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SOLUTION TO THE REFLECTION EQUATION RELATED TO THE ıQUANTUM GROUP OF TYPE AII

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Abstract. A solution to the reflection equation associated to a coideal subalgebra of $U_q(A^{(1)}_{2n-1})$ of type AII in the symmetric tensor representations is presented. If parameters of the coideal subalgebra are suitably chosen, the K matrix does not depend on the quantum parameter q and still agrees with a solution in [8] at $q = 0$.

1. Introduction

Reflection equation assures the integrability in one-dimensional quantum systems or two-dimensional statistical models with boundaries. In the context of quantum integrability, it is an equation involving two kinds of linear operators, called quantum R and K matrices, on the twofold tensor product of vector spaces. The mathematical framework to construct its solution lies in considering a pair of a quantum group and its coideal subalgebra. They are called a quantum symmetric pair [10] or an ıquantum group [3] and known to be classified by Satake diagrams [10, 7]. In such a situation, R and K matrices contain the quantum parameter q. Moreover, if the representations have crystal bases in the sense of Kashiwara [6], one can take the limit where q goes to 0, and we obtain bijections between sets that still satisfy a combinatorial version of the reflection equation.

In [8], from the motivation of constructing a so-called box-ball system with boundary, we found three solutions of the combinatorial K matrix where the combinatorial R matrix in the reflection equation comes from the crystal basis of the symmetric tensor representation of the quantum affine algebra of type A. See (2.10)-(2.12) of [8]. They were called "Rotateleft", "Switch$_{12}$" and "Switch$_{1n}$". However, their quantum versions, namely, solutions of quantum K matrices, were not found for a long time. Only recently, in [9] the solution corresponding to "Rotateleft" were found. The purpose of this note is to find the origin of the other two solutions "Switch$_{12}$" and "Switch$_{1n}$" from the list of ıquantum groups. The correct one was found to be the affine version of type AII. See e.g. [10, 7, 12]. Rather surprisingly, if we choose parameters in our ıquantum group suitably, the K matrices does not depend on q, although the R matrices do.

There are many ıquantum groups other than affine type AII which we dealt with in this note, and there also exists a notion of the universal (or quasi) Kmatrix [2, 3, 4, 1] as with the universal (quasi) R matrix of a quantum group. We hope to report more solutions of the reflection equation that become combinatorial upon taking the limit $q \to 0$ in near future.

2. $U_q(A^{(1)}_{2n-1})$ and relevant R matrices

2.1. $U_q(A^{(1)}_{2n-1})$ and relevant representations.

Let $\mathbf{U} = U_q(A^{(1)}_{2n-1})$ be the Drinfeld-Jimbo quantum affine algebra (without the derivation operator). In this note, we assume $n \geq 2$. $\mathbf{U}$ is generated by $e_i, f_i, k_i^{\pm1}$ $(i \in \mathbb{Z}_{2n})$ obeying the relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1,\ \ [k_i, k_j] = 0,\ \ k_i e_j k_i^{-1} = q^{a_{ij}} e_j,\ \ k_i f_j k_i^{-1} = q^{-a_{ij}} f_j,\ \ [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}},$$
$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu e_i^{(1-a_{ij}-\nu)} e_j e_i^{(\nu)} = 0, \qquad \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu f_i^{(1-a_{ij}-\nu)} f_j f_i^{(\nu)} = 0\ \ (i \neq j), \tag{1}$$

where $e_i^{(\nu)} = e_i^\nu/[\nu]!$, $f_i^{(\nu)} = f_i^\nu/[\nu]!$ and $[m]! = \prod_{j=1}^m [j]$. The Cartan matrix $(a_{ij})_{i,j\in\mathbb{Z}_{2n}}$ is given by $a_{ij} = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}$. It is well known that $\mathbf{U}$ is a Hopf algebra. We employ the coproduct Δ of the form

$$\Delta(k_i^{\pm1}) = k_i^{\pm1} \otimes k_i^{\pm1}, \quad \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i. \tag{2}$$

