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Extremal solutions to Liouville-Gelfand type elliptic problems with nonlinear Neumann boundary conditions

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Abstract. Consider the Liouville-Gelfand type problems with nonlinear Neumann boundary conditions

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda f(u) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \ge 2$, is a smooth bounded domain, $f : [0, +\infty) \to (0, +\infty)$ is a smooth, strictly positive, convex, increasing function with superlinear at $+\infty$, and $\lambda > 0$ is a parameter. In this paper, after introducing a suitable notion of weak solutions, we prove several properties of extremal solutions u^* corresponding to $\lambda = \lambda^*$, called an extremal parameter, such as regularity, uniqueness, and the existence of weak eigenfunctions associated to the linearized extremal problem.

Keywords: Extremal solutions, Weak solutions, Nonlinear Neumann boundary conditions.

2010 Mathematics Subject Classifications: 35J20, 35J25. 35J60.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ be a smooth bounded domain and let ν denote the unit outer normal to $\partial\Omega$. Consider the Liouville-Gelfand type problems with nonlinear Neumann boundary conditions

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda f(u) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter. Throughout the paper, the nonlinearity $f : [0, +\infty) \to (0, +\infty)$ is assumed to satisfy

$$f \in C^1([0, +\infty)),\ f(0) > 0, \text{ convex, increasing,} \tag{1.2}$$

$$\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty. \tag{1.3}$$

Then maximum principle implies that solutions are positive on $\overline{\Omega}$.

Problem (1.1) may be considered as a variant of the well-studied problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\lambda > 0$ and f is assumed to satisfy (1.2), (1.3). For the problem (1.4), the notion of suitable weak solutions, the uniqueness and the regularity of extremal solutions, and the existence of the weak eigenfunction corresponding to zero eigenvalue of the linearized problem around the weak extremal solution, have been studied so far, see [4], [6], [13], [17], [8], and the references therein. Main purpose of this paper is to establish several facts for the problem (1.1), known to be true for (1.4). For other type of variants of the problem (1.4), see [3], [11].

Now, it is classic that the following proposition holds for the problem (1.1). The proof will be obtained by a slight modification of that of the similar proposition for the problem (1.4), see [9], [16], [10], [12], so we omit it.

Proposition 1 *Define*

$$\lambda^* = \sup\{\lambda > 0 :\ (1.1)_\lambda \textit{ admits a classical solution} \in C^2(\overline{\Omega})\}. \tag{1.5}$$

Then we have $0 < \lambda^* < \infty$ *and*

(i) For every $\lambda \in (0, \lambda^*)$, $(1.1)_\lambda$ *has a positive, classical, minimal solution* u_λ *in the sense that* $u_\lambda(x) \le u(x) (\forall x \in \overline{\Omega})$ *for any other solution* u *to* $(1.1)_\lambda$. *This is the unique strictly stable solution of* $(1.1)_\lambda$, *in the sense that*

$$\int_\Omega (|\nabla\varphi|^2 + \varphi^2) dx > \lambda \int_{\partial\Omega} f'(u_\lambda)\varphi^2 ds_x \tag{1.6}$$

holds for every $\varphi \in C^1(\overline{\Omega}), \varphi \not\equiv 0$.

(ii) The map $\lambda \mapsto u_\lambda(x)$ is continuous and increasing for any $x \in \overline{\Omega}$.

Motivated by the work by P. Quittner and W. Reichel [18], see also J. Dávila [10], we define the notion of weak solutions of (1.1) as follows.

Definition 2 *([18]) Let $L^1(\Omega\times\partial\Omega)$ denote the space of measurable functions u on $\overline{\Omega}$ such that its pointwise restrictions $u|_\Omega$ and $u|_{\partial\Omega}$ satisfy $(u|_\Omega, u|_{\partial\Omega}) \in L^1(\Omega) \times L^1(\partial\Omega)$. $L^1(\Omega \times \partial\Omega)$ is a Banach space with the norm*

$$\|u\|_{L^1(\Omega\times\partial\Omega)} = \|u|_\Omega\|_{L^1(\Omega)} + \|u|_{\partial\Omega}\|_{L^1(\partial\Omega)}.$$

As is remarked in [18], $u|_\Omega$ and $u|_{\partial\Omega}$ are not generally related with each other for $u \in L^1(\Omega \times \partial\Omega)$. The space $L^1(\Omega \times \partial\Omega)$ is isomorphic to $L^1(\Omega) \times L^1(\partial\Omega)$. We use the notation $u = (u_1, u_2) \in L^1(\Omega) \times L^1(\partial\Omega)$ for $u \in L^1(\Omega \times \partial\Omega)$, where $u_1 = u|_\Omega$, $u_2 = u|_{\partial\Omega}$.

Definition 3 *Let $h \in L^1(\partial\Omega)$. We call a function $u = (u_1, u_2) \in L^1(\Omega) \times L^1(\partial\Omega)$ is a weak solution to*

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = h & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

if it holds

$$\int_\Omega (-\Delta\varphi + \varphi) u_1 dx = \int_{\partial\Omega} (h\varphi - \frac{\partial \varphi}{\partial \nu} u_2) ds_x \tag{1.8}$$

for any $\varphi \in C^2(\overline{\Omega})$. Also a function $u = (u_1, u_2) \in L^1(\Omega) \times L^1(\partial\Omega)$ is called a weak solution to $(1.1)_\lambda$ if $f(u_2) \in L^1(\partial\Omega)$ and

$$\int_\Omega (-\Delta\varphi + \varphi) u_1 dx = \int_{\partial\Omega} (\lambda f(u_2)\varphi - \frac{\partial \varphi}{\partial \nu} u_2) ds_x \tag{1.9}$$

holds for any $\varphi \in C^2(\overline{\Omega})$.

Remark 4 *In some parts of the paper, admitting some ambiguity, we will identify u_1 or u_2 with u for $u \in L^1(\Omega \times \partial\Omega)$.*

Remark 5 *If $u \in H^1(\Omega)$ is an energy solution to (1.1), that is, $f(\gamma(u)) \in L^1(\partial\Omega)$ and*

$$\int_\Omega (\nabla u \cdot \nabla \varphi + u\varphi) dx = \int_{\partial\Omega} \lambda f(\gamma(u))\varphi ds_x$$

holds for any $\varphi \in C^1(\overline{\Omega})$*, then* u *is a weak solution in the sense of Definition 3 for* $u_1 = u$ *and* $u_2 = \gamma(u)$*, here* $\gamma(u) \in H^{1/2}(\partial\Omega)$ *is the usual trace of* H^1 *function* u *on* $\partial\Omega$*. In the following, we denote again* $\gamma(u) = u|_{\partial\Omega}$*, or simply* u*, for a Sobolev function* u*.*

By Proposition 1, we may define a function

$$u^*(x) = \lim_{\lambda\uparrow\lambda^*} u_\lambda(x), \quad x \in \overline{\Omega}. \tag{1.10}$$

Then $u^* = ((u^*)_1, (u^*)_2) = (u^*|_\Omega, u^*|_{\partial\Omega})$ becomes a weak solution of $(1.1)_{\lambda^*}$ in the sense of Definition 3. Indeed, let $\lambda_1 > 0$ denote the first eigenvalue of the Steklov type eigenvalue problem

$$\begin{cases} -\Delta\varphi + \varphi = 0 & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = \lambda\varphi & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

and φ_1 the first eigenfunction. It is known that λ_1 is simple, isolated and φ_1 can be chosen positive (see [19]). Multiplying φ_1 to the equation of u_λ, we have

$$\lambda_1 \int_{\partial\Omega} u_\lambda\varphi_1 ds_x = \lambda \int_{\partial\Omega} f(u_\lambda)\varphi_1 ds_x.$$

Since f satisfies the assumption (1.3), there exists a $C > 0$ such that $f(t) \geq \frac{4\lambda_1 t}{\lambda^*} - C$ for every $t > 0$. Thus when $\lambda \in (\lambda^*/2, \lambda^*)$, it holds

$$\begin{aligned} &\lambda_1 \int_{\partial\Omega} u_\lambda\varphi_1 ds_x = \lambda \int_{\partial\Omega} f(u_\lambda)\varphi_1 ds_x \\ &\geq 2\lambda_1 \int_{\partial\Omega} u_\lambda\varphi_1 ds_x - C,' \quad (C' = \frac{C\lambda^*}{2}\int_{\partial\Omega}\varphi_1 ds_x). \end{aligned}$$

Thus we have $\int_{\partial\Omega} u_\lambda\varphi_1 ds_x \leq C$ and also $\int_{\partial\Omega} f(u_\lambda)\varphi_1 ds_x \leq C$, where C is independent of $\lambda \in (\lambda^*/2, \lambda^*)$. Thus by Fatou's lemma and the fact $\varphi_1|_{\partial\Omega} \geq c_0 > 0$ for some $c_0 > 0$, we obtain that $u^* \in L^1(\partial\Omega)$ and $f(u^*) \in L^1(\partial\Omega)$. To see that $u^* \in L^1(\Omega)$, let $\zeta \in C^2(\overline{\Omega})$ be the solution of

$$\begin{cases} -\Delta\zeta + \zeta = 1 & \text{in } \Omega, \\ \frac{\partial\zeta}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying ζ to the equation of u_λ, we have

$$\int_\Omega u_\lambda dx = \lambda \int_{\partial\Omega} f(u_\lambda)\zeta ds_x \leq C.$$

Fatou's lemma again confirms that $u^* \in L^1(\Omega)$. Since u_λ satisfies

$$\int_\Omega (-\Delta\varphi + \varphi) u_\lambda dx = \int_{\partial\Omega} (\lambda f(u_\lambda)\varphi - \frac{\partial\varphi}{\partial\nu} u_\lambda) ds_x$$

for any $\varphi \in C^2(\overline{\Omega})$, letting $\lambda \uparrow \lambda^*$ and using Lebesgue's dominated convergence theorem on Ω and $\partial\Omega$, we obtain that u^* is a weak solution of $(1.1)_{\lambda^*}$. In the following, we call u^* the *extremal solution* of (1.1).

Compared with the well-studied problem (1.4), the problem (1.1) has several technical difficulties. For example, on $\Omega = B$, a ball, explicit singular extremal solutions are known for some specific nonlinearities such as $f(u) = e^u$ or $f(u) = (1+u)^p$ for (1.4). However, we lose such explicit examples for the problem (1.1). Also, for the problem (1.4), Hardy inequality

$$\left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx \le \int_\Omega |\nabla u|^2 dx, \quad \forall u \in H_0^1(\Omega)$$

plays an important role when one studies the stability properties for the singular extremal solutions, see [6]. We can not use the arguments in [6] directly for (1.1).

In [4], [14], [15], the corresponding parabolic problems for (1.4) have been treated and the global behavior of solutions, complete blow up phenomena, and the instability of singular extremal solutions are studied. Corresponding studies for the problem (1.1) will be future works.

The organization of the paper is as follows. In §2, we collect lemmas which will be used in the later sections. Several facts analogous to those established by Brezis, Cazenave, Martel and Ramiandrisoa [4] for (1.4) will be proved. In §3, we treat the regularity property of the extremal solution to (1.1), as in Nedev [17], see also Dávila [10]. In §4, similarly to the result by Martel [13], the uniqueness of the extremal solution among weak solutions will be proved. In §5, we study the existence of weak eigenfunctions corresponding to zero eigenvalue of the linearized eigenvalue problem around the extremal solution. Corresponding result for the problem (1.4) has been studied by Cábre and Martel [8].

