{j} \rangle) = \sum_{\mathbf{a}, \mathbf{b}} A_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}}(z) | \mathbf{a} \rangle \otimes | \mathbf{b} \rangle, \quad (4.4)$$

$$\varrho^{s,t}(z)^{-1} y^{\delta_{r,0}} S^{s,t}(z) (f_r \otimes 1 + k_r^{-1} \otimes \tilde{f}_r) (| \mathbf{i} \rangle \otimes | \mathbf{j} \rangle) = \sum_{\mathbf{a}, \mathbf{b}} B_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}}(z) | \mathbf{a} \rangle \otimes | \mathbf{b} \rangle, \quad (4.5)$$

where we have removed the normalization factor $\varrho^{s,t}(z)$ (see (??)) for simplicity and multiplied $y^{\delta_{r,0}}$ to confine the dependence on x and y to the ratio $z = x/y$. We are to show the equality of the matrix elements $A_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}}(z) = B_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}}(z)$.

For illustration let us consider the case $(s, t) = (1, 1)$ and $r = n$. Then f_n , \tilde{f}_n and k_n are given by Proposition ?? and they only touch the n th component in $| \mathbf{i} \rangle$ and $| \mathbf{j} \rangle$. The transition of these

components in (??) is traced as follows.

$$\begin{array}{ccc}
 & |i_n\rangle \otimes |j_n\rangle & \\
 \mathcal{R}_{i_n, j_n, c_n}^{a_n, b_n-1, c_{n-1}} \swarrow & S^{1,1}(z) & \searrow \mathcal{R}_{i_n, j_n, c_n}^{a_n-1, b_n, c_{n-1}} \\
 & & \\
 |a_n\rangle \otimes |b_n-1\rangle & & |a_n-1\rangle \otimes |b_n\rangle \\
 \swarrow 1 \otimes f_n & & \searrow \tilde{f}_n \otimes k_n^{-1} \\
 & & \\
 & |a_n\rangle \otimes |b_n\rangle & \\
 \swarrow 1 & & \searrow q^{b_n}
 \end{array}$$

By this diagram we mean that the substitution of (??) into (??) yields

$$\begin{aligned}
 A_{i,j}^{\mathbf{a},\mathbf{b}}(z) &= \sum_{c_0, \dots, c_n} \frac{z^{c_0}}{(q)_{c_n}} X(c_0, \dots, c_{n-1}) \left(\mathcal{R}_{i_n, j_n, c_n}^{a_n, b_n-1, c_{n-1}} + q^{b_n} \mathcal{R}_{i_n, j_n, c_n}^{a_n-1, b_n, c_{n-1}} \right) \\
 &= \sum_{c_0, \dots, c_n} \frac{z^{c_0}}{(q)_{c_n}} X(c_0, \dots, c_{n-1}) \left((1 - q^{c_n}) \mathcal{R}_{i_n, j_n, c_{n-1}}^{a_n, b_n-1, c_{n-1}} + q^{b_n} \mathcal{R}_{i_n, j_n, c_n}^{a_n-1, b_n, c_{n-1}} \right) \quad (4.6)
 \end{aligned}$$

for some X which is independent of z . To get the second line we have just changed the dummy summation variable c_n in the first term into $c_n - 1$. This has the effect of letting the two terms have the identical constraint $b_n + c_{n-1} = j_n + c_n$. See (??). Similarly the diagram for the matrix element $B_{i,j}^{\mathbf{a},\mathbf{b}}(z)$ (??) with $r = n$ looks as

$$\begin{array}{ccc}
 & |i_n\rangle \otimes |j_n\rangle & \\
 \swarrow 1 & & \searrow q^{i_n} \\
 & f_n \otimes 1 & k_n^{-1} \otimes \tilde{f}_n \\
 & & \\
 |i_n+1\rangle \otimes |j_n\rangle & & |i_n\rangle \otimes |j_n+1\rangle \\
 \swarrow \mathcal{R}_{i_n+1, j_n, c_n}^{a_n, b_n, c_{n-1}} & S^{1,1}(z) & \searrow \mathcal{R}_{i_n, j_n+1, c_n}^{a_n, b_n, c_{n-1}} \\
 & & \\
 & |a_n\rangle \otimes |b_n\rangle &
 \end{array}$$

Thus we get

$$\begin{aligned}
 B_{i,j}^{\mathbf{a},\mathbf{b}}(z) &= \sum_{c_0, \dots, c_n} \frac{z^{c_0}}{(q)_{c_n}} X(c_0, \dots, c_{n-1}) \left(\mathcal{R}_{i_n+1, j_n, c_n}^{a_n, b_n, c_{n-1}} + q^{i_n} \mathcal{R}_{i_n, j_n+1, c_n}^{a_n, b_n, c_{n-1}} \right) \\
 &= \sum_{c_0, \dots, c_n} \frac{z^{c_0}}{(q)_{c_n}} X(c_0, \dots, c_{n-1}) \left(\mathcal{R}_{i_n+1, j_n, c_n}^{a_n, b_n, c_{n-1}} + q^{i_n} (1 - q^{c_n}) \mathcal{R}_{i_n, j_n+1, c_{n-1}}^{a_n, b_n, c_{n-1}} \right), \quad (4.7)
 \end{aligned}$$

where X is exactly the same function as the one in (??). In (??) and (??), the z -dependence is solely by z^{c_0} hence the two c_0 's must be identified. Then from (??) and $\mathcal{R}_{i,j,k}^{a,b,c} \propto \delta_{j+k}^{b+c}$ it follows that all the c_i 's appearing in (??) and (??) are identical. Therefore the proof of $A_{i,j}^{\mathbf{a},\mathbf{b}}(z) = B_{i,j}^{\mathbf{a},\mathbf{b}}(z)$ is reduced to

$$(1 - q^{c_n}) \mathcal{R}_{i_n, j_n, c_{n-1}}^{a_n, b_n-1, c_{n-1}} + q^{b_n} \mathcal{R}_{i_n, j_n, c_n}^{a_n-1, b_n, c_{n-1}} - \mathcal{R}_{i_n+1, j_n, c_n}^{a_n, b_n, c_{n-1}} - q^{i_n} (1 - q^{c_n}) \mathcal{R}_{i_n, j_n+1, c_{n-1}}^{a_n, b_n, c_{n-1}} = 0.$$

But this is just (??), which completes the proof of (??) for $(s, t) = (1, 1)$ and $r = n$.