We will be concerned with the two irreducible representations of $\mathbf{U}$ labeled with a positive integer l:

$$\pi_{l,x} : \mathbf{U} \to \mathrm{End}(V_{l,x}), \quad V_{l,x} = \bigoplus_{\alpha \in B_l} \mathbb{Q}(q) v_\alpha, \tag{3}$$

$$\pi^*_{l,x} : \mathbf{U} \to \mathrm{End}(V^*_{l,x}), \quad V^*_{l,x} = \bigoplus_{\alpha \in B_l} \mathbb{Q}(q) v^*_\alpha, \tag{4}$$

where x is a spectral parameter in $\mathbb{Q}(q)$ and

$$B_l = \{\alpha = (\alpha_1, \dots, \alpha_{2n}) \in \mathbb{Z}^{2n}_{\geq 0} \mid |\alpha| = l\}. \tag{5}$$

Here $|\alpha| = \sum_{i=1}^{2n} \alpha_i$. The actions of the generators of $\mathbf{U}$ on these representations are given by

$$e_i v_\alpha = x^{\delta_{i,0}}[\alpha_{i+1}] v_{\alpha+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}}, \qquad e_i v^*_\alpha = x^{\delta_{i,0}}[\alpha_i] v^*_{\alpha-\boldsymbol{e}_i+\boldsymbol{e}_{i+1}}, \tag{6}$$

$$f_i v_\alpha = x^{-\delta_{i,0}}[\alpha_i] v_{\alpha-\boldsymbol{e}_i+\boldsymbol{e}_{i+1}}, \qquad f_i v^*_\alpha = x^{-\delta_{i,0}}[\alpha_{i+1}] v^*_{\alpha+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}}, \tag{7}$$

$$k_i v_\alpha = q^{\alpha_i-\alpha_{i+1}} v_\alpha, \qquad k_i v^*_\alpha = q^{-\alpha_i+\alpha_{i+1}} v^*_\alpha. \tag{8}$$

Here $\boldsymbol{e}_i$ is the i-th standard basis vector and the index j of the Chevalley generators or α should be understood as elements of $\mathbb{Z}_{2n}$. $V_{l,x}$ is the l-th symmetric tensor representation of $\mathbf{U}$. $V^*_{l,x}$ is constructed on the dual space of $V_{l,x}$ by using the anti-automorphism $*$ of $\mathbf{U}$ defined on the generators as

$$e_i^* = e_i, \quad f_i^* = f_i, \quad k_i^* = k_i^{-1},$$

and by defining actions on $V^*_{l,x}$ as $\langle uv^*, v\rangle = \langle v^*, u^* v\rangle$ for $u \in \mathbf{U}, v \in V_{l,x}, v^* \in V^*_{l,x}$. Our basis $\{v^*_\alpha\}$ of $V^*_{l,x}$ is changed from the dual basis of $\{v_\alpha\}$ by multiplying $\prod_j [\alpha_j]!^{-1}$ on each dual basis vector, so it turns out that when $x = 1$ both $\{v_\alpha\}$ and $\{v^*_\alpha\}$ are upper crystal bases [6]. At $q = 0$, the former gives the crystal B_l and the latter its dual $B_l^\vee$ in [8].

2.2. R **matrices.** We consider the following three R matrices R, R^*, R^{**} that are defined as intertwiners between the tensor product representations below.

$$R(x/y) : V_{l,x} \otimes V_{m,y} \to V_{m,y} \otimes V_{l,x}, \quad (\pi_{m,y} \otimes \pi_{l,x})\Delta(u) R(x/y) = R(x/y)(\pi_{l,x} \otimes \pi_{m,y})\Delta(u), \tag{9}$$

$$R^*(x/y) : V^*_{l,x} \otimes V_{m,y} \to V_{m,y} \otimes V^*_{l,x}, \quad (\pi_{m,y} \otimes \pi^*_{l,x})\Delta(u) R^*(x/y) = R^*(x/y)(\pi^*_{l,x} \otimes \pi_{m,y})\Delta(u), \tag{10}$$

$$R^{**}(x/y) : V^*_{l,x} \otimes V^*_{m,y} \to V^*_{m,y} \otimes V^*_{l,x}, \quad (\pi^*_{m,y} \otimes \pi^*_{l,x})\Delta(u) R^{**}(x/y) = R^{**}(x/y)(\pi^*_{l,x} \otimes \pi^*_{m,y})\Delta(u), \tag{11}$$

where $u \in \mathbf{U}$. They satisfy the Yang-Baxter equations:

$$(1 \otimes R(x))(R(xy) \otimes 1)(1 \otimes R(y)) = (R(y) \otimes 1)(1 \otimes R(xy))(R(x) \otimes 1), \tag{12}$$

$$(1 \otimes R^*(x))(R^*(xy) \otimes 1)(1 \otimes R(y)) = (R(y) \otimes 1)(1 \otimes R^*(xy))(R^*(x) \otimes 1), \tag{13}$$

$$(1 \otimes R^{**}(x))(R^*(xy) \otimes 1)(1 \otimes R^*(y)) = (R^*(y) \otimes 1)(1 \otimes R^*(xy))(R^{**}(x) \otimes 1), \tag{14}$$

$$(1 \otimes R^{**}(x))(R^{**}(xy) \otimes 1)(1 \otimes R^{**}(y)) = (R^{**}(y) \otimes 1)(1 \otimes R^{**}(xy))(R^{**}(x) \otimes 1). \tag{15}$$