2 Preliminaries

In this section, we prepare several useful lemmas in the sequel of the paper.

Lemma 6 *Given $h \in L^1(\partial\Omega)$, there exists a unique weak solution $u = (u_1, u_2) \in L^1(\Omega) \times L^1(\partial\Omega)$ to (1.7) in the sense of (1.8). Moreover, it holds*

$$\|u_1\|_{L^1(\Omega)} + \|u_2\|_{L^1(\partial\Omega)} \le C\|h\|_{L^1(\partial\Omega)}. \tag{2.1}$$

for some $C > 0$ independent of u and h. Also if $h \ge 0$ on $\partial\Omega$, then $u_1, u_2 \ge 0$.

Proof. We prove the uniqueness first. Let $u = (u_1, u_2)$, $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ be weak solutions. Then $w = (w_1, w_2), w_1 = u_1 - \tilde{u}_1, w_2 = u_2 - \tilde{u}_2$, satisfies

$$\int_\Omega (-\Delta\varphi + \varphi) w_1 dx = \int_{\partial\Omega} (-\frac{\partial\varphi}{\partial\nu}) w_2 ds_x$$

for any $\varphi \in C^2(\overline{\Omega})$. Given $\zeta \in C^\infty(\partial\Omega)$, let $\varphi \in C^2(\overline{\Omega})$ be the solution to

$$\begin{cases} -\Delta\varphi + \varphi = 0 & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = \zeta & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Then we have

$$\int_{\partial\Omega} w_2 \zeta ds_x = 0$$

for such ζ, hence $w_2 = 0$ a.e. on $\partial\Omega$. Similarly, for given $\eta \in C_0^\infty(\Omega)$, let $\varphi \in C^2(\overline{\Omega})$ be the solution to

$$\begin{cases} -\Delta\varphi + \varphi = \eta & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

then we have

$$\int_\Omega w_1 \eta dx = 0$$

and conclude $w_1 = 0$ a.e. on Ω.

To prove the a priori estimate (2.1), let $\varphi_+, \varphi_- \in C^2(\overline{\Omega})$ be the solution to

$$\begin{cases} -\Delta\varphi + \varphi = \pm 1 & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega \end{cases}$$

respectively. By the definition of the weak solution, we have

$$\int_\Omega u_1 dx = \int_\Omega (-\Delta\varphi_+ + \varphi_+) u_1 dx = \int_{\partial\Omega} h\varphi_+ ds_x \le \max_{\partial\Omega} |\varphi_+| \|h\|_{L^1(\partial\Omega)},$$

and similarly

$$\int_\Omega (-u_1)dx = \int_\Omega (-\Delta\varphi_- + \varphi_-)u_1 dx = \int_{\partial\Omega} h\varphi_- ds_x \le \max_{\partial\Omega} |\varphi_-| \|h\|_{L^1(\partial\Omega)}.$$

By the maximum principle, $\max_{x\in\partial\Omega} |\varphi_\pm(x)| \le 1$, thus we have $\|u_1\|_{L^1(\Omega)} \le \|h\|_{L^1(\partial\Omega)}$. Similarly, we have $\|u_2\|_{L^1(\partial\Omega)} \le C\|h\|_{L^1(\partial\Omega)}$ if we take test functions ψ_+, ψ_- as the solutions to

$$\begin{cases} -\Delta\psi + \psi = 0 & \text{in } \Omega, \\ \frac{\partial\psi}{\partial\nu} = \pm 1 & \text{on } \partial\Omega \end{cases}$$

respectively.

To prove the existence, put

$$h_m(x) = \begin{cases} m & \text{if } h(x) \ge m, \\ h(x) & \text{if } |h(x)| \le m, \\ -m & \text{if } h(x) \le -m. \end{cases} \tag{2.4}$$

for $m \in \mathbb{N}$. Since $h_m \in L^\infty(\partial\Omega)$, there exists $u_m \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u_m + u_m = 0 & \text{in } \Omega, \\ \frac{\partial u_m}{\partial\nu} = h_m & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

By the estimate (2.1), we have

$$\|u_m - u_n\|_{L^1(\Omega)} + \|u_m|_{\partial\Omega} - u_n|_{\partial\Omega}\|_{L^1(\partial\Omega)} \le C\|h_m - h_n\|_{L^1(\partial\Omega)} = o(1), \quad (m, n \to \infty).$$

Thus $\{u_m\}_{m\in\mathbb{N}}$ and $\{u_m|_{\partial\Omega}\}_{m\in\mathbb{N}}$ are Cauchy sequences in $L^1(\Omega)$ and $L^1(\partial\Omega)$ respectively. Then there exist $u_1 \in L^1(\Omega), u_2 \in L^1(\partial\Omega)$ such that

$$u_m \to u_1 \quad \text{in } L^1(\Omega), \quad u_m|_{\partial\Omega} \to u_2 \quad \text{in } L^1(\partial\Omega).$$

Since u_m satisfies

$$\int_\Omega (-\Delta\varphi + \varphi)u_m dx = \int_{\partial\Omega} \left(h\varphi - \frac{\partial\varphi}{\partial\nu} u_m|_{\partial\Omega} \right) ds_x$$

for any $\varphi \in C^2(\overline{\Omega})$, we easily see that $(u_1, u_2) \in L^1(\Omega) \times L^1(\partial\Omega)$ is a weak solution of (1.7) by letting $m \to \infty$.

Lastly, if $h \ge 0$ on $\partial\Omega$, we have $h_m \ge 0$ and the maximum principle implies that $u_m \ge 0$ on $\overline{\Omega}$. Thus $u_1 \ge 0$ on Ω and $u_2 \ge 0$ on $\partial\Omega$. $\square$

Lemma 7 *Let $h \in L^1(\partial\Omega)$ and let $u = (u_1, u_2) \in L^1(\Omega) \times L^1(\partial\Omega)$ be the weak solution to (1.7) in the sense of (1.8). Let $\Phi \in C^2(\mathbb{R})$ be concave, with Φ' bounded and $\Phi(0) = 0$. Then $v = (v_1, v_2) = (\Phi(u_1), \Phi(u_2)) \in L^1(\Omega) \times L^1(\partial\Omega)$ is a weak supersolution to*

$$\begin{cases} -\Delta v + v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \Phi'(u_2)h & \text{on } \partial\Omega, \end{cases}$$

in the sense that

$$\int_\Omega (-\Delta\psi + \psi) v_1 dx \geq \int_{\partial\Omega} \left\{ \Phi'(u_2)h\psi - \frac{\partial\psi}{\partial\nu} v_2 \right\} ds_x$$

for any $\psi \in C^2(\overline{\Omega})$, $\psi \geq 0$ on $\overline{\Omega}$.

Proof. For $h \in L^1(\partial\Omega)$ and $m \in \mathbb{N}$, define h_m as before in (2.4) and let $u_m \in H^1(\Omega)$ be an energy solution of (2.5). By Lemma 6, we know $u_m \to u_1$ in $L^1(\Omega)$, $u_m|_{\partial\Omega} \to u_2$ in $L^1(\partial\Omega)$, where $u = (u_1, u_2)$ is a weak solution of (1.7) in the sense of (1.8). From

$$\begin{aligned} &\|\Phi(u_1) - \Phi(u_m)\|_{L^1(\Omega)} \leq \|\Phi'\|_{L^\infty(\mathbb{R})} \|u_1 - u_m\|_{L^1(\Omega)}, \\ &\|\Phi(u_2) - \Phi(u_m|_{\partial\Omega})\|_{L^1(\partial\Omega)} \leq \|\Phi'\|_{L^\infty(\mathbb{R})} \|u_2 - u_m|_{\partial\Omega}\|_{L^1(\partial\Omega)}, \end{aligned}$$

we obtain $v = (v_1, v_2) = (\Phi(u_1), \Phi(u_2)) \in L^1(\Omega) \times L^1(\partial\Omega)$. Since u_m satisfies

$$\int_\Omega (\nabla\varphi \cdot \nabla u_m + \varphi u_m)\, dx = \int_{\partial\Omega} h_m \varphi ds_x$$

for any $\varphi \in C^2(\overline{\Omega})$, by density argument, this holds true for any $\varphi \in H^1(\Omega)$. We take $\varphi = \Phi'(u_m)\psi$ with $\psi \in C^2(\overline{\Omega})$, $\psi \geq 0$ on $\overline{\Omega}$, then we find that

$$\int_\Omega \left\{ \nabla\left(\Phi'(u_m)\psi\right) \cdot \nabla u_m + \Phi'(u_m)\psi u_m \right\} dx = \int_{\partial\Omega} h_m \Phi'(u_m|_{\partial\Omega})\psi ds_x.$$

Noting $\Phi'' \leq 0$ and $\psi \geq 0$, we have

$$\int_\Omega \left\{ \nabla\left(\Phi(u_m)\right) \cdot \nabla\psi + \Phi'(u_m)\psi u_m \right\} dx \geq \int_{\partial\Omega} h_m \Phi'(u_m|_{\partial\Omega})\psi ds_x.$$

Now, since Φ is concave with $\Phi(0) = 0$, we have $\frac{\Phi(u_m)}{u_m} \geq \Phi'(u_m)$. Therefore,

$$\int_\Omega \{\nabla(\Phi(u_m)) \cdot \nabla\psi + \Phi(u_m)\psi\}\,dx \geq \int_{\partial\Omega} h_m \Phi'(u_m|_{\partial\Omega})\psi ds_x$$

holds for any $\psi \in C^2(\overline{\Omega})$, $\psi \geq 0$. Integration by parts leads to

$$\int_\Omega (-\Delta\psi + \psi)\,\Phi(u_m)dx \geq \int_{\partial\Omega} \left\{ h_m \Phi'(u_m|_{\partial\Omega})\psi - \Phi(u_m|_{\partial\Omega})\frac{\partial\psi}{\partial\nu} \right\} ds_x.$$

Passing to the limit with the estimates

$$\begin{aligned}
&\left| \int_\Omega (-\Delta\psi + \psi)(\Phi(u_1) - \Phi(u_m))dx \right| \\
&\leq \| - \Delta\psi + \psi\|_{L^\infty(\Omega)} \|\Phi'\|_{L^\infty(\mathbb{R})} \|u_m - u_1\|_{L^1(\Omega)} = o(1), \\
&\left| \int_{\partial\Omega} (\Phi(u_2) - \Phi(u_m|_{\partial\Omega}))\frac{\partial\psi}{\partial\nu} ds_x \right| \leq \|\nabla\psi\|_{L^\infty(\partial\Omega)} \|\Phi'\|_{L^\infty(\mathbb{R})} \|u_m|_{\partial\Omega} - u_2\|_{L^1(\partial\Omega)} = o(1), \\
&\int_{\partial\Omega} h_m \Phi'(u_m|_{\partial\Omega})\psi ds_x \to \int_{\partial\Omega} h\Phi'(u_2)\psi ds_x,
\end{aligned}$$

we confirm that $v = \Phi(u) = (\Phi(u_1), \Phi(u_2))$ is the desired weak supersolution. Note that $h_m \to h$ in $L^1(\partial\Omega)$ strongly. Thus the last estimate is assured by the Lebesgue dominated convergence theorem, since a.e. convergence along a subsequence and the estimate $|h_m \Phi'(u_m|_{\partial\Omega})\psi| \leq \|\Phi'\|_{L^\infty(\mathbb{R})} \|\psi\|_{L^\infty(\partial\Omega)} |h| \in L^1(\partial\Omega)$ hold true. $\square$

Lemma 8 *Assume* $(1.1)_\lambda$ *has a weak supersolution* $\overline{w} = (\overline{w}_1, \overline{w}_2) \in L^1(\Omega) \times L^1(\partial\Omega)$, *in the sense that* $f(\overline{w}_2) \in L^1(\partial\Omega)$ *and*

$$\int_\Omega (-\Delta\varphi + \varphi)\overline{w}_1 dx \geq \int_{\partial\Omega} \left\{ \lambda f(\overline{w}_2)\varphi - \frac{\partial\varphi}{\partial\nu}\overline{w}_2 \right\} ds_x$$

for any $\varphi \in C^2(\overline{\Omega})$, $\varphi \geq 0$ *on* $\overline{\Omega}$. *Then* $(1.1)_\lambda$ *has a weak solution* $u = (u_1, u_2) \in L^1(\Omega) \times L^1(\partial\Omega)$.