The essential feature in the above proof is that (??) is reduced, upon substitution of (??), to a *local* relation in the sequence of \mathcal{R} 's in (??) with *no* sum over c_0, \dots, c_n . Another useful fact is that the actions of f_r 's are identical in Proposition ??, ?? and ?? if the vicinity of the vertex r of the corresponding Dynkin diagrams has the same shape. From these considerations, one can attribute the full proof of (??) to the following cases.

- (i) $r = n$ with $(s, t) = (1, 1)$. This concerns $D_{n+1}^{(2)}$. We have just finished the proof.
- (ii) $r = 0$ with $(s, t) = (1, 1), (1, 2)$. This covers $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}$.
- (iii) $r = n$ with $(s, t) = (1, 2), (2, 2)$. This covers $A_{2n}^{(2)}$ and $C_n^{(1)}$.
- (iv) $r = 0$ with $(s, t) = (2, 2)$. This concerns $C_n^{(1)}$.

(v) $1 \leq r \leq n-1$ for any (s, t) . This covers $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$ and $C_n^{(1)}$.

In what follows, we present the expressions like (??) and (??) for each case and show how they are identified by using the formulas in Appendix ???. In the cases (ii) and (iv), one needs to cope with the spectral parameter $z = x/y$ by shifting c_0 appropriately. The case (v) is peculiar in that it requires a proof of a *quadratic* relation of \mathcal{R} .

(ii) The $A_{i,j}^{\mathbf{a},\mathbf{b}}(z)$ and $B_{i,j}^{\mathbf{a},\mathbf{b}}(z)$ relevant to $(s, t) = (1, 1), (1, 2)$ are expressed as

$$A_{i,j}^{\mathbf{a},\mathbf{b}}(z) = \sum_{c_0, \dots, c_n} \frac{z^{c_0} (q^2)_{c_0}}{(q)_{c_0}} \left([b_1 + 1] \mathcal{R}_{i_1, j_1, c_1}^{a_1, b_1+1, c_0} + q^{-b_1} (1 + q^{c_0+1}) [a_1 + 1] \mathcal{R}_{i_1, j_1, c_1}^{a_1+1, b_1, c_0+1} \right) Y_t(c_1, \dots, c_n),$$

$$B_{i,j}^{\mathbf{a},\mathbf{b}}(z) = \sum_{c_0, \dots, c_n} \frac{z^{c_0} (q^2)_{c_0}}{(q)_{c_0}} \left((1 + q^{c_0+1}) [i_1] \mathcal{R}_{i_1-1, j_1, c_1}^{a_1, b_1, c_0+1} + q^{-i_1} [j_1] \mathcal{R}_{i_1, j_1-1, c_1}^{a_1, b_1, c_0} \right) Y_t(c_1, \dots, c_n)$$

for some Y_t which is independent of z . These expressions are identified by (??).

(iii) The $A_{i,j}^{\mathbf{a},\mathbf{b}}(z)$ and $B_{i,j}^{\mathbf{a},\mathbf{b}}(z)$ relevant to $(s, t) = (1, 2), (2, 2)$ are expressed as

$$A_{i,j}^{\mathbf{a},\mathbf{b}}(z) = \sum_{c_0, \dots, c_n} \frac{z^{c_0}}{(q^4)_{c_n}} Z_s(c_0, \dots, c_{n-1}) \left((1 - q^{4c_n}) \mathcal{R}_{i_n, j_n, 2c_n-2}^{a_n, b_n-2, c_{n-1}} + q^{2b_n} \mathcal{R}_{i_n, j_n, 2c_n}^{a_n-2, b_n, c_{n-1}} \right),$$

$$B_{i,j}^{\mathbf{a},\mathbf{b}}(z) = \sum_{c_0, \dots, c_n} \frac{z^{c_0}}{(q^4)_{c_n}} Z_s(c_0, \dots, c_{n-1}) \left(\mathcal{R}_{i_n+2, j_n, 2c_n}^{a_n, b_n, c_{n-1}} + q^{2i_n} (1 - q^{4c_n}) \mathcal{R}_{i_n, j_n+2, 2c_n-2}^{a_n, b_n, c_{n-1}} \right)$$

for some Z_s which is independent of z . These expressions are identified by (??).

(iv) The $A_{i,j}^{\mathbf{a},\mathbf{b}}(z)$ and $B_{i,j}^{\mathbf{a},\mathbf{b}}(z)$ relevant to $(s, t) = (2, 2)$ are expressed as

$$A_{i,j}^{\mathbf{a},\mathbf{b}}(z) = \sum_{c_0, \dots, c_n} \frac{z^{c_0} (q^2)_{2c_0}}{(q^4)_{c_0}} \left([b_1 + 2] [b_1 + 1] \mathcal{R}_{i_1, j_1, c_1}^{a_1, b_1+2, 2c_0} \right. \\ \left. + q^{-2b_1} (1 - q^{4c_0+2}) [a_1 + 2] [a_1 + 1] \mathcal{R}_{i_1, j_1, c_1}^{a_1+2, b_1, 2c_0+2} \right) W(c_1, \dots, c_n),$$

$$B_{i,j}^{\mathbf{a},\mathbf{b}}(z) = \sum_{c_0, \dots, c_n} \frac{z^{c_0} (q^2)_{2c_0}}{(q^4)_{c_0}} \left((1 - q^{4c_0+2}) [i_1] [i_1 - 1] \mathcal{R}_{i_1-2, j_1, c_1}^{a_1, b_1, 2c_0+2} \right. \\ \left. + q^{-2i_1} [j_1] [j_1 - 1] \mathcal{R}_{i_1, j_1-2, c_1}^{a_1, b_1, 2c_0} \right) W(c_1, \dots, c_n)$$

for some W which is independent of z . These expressions are identified by (??).