3. Reflection equation and its solution

3.1. **Coideal subalgebra.** We consider two coideal subalgebras $\mathbf{U}^\imath_\varepsilon$ $(\varepsilon = 0, 1)$ of $\mathbf{U}$. Set $I = \{0, 1, \dots, 2n-1\}$. An element of I is considered to correspond to a vertex of the Dynkin diagram of $A^{(1)}_{2n-1}$. In view of this, we identify I with $\mathbb{Z}_{2n}$. For each $\varepsilon = 0, 1$, set

$$I_\circ = \{\varepsilon, 2+\varepsilon, \dots, 2n-2+\varepsilon\}, \quad I_\bullet = I \setminus I_\circ.$$

We define two subalgebras $\mathbf{U}^\imath_\varepsilon$ of $\mathbf{U}$ for $\varepsilon = 0, 1$. Each one is generated by $e_i, f_i, k_i\, (i \in I_\bullet), b_i\, (i \in I_\circ)$ where

$$b_i = f_i + \gamma_i T_{w_\bullet}(e_i) k_i^{-1},$$
$$T_{w_\bullet}(e_i) = e_{i+1}e_{i-1}e_i - q^{-1}(e_{i+1}e_ie_{i-1} + e_{i-1}e_ie_{i+1}) + q^{-2}e_ie_{i-1}e_{i+1}.$$

Here γ_i is a constant. Then, we have the following facts which are well known. See [10, 7, 12] for instance.

Proposition 1. *For $i \in I_\circ$, $e_{i\pm1}b_i = b_ie_{i\pm1}$.*

Proposition 2. *$\mathbf{U}^\imath_\varepsilon$ is a right coideal subalgebra of $\mathbf{U}$. Namely, we have $\Delta(\mathbf{U}^\imath_\varepsilon) \subset \mathbf{U}^\imath_\varepsilon \otimes \mathbf{U}$.*

We also use the following result later.

TABLE 1. Satake diagrams of $\mathbf{U}_0^\imath$ and $\mathbf{U}_1^\imath$

Lemma 3. *For $i \in I_\circ$, the action of b_i on $V_{l,x}$ or $V_{l,x}^*$ is given by*

$$b_i v_\alpha = x^{-\delta_{i,0}}[\alpha_i] v_{\alpha-\boldsymbol{e}_i+\boldsymbol{e}_{i+1}} - x^{\delta_{i,0}+\delta_{i,1}+\delta_{i,-1}} q^{-1}\gamma_i[\alpha_{i+2}] v_{\alpha+\boldsymbol{e}_{i-1}-\boldsymbol{e}_{i+2}},$$
$$b_i v^*_\alpha = x^{-\delta_{i,0}}[\alpha_{i+1}] v^*_{\alpha+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}} - x^{\delta_{i,0}-\delta_{i,1}-\delta_{i,-1}} q^{-1}\gamma_i[\alpha_{i-1}] v^*_{\alpha-\boldsymbol{e}_{i-1}+\boldsymbol{e}_{i+2}}.$$

3.2. K **matrix and the reflection equation.** For each $\varepsilon = 0,1$, consider a linear map $K(x): V_{l,x} \to V^*_{l,x^{-1}}$ satisfying

$$K(x)\pi_{l,x}(a) = \pi^*_{l,x^{-1}}(a)K(x) \quad \text{for any } a \in \mathbf{U}^\imath_\varepsilon. \tag{16}$$

To describe the solution, we introduce a particular permutation $\sigma^{(\varepsilon)}$ of entries of α for $\varepsilon = 0,1$. $\sigma^{(\varepsilon)}$ switches α_{i-1} and α_i whenever $i \equiv \varepsilon \pmod 2$. For instance, when $n = 3$ we have

$$\sigma^{(0)}(\alpha) = (\alpha_2,\alpha_1,\alpha_4,\alpha_3,\alpha_6,\alpha_5), \quad \sigma^{(1)}(\alpha) = (\alpha_6,\alpha_3,\alpha_2,\alpha_5,\alpha_4,\alpha_1).$$