Proof. Proof consists of a standard monotone iteration argument in our context. Define $w^{(1)} = (w_1^{(1)}, w_2^{(1)}) = \overline{w} = (\overline{w}_1, \overline{w}_2) \in L^1(\Omega) \times L^1(\partial\Omega)$. By the definition, we have $f(w_2^{(1)}) \in L^1(\partial\Omega)$. Let $w^{(2)} = (w_1^{(2)}, w_2^{(2)})$ be the unique weak solution of

$$\begin{cases} -\Delta w_1^{(2)} + w_1^{(2)} = 0 & \text{in } \Omega, \\ \frac{\partial w_2^{(2)}}{\partial \nu} = \lambda f(w_2^{(1)}) & \text{on } \partial\Omega \end{cases}$$

obtained by Lemma 6. Thus,

$$\int_\Omega (-\Delta\varphi + \varphi)(w_1^{(1)} - w_1^{(2)})dx \geq \int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu}(w_2^{(2)} - w_2^{(1)})ds_x$$

holds for any $\varphi \in C^2(\overline{\Omega})$, $\varphi \geq 0$ on $\overline{\Omega}$. As before, for given $\eta \in C_0^\infty(\Omega)$ $\eta \geq 0$ on Ω, take $\varphi \in C^2(\overline{\Omega})$ as the solution of (2.3). Then we have

$$\int_\Omega (w_1^{(1)} - w_1^{(2)})\eta dx \geq 0,$$

and since $\eta \in C_0^\infty(\Omega)$, $\eta \geq 0$ can be chosen arbitrary, we conclude that $w_1^{(1)} \geq w_1^{(2)}$ a.e. on Ω. Similarly, for any $\zeta \in C^\infty(\partial\Omega)$, $\zeta \geq 0$ on $\partial\Omega$, let $\varphi \in C^2(\overline{\Omega})$ be the solution to (2.2). Then we have

$$0 \geq \int_{\partial\Omega} \zeta(w_2^{(2)} - w_2^{(1)})ds_x,$$

which implies that $w_2^{(2)} \leq w_2^{(1)}$ a.e. on $\partial\Omega$. By induction, we obtain

$$\begin{aligned} \overline{w}_1 = w_1^{(1)} \geq w_1^{(2)} \geq \cdots \geq w_1^{(n)} \geq \cdots, \quad & a.e. \text{ on } \Omega, \\ \overline{w}_2 = w_2^{(1)} \geq w_2^{(2)} \geq \cdots \geq w_2^{(n)} \geq \cdots, \quad & a.e. \text{ on } \partial\Omega. \end{aligned}$$

By Lemma 6, we know $w_1^{(n)} \geq 0$ and $w_2^{(n)} \geq 0$. By the monotone convergence theorem, $w_1^{(n)}$ and $w_2^{(n)}$ converges to u_1, u_2 respectively in $L^1(\Omega)$ and $L^1(\partial\Omega)$. Since f is increasing, we have also $f(w_2^{(n)}) \leq f(w_2^{(1)}) \in L^1(\partial\Omega)$ for any $n \in \mathbb{N}$, which leads to $f(u_2) \in L^1(\partial\Omega)$. Finally, it is easy to check that $u = (u_1, u_2)$ is a desired weak solution to $(1.1)_\lambda$. □

Main result in this section is the following nonexistence result for $(1.1)_\lambda$ above the extremal parameter λ^*. See [4] Corollary 2, or [10] Theorem 3.8.

Theorem 9 *Assume (1.2). If $\lambda > \lambda^*$, then there is no solution to $(1.1)_\lambda$, even in the weak sense in Definition 3.*

Actually, we prove the following proposition. Theorem 9 is an easy consequence of this proposition and the definition of λ^* (1.5).

Proposition 10 *Let* $\lambda > 0$ *and assume that there exists a weak solution* $u = (u_1, u_2) \in L^1(\Omega) \times L^1(\partial\Omega)$ *to* $(1.1)_\lambda$. *Then for any* $\alpha \in (0,1)$, *the problem*

$$\begin{cases} -\Delta u + u = 0 & \textit{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \alpha\lambda f(u) & \textit{on } \partial\Omega, \end{cases}$$

has a classical solution.

Proof. Let $u = (u_1, u_2) \in L^1(\Omega) \times L^1(\partial\Omega)$, $u_1, u_2 \geq 0$ be a weak solution to $(1.1)_\lambda$. Given $\alpha \in (0,1)$, define

$$H(t) = \int_0^t \frac{ds}{\lambda f(s)} \tag{2.6}$$

and

$$\Phi(t) = H^{-1}(\alpha H(t)) \tag{2.7}$$

for $t \geq 0$. Then by an easy observation, we see

(i) $0 = \Phi(0) \leq \Phi(t) \leq t$ for all $t \geq 0$,

(ii) Φ is increasing, concave, $\Phi'(t) \leq 1$ for all $t \geq 0$,

(iii) if $\lim_{t\to+\infty} H(t)$ is finite, then $\lim_{t\to+\infty} \Phi(t)$ is also finite,

see [4]:Lemma 4. Also simple calculation shows

$$\lambda\Phi'(t) f(t) = \alpha\lambda f(\Phi(t)) \tag{2.8}$$

holds. Thus, by Lemma 7 and the relation (2.8), we see that $v = (v_1, v_2) = (\Phi(u_1), \Phi(u_2)) \in L^1(\Omega) \times L^1(\partial\Omega)$ satisfies

$$\int_\Omega (-\Delta\varphi + \varphi) v_1 dx \geq \int_{\partial\Omega} \left\{ \alpha\lambda f(v_2)\varphi - \frac{\partial\psi}{\partial\nu} v_2 \right\} ds_x$$

for any $\varphi \in C^2(\overline{\Omega})$, $\varphi \geq 0$ on $\overline{\Omega}$. That is, $v = (v_1, v_2)$ is a weak supersolution to $(1.1)_{\alpha\lambda}$.

Suppose first that

$$\int_0^\infty \frac{ds}{f(s)} < +\infty.$$

In this case, by (iii) above, we have $\Phi(\infty) < \infty$, which implies $(v_1, v_2) = (\Phi(u_1), \Phi(u_2))$ be a bounded weak supersolution to $(1.1)_{\alpha\lambda}$. By Lemma 8, we have a weak solution to $(1.1)_{\alpha\lambda}$, which is bounded, hence classical solution. This proves Proposition in this case.

Next, consider the case

$$\int_0^\infty \frac{ds}{f(s)} = +\infty.$$

In this case, we set $v^{(1)} = (v_1^{(1)}, v_2^{(1)}) = (\Phi(u_1), \Phi(u_2)) \in L^1(\Omega) \times L^1(\partial\Omega)$. Then by (i), we have $0 \leq v_i^{(1)} \leq u_i$ for $i = 1, 2$, and since H is concave,

$$H(u_i) - H(v_i^{(1)}) \leq H'(v_i^{(1)})(u_i - v_i^{(1)})$$

holds. By the definitions (2.6) and (2.7), we have

$$H(v_2^{(1)}) = \alpha H(u_2), \quad \text{and} \quad H'(v_2^{(1)}) = \frac{1}{\lambda f(v_2^{(1)})}.$$

Thus, we obtain

$$\lambda(1-\alpha)H(u_2) \leq \frac{u_2 - v_2^{(1)}}{f(v_2^{(1)})},$$

hence by the assumption $H(\infty) = +\infty$,

$$\lambda(1-\alpha)f(v_2^{(1)}) \leq \frac{u_2}{H(u_2)} \leq C(1 + u_2) \in L^1(\partial\Omega),$$

which leads to $f(v_2^{(1)}) \in L^1(\partial\Omega)$. By Lemma 7, $v^{(1)} = (v_1^{(1)}, v_2^{(1)}) \in L^1(\Omega) \times L^1(\partial\Omega)$ is a weak supersolution of $(1.1)_{\alpha\lambda}$. Therefore by Lemma 8, we obtain a weak solution $u^{(1)} = (u_1^{(1)}, u_2^{(1)}) \in L^1(\Omega) \times L^1(\partial\Omega)$ with the property $u_i^{(1)} \leq v_i^{(1)}$ for $i = 1, 2$. Also since f is positive and increasing, $0 \leq f(u_2^{(1)}) \leq f(v_2^{(1)}) \in L^1(\partial\Omega)$. Hence by the elliptic L^1 estimate of Brezis and Strauss [5], we have $u_1^{(1)} \in W^{1,q}(\Omega)$ for any $1 \leq q < \frac{N}{N-1}$ and $u_2^{(1)} \in L^p(\partial\Omega)$ for any $1 \leq p < \frac{N-1}{N-2}$ (for any $p < \infty$ if $N = 2$). Now, set $v^{(2)} = (v_1^{(2)}, v_2^{(2)}) = (\Phi(u_1^{(1)}), \Phi(u_2^{(1)}))$ and repeat the procedure. We confirm that $v^{(2)}$ is a weak supersolution to $(1.1)_{\alpha^2\lambda}$, and there exists a weak solution $u^{(2)} = (u_1^{(2)}, u_2^{(2)})$ to $(1.1)_{\alpha^2\lambda}$ with the property that $0 \le f(u_2^{(2)}) \le f(v_2^{(2)})$ a.e. on $\partial\Omega$,

$$\lambda(1-\alpha^2)f(v_2^{(2)}) \le \frac{u_2^{(1)}}{H(u_2^{(1)})} \le C(1+u_2^{(1)}) \in L^p(\partial\Omega),$$

in particular, $f(u_2^{(2)}) \in L^p(\partial\Omega)$ for any $1 \le p < \frac{N-1}{N-2}$ (for any $p < \infty$ if $N = 2$). Then elliptic L^p estimates ([1, 2]) that $u_1^{(2)} \in W^{1,q}(\Omega)$ for any $q < \frac{N}{N-2}$ and the trace Sobolev embedding implies $u_2^{(2)} \in L^p(\partial\Omega)$ for any $p < \frac{N-1}{N-3}$ (for any $p < \infty$ if $N = 3$). By iteration, we find a weak solution $u^{(k)} = (u_1^{(k)}, u_2^{(k)})$ to the problem

$$\begin{cases} -\Delta u^{(k)} + u^{(k)} = 0 & \text{in } \Omega, \\ \frac{\partial u^{(k)}}{\partial \nu} = \alpha^k \lambda f(u^{(k)}) & \text{on } \partial\Omega, \end{cases}$$

with the property that

$$u_1^{(k)} \in W^{1,q}(\Omega), \quad \forall q < \frac{N}{N-k}, \quad u_2^{(k)} \in L^p(\partial\Omega), \quad \forall p < \frac{N-1}{N-(k+1)}.$$

Thus after iterating N times, we obtain that $u_1^{(k)} \in L^\infty(\Omega)$ and $u_2^{(k)} \in L^\infty(\partial\Omega)$. That is, $u^{(k)}$ is a bounded, hence classical solution to $(1.1)_{\alpha^k\lambda}$. Since $\alpha \in (0,1)$ is arbitrary, we complete the proof. □

3 Regularity of extremal solutions

In this section, we prove the extremal solution u^* to our problem (1.1) is bounded for $N = 2$. We follow the argument by Nedev [17], in which the extremal solution of (1.4) is bounded (hence regular by usual elliptic estimates) when $N \le 3$. Recently, this result for the extremal solution of (1.4) is improved to $N = 4$ by Villegas [21], which uses a key estimate by X. Cábre [7].