(v) The f_r and k_r with $1 \leq r \leq n-1$ concern the r th and the $(r+1)$ th components of $F^{\otimes n}$ only. The diagram for (??) tracing them looks as

$$\begin{array}{ccc} & |i_r, i_{r+1}\rangle \otimes |j_r, j_{r+1}\rangle & \\ & \swarrow S^{s,t}(z) \quad S^{s,t}(z) \searrow & \\ |a_r, a_{r+1}\rangle \otimes |b_r-1, b_{r+1}+1\rangle & & |a_r-1, a_{r+1}+1\rangle \otimes |b_r, b_{r+1}\rangle \\ & \swarrow 1 \otimes f_r \quad \tilde{f}_r \otimes k_r^{-1} \searrow & \\ [b_{r+1}+1] & & q^{b_r-b_{r+1}} [a_{r+1}+1] \\ & |a_r, a_{r+1}\rangle \otimes |b_r, b_{r+1}\rangle & \end{array}$$

Thus we have

$$A_{i,j}^{\mathbf{a},\mathbf{b}}(z) = \sum_{c_0, \dots, c_n} z^{c_0} U_{s,t}(c_0, \dots, c_{r-1}, c_{r+2}, \dots, c_n) \\ \times \left([b_{r+1}+1] \mathcal{R}_{i_r, j_r, c_r-1}^{a_r, b_r-1, c_{r-1}} \mathcal{R}_{i_{r+1}, j_{r+1}, c_{r+1}}^{a_{r+1}, b_{r+1}+1, c_r-1} + q^{b_r-b_{r+1}} [a_{r+1}+1] \mathcal{R}_{i_r, j_r, c_r}^{a_r-1, b_r, c_{r-1}} \mathcal{R}_{i_{r+1}, j_{r+1}, c_{r+1}}^{a_{r+1}+1, b_{r+1}, c_r} \right) \quad (4.8)$$

for some $U_{s,t}$ which is independent of z . We have shifted c_r to $c_r - 1$ in the first term by the reason similar to (??) and (??). Similarly the diagram for (??) looks as

$$\begin{array}{ccc}
 & |i_r, i_{r+1}\rangle \otimes |j_r, j_{r+1}\rangle & \\
 [i_{r+1}] \swarrow & & \searrow q^{i_r - i_{r+1}} [j_{r+1}] \\
 f_r \otimes 1 & & k_r^{-1} \otimes \tilde{f}_r \\
 \swarrow & & \searrow \\
 |i_{r+1}, i_{r+1}-1\rangle \otimes |j_r, j_{r+1}\rangle & & |i_r, i_{r+1}\rangle \otimes |j_{r+1}, j_{r+1}-1\rangle \\
 \searrow S^{s,t}(z) & & \swarrow S^{s,t}(z) \\
 & |a_r, a_{r+1}\rangle \otimes |b_r, b_{r+1}\rangle &
 \end{array}$$

This leads to the expression

$$\begin{aligned}
 B_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}}(z) &= \sum_{c_0, \dots, c_n} z^{c_0} U_{s,t}(c_0, \dots, c_{r-1}, c_{r+2}, \dots, c_n) \\
 &\times \left([i_{r+1}] \mathcal{R}_{i_{r+1}, j_r, c_r}^{a_r, b_r, c_{r-1}} \mathcal{R}_{i_{r+1}-1, j_{r+1}, c_{r+1}}^{a_{r+1}, b_{r+1}, c_r} + q^{i_r - i_{r+1}} [j_{r+1}] \mathcal{R}_{i_r, j_{r+1}, c_{r-1}}^{a_r, b_r, c_{r-1}} \mathcal{R}_{i_{r+1}, j_{r+1}-1, c_{r+1}}^{a_{r+1}, b_{r+1}, c_r} \right)
 \end{aligned} \tag{4.9}$$

with the same $U_{s,t}$ as (??). This time the shift of c_r to $c_r - 1$ has been done in the second term. Now that all the c_i 's can be identified in (??) and (??), their equality follows from (??). This completes the proof of (??). The relation (??) can be verified similarly by using (??), (??), (??) and (??). \square

APPENDIX A. BRIEF GUIDE TO 3D R

A.1. Origin in quantized coordinate ring. Let us summarize the basic facts on the 3d R \mathcal{R} that has played a central role in the paper from the viewpoint of the quantized coordinate ring $A_q(sl_3)$ following [?, ?]. See also [?, ?, ?, ?] for more aspects. The $A_q(sl_3)$ is a Hopf algebra generated by $T = (t_{ij})_{1 \leq i, j \leq 3}$ satisfying the relations

$$\begin{aligned}
 [t_{ik}, t_{jl}] &= \begin{cases} 0 & (i < j, k > l), \\ (q - q^{-1}) t_{jk} t_{il} & (i < j, k < l), \end{cases} \\
 t_{ik} t_{jk} &= q t_{jk} t_{ik} \quad (i < j), \quad t_{ki} t_{kj} = q t_{kj} t_{ki} \quad (i < j).
 \end{aligned}$$