Proposition 4. *For each $\varepsilon = 0,1$, the intertwining relation* (16) *has a solution if and only if*

$$\prod_{j \in I_\circ} \gamma_j = (-q)^n,$$

in which case the solution is unique up to scalar multiple and given by

$$K(x)v_\alpha = x^{\varepsilon(\alpha_1-\alpha_{2n})} \prod_{j=\varepsilon,2+\varepsilon,\dots,2n-2+\varepsilon} (-q^{-1}\gamma_j)^{-\sum_{i=1+\varepsilon}^{j}\alpha_i} v^*_{\sigma^{(\varepsilon)}(\alpha)}.$$

Proof. In the proof we assume $i \in I_\bullet, j \in I_\circ$. Define K_α^β by $K(x)v_\alpha = \sum_\beta K_\alpha^\beta v_\beta^*$. Note that K_α^β also depends on x. Comparing the coefficients of v^*_β in $K(x)\pi_{l,x}(a)v_\alpha = \pi^*_{l,x^{-1}}(a)K(x)v_\alpha$ with k_i, e_i, f_i, b_j we obtain

$$K_\alpha^\beta \neq 0 \quad \Rightarrow \quad \alpha_i - \alpha_{i+1} = -\beta_i + \beta_{i+1}, \tag{17}$$
$$[\beta_i+1]K_\alpha^{\beta+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}} = x^{2\delta_{i,0}}[\alpha_{i+1}]K^\beta_{\alpha+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}}, \tag{18}$$
$$[\alpha_i+1]K_\alpha^{\beta+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}} = x^{2\delta_{i,0}}[\beta_{i+1}]K^\beta_{\alpha+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}}, \tag{19}$$
$$\begin{aligned} &x^{\delta_{j,0}}[\beta_{j+1}+1]K_\alpha^{\beta-\boldsymbol{e}_j+\boldsymbol{e}_{j+1}} - x^{-\delta_{j,0}-\delta_{j,1}-\delta_{j,-1}}q^{-1}\gamma_j[\beta_{j-1}+1]K_\alpha^{\beta+\boldsymbol{e}_{j-1}-\boldsymbol{e}_{j+2}} \\ &\qquad = x^{-\delta_{j,0}}[\alpha_j]K^\beta_{\alpha-\boldsymbol{e}_j+\boldsymbol{e}_{j+1}} - x^{\delta_{j,0}+\delta_{j,1}+\delta_{j,-1}}q^{-1}\gamma_j[\alpha_{j+2}]K^\beta_{\alpha+\boldsymbol{e}_{j-1}-\boldsymbol{e}_{j+2}}. \end{aligned} \tag{20}$$

Since we look for a nontrivial solution, we assume the right hand side of (17). This condition together with (18),(19) implies

$$\alpha_i = \beta_{i+1}, \quad \beta_i = \alpha_{i+1} \tag{21}$$

or equivalently $\beta = \sigma^{(\varepsilon)}(\alpha)$. Then by taking β to $\sigma^{(\varepsilon)}(\alpha) - \boldsymbol{e}_i + \boldsymbol{e}_{i+1}$ in (18) or (19), we have

$$K_\alpha^{\sigma^{(\varepsilon)}(\alpha)} = x^{2\delta_{i,0}} K_{\alpha+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}}^{\sigma^{(\varepsilon)}(\alpha+\boldsymbol{e}_i-\boldsymbol{e}_{i+1})}. \tag{22}$$

Similarly, assuming (21), (20) reduces to

$$\begin{aligned} &x^{\delta_{j,0}}[\alpha_{j+2}](K_\alpha^{\beta-\boldsymbol{e}_j+\boldsymbol{e}_{j+1}} + x^{\delta_{j,1}+\delta_{j,-1}}q^{-1}\gamma_j K^\beta_{\alpha+\boldsymbol{e}_{j-1}-\boldsymbol{e}_{j+2}}) \\ &\qquad = x^{-\delta_{j,0}}[\alpha_j](K^\beta_{\alpha-\boldsymbol{e}_j+\boldsymbol{e}_{j+1}} + x^{-\delta_{j,1}-\delta_{j,-1}}q^{-1}\gamma_j K_\alpha^{\beta+\boldsymbol{e}_{j-1}-\boldsymbol{e}_{j+2}}). \end{aligned}$$