Theorem 11 *Let $u^* = ((u^*)_1, (u^*)_2)$ be the extremal solution to $(1.1)_{\lambda^*}$. Assume $f \in C^2([0,+\infty))$ satisfies (1.2), (1.3). Then we have:*

(i) If $N = 2$, then $u^* = ((u^*)_1, (u^*)_2) \in L^\infty(\Omega) \times L^\infty(\partial\Omega)$.

(ii) *If* $N \geq 3$, *then* $(u^*)_2 \in L^p(\partial\Omega)$ *for* $1 \leq p < \frac{N-1}{N-3}$ *(for any* $1 \leq p < \infty$ *when* $N = 3$*). If* $N \geq 2$, *then* $f((u^*)_2) \in L^p(\partial\Omega)$ *for any* $1 \leq p < \frac{N-1}{N-2}$ *(for any* $1 \leq p < \infty$ *when* $N = 2$*).*

(iii) $(u^*)_1 \in W^{1,\gamma}(\Omega)$ *for any* $1 \leq \gamma < \frac{N}{N-2}$ *when* $N \geq 3$ *(for any* $1 \leq \gamma < \infty$ *when* $N = 2$*). In particular,* $(u^*)_1 \in H^1(\Omega)$ *if* $N \leq 3$.

Proof. We obtain several estimates of minimal solutions u_λ to $(1.1)_\lambda$ which are independent of $\lambda \in (0, \lambda^*)$. Following Nedev [17], see also [10], we put

$$g(t) = \int_0^t \{f'(s)\}^2 ds, \quad t \geq 0.$$

Since f is C^2, g is also a C^2 function. Multiplying $g(u_\lambda) \in C^2(\overline{\Omega})$ to the equation of $(1.1)_\lambda$ satisfied by u_λ and integrating, we obtain

$$\int_\Omega f'(u_\lambda)^2 |\nabla u_\lambda|^2 dx = \lambda \int_{\partial\Omega} f(u_\lambda) g(u_\lambda) ds_x - \int_\Omega u_\lambda g(u_\lambda) dx. \tag{3.1}$$

Recall the stability of u_λ:

$$\int_\Omega (|\nabla\varphi|^2 + \varphi^2) dx \geq \lambda \int_{\partial\Omega} f'(u_\lambda)\varphi^2 ds_x$$

holds for every $\varphi \in C^1(\overline{\Omega})$. Applying this inequality to $\varphi = \tilde{f}(u_\lambda)$, $\tilde{f}(t) = f(t) - f(0)$, we obtain

$$\lambda \int_{\partial\Omega} f'(u_\lambda)\tilde{f}(u_\lambda)^2 ds_x \leq \int_\Omega \left(f'(u_\lambda)^2 |\nabla u_\lambda|^2 + \tilde{f}(u_\lambda)^2 \right) dx. \tag{3.2}$$

By (3.1) and (3.2), we have

$$\begin{aligned} \lambda \int_{\partial\Omega} \left\{ f'(u_\lambda)\tilde{f}(u_\lambda)^2 - \tilde{f}(u_\lambda) g(u_\lambda) \right\} ds_x &\leq \lambda f(0) \int_{\partial\Omega} g(u_\lambda) ds_x \\ &\quad + \int_\Omega \left(\tilde{f}(u_\lambda)^2 - u_\lambda g(u_\lambda) \right) dx. \end{aligned} \tag{3.3}$$

Let

$$h(t) = \int_0^t f'(s)(f'(t) - f'(s)) ds.$$

Then we see $h(t) \geq 0$ and $f'(t)\tilde{f}(t)^2 - \tilde{f}(t)g(t) = \tilde{f}(t)h(t)$. Also if we put

$$A(t) = \tilde{f}(t)^2 - tg(t),$$

then we see $A \in C^2([0,\infty))$, $A(0) = 0$, $A'(0) = 0$ and

$$A''(t) = 2f''(t)\{\tilde{f}(t) - tf'(t)\} \leq 0,$$

since by the convexity of f and the assumption $f \in C^2$, we have $f''(t) \geq 0$ and $f'(t) \geq \frac{f(t)-f(0)}{t}$ for $t > 0$. Thus we obtain $A(t) \leq 0$ for all $t \geq 0$, which leads to

$$\lambda \int_{\partial\Omega} \tilde{f}(u_\lambda)h(u_\lambda)ds_x \leq \lambda f(0) \int_{\partial\Omega} g(u_\lambda)ds_x \tag{3.4}$$

from (3.3). By the same argument in [17], we have

$$\lim_{t\to\infty} \frac{h(t)}{f'(t)} = +\infty \tag{3.5}$$

and

$$g(t) = \int_0^t \{f'(s)\}^2 ds \leq \int_0^t f'(s)f'(t)ds \leq f'(t)\tilde{f}(t),$$

which with (3.5) implies

$$\lim_{t\to\infty} \frac{\tilde{f}(t)h(t)}{g(t)} = +\infty. \tag{3.6}$$

From (3.4),(3.5) and (3.6), we have, as in [17],

$$\int_{\partial\Omega} g(u_\lambda)ds_x \leq C, \quad \int_{\partial\Omega} \tilde{f}(u_\lambda)h(u_\lambda)ds_x \leq C \tag{3.7}$$

and also

$$\int_{\partial\Omega} \tilde{f}(u_\lambda)f'(u_\lambda)ds_x \leq C, \quad \int_{\partial\Omega} \frac{\tilde{f}(u_\lambda)^2}{u_\lambda}ds_x \leq C \tag{3.8}$$

for $C > 0$ independent of λ. We prove here (3.7) only. Indeed, by (3.6), there exists $T > 0$ such that $h(t)\tilde{f}(t) \geq 2f(0)g(t)$ for all $t > T$. Let

$$\partial\Omega_{\lambda,T} = \{x \in \partial\Omega \mid u_\lambda(x) > T\}.$$

Then we have

$$2f(0)\int_{\partial\Omega_{\lambda,T}} g(u_\lambda)ds_x \leq \int_{\partial\Omega_{\lambda,T}} \tilde{f}(u_\lambda)h(u_\lambda)ds_x \leq f(0)\int_{\partial\Omega} g(u_\lambda)ds_x$$

by (3.4). This implies

$$\int_{\partial\Omega_{\lambda,T}} g(u_\lambda)ds_x \le \int_{\partial\Omega\setminus\partial\Omega_{\lambda,T}} g(u_\lambda)ds_x \le g(T)|\partial\Omega|$$

and

$$\int_{\partial\Omega} g(u_\lambda)ds_x \le 2g(T)|\partial\Omega|.$$

Backing to (3.4), we have (3.7).

From (3.8) and the assumption $\lim_{t\to\infty}\frac{\tilde{f}(t)}{t} = +\infty$, we obtain that

$$\int_{\partial\Omega} f(u_\lambda)ds_x \le C$$

for some $C > 0$ independent of λ. By the elliptic L^1 estimate of Brezis and Strauss [5], we have

$$\begin{aligned} &u_\lambda \in W^{1,q}(\Omega) \quad \text{for any } 1 \le q < \frac{N}{N-1}, \quad \text{and} \\ &u_\lambda \in L^p(\partial\Omega) \quad \text{for any } 1 \le p < \frac{N-1}{N-2}, \ (1 \le p < \infty \text{ if } N = 2). \end{aligned}$$

Let $\alpha \in (0,1)$ and define

$$\begin{aligned} &A = \{x \in \partial\Omega \mid \tilde{f}(u_\lambda(x)) \le u_\lambda(x)^{1/\alpha}\}, \\ &B = \{x \in \partial\Omega \mid \tilde{f}(u_\lambda(x))^2/u_\lambda(x) > \tilde{f}(u_\lambda(x))^{2-\alpha}\}. \end{aligned}$$

Then by (3.8), we have

$$\int_B \tilde{f}(u_\lambda(x))^{2-\alpha}ds_x \le C$$

and

$$\int_A \tilde{f}(u_\lambda)^p ds_x \le \int_A u_\lambda^{p/\alpha}ds_x \le C$$

if $p/\alpha < \frac{N-1}{N-2}$. Choosing $\alpha \in (0,1)$ such that $2-\alpha = \frac{\alpha(N-1)}{N-2}$, i.e., $\alpha = \frac{2(N-2)}{2N-3}$ ($\alpha > 0$ is any small if $N = 2$), we see that

$$\int_{\partial\Omega} \tilde{f}(u_\lambda)^p ds_x \le C \quad \text{for } 1 \le p < \frac{\alpha(N-1)}{N-2} = \frac{2N-1}{2N-3}.$$

Then, elliptic L^p estimate ([1], [2]) implies

$$u_\lambda \in W^{1,\gamma}(\Omega) \quad \text{for } \gamma = \frac{Np}{N-1},\ 1 \le p < \frac{2(N-1)}{2N-3},$$

and by the trace Sobolev embedding $W^{1,\gamma}(\Omega) \hookrightarrow L^{\frac{(N-1)\gamma}{N-\gamma}}(\partial\Omega)$,

$$u_\lambda \in L^p(\partial\Omega) \quad \text{for any } 1 \le p < \frac{2(N-1)}{2N-5},\ (1 \le p \le \infty \text{ if } N = 2).$$