The coproduct is given by $\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}$. Let $F = \bigoplus_{m \geq 0} \mathbb{Q}(q) |m\rangle$ be the Fock space as in the main text. The $A_q(sl_3)$ has irreducible representations $\pi_i : A_q(sl_3) \rightarrow \text{End}(F)$ ($i = 1, 2$) as

$$\begin{aligned}
 \pi_1(T) &= \begin{pmatrix} \mu_1 \mathbf{a}^- & \alpha_1 \mathbf{k} & 0 \\ -q \alpha_1^{-1} \mathbf{k} & \mu_1^{-1} \mathbf{a}^+ & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \pi_2(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 \mathbf{a}^- & \alpha_2 \mathbf{k} \\ 0 & -q \alpha_2^{-1} \mathbf{k} & \mu_2^{-1} \mathbf{a}^+ \end{pmatrix}, \\
 \mathbf{k} |m\rangle &= q^m |m\rangle, \quad \mathbf{a}^+ |m\rangle = |m+1\rangle, \quad \mathbf{a}^- |m\rangle = (1 - q^{2m}) |m-1\rangle.
 \end{aligned}$$

The parameters μ_i, α_i are set to be 1 in the sequel as they do not influence the construction in a nontrivial way. The π_1 and π_2 are called fundamental representations. Let π_{121} and π_{212} be their tensor product representations on $F^{\otimes 3}$ obtained by evaluating the coproduct $\Delta(t_{ij}) = \sum_{k,l} t_{ik} \otimes t_{kl} \otimes t_{lj}$ by $\pi_1 \otimes \pi_2 \otimes \pi_1$ and $\pi_2 \otimes \pi_1 \otimes \pi_2$, respectively. It is known that they are both irreducible and equivalent. Thus there is the unique map $\Phi \in \text{End}(F^{\otimes 3})$ satisfying the intertwining relation $\Phi \circ \pi_{121} = \pi_{212} \circ \Phi$ up to normalization. Let $\sigma \in \text{End}(F^{\otimes 3})$ be the reversal of the components $\sigma(u \otimes v \otimes w) = w \otimes v \otimes u$. Then the 3d R is identified as $\mathcal{R} = \Phi \circ \sigma$ with the normalization $\mathcal{R}(|0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle$. In short 3d R \mathcal{R} is the intertwiner of $A_q(sl_3)$. The tetrahedron equation (??) is a corollary of the fact that the similar intertwiner for $A_q(sl_4)$ can be constructed as quartic products of \mathcal{R} in two different forms [?]. By investigating the intertwining relation one can show (cf. [?, Proposition 2.4])

$$\mathcal{R}_{i,j,k}^{a,b,c} = \mathcal{R}_{k,j,i}^{c,b,a}, \quad \mathcal{R}_{i,j,k}^{a,b,c} = \frac{(q^2)_i (q^2)_j (q^2)_k}{(q^2)_a (q^2)_b (q^2)_c} \mathcal{R}_{a,b,c}^{i,j,k}. \tag{A.1}$$

A.2. Intertwining relations of 3d R . By the definition the 3d R satisfies the intertwining relation $\mathcal{R} \circ \pi'_{121} = \pi_{212} \circ \mathcal{R}$, where $\pi'_{121} = \sigma \circ \pi_{121} \circ \sigma$. Evaluating the generator t_{rs} on the both sides and picking the matrix elements for $|i\rangle \otimes |j\rangle \otimes |k\rangle \mapsto |a\rangle \otimes |b\rangle \otimes |c\rangle$ by using (??) leads to useful recursion relations on $\mathcal{R}_{i,j,k}^{a,b,c}$:

$$\begin{aligned}
t_{11} &: q^{i+k+1} (1-q^{2j}) \mathcal{R}_{i,j-1,k}^{a,b,c} - (1-q^{2i}) (1-q^{2k}) \mathcal{R}_{i-1,j,k-1}^{a,b,c} + (1-q^{2b+2}) \mathcal{R}_{i,j,k}^{a,b+1,c} = 0, \\
t_{12} &: q^k (1-q^{2j}) \mathcal{R}_{i+1,j-1,k}^{a,b,c} + q^i (1-q^{2k}) \mathcal{R}_{i,j,k-1}^{a,b,c} - q^b (1-q^{2c+2}) \mathcal{R}_{i,j,k}^{a,b,c+1} = 0, \\
t_{21} &: q^i (1-q^{2j}) \mathcal{R}_{i,j-1,k+1}^{a,b,c} + q^k (1-q^{2i}) \mathcal{R}_{i-1,j,k}^{a,b,c} - q^b (1-q^{2a+2}) \mathcal{R}_{i,j,k}^{a+1,b,c} = 0, \\
t_{22} &: q (q^{a+c} - q^{i+k}) \mathcal{R}_{i,j,k}^{a,b,c} + (1-q^{2j}) \mathcal{R}_{i+1,j-1,k+1}^{a,b,c} - (1-q^{2a+2}) (1-q^{2c+2}) \mathcal{R}_{i,j,k}^{a+1,b-1,c+1} = 0, \\
t_{23} &: q^j \mathcal{R}_{i,j,k+1}^{a,b,c} - q^a \mathcal{R}_{i,j,k}^{a,b,c-1} - q^c (1-q^{2a+2}) \mathcal{R}_{i,j,k}^{a+1,b-1,c} = 0, \\
t_{32} &: q^c \mathcal{R}_{i,j,k}^{a-1,b,c} - q^j \mathcal{R}_{i+1,j,k}^{a,b,c} + q^a (1-q^{2c+2}) \mathcal{R}_{i,j,k}^{a,b-1,c+1} = 0, \\
t_{33} &: q^{a+c+1} \mathcal{R}_{i,j,k}^{a,b-1,c} - \mathcal{R}_{i,j,k}^{a-1,b,c-1} + \mathcal{R}_{i,j+1,k}^{a,b,c} = 0.
\end{aligned} \tag{A.2}$$

We have skipped t_{13} and t_{31} as they just give $(q^{i+j} - q^{a+b}) \mathcal{R}_{i,j,k}^{a,b,c} = (q^{j+k} - q^{b+c}) \mathcal{R}_{i,j,k}^{a,b,c} = 0$, which is the origin of the conservation law (??). The formula (??) was derived by solving (??) and (??) [?]. The relation for t_{rs} here is transformed into the one for $t_{4-s,4-r}$ by using the latter property in (??).