If $\beta = \sigma^{(\varepsilon)}(\alpha) + \boldsymbol{e}_j - \boldsymbol{e}_{j+1}$, the right hand side vanishes, whereas if $\beta = \sigma^{(\varepsilon)}(\alpha - \boldsymbol{e}_j + \boldsymbol{e}_{j+1})$, the left one does. Under (22), both conditions reduce to

$$K_{\alpha-\boldsymbol{e}_{j-1}+\boldsymbol{e}_{j+1}}^{\sigma^{(\varepsilon)}(\alpha-\boldsymbol{e}_{j-1}+\boldsymbol{e}_{j+1})} / K_\alpha^{\sigma^{(\varepsilon)}(\alpha)} = -x^{\delta_{j,1}-\delta_{j,-1}}q^{-1}\gamma_j. \tag{23}$$

Replacing α with $\alpha + \boldsymbol{e}_{j-1}$ in this equation and multiplying it for $j = \varepsilon, 2+\varepsilon, \dots, 2n-2+\varepsilon$, we obtain the condition for K to exist. Set $K_\alpha^{\sigma^{(\varepsilon)}(\alpha)} = x^{\varepsilon(\alpha_1-\alpha_{2n})}\mathcal{K}_\alpha$. Then (22) and (23) reduce to

$$\mathcal{K}_\alpha = \mathcal{K}_{\alpha+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}}, \quad \mathcal{K}_{\alpha-\boldsymbol{e}_{j-1}+\boldsymbol{e}_{j+1}}/\mathcal{K}_\alpha = -q^{-1}\gamma_j.$$

Recall $i \in I_\bullet, j \in I_\circ$. From the first relation, one notices that $\mathcal{K}_\alpha$ depends only on $\alpha_i + \alpha_{i+1}$ $(i \in I_\bullet)$. The second one then determines $\mathcal{K}_\alpha$ uniquely up to scalar multiple. □

Remark 5. The K matrix corresponding to the same Satake diagram to ours but in the vector representation V_1 was considered in [11, Result 9.9].

In view of this proposition, we set $\gamma_j = -q$ for any $j \in I_\circ$ later in this note.

Theorem 6. *The reflection equation*

$$K_1(x)R^*((xy)^{-1})K_1(y)R(xy^{-1}) = R^{**}(xy^{-1})K_1(y)R^*((xy)^{-1})K_1(x) \tag{24}$$

holds as a linear map $V_{l,x} \otimes V_{m,y} \to V^*_{l,x^{-1}} \otimes V^*_{m,y^{-1}}$. *Here* $K_1(x) = K(x) \otimes 1$.

The proof is completely the same as that in [9, Theorem 1], although a similar reasoning goes back to [5] at the latest, under the assumption that $V_{l,x} \otimes V_{m,y}$ is irreducible as a $\mathbf{U}^\imath_\varepsilon$-module, which is shown in next section.

4. Proof of the irreducibility of $V_{l,x} \otimes V_{m,y}$

To show that the reflection equation holds (Theorem 6), we need to prove

Theorem 7. *As a* $\mathbf{U}^\imath_\varepsilon$*-module,* $V_{l,x} \otimes V_{m,y}$ *is irreducible.*

Actually, even when the spectral parameters x, y are specialized to 1, it is irreducible as we will see below. Hence, in this section we set $x = y = 1$, since it is enough to show the theorem. $V_{l,1}$ will be denoted by V_l. We can also restrict our proof to the $\varepsilon = 0$ case, since the consideration for the $\varepsilon = 1$ case is just the repetition by shifting the index i of the generators or the entries of α. Finally, in view of Proposition 4, we specialize γ_i for $i \in I_\circ$ to be $-q$.