Now, we use a bootstrap argument. Assume we obtain that $u_\lambda \in L^p(\partial\Omega)$ for $p < p_0$. We choose $\alpha \in (0,1)$ as $2 - \alpha = \alpha p_0$, i.e., $\alpha = \frac{2}{1+p_0}$. Then elliptic L^p estimate and trace Sobolev embedding imply that $u_\lambda \in W^{1,\gamma}(\Omega) \hookrightarrow L^{\frac{(N-1)\gamma}{N-\gamma}}(\partial\Omega)$, where $\gamma = \frac{Np}{N-1}$ so $\frac{(N-1)\gamma}{N-\gamma} = \frac{(N-1)p}{N-1-p}$. Also $\|f(u_\lambda)\|_{L^p(\partial\Omega)} \le C$ for any $p < \alpha p_0$. Note that when $p < \frac{2p_0}{1+p_0}$, then $\frac{(N-1)p}{N-1-p} < \frac{2(N-1)p_0}{N-1+(N-3)p_0}$. Let us define

$$p_\infty = \frac{2(N-1)p_\infty}{N-1+(N-3)p_\infty},$$

that is, $p_\infty = \frac{N-1}{N-3}$. Then we obtain $\|u_\lambda\|_{L^p(\partial\Omega)} \le C$ independent of λ for any $p < \frac{N-1}{N-3}$ and also $\|f(u_\lambda)\|_{L^p(\partial\Omega)} \le C$ for any $p < \frac{2p_\infty}{1+p_\infty} = \frac{N-1}{N-2}$. Thus by elliptic estimates, we have $u_\lambda \in W^{1,\gamma}(\Omega)$ for $\gamma = \frac{Np}{N-1}$, $p < \frac{N-1}{N-2}$. Thus $u_\lambda \in W^{1,\gamma}(\Omega)$ for any $\gamma < \frac{N}{N-2}$ when $N \ge 3$. □

For typical nonlinearities such as $f(u) = e^u$ or $f(u) = (1+u)^p$ for $p > 1$, we improve the above result as follows:

Proposition 12 *Let u^* be the extremal solution to $(1.1)_{\lambda^*}$ with $f(u) = e^u$. Then if $N \le 5$, we have $u^* \in L^\infty(\Omega)$.*

Proposition 13 *Let u^* be the extremal solution to $(1.1)_{\lambda^*}$ with $f(u) = (1+u)^p$ for $p > 1$. Define*

$$N_p = 4 + 2\left(\frac{1}{p-1} + \sqrt{1 + \frac{1}{p-1}}\right). \tag{3.9}$$

Then if $N < N_p$, we have $u^* \in L^\infty(\Omega)$.

In particular, if $N \le 6$, or $N \ge 7$ and $1 < p < p_+(N) := \frac{N^2-6N+6+2\sqrt{2N-3}}{(N-2)(N-6)}$, then $u^* \in L^\infty(\Omega)$.

Note that for our problem (1.1), we do not know any information of the explicit singular extremal solutions even when $\Omega = B$ is a ball and f is one of the above nonlinearities.

Proof of Proposition 12. We follow the arguments in [9], [16] with some modifications for out context. Recall the minimal solution u_λ satisfies the stability inequality

$$\int_\Omega (|\nabla\varphi|^2 + \varphi^2)dx \geq \lambda \int_{\partial\Omega} e^{u_\lambda}\varphi^2 ds_x, \quad \forall\varphi \in C^1(\overline{\Omega})$$

and the weak form of the equation

$$\int_\Omega (\nabla\psi \cdot \nabla u_\lambda + u_\lambda\psi)dx = \lambda \int_{\partial\Omega} e^{u_\lambda}\psi ds_x, \quad \forall\psi \in C^1(\overline{\Omega}).$$

We put $\varphi = e^{tu_\lambda}$ and $\psi = e^{2tu_\lambda}$, where $t > 0$. Testing with them we have

$$\int_\Omega (t^2 e^{2tu_\lambda}|\nabla u_\lambda|^2 + e^{2tu_\lambda})dx \geq \lambda \int_{\partial\Omega} e^{(2t+1)u_\lambda} ds_x$$

and

$$\int_\Omega (2te^{2tu_\lambda}|\nabla u_\lambda|^2 + u_\lambda e^{2tu_\lambda})dx = \lambda \int_{\partial\Omega} e^{(2t+1)u_\lambda} ds_x.$$

Combining these, we obtain

$$2\int_\Omega e^{2tu_\lambda}dx - t\int_\Omega u_\lambda e^{2tu_\lambda}dx \geq \lambda(2-t)\int_{\partial\Omega} e^{(2t+1)u_\lambda} ds_x.$$

Since $(0, +\infty) \ni s \mapsto (2 - ts)e^{2ts}$ is bounded from above for $t > 0$, the left hand side is bounded when $\lambda \uparrow \lambda^*$. Thus for any $0 < t < 2$, we have e^{u_λ} is uniformly bounded in $L^{2t+1}(\partial\Omega)$, and the elliptic estimate implies that $\|u_\lambda\|_{W^{1,\frac{N}{N-1}(2t+1)}(\Omega)} \leq C$ uniformly in λ. Sobolev embedding assures that $\|u_\lambda\|_{L^\infty(\Omega)} \leq C$ uniformly in λ if $2t + 1 > N - 1$. Since t can be chosen arbitrary near to 2, this shows that $u^* \in L^\infty(\Omega)$ if $N < 6$. □

Proof of Proposition 13. Again, minimal solution u_λ satisfies the stability inequality

$$\int_\Omega (|\nabla\varphi|^2 + \varphi^2)dx \geq \lambda \int_{\partial\Omega} p(1 + u_\lambda)^{p-1}\varphi^2 ds_x, \quad \forall\varphi \in C^1(\overline{\Omega})$$

and the weak form of the equation

$$\int_\Omega (\nabla\psi\cdot\nabla u_\lambda + u_\lambda\psi)dx = \lambda\int_{\partial\Omega}(1+u_\lambda)^p\psi ds_x, \quad \forall\psi\in C^1(\overline{\Omega}).$$

In this case, choosing $\varphi = (1+u_\lambda)^{tp+\frac{1}{2}}$ and $\psi = (1+u_\lambda)^{2tp}$ for $t>0$, we have

$$\int_\Omega \left\{(tp+\frac{1}{2})^2(1+u_\lambda)^{2tp-1}|\nabla u_\lambda|^2 + (1+u_\lambda)^{2tp+1}\right\}dx \geq \lambda p\int_{\partial\Omega}(1+u_\lambda)^{p(2t+1)}ds_x$$

and

$$\int_\Omega \left\{2tp(1+u_\lambda)^{2tp-1}|\nabla u_\lambda|^2 + u_\lambda(1+u_\lambda)^{2tp}\right\}dx = \lambda\int_{\partial\Omega}(1+u_\lambda)^{p(2t+1)}ds_x.$$

Combining these, we have

$$\begin{aligned}&\int_\Omega \left\{(tp+\frac{1}{2})^2(1+u_\lambda)^{2tp} - (tp-\frac{1}{2})^2(1+u_\lambda)^{2tp+1}\right\}dx \\ &\geq \lambda\left\{2tp^2-(tp+\frac{1}{2})^2\right\}\int_{\partial\Omega}(1+u_\lambda)^{p(2t+1)}ds_x.\end{aligned}$$

Since $(0,+\infty)\ni s\mapsto A(1+s)^{2tp}-B(1+s)^{2tp+1}$ is bounded from above for $A,B>0$, the left hand side is bounded when $\lambda\uparrow\lambda^*$. Therefore, we have a uniform bound $\|(1+u_\lambda)^p\|_{L^{2t+1}(\partial\Omega)}\leq C$ when $2tp^2-(tp+\frac{1}{2})^2>0$. This quadratic inequality with respect to t is equivalent to that $t\in\left(\frac{2p-1-2\sqrt{p(p-1)}}{2p}, \frac{2p-1+2\sqrt{p(p-1)}}{2p}\right)$, that is,

$$\frac{3p-1-2\sqrt{p(p-1)}}{p} < 2t+1 < \frac{3p-1+2\sqrt{p(p-1)}}{p}.$$

Now, we use a bootstrap argument. If $(1+u_\lambda)^p\in L^q(\partial\Omega)$, elliptic estimate and trace Sobolev embedding imply that $u_\lambda\in W^{1,\frac{N}{N-1}q}(\Omega)\hookrightarrow L^{\frac{(N-1)q}{N-1-q}}(\partial\Omega)$. Define $\{q_k\}_{k\in\mathbb{N}}$ as

$$\begin{cases} q_1 = 2t+1, \\ \frac{1}{q_{k+1}} = p\left(\frac{1}{q_k}-\frac{1}{N-1}\right), \quad (k=1,2,\cdots). \end{cases}$$

We easily obtain that $\frac{1}{q_k} = p^{k-1}\left(\frac{1}{q_1} - \frac{p}{(N-1)(p-1)}\right) + \frac{p}{(N-1)(p-1)}$, hence if

$$\frac{1}{q_1} < \frac{p}{(N-1)(p-1)}, \tag{3.10}$$

then there exists some $k \in \mathbb{N}$ such that $\frac{1}{q_{k+1}} < 0$, which implies $q_k > N-1$ and $u_\lambda \in W^{1,\frac{N}{N-1}q_k}(\Omega) \hookrightarrow L^\infty(\Omega)$, which ends the proof. Since $q_1 = 2t+1$ can be chosen arbitrary close to the number $\frac{3p-1+2\sqrt{p(p-1)}}{p}$, (3.10) is satisfied when

$$(N-1)(1-\frac{1}{p}) < \frac{3p-1+2\sqrt{p(p-1)}}{p},$$

which is equivalent to $N < N_p$ where N_p is defined in (3.9). Since N_p is decreasing with respect to p and $N_p \to 6$ as $p \to \infty$, we have $N_p > 6$ for any $p > 1$. Also we can check that the inequality $N < N_p$ is equivalent to $1 < p < p_+(N) = \frac{N^2-6N+6+2\sqrt{2N-3}}{(N-2)(N-6)}$ when $N \geq 7$. This proves Proposition 13. $\square$

4 Uniqueness of weak extremal solutions

In this section, following the argument of Martel [13], see also [10], we show the uniqueness of extremal solution even in the weak sense, as described below.

Theorem 14 *Assume $f \in C^2([0,+\infty))$ satisfies (1.2), (1.3). Let λ^* be defined in (1.5). Assume $(1.1)_{\lambda^*}$ has a weak supersolution $w = (w_1, w_2) \in L^1(\Omega) \times L^1(\partial\Omega)$, in the sense that $f(w_2) \in L^1(\partial\Omega)$ and*

$$\int_\Omega (-\Delta\varphi + \varphi) w_1 dx \geq \int_{\partial\Omega} \left\{ \lambda^* f(w_2)\varphi - \frac{\partial \varphi}{\partial \nu} w_2 \right\} ds_x$$

for any $\varphi \in C^2(\overline{\Omega})$, $\varphi \geq 0$ on $\overline{\Omega}$. Then $(w_1, w_2) = ((u^*)_1, (u^*)_2)$, where u^* is defined by (1.10). As a consequence, the extremal solution u^* is the unique weak solution to $(1.1)_{\lambda^*}$.

Proof. By assumption and Lemma 8, there exists a weak solution u to $(1.1)_{\lambda^*}$. We argue by contradiction and assume that $u \not\equiv u^*$, $u > u^*$ in $\overline{\Omega}$. We divide the proof into several steps.

Step 1. There exists a strict supersolution v to $(1.1)_{\lambda^*}$.