It is known that $\mathcal{R} = \mathcal{R}^{-1}$ hence $\pi'_{121} \circ \mathcal{R} = \mathcal{R} \circ \pi_{212}$ also holds. The recursion relations corresponding to this read

$$t_{11} : q^{a+c+1} (1-q^{2b+2}) \mathcal{R}_{i,j,k}^{a,b+1,c} + (1-q^{2j}) \mathcal{R}_{i,j-1,k}^{a,b,c} - (1-q^{2a+2}) (1-q^{2c+2}) \mathcal{R}_{i,j,k}^{a+1,b,c+1} = 0, \tag{A.4}$$

$$t_{12} : q^j (1-q^{2k}) \mathcal{R}_{i,j,k-1}^{a,b,c} - q^c (1-q^{2b+2}) \mathcal{R}_{i,j,k}^{a-1,b+1,c} - q^a (1-q^{2c+2}) \mathcal{R}_{i,j,k}^{a,b,c+1} = 0,$$

$$t_{21} : q^j (1-q^{2i}) \mathcal{R}_{i-1,j,k}^{a,b,c} - q^a (1-q^{2b+2}) \mathcal{R}_{i,j,k}^{a,b+1,c-1} - q^c (1-q^{2a+2}) \mathcal{R}_{i,j,k}^{a+1,b,c} = 0, \tag{A.5}$$

$$t_{22} : (1-q^{2b+2}) \mathcal{R}_{i,j,k}^{a-1,b+1,c-1} - (1-q^{2i}) (1-q^{2k}) \mathcal{R}_{i-1,j+1,k-1}^{a,b,c} - q (q^{a+c} - q^{i+k}) \mathcal{R}_{i,j,k}^{a,b,c} = 0,$$

$$t_{23} : q^i \mathcal{R}_{i,j,k+1}^{a,b,c} - q^b \mathcal{R}_{i,j,k}^{a,b,c-1} + q^k (1-q^{2i}) \mathcal{R}_{i-1,j+1,k}^{a,b,c} = 0,$$

$$t_{32} : q^b \mathcal{R}_{i,j,k}^{a-1,b,c} - q^i (1-q^{2k}) \mathcal{R}_{i,j+1,k-1}^{a,b,c} - q^k \mathcal{R}_{i+1,j,k}^{a,b,c} = 0, \tag{A.6}$$

$$t_{33} : q^{i+k+1} \mathcal{R}_{i,j+1,k}^{a,b,c} + \mathcal{R}_{i,j,k}^{a,b-1,c} - \mathcal{R}_{i+1,j,k+1}^{a,b,c} = 0. \tag{A.7}$$

Again we have omitted t_{13} and t_{31} leading to the conservation law. The relation for t_{rs} here is transformed into the one for $t_{4-s,4-r}$ by combining the two properties in (??).

Although not all of the above recursion relations are necessary in this paper, we have listed them for convenience in possible future works.

A.3. Lemma. Now we collect the relations necessary in the proof of Theorem ???. We recall that $[m] = [m]_q$ is defined in the end of Section ???.

Lemma 15. *The following relations hold:*

$$(1-q^k) \mathcal{R}_{i,j,k-1}^{a,b-1,c} + q^b \mathcal{R}_{i,j,k}^{a-1,b,c} - \mathcal{R}_{i+1,j,k}^{a,b,c} - q^i (1-q^k) \mathcal{R}_{i,j+1,k-1}^{a,b,c} = 0, \tag{A.8}$$

$$[b+1] \mathcal{R}_{i,j,k}^{a,b+1,c} + q^{-b} (1+q^{c+1}) [a+1] \mathcal{R}_{i,j,k}^{a+1,b,c+1} - (1+q^{c+1}) [i] \mathcal{R}_{i-1,j,k}^{a,b,c+1} - q^{-i} [j] \mathcal{R}_{i,j-1,k}^{a,b,c} = 0, \tag{A.9}$$

$$(1-q^{2k}) \mathcal{R}_{i,j,k-2}^{a,b-2,c} + q^{2b} \mathcal{R}_{i,j,k}^{a-2,b,c} - \mathcal{R}_{i+2,j,k}^{a,b,c} - q^{2i} (1-q^{2k}) \mathcal{R}_{i,j+2,k-2}^{a,b,c} = 0, \tag{A.10}$$

$$\begin{aligned}
& [b+2][b+1] \mathcal{R}_{i,j,k}^{a,b+2,c} + q^{-2b} (1-q^{2c+2}) [a+2][a+1] \mathcal{R}_{i,j,k}^{a+2,b,c+2} \\
& - (1-q^{2c+2}) [i][i-1] \mathcal{R}_{i-2,j,k}^{a,b,c+2} - q^{-2i} [j][j-1] \mathcal{R}_{i,j-2,k}^{a,b,c} = 0.
\end{aligned} \tag{A.11}$$

Proof. Regard (??) as a recursion increasing a by one keeping b and c . Similarly (??) provides a recursion increasing b by one keeping a and c . One can apply them to the first two terms in (??) to bring all the terms into the form $\mathcal{R}_{\bullet,\bullet,\bullet}^{a,b,c}$. The result turns out to be identically zero. The equality (??) is shown in the same way by applying each recursion *twice* to the first two terms

therein. Finally (??) and (??) are derived by applying the latter relation in (??) to (??) and (??), respectively. \square

Lemma 16. *The following quadratic relation among \mathcal{R} holds:*

$$\begin{aligned} & [b'+1]\mathcal{R}_{i,j,k-1}^{a,b-1,c}\mathcal{R}_{i',j',k'}^{a',b'+1,k-1} + q^{b-b'}[a'+1]\mathcal{R}_{i,j,k}^{a-1,b,c}\mathcal{R}_{i',j',k'}^{a'+1,b',k} \\ & - [i']\mathcal{R}_{i+1,j,k}^{a,b,c}\mathcal{R}_{i'-1,j',k'}^{a',b',k} - q^{i-i'}[j']\mathcal{R}_{i,j+1,k-1}^{a,b,c}\mathcal{R}_{i',j'-1,k'}^{a',b',k-1} = 0. \end{aligned} \quad (\text{A.12})$$

Proof. From (??) and (??) one has

$$\begin{aligned} \mathcal{R}_{i,j,k-1}^{a,b-1,c} &= \mathcal{R}_{i+1,j,k}^{a,b,c} - q^{i+k}\mathcal{R}_{i,j+1,k-1}^{a,b,c}, \\ [j']\mathcal{R}_{i',j'-1,k'}^{a',b',k-1} &= q^{a'-j'+1}[a'+1](1-q^{2k})\mathcal{R}_{i',j',k'}^{a'+1,b',k} - q^{i'+k}[b'+1]\mathcal{R}_{i',j',k'}^{a',b'+1,k-1}. \end{aligned}$$

By substituting them to the first and the last \mathcal{R} in (??), the LHS becomes

$$\begin{aligned} & [b'+1](\mathcal{R}_{i+1,j,k}^{a,b,c} - q^{i+k}\mathcal{R}_{i,j+1,k-1}^{a,b,c})\mathcal{R}_{i',j',k'}^{a',b'+1,k-1} + q^{b-b'}[a'+1]\mathcal{R}_{i,j,k}^{a-1,b,c}\mathcal{R}_{i',j',k'}^{a'+1,b',k} \\ & - [i']\mathcal{R}_{i+1,j,k}^{a,b,c}\mathcal{R}_{i'-1,j',k'}^{a',b',k} - q^{i-i'}\mathcal{R}_{i,j+1,k-1}^{a,b,c}(q^{a'-j'+1}[a'+1](1-q^{2k})\mathcal{R}_{i',j',k'}^{a'+1,b',k} - q^{i'+k}[b'+1]\mathcal{R}_{i',j',k'}^{a',b'+1,k-1}). \end{aligned}$$

The contributions from the underlined terms cancel. The remaining four terms are grouped as

$$\begin{aligned} & \mathcal{R}_{i+1,j,k}^{a,b,c}([b'+1]\mathcal{R}_{i',j',k'}^{a',b'+1,k-1} - [i']\mathcal{R}_{i'-1,j',k'}^{a',b',k}) \\ & + q^{-b'}[a'+1](q^b\mathcal{R}_{i,j,k}^{a-1,b,c} - q^{i+\phi}(1-q^{2k})\mathcal{R}_{i,j+1,k-1}^{a,b,c})\mathcal{R}_{i',j',k'}^{a'+1,b',k} \end{aligned} \quad (\text{A.13})$$

with $\phi = a' + b' - i' - j' + 1$ which is zero due the conservation law for $\mathcal{R}_{i',j',k'}^{a'+1,b',k}$. The combination in the first parenthesis in (??) is equal to $-q^{k-b'}[a'+1]\mathcal{R}_{i',j',k'}^{a'+1,b',k}$ due to (??) and $\phi = 0$. The one in the second parenthesis is equal to $q^k\mathcal{R}_{i+1,j,k}^{a,b,c}$ by (??). Thus (??) vanishes. \square

We note that \mathcal{R} satisfies further relations

$$q^{a+1}(1+q^c)\mathcal{R}_{i,j,k}^{a,b-1,c} - \mathcal{R}_{i,j,k}^{a-1,b,c-1} - q^{j+1}\mathcal{R}_{i+1,j,k}^{a,b,c-1} + (1+q^c)\mathcal{R}_{i,j+1,k}^{a,b,c} = 0, \quad (\text{A.14})$$

$$\mathcal{R}_{i,j,k}^{a-2,b,c} + q^{2a+2}(1-q^{2c+2})\mathcal{R}_{i,j,k}^{a,b-2,c+2} - (1-q^{2c+2})\mathcal{R}_{i,j+2,k}^{a,b,c+2} - q^{2j+2}\mathcal{R}_{i+2,j,k}^{a,b,c} = 0, \quad (\text{A.15})$$

$$\begin{aligned} & [a+1]\mathcal{R}_{i,j,k}^{a+1,b,c}\mathcal{R}_{i',j',k'}^{a'-1,b',k} + q^{-a+a'}[b+1]\mathcal{R}_{i,j,k+1}^{a,b+1,c}\mathcal{R}_{i',j',k'}^{a',b'-1,k+1} \\ & - [j]\mathcal{R}_{i,j-1,k+1}^{a,b,c}\mathcal{R}_{i',j'+1,k'}^{a',b',k+1} - q^{-j+j'}[i]\mathcal{R}_{i-1,j,k}^{a,b,c}\mathcal{R}_{i'+1,j',k'}^{a',b',k} = 0. \end{aligned} \quad (\text{A.16})$$

They are proved similarly to Lemma ?? and Lemma ??.