Indeed, the convexity of f implies that $u_t = tu^* + (1-t)u$ is a supersolution of $(1.1)_{\lambda^*}$ for any $t \in (0,1)$. Suppose on the contrary that u_t is a solution to $(1.1)_{\lambda^*}$ for all $t \in (0,1)$. This implies that there is a set $\mathcal{N} \subset \partial\Omega$ with $(N-1)$-dimensional measure 0 such that $f(u_t(x)) = tf(u^*(x)) + (1-t)f(u(x))$ for any $x \in \partial\Omega \setminus \mathcal{N}$ and for all $t \in (0,1)$. Thus f is linear on the interval $[u^*(x), u(x)]$ for such x. By the same argument of [10] p.148, which uses the regularity of the extremal solution u^* as described in Theorem 11, we obtain that $u^*(\partial\Omega)$ is dense in the interval $[\operatorname{ess\,inf}_{\partial\Omega} u^*, \operatorname{ess\,sup}_{\partial\Omega} u^*]$ and $\cup_{x\in\partial\Omega\setminus\mathcal{N}}[u^*(x), u(x)]$ is also an interval. This implies u^* is a solution to

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda^*(au + b) & \text{on } \partial\Omega \end{cases}$$

for some $a, b \in \mathbb{R}$. Assumption $f(0) > 0$ implies $b > 0$. In this linear case, we easily see that $\lambda^* = \frac{\lambda_1}{a}$, where λ_1 is the first eigenvalue of the problem (1.11). Regularity theory assures that u^* is a classical solution. Thus if we multiply the equation by φ_1 the first eigenfunction of (1.11) with the normalization $\int_{\partial\Omega} \varphi_1 ds_x = 1$, we have

$$\int_{\partial\Omega} \lambda^*(au + b)\varphi_1 ds_x = \int_{\partial\Omega} \lambda_1 u\varphi_1 ds_x.$$

Thus we obtain $b = 0$, a contradiction.

Step 2. There is an $\varepsilon > 0$ such that

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda^* f(u) + \varepsilon & \text{on } \partial\Omega \end{cases}$$

has a weak supersolution w.

Indeed, by Step 1, we have a strict supersolution v to $(1.1)_{\lambda^*}$. Let V be the solution to the linear problem

$$\begin{cases} -\Delta V + V = 0 & \text{in } \Omega, \\ \frac{\partial V}{\partial \nu} = \lambda^* f(v) & \text{on } \partial\Omega, \end{cases}$$

and ψ is a solution of

$$\begin{cases} -\Delta \psi + \psi = 0 & \text{in } \Omega, \\ \frac{\partial \psi}{\partial \nu} = 1 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Then the maximum principle implies $v - V \geq \varepsilon\psi$ on $\overline{\Omega}$ for sufficiently small $\varepsilon > 0$. Define $w = V + \varepsilon\psi$. Then we see $w \leq v$ and

$$\frac{\partial w}{\partial \nu} = \lambda^* f(v) + \varepsilon \geq \lambda^* f(w) + \varepsilon$$

by the monotonicity of f. Thus w is a weak supersolution.

Step 3. Let $\varepsilon_1 \in (0, \varepsilon)$, where $\varepsilon > 0$ is a constant in Step 2. Then there exists a bounded (classical) solution to

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda^* f(u) + \varepsilon_1 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

The proof of this fact is quite similar to that of Proposition 10. Indeed, let us define

$$H_\varepsilon(t) = \int_0^t \frac{ds}{\lambda^* f(s) + \varepsilon}$$

and

$$\Phi(t) = H_{\varepsilon_1}^{-1}(H_\varepsilon(t))$$

for $t \geq 0$. Put $v = \Phi(w)$ where w is a weak supersolution in Step 2. It is enough to consider the case when $\int_0^\infty \frac{ds}{f(s)} = +\infty$, because otherwise, we find as before that $v = \Phi(w)$ is a bounded weak supersolution to (4.2) and Lemma 8 yields the result. We see $v \leq w$ and since H_ε is concave,

$$\frac{H_\varepsilon(w) - H_\varepsilon(v)}{w - v} \leq H_\varepsilon'(v) = \frac{1}{\lambda^* f(v) + \varepsilon}.$$

Also since $H_\varepsilon(w) = H_{\varepsilon_1}(v)$, we have

$$\begin{aligned} H_\varepsilon(w) - H_\varepsilon(v) = H_{\varepsilon_1}(v) - H_\varepsilon(v) &= \int_0^v \left(\frac{1}{\lambda^* f(s) + \varepsilon_1} - \frac{1}{\lambda^* f(s) + \varepsilon} \right) ds \\ &\geq (\varepsilon - \varepsilon_1) \int_0^v \frac{1}{(\lambda^* f(s) + 1)^2} ds. \end{aligned}$$

From these, we obtain $\lambda^* f(v) + \varepsilon \leq \frac{C(1+w)}{\varepsilon - \varepsilon_1} \in L^1(\partial\Omega)$. This and the bootstrap argument as in Proposition 10 yield the proof of Step 3.

Let u be the bounded solution obtained in Step 3 and let $\lambda' > \lambda^*$. Define

$$U = \frac{\lambda'}{\lambda^*} u - \varepsilon_1 \psi$$

where ψ is a solution to (4.1). Then we see $\frac{\partial U}{\partial \nu} = \lambda' f(u) + \frac{\lambda'}{\lambda^*}\varepsilon_1 - \varepsilon_1 \geq \lambda' f(u)$ on $\partial\Omega$. Choose $\frac{\lambda'}{\lambda^*} > 1$ sufficiently close to 1 in order to assure $U \leq u$ on $\overline{\Omega}$ (note that u is bounded), then we see U is a bounded supersolution to (1.1) for $\lambda = \lambda'$. By Lemma 8, we have a classical solution to $(1.1)_{\lambda'}$ for $\lambda' > \lambda^*$, contradicting to the definition of λ^*. □

As an application of Theorem 14, we show a characterization of the unbounded extremal solutions in the energy class $H^1(\Omega)$.

Theorem 15 *Let $u \in H^1(\Omega)$, $u \notin L^\infty(\partial\Omega)$, be a singular weak solution to $(1.1)_\lambda$ where f is as in Theorem 14. Then the followings are equivalent:*

(i) $f'(u) \in L^1(\partial\Omega)$ *and*

$$\int_\Omega (|\nabla\varphi|^2 + \varphi^2)dx \geq \lambda \int_{\partial\Omega} f'(u)\varphi^2 ds_x$$

holds for every $\varphi \in C^1(\overline{\Omega}), \varphi \not\equiv 0$.

(ii) $\lambda = \lambda^*$ *and* $u = u^*$.

Proof. The implication $(ii) \Longrightarrow (i)$ follows easily by the stability property of the minimal solutions u_λ and Fatou's lemma.

Let us prove $(i) \Longrightarrow (ii)$. Since no solution exists for $\lambda > \lambda^*$ by Theorem 9, we have $\lambda \leq \lambda^*$. Assume the contrary that $\lambda < \lambda^*$. By the density argument, we can take the test function $\varphi = u - u_\lambda \in H^1(\Omega)$. Note that here we have used the assumption $u \in H^1(\Omega)$. Also the assumption $u \notin L^\infty(\partial\Omega)$ implies that $u - u_\lambda \not\equiv 0$. Combining the equation satisfied by $u - u_\lambda$ with (i), we get

$$\begin{aligned}\lambda \int_{\partial\Omega} (f(u) - f(u_\lambda))(u - u_\lambda) ds_x &= \int_\Omega (|\nabla(u - u_\lambda)|^2 + (u - u_\lambda)^2)dx \\ &\geq \lambda \int_{\partial\Omega} f'(u)(u - u_\lambda)^2 ds_x,\end{aligned}$$

which implies

$$\lambda \int_{\partial\Omega} (u - u_\lambda)(f(u) - f(u_\lambda) - f'(u)(u - u_\lambda)) ds_x \geq 0.$$

Since the integrand is non positive by the convexity of f, we conclude that $f(u) = f(u_\lambda) + f'(u)(u - u_\lambda)$ a.e. on $\partial\Omega$. This implies that f is linear on intervals of the form $[u_\lambda(x), u(x)]$ for a.e. $x \in \partial\Omega$. Now, since u is unbounded on $\partial\Omega$, the union of these intervals is an interval of the form $[A, +\infty]$ and f is linear on this interval. But this contradicts to the superlinearity at ∞ of f in (1.3). Thus we have $\lambda = \lambda^*$. Finally, by the uniqueness of extremal solution u^* in Theorem 14, we conclude $u = u^*$. $\square$

5 Weak eigenfunctions for the extremal linearized problem

In this section, we prove the following theorem, which is a natural extension of the result by Cabré and Martel [8] to our case.

Theorem 16 *Let f be as in Theorem 14. Then there exists a function $\varphi \geq 0$, $\varphi \not\equiv 0$, such that $\varphi \in W^{1,q}(\Omega)$ for any $1 \leq q < \frac{N}{N-1}$, $f'(u^*)\varphi \in L^1(\partial\Omega)$ and*

$$\int_\Omega (-\Delta\zeta + \zeta)\varphi dx = \int_{\partial\Omega} \left\{ \lambda^* f'(u^*)\varphi\zeta - \frac{\partial\zeta}{\partial\nu}\varphi \right\} ds_x$$

for all $\zeta \in C^2(\overline{\Omega})$. That is, there exists a weak solution to the linearized eigenvalue problem around the extremal solution u^*:

$$\begin{cases} -\Delta\varphi + \varphi = 0 & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = \lambda^* f'(u^*)\varphi + \mu\varphi & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

for $\mu = 0$.

First, we need a lemma.

Lemma 17 *Let $\{u_n\} \subset C^2(\overline{\Omega})$ be a sequence of functions such that*

$$\begin{cases} -\Delta u_n + u_n = 0 & \text{in } \Omega, \\ \frac{\partial u_n}{\partial\nu} \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Assume $\|u_n\|_{L^1(\partial\Omega)} \le C$ *for some* $C > 0$ *independent of* n*. Then there exists a subsequence (denoted again by* u_n*) and* $u \in W^{1,q}(\Omega)$ *such that*

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{weakly in } W^{1,q}(\Omega),\ 1 < q < \frac{N}{N-1},\\
&u_n \to u \quad \text{strongly in } L^p(\partial\Omega),\ 1 \le p < \frac{N-1}{N-2}.
\end{aligned}$$

Moreover, for any $1 \le p < \frac{N-1}{N-2}$*, there exists a constant* $C_p > 0$ *depending only on* p *such that*

$$\|u_n\|_{L^p(\partial\Omega)} \le C_p \|u_n\|_{L^1(\partial\Omega)} \tag{5.2}$$

holds true for any $n \in \mathbb{N}$*.*

Proof. First we prove the a priori estimate (5.2) by a duality argument. For $\eta \in C^\infty(\partial\Omega)$ be given, let $\zeta \in C^2(\overline{\Omega})$ be a solution to

$$\begin{cases} -\Delta\zeta + \zeta = 0 & \text{in } \Omega,\\ \frac{\partial\zeta}{\partial\nu} = \eta & \text{on } \partial\Omega. \end{cases}$$

Let $p \in (1, \frac{N-1}{N-2})$. Then the Hölder conjugate exponent $p' = \frac{p}{p-1} > N-1$. Elliptic estimate ([1, 2]) implies that $\|\zeta\|_{W^{1,\gamma}(\Omega)} \le C\|\eta\|_{L^{p'}(\partial\Omega)}$ where $\gamma = \frac{Np'}{N-1} > N$. Since $W^{1,\gamma}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$ for $\alpha = 1 - \frac{N}{\gamma} \in (0,1)$, we have $|\zeta|(x) \le C\|\eta\|_{L^{p'}(\partial\Omega)}\varphi_1(x)$, $x \in \overline{\Omega}$, where φ_1 denotes the first eigenfunction of the problem (1.11). By Green's identity, we have

$$\begin{aligned}
&\left|\int_{\partial\Omega} u_n \eta ds_x\right| = \left|\int_{\partial\Omega} u_n \frac{\partial\zeta}{\partial\nu} ds_x\right|\\
&= \left|\int_{\partial\Omega}\left(\frac{\partial u_n}{\partial\nu}\right)\zeta ds_x - \int_\Omega (\Delta\zeta - \zeta)u_n - (\Delta u_n - u_n)\zeta dx\right|\\
&\le \int_{\partial\Omega}\left(\frac{\partial u_n}{\partial\nu}\right)|\zeta| ds_x \le C\|\eta\|_{L^{p'}(\partial\Omega)}\int_{\partial\Omega}\left(\frac{\partial u_n}{\partial\nu}\right)\varphi_1 ds_x\\
&= C\|\eta\|_{L^{p'}(\partial\Omega)}\lambda_1\int_{\partial\Omega} u_n\varphi_1 ds_x \le C\|\eta\|_{L^{p'}(\partial\Omega)}\lambda_1\|\varphi_1\|_{L^\infty(\partial\Omega)}\|u_n\|_{L^1(\partial\Omega)}.
\end{aligned}$$

Since $\eta \in C^\infty(\partial\Omega)$ is arbitrary, we obtain (5.2) by duality.

Now, let $\psi > 0$ be the solution of (4.1). Then we have

$$\int_{\partial\Omega}\left(\frac{\partial u_n}{\partial\nu}\right)\psi ds_x = \int_{\partial\Omega}\left(\frac{\partial\psi}{\partial\nu}\right)u_n ds_x = \int_{\partial\Omega} u_n ds_x,$$

hence $\|\frac{\partial u_n}{\partial \nu}\|_{L^1(\partial\Omega)} \le \frac{\|u_n\|_{L^1(\partial\Omega)}}{\min_{\partial\Omega}\psi} \le C$, where C is independent of n by the assumption. Thus, by Brezis-Strauss estimate [5], we confirm that $\|u_n\|_{W^{1,q}(\Omega)} \le C\|\frac{\partial u_n}{\partial \nu}\|_{L^1(\partial\Omega)} \le C$ for any $1 \le q < \frac{N}{N-1}$ and there exists a subsequence such that $u_n \rightharpoonup u$ in $W^{1,q}(\Omega)$ for $1 < q < \frac{N}{N-1}$ for some $u \in W^{1,q}(\Omega)$. Since the trace Sobolev embedding $W^{1,q}(\Omega) \hookrightarrow L^p(\partial\Omega)$ is compact if $1 \le p < \frac{(N-1)q}{N-q}$, we conclude that $\|u_n - u\|_{L^p(\partial\Omega)} \to 0$ for $1 \le p < \frac{N-1}{N-2}$. □

Now, we prove Theorem 16.

Proof. As in [8], we divide the proof into several steps.

Step 1. For $n \in \mathbb{N}$, define a sequence of functions which are asymptotically linear approximations of f as

$$f_n(s) = \begin{cases} f(s) & \text{if } s \le n, \\ f(n) + f'(n)(s-n) & \text{if } s > n, \end{cases}$$

and consider the approximated problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda f_n(u) & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Denote

$$\lambda_n^* = \sup\{\lambda > 0 : \ (5.3)_\lambda \text{ admits a minimal classical solution} \in C^2(\overline{\Omega})\},$$

and let $u_{n,\lambda}$ be a classical minimal solution to (5.3) for $\lambda < \lambda_n^*$. Note that $f_n(0) > 0$, increasing and convex, the above extremal parameter λ_n^* is finite and the existence of minimal solution is assured by the standard method. Note also that $f_n \le f_{n+1} \le f$, hence $\lambda^* \le \lambda_{n+1}^* \le \lambda_n^*$ for any $n \in \mathbb{N}$. Though f_n does not satisfy the superlinear condition at ∞, we claim that the pointwise limit

$$u_n^*(x) = \lim_{\lambda \uparrow \lambda_n^*} u_{n,\lambda}(x)$$

is a classical solution of $(5.3)_{\lambda_n^*}$ for n large. Indeed, take $\lambda \in (\lambda^*/2, \lambda_n^*)$ and let $u_{n,\lambda}$ be the minimal solution to $(5.3)_\lambda$. Multiplying the equation satisfied by $u_{n,\lambda}$ by φ_1, where φ_1 is the first eigenfunction of (1.11), which is normalized as $\int_{\partial\Omega} \varphi_1 ds_x = 1$, we obtain that

$$\begin{aligned} \lambda_1 \int_{\partial\Omega} \varphi_1 u_{n,\lambda} ds_x &= \lambda \int_{\partial\Omega} f_n(u_{n,\lambda}) \varphi_1 ds_x \\ &\geq \lambda f_n \left(\int_{\partial\Omega} \varphi_1 u_{n,\lambda} ds_x \right) > \frac{\lambda^*}{2} f_n \left(\int_{\partial\Omega} \varphi_1 u_{n,\lambda} ds_x \right). \end{aligned}$$

Here we have used Jensen's inequality for convex functions f_n. Thus we have

$$a_{n,\lambda} \geq \left(\frac{\lambda^*}{2\lambda_1} \right) f_n(a_{n,\lambda})$$

where we put $a_{n,\lambda} = \int_{\partial\Omega} \varphi_1 u_{n,\lambda} ds_x$. On the other hand, (1.3) implies that $f(n) > \left(\frac{2\lambda_1}{\lambda^*}\right) n$ and $f'(n) > \left(\frac{2\lambda_1}{\lambda^*}\right)$ for n sufficiently large. Assume the contrary that $f_n(a_{n,\lambda}) = f'(n)(a_{n,\lambda} - n) + f(n)$ for some $n \in \mathbb{N}$ sufficiently large. Then we have, since $a_{n,\lambda} > n$,

$$\begin{aligned} a_{n,\lambda} &\geq \left(\frac{\lambda^*}{2\lambda_1} \right) f_n(a_{n,\lambda}) = \left(\frac{\lambda^*}{2\lambda_1} \right) \{ f'(n)(a_{n,\lambda} - n) + f(n) \} \\ &> a_{n,\lambda} - n + n = a_{n,\lambda}, \end{aligned}$$

which is a contradiction. Thus we conclude there exists $n_0 \in \mathbb{N}$ such that $f_n(a_{n,\lambda}) = f(a_{n,\lambda})$, and $a_n \geq \left(\frac{\lambda^*}{2\lambda_1} \right) f(a_{n,\lambda})$ for $n \geq n_0$. Now, by the assumption f, we have $C > 0$ such that $f(s) \geq \frac{4\lambda_1}{\lambda^*} s - C$ holds for any $s > 0$. From this and the former estimate, we have $a_{n,\lambda} \leq \left(\frac{\lambda^*}{2\lambda_1} \right) C$ for $n \geq n_0$. This implies that

$$\|u_{n,\lambda}\|_{L^1(\partial\Omega)} \leq C \text{ for any } n \geq n_0 \text{ and any } \lambda \in (\lambda^*/2, \lambda_n^*). \tag{5.4}$$

At this stage, we can invoke Lemma 17 to confirm that

$$\|u_{n,\lambda}\|_{W^{1,q}(\Omega)} \leq C \ (1 \leq q < \frac{N}{N-1}) \quad \text{and} \quad \|u_{n,\lambda}\|_{L^p(\partial\Omega)} \leq C \ (1 \leq p < \frac{N-1}{N-2})$$

for any $n \geq n_0$ and $\lambda \in (\lambda^*/2, \lambda_n^*)$. Now, since $f_n(s)$ is linear for s large, we have $f_n(u_{n,\lambda}) \in L^p(\partial\Omega)$ and $\|f_n(u_{n,\lambda})\|_{L^p(\partial\Omega)} \leq C(n)$ for $1 \leq p < \frac{N-1}{N-2}$. Thus the elliptic L^p estimate: $\|u_{n,\lambda}\|_{W^{1,\gamma}(\Omega)} \leq C \|f_n(u_{n,\lambda})\|_{L^p(\partial\Omega)}$ where $\gamma = \frac{Np}{N-1} < \frac{N}{N-2}$, and the trace Sobolev embedding: $W^{1,\gamma}(\Omega) \hookrightarrow L^{\frac{(N-1)\gamma}{N-\gamma}}(\partial\Omega)$, imply that

$\|u_{n,\lambda}\|_{L^p(\partial\Omega)} \le C(n)$ for any $1 \le p < \frac{N-1}{N-3}$. We can continue this bootstrap procedure. Finally, we have $\|u_{n,\lambda}\|_{C^2(\overline{\Omega})} \le C(n)$ uniformly in $\lambda \in (\lambda^*/2, \lambda_n^*)$. Thus, letting $\lambda \uparrow \lambda_n^*$, we see that $u_{n,\lambda} \to u_n^*$ in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and $u_n^* \in C^2(\overline{\Omega})$ is a classical solution of

$$\begin{cases} -\Delta u_n^* + u_n^* = 0 & \text{in } \Omega, \\ \frac{\partial u_n^*}{\partial \nu} = \lambda_n^* f_n(u_n^*) & \text{on } \partial\Omega. \end{cases}$$

This proves the claim.

Now, the facts that u_n^* is classical and there is no classical solution of $(5.3)_\lambda$ for $\lambda > \lambda_n^*$, the linearized problem around u_n^* must have zero eigenvalue. Thus, there exists $\varphi_n > 0$ with $\int_{\partial\Omega} \varphi_n ds_x = 1$ such that

$$\begin{cases} -\Delta \varphi_n + \varphi_n = 0 & \text{in } \Omega, \\ \frac{\partial \varphi_n}{\partial \nu} = \lambda_n^* f_n'(u_n^*)\varphi_n & \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

Step 2. By letting $\lambda \uparrow \lambda_n^*$ in (5.4), we have $\|u_n^*\|_{L^1(\partial\Omega)} \le C$. Also recall our normalization $\|\varphi_n\|_{L^1(\partial\Omega)} = 1$. Thus by Lemma 17, we see that there exist $w, \varphi \in L^1(\Omega)$, $\varphi \ge 0$ a.e. such that

$$\begin{aligned} &u_n^* \rightharpoonup w, \quad \varphi_n \rightharpoonup \varphi \quad \text{weakly in } W^{1,q}(\Omega), \\ &u_n^* \to w, \quad \varphi_n \to \varphi \quad \text{strongly in } L^p(\partial\Omega) \text{ and a.e. on } \partial\Omega \end{aligned} \tag{5.6}$$

for any $1 < q < \frac{N}{N-1}$ and $1 \le p < \frac{N-1}{N-2}$. In particular, we have $\int_{\partial\Omega} \varphi ds_x = 1$, which implies $\varphi \not\equiv 0$ on $\partial\Omega$.