APPENDIX B. TRACE REDUCTION OF TETRAHEDRON EQUATION AND q -OSCILLATOR REPRESENTATION OF $U_q(A_{n-1}^{(1)})$

For comparison we include an exposition of type A case which is known to be related to the trace reduction of the tetrahedron equation to the Yang-Baxter equation [?].

Let \mathbf{h} be the operator on F acting as $\mathbf{h}|m\rangle = m|m\rangle$. Then the conservation law (??) implies the commutativity $[\mathcal{R}_{1,2,3}, x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_3}] = 0$. Multiplying $\mathcal{R}_{4,5,6}^{-1}x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6}$ from the left to (??) and taking the trace over $\overset{4}{F} \otimes \overset{5}{F} \otimes \overset{6}{F}$, one finds that $S_{\alpha,\beta}^{\text{tr}}(z) = \text{Tr}_3(z^{\mathbf{h}_3}\mathcal{R}_{\alpha_1,\beta_1,3}\mathcal{R}_{\alpha_2,\beta_2,3}\cdots\mathcal{R}_{\alpha_n,\beta_n,3}) \in \text{End}(\overset{\alpha}{F} \otimes \overset{\beta}{F})$ satisfies the Yang-Baxter equation (??). The matrix elements are given by

$$\begin{aligned} S^{\text{tr}}(z)(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle) &= \sum_{\mathbf{a}, \mathbf{b}} S^{\text{tr}}(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle, \\ S^{\text{tr}}(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} &= \sum_{c_0, \dots, c_{n-1} \geq 0} z^{c_0} \mathcal{R}_{i_1, j_1, c_1}^{a_1, b_1, c_0} \mathcal{R}_{i_2, j_2, c_2}^{a_2, b_2, c_1} \cdots \mathcal{R}_{i_{n-1}, j_{n-1}, c_{n-1}}^{a_{n-1}, b_{n-1}, c_{n-2}} \mathcal{R}_{i_n, j_n, c_0}^{a_n, b_n, c_{n-1}}, \end{aligned}$$

which is the trace version of the formula (??)–(??). For instance one has

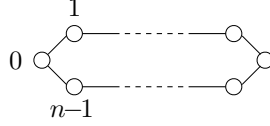
$$\begin{aligned} S^{\text{tr}}(z)_{\mathbf{a}, \mathbf{0}}^{\mathbf{a}, \mathbf{0}} &= \frac{1}{1 - zq^{|\mathbf{a}|}}, \\ S^{\text{tr}}(z)_{m\mathbf{e}_k, l\mathbf{e}_k}^{m\mathbf{e}_k, l\mathbf{e}_k} &= (-q)^{l-m} S^{\text{tr}}(z)_{l\mathbf{e}_k, m\mathbf{e}_k}^{l\mathbf{e}_k, m\mathbf{e}_k} = z^l \frac{(q^{m-l+2}z^{-1}; q^2)_l}{(q^{m-l}z; q^2)_{l+1}} \end{aligned} \quad (\text{B.1})$$

for any k . See Section ?? for the notation. The conservation law takes the form

$$S^{\text{tr}}(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} = 0 \quad \text{unless } |\mathbf{a}| = |\mathbf{i}|, |\mathbf{b}| = |\mathbf{j}|, \quad (\text{B.2})$$

therefore $S^{\text{tr}}(z)$ splits into infinitely many irreducible components.

The $S^{\text{tr}}(z)$ stems from the q -oscillator representation of $U_q(A_{n-1}^{(1)})$ ($n \geq 2$), which we shall now explain. The algebra $U_q(A_{n-1}^{(1)})$ is defined by (??) with n replaced by $n-1$, $a_{ij} = 2\delta_{i,j} - \delta_{|i-j|,1} - \delta_{|i-j|,n}$ and $q_i = q$ for $0 \leq i \leq n-1$. For $n \geq 3$ the Dynkin diagram has circle shape:



It is easy to see that the action of the generators

$$\begin{aligned} e_j |\mathbf{m}\rangle &= x^{\delta_{j,0}} [m_j] |\mathbf{m} - \mathbf{e}_j + \mathbf{e}_{j+1}\rangle, \\ f_j |\mathbf{m}\rangle &= x^{-\delta_{j,0}} [m_{j+1}] |\mathbf{m} + \mathbf{e}_j - \mathbf{e}_{j+1}\rangle, \\ k_j |\mathbf{m}\rangle &= q^{-m_j + m_{j+1}} |\mathbf{m}\rangle \end{aligned} \quad (\text{B.3})$$

defines a $U_q(A_{n-1}^{(1)})$ module structure on $F^{\otimes n}$. Here the indices are to be understood mod n and x is a nonzero parameter. The representation (??) essentially goes back to [?].

Denote the representation space by $V_x = F^{\otimes n}[x, x^{-1}]$. It decomposes as

$$V_x = \bigoplus_{l \geq 0} V_{x,l}, \quad V_{x,l} = \bigoplus_{\mathbf{m} \in (\mathbb{Z}_{\geq 0})^n, |\mathbf{m}|=l} \mathbb{Q}(q) |\mathbf{m}\rangle,$$

where the symbol $|\mathbf{m}\rangle$ is defined under (??). The component $V_{x,l}$ is isomorphic, as a module over the classical subalgebra $U_q(A_{n-1}) = \langle e_i, f_i, k_i^{\pm 1} \rangle_{1 \leq i < n}$, to the highest weight representation with highest weight $l\varpi_{n-1}$ ³. Its highest weight vector is $|\mathbf{l}\mathbf{e}_n\rangle$. Let $R = R_{l,m}(z) \in \text{End}(V_{x,m} \otimes V_{y,l})$ ($z = x/y$) be the quantum R matrix. Namely R satisfies (??) for $U_q = U_q(A_{n-1}^{(1)})$. We normalize it by $R_{m,l}(z)(|m\mathbf{e}_n\rangle \otimes |\mathbf{l}\mathbf{e}_n\rangle) = z^l \frac{(q^{m-l+2}z^{-1}; q^2)_l}{(q^{m-l}z; q^2)_{l+1}} |m\mathbf{e}_n\rangle \otimes |\mathbf{l}\mathbf{e}_n\rangle$. Up to the normalization of R matrices and conventional difference, the following equality was announced in [?].