We prove that $\lambda_n^* \downarrow \lambda^*$ as $n \to \infty$ and $w = u^*$. First, we show that the weak limit $w \in W^{1,q}(\Omega)$ is a weak supersolution when considered as $w = (w_1, w_2) \in L^1(\Omega) \times L^1(\partial\Omega)$, where $w_1 = w|_\Omega$, and $w_2 = w|_{\partial\Omega}$ is the usual trace of a Sobolev function w on $\partial\Omega$. Indeed, put $\overline{\lambda} = \inf_{n \ge n_0} \lambda_n^*$. Since $\lambda_n^* \ge \lambda^*$ for any $n \ge n_0$, we have $\overline{\lambda} \ge \lambda^*$. For all $\zeta \in C^2(\overline{\Omega})$, $\zeta \ge 0$, we observe that

$$\begin{aligned} \int_\Omega (-\Delta\zeta + \zeta)\, u_n^* dx &= \lambda_n^* \int_{\partial\Omega} f_n(u_n^*)\zeta ds_x - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \nu} u_n^* ds_x \\ &\ge \overline{\lambda} \int_{\partial\Omega} f_n(u_n^*)\zeta ds_x - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \nu} u_n^* ds_x. \end{aligned}$$

Using $u_n^* \to w$ in $L^1(\Omega)$ on the left side hand and Fatou's lemma on the right hand side, we have

$$\begin{aligned}\int_\Omega (-\Delta\zeta + \zeta)\, w dx &\geq \overline{\lambda} \int_{\partial\Omega} f(w)\zeta ds_x - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} w ds_x \\ &\geq \lambda^* \int_{\partial\Omega} f(w)\zeta ds_x - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} w ds_x, \quad \forall\zeta \in C^2(\overline{\Omega}), \zeta \geq 0.\end{aligned}$$

This implies also $f(w) \in L^1(\partial\Omega)$ if we take $\zeta \equiv 1$. Thus, we conclude that w is a weak supersolution to $(1.1)_{\lambda^*}$ Then by Theorem 14, we conclude that $\overline{\lambda} = \lambda^*$ and $w = u^*$.

Step 3. Let φ_n be as in (5.5). We claim that $\lambda_n^* f_n'(u_n^*)\varphi_n \to \lambda^* f'(u^*)\varphi$ strongly in $L^1(\partial\Omega)$ as $n \to \infty$.

If this claim is proved, then we pass to the limit $n \to \infty$ in the weak formulation of (5.5):

$$\int_\Omega (-\Delta\zeta + \zeta)\, \varphi_n dx = \int_{\partial\Omega} \lambda_n^* f_n'(u_n^*)\varphi_n\zeta - \frac{\partial\zeta}{\partial\nu}\varphi_n ds_x, \quad \forall\zeta \in C^2(\overline{\Omega}),$$

and conclude that φ is a weak solution of

$$\begin{cases} -\Delta\varphi + \varphi = 0 & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = \lambda^* f'(u^*)\varphi & \text{on } \partial\Omega \end{cases}$$

in the sense of Definition 3. Recall $\varphi \in W^{1,q}(\Omega)$ for any $1 \leq q < \frac{N}{N-1}$. Thus the proof of Theorem 16 is finished.

To prove the strong convergence $\lambda_n^* f_n'(u_n^*)\varphi_n \to \lambda^* f'(u^*)\varphi$ in $L^1(\partial\Omega)$, we invoke Vitali's Convergence Theorem. First, by (5.6), we see

$$\lambda_n^* f_n'(u_n^*(x))\varphi_n(x) \to \lambda^* f'(u^*(x))\varphi(x) \quad \text{a.e. } x \in \partial\Omega$$

for a subsequence. To prove the uniformly absolute continuous property of the sequence $\{\lambda_n^* f_n'(u_n^*)\varphi_n\}_{n\in\mathbb{N}}$, let $A \subset \partial\Omega$ and $\varepsilon > 0$ be given arbitrary. Convexity of f_n implies

$$f_n\left(\frac{\chi_A(x)}{\varepsilon}\right) \geq f_n(u_n^*(x)) + f_n'(u_n^*(x))\left(\frac{\chi_A(x)}{\varepsilon} - u_n^*(x)\right)$$

a.e. $x \in \partial\Omega$, here χ_A is the characteristic function of A. Also from the equations satisfied by φ_n and u_n^*,

$$\int_{\partial\Omega} f_n(u_n^*)\varphi_n ds_x = \int_{\partial\Omega} f_n'(u_n^*)u_n^*\varphi_n ds_x$$

holds. Thus

$$\begin{aligned}\int_{\partial\Omega} f_n'(u_n^*)\frac{\chi_A}{\varepsilon}\varphi_n ds_x &\le \int_{\partial\Omega} f_n\left(\frac{\chi_A}{\varepsilon}\right)\varphi_n ds_x + \int_{\partial\Omega} f_n'(u_n^*)u_n^*\varphi_n ds_x - \int_{\partial\Omega} f_n(u_n^*)\varphi_n ds_x \\ &\le \int_{\partial\Omega} f_n\left(\frac{\chi_A}{\varepsilon}\right)\varphi_n ds_x \\ &= \int_{\partial\Omega}\left\{f_n\left(\frac{\chi_A}{\varepsilon}\right) - f(0)\right\}\varphi_n ds_x + \int_{\partial\Omega} f(0)\varphi_n ds_x \\ &\le \int_{\partial\Omega} f\left(\frac{1}{\varepsilon}\right)\varphi_n\chi_A ds_x + f(0) \\ &\le f\left(\frac{1}{\varepsilon}\right)|A|^{\frac{1}{p'}}\|\varphi_n\|_{L^p(\partial\Omega)} + f(0) \\ &\le Cf\left(\frac{1}{\varepsilon}\right)|A|^{\frac{1}{p'}} + f(0)\end{aligned}$$

for any $1 \le p < \frac{N-1}{N-2}$, here $|A|$ denotes the $(N-1)$ dimensional measure of $A \subset \partial\Omega$. Note that

$$\left\{f_n\left(\frac{\chi_A(x)}{\varepsilon}\right) - f(0)\right\}\varphi_n(x) \le f\left(\frac{1}{\varepsilon}\right)\varphi_n(x)\chi_A(x) \quad \text{a.e. on } \partial\Omega$$

and $\|\varphi_n\|_{L^p(\partial\Omega)} \le C$ for some $C > 0$ independent of n by (5.6). Define

$$\delta = \delta(\varepsilon) = \left(\frac{f(0)}{f(\frac{1}{\varepsilon})C}\right)^{p'}.$$

Then above calculation shows that for any $\varepsilon > 0$, if $A \subset \partial\Omega$ satisfies that $|A| < \delta(\varepsilon)$, we obtain $\int_A f_n'(u_n^*)\varphi_n ds_x \le 2f(0)\varepsilon$. Thus the uniform absolutely continuity of the sequence $\{\lambda_n^* f_n'(u_n^*)\varphi_n\}_{n\in\mathbb{N}}$ is confirmed. Also if we take $E \subset \partial\Omega$ such that $|\partial\Omega \setminus E| < \delta$ where δ is as above, we obtain the uniform integrability of $\{\lambda_n^* f_n'(u_n^*)\varphi_n\}_{n\in\mathbb{N}}$: for any $\varepsilon > 0$, there exists $E \subset \partial\Omega$ such that $\int_{\partial\Omega\setminus E}\lambda_n^* f_n'(u_n^*)\varphi_n ds_x \le C\varepsilon$. Therefore, Vitali's Convergence Theorem assures the claim. □

Phenomena of continuum spectrum for the extremal eigenvalue problem (5.1) has been studied also, see [20].

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References

[1] S. Agmon, A. Douglis, and L. Nirenberg: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I,* Comm. Pure Appl. Math., **12**, 623–727, (1959)

[2] S. Agmon, A. Douglis, and L. Nirenberg: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, II,* Comm. Pure Appl. Math., **17**, 35–92, (1964)

[3] E. Berchio, F. Gazzola, and D. Pierotti: *Gelfand type elliptic problems under Steklov boundary conditions,* Ann. Inst. H. Poincaré Anal. Non Linéaire **27**, 315–335, (2010)

[4] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa: *Blow up for* $u_t - \Delta u = g(u)$ *revisited,* Adv. Differential Equations. **1**, 73–90, (1996)

[5] H. Brezis, and W. A. Strauss: *Semi-linear elliptic equations in* L^1*,* J. Math. Soc. Japan., **25**, 565–590, (1973)

[6] H. Brezis, and J. L. Vázquez: *Blow-up solutions of some nonlinear elliptic problems,* Rev. Mat. Univ. Compl. Madrid, **10**, 443–469, (1997)

[7] X. Cabré: *Regularity of minimizers of semilinear elliptic problems up to dimension 4,* Comm. Pure Appl. Math. **63**, 1362–1380, (2010)

[8] X. Cabré, and Y. Martel: *Weak eigenfunctions for the linearization of extremal elliptic problems,* J. Funct. Anal. **156**, 30–56, (1998)

[9] M.G. Crandall, and R.H. Rabinowitz: *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems,* Arch. Rational Mech. Anal. **58**, 207–218, (1975)

[10] J. Dávila: *Singular solutions of semi-linear elliptic problems,* Handbook of differential equations: stationary partial differential equations. Vol. VI, 83–176, (2008)

[11] J. Dávila, L. Dupaigne, and M. Montenegro: *The extremal solution of a boundary reaction problem,* Commun. Pure Appl. Anal. **7**, 795–817, (2008)

[12] L. Dupaigne: *Stable Solutions of Elliptic Partial Differential Equations,* Monographs and Surveys in Pure and Applied Mathematics 143, Chapman & Hall/CRC Press, xiv+321 pp. (2011)

[13] Y. Martel: *Uniqueness of weak extremal solutions of nonlinear elliptic problems,* Houston J. Math. **23**, 161–168, (1997)

[14] Y. Martel: *Complete blow up and global behaviour of solutions* $u_t - \Delta u = g(u)$*,* Ann. Inst. H. Poincaré Anal. Non Linéaire. **15**, 687–723, (1998)

[15] Y. Martel: *Dynamical instability of singular extremal solutions of nonlinear elliptic problems,* Adv. Math. Sci. Appl. **9**, 163–181, (1999)

[16] F. Mignot, and J.P. Puel: *Sur une classe de problèmes non linéaires avec nonlinéairité positive, croissante, convexe,* Comm. Partial Differential Equations. **5**, 791–836, (1980)

[17] G. Nedev: *Regularity of the extremal solution of semilinear elliptic equations,* C. R. Acad. Sci. Paris Sér.I Math. **330**, 997–1002, (2000)

[18] P. Quittner, and W. Reichel: *Very weak solutions to elliptic equations with nonlinear Neumann boundary conditions,* Calc. Var. **32**, 429–452, (2008)

[19] J. D. Rossi: *Elliptic problems with nonlinear boundary conditions and the Sobolev trace theorem,* Handbook of differential equations: stationary partial differential equations. Vol. II, 311–406, (2005)

[20] F. Takahashi: *Continuum spectrum for the linearized extremal eigenvalue problem with boundary reactions,* to appear in the Proceedings of the conference Equadiff 13.

[21] S. Villegas: *Boundedness of extremal solutions in dimension 4,* Adv. Math. **235**, 126–133, (2013)