Proposition 17.

$$S^{\text{tr}}(z) = \bigoplus_{m,l \geq 0} R_{m,l}(z).$$

Proof. The conservation law (??) tells that $S^{\text{tr}}(z)$ satisfies (??) and splits in the same pattern as the RHS. By (??) the both sides have the same normalization on $|m\mathbf{e}_n\rangle \otimes |\mathbf{l}\mathbf{e}_n\rangle$. Thus it suffices to show $(1 \otimes f_r + f_r \otimes k_r^{-1})S^{\text{tr}}(z) = S^{\text{tr}}(z)(f_r \otimes 1 + k_r^{-1} \otimes f_r)$ for $0 \leq r \leq n-1$. As the case (v) in the proof of Theorem ??, this reduces exactly to Lemma ?? including $r = 0$ case. \square

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³ ϖ_j denotes the j th fundamental weight. By a conventional reason the highest weight here is a dual of $l\varpi_1$ in [?].

REFERENCES

- [1] R. J. Baxter, *Exactly solved models in statistical mechanics*, Dover (2007).
- [2] V. V. Bazhanov, V. V. Mangazeev and S. M. Sergeev, *Quantum geometry of 3-dimensional lattices*, J. Stat. Mech. P07004 (2008).
- [3] V. V. Bazhanov, V. V. Mangazeev and S. M. Sergeev, *An integrable 3D lattice model with positive Boltzmann weights*, J. Phys. A: Math. Theor. **46** 465206 (2013).
- [4] V. V. Bazhanov and S. M. Sergeev, *Zamolodchikov's tetrahedron equation and hidden structure of quantum groups*, J. Phys. A: Math. Theor. **39** 3295–3310 (2006).
- [5] V. G. Drinfeld, *Quantum groups*, in Proceedings of the International Congress of Mathematicians, Vols. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, 798–820.
- [6] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and Its Applications (G.C.Rota, ed.) **35**, Cambridge Univ. press (1990).
- [7] M. Jimbo, *A q -difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** 63–69 (1985).
- [8] T. Hayashi, *Q -analogues of Clifford and Weyl algebras—spinor and oscillator representations of quantum enveloping algebras*, Comm. Math. Phys. **127** 129–144 (1990).
- [9] V. G. Kac, *Infinite dimensional Lie algebras*, third ed., Cambridge University Press (1990).
- [10] M. M. Kapranov and V. A. Voevodsky, *2-Categories and Zamolodchikov tetrahedron equations*, Proc. Symposia in Pure Math. **56** 177–259 (1994).
- [11] R. M. Kashaev and A. Yu. Volkov, *From the tetrahedron equation to universal R -matrices*, L. D. Faddeev's Seminar on Mathematical Physics, 79–89, Amer. Math. Soc. Transl. Ser. 2, 201, Amer. Math. Soc., Providence, RI, 2000.
- [12] A. Kuniba and M. Okado, *Tetrahedron and 3D reflection equations from quantized algebra of functions*, J. Phys. A: Math.Theor. **45** (2012) 465206 (27pp).
- [13] A. Kuniba and M. Okado, *Tetrahedron equation and quantum R matrices for infinite dimensional modules of $U_q(A_1^{(1)})$ and $U_q(A_2^{(2)})$* , J. Phys. A: Math.Theor. **46** (2013) 485203 (12pp).
- [14] A. Kuniba, M. Okado and Y. Yamada, *A common structure in PBW bases of the nilpotent subalgebra of $U_q(\mathfrak{g})$ and quantized algebra of functions*, SIGMA 9 (2013), 049, 23 pages.
- [15] A. Kuniba and S. Sergeev, *Tetrahedron equation and quantum R matrices for spin representations of $B_n^{(1)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$* , Commun. Math. Phys. **324** 695–713 (2013).
- [16] T. Masuda, K. Mimachi, Y. Nakagami, M. Noumi, Y. Saburi and K. Ueno, *Unitary representations of the quantum group $SU_q(1, 1)$: II - Matrix elements of unitary representations and the basic hypergeometric functions*, Lett. Math. Phys. **19** 195–204 (1990).
- [17] M. Okado, *Quantum R matrices related to the spin representations of B_n and D_n* , Commun. Math. Phys. **134** 467–486 (1990).
- [18] W. Pusz, *Irreducible unitary representations of quantum Lorentz group*, Commun. Math. Phys. **152** 591–626 (1993).
- [19] S. M. Sergeev, *Two-dimensional R -matrices—descendants of three-dimensional R -matrices*, Modern Phys. Lett. A **12** 1393–1410 (1997).
- [20] S. M. Sergeev, *Tetrahedron equations and nilpotent subalgebras of $U_q(\mathfrak{sl}_n)$* , Lett. Math. Phys. **83** 231–235 (2008).
- [21] A. B. Zamolodchikov, *Tetrahedra equations and integrable systems in three-dimensional space*, Soviet Phys. JETP **79** 641–664 (1980).

E-mail address: atsuo@gokutan.c.u-tokyo.ac.jp

INSTITUTE OF PHYSICS, GRADUATE SCHOOL OF ARTS AND SCIENCES, UNIVERSITY OF TOKYO, KOMABA, TOKYO 153-8902, JAPAN

E-mail address: okado@sci.osaka-cu.ac.jp

DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY, 3-3-138, SUGIMOTO, SUMIYOSHI-KU, OSAKA, 558-8585, JAPAN