4.1. Representation theory of $U_q(sl_2)$.

$U_q(sl_2)$ is the subalgebra of $\mathbf{U}$ generated only by e_1, f_1, k_1. Its irreducible representations are parametrized by their dimensions which run positive integers. Let U_l be the $(l+1)$-dimensional module of $U_q(sl_2)$. As a basis of U_l, one can take $\{v_\alpha \,|\, |\alpha| = l\}$ in (3) with $n = 1$. The actions of the generators e_1, f_1, k_1 are given by (6)-(8). It is well known that $U_l \otimes U_m$ decomposes into $\min(l, m) + 1$ components as

$$U_l \otimes U_m \simeq \bigoplus_{j=0}^{\min(l,m)} U_{l+m-2j}$$

where a highest weight vector of U_{l+m-2j} is given by

$$w_j^{(l,m)} = \sum_{p=0}^{j} (-1)^p q^{p(l-p+1)} \begin{bmatrix} j \\ p \end{bmatrix} v_{(l-p,p)} \otimes v_{(m-j+p,j-p)}. \tag{25}$$

Here $\begin{bmatrix} j \\ p \end{bmatrix}$ is the q-binomial coefficient defined by $\frac{[j]!}{[p]![j-p]!}$.

Now consider the subalgebra $\mathbf{U}(I_\bullet)$ of $\mathbf{U}^\imath$ generated by $e_i, f_i, k_i \, (i \in I_\bullet)$. Recall $I_\bullet = \{1, 3, \dots, 2n-1\}$. $\mathbf{U}(I_\bullet)$ is isomorphic to $U_q(sl_2)^{\otimes n}$. We want to construct a basis of $V_l \otimes V_m$ using its $\mathbf{U}(I_\bullet)$-module structure. To parametrize the highest weight vectors of $V_l \otimes V_m$, we introduce n-tuples of nonnegative integers $\boldsymbol{l} = (l_1, \dots, l_n), \boldsymbol{m} = (m_1, \dots, m_n)$ such that $|\boldsymbol{l}| = l, |\boldsymbol{m}| = m$. Here we use the notation $|\boldsymbol{l}|$ to signify the sum of entries of the vector $\boldsymbol{l}$ irrespective of the number of entries. Let

$$\iota : \bigoplus_{\boldsymbol{l},\boldsymbol{m}} (U_{l_1} \otimes U_{m_1}) \otimes \cdots \otimes (U_{l_n} \otimes U_{m_n}) \longrightarrow V_l \otimes V_m$$

be the linear map sending $(v_{(\alpha_1,\alpha_2)} \otimes v_{(\beta_1,\beta_2)}) \otimes \cdots \otimes (v_{(\alpha_{2n-1},\alpha_{2n})} \otimes v_{(\beta_{2n-1},\beta_{2n})})$ to $v_\alpha \otimes v_\beta$. Note that $U_{l_i} \otimes U_{m_i}$ is the tensor product of the irreducible highest weight modules U_{l_i}, U_{m_i} of the i-th $U_q(sl_2)$ of $U_q(sl_2)^{\otimes n}$ generated by $e_{2i-1}, f_{2i-1}, k_{2i-1}$. Since $U_q(sl_2)$ in different positions commute with each other, one obtains the following proposition.

Proposition 8. *For any* $\boldsymbol{l}, \boldsymbol{m}$ *and* $\boldsymbol{j} = (j_1, \dots, j_n)$ *such that* $0 \le j_i \le \min(l_i, m_i)$ *for* $1 \le i \le n$,

$$\boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} = \iota(w_{j_1}^{(l_1,m_1)} \otimes \cdots \otimes w_{j_n}^{(l_n,m_n)})$$

is a $\mathbf{U}(I_\bullet)$*-highest weight vector, and we have* $\bigoplus_{\boldsymbol{l},\boldsymbol{m},\boldsymbol{j}} \mathbf{U}(I_\bullet)\boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} = V_l \otimes V_m$.

4.2. **Necessary formulas.** In what follows, we assume $i \in I_\circ = \{0, 2, \dots, 2n-2\}$ and set $i = 2s$. By abuse of notation, we denote by $\boldsymbol{e}_s$ $(s = 1, \dots, n)$ the s-th standard basis vector of the n-dimensional space, although we have used it in section 2 for the $2n$-dimensional space. $\boldsymbol{e}_0$ should be understood as $\boldsymbol{e}_n$. For the action of $\mathbf{U}$ on the tensor product, we abbreviate Δ.

Proposition 9. *On $V_l \otimes V_m$, we have*

$$b_i \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} = D'_1 \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s}^{(\boldsymbol{l}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{m})} + D'_2 \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s}^{(\boldsymbol{l},\boldsymbol{m}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})} + D'_3 \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1},\boldsymbol{m})} + D'_4 \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l},\boldsymbol{m}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1})},$$

where

$$D'_1 = -q^{-j_s-j_{s+1}+l_s+m_{s+1}+1}[j_s], \quad D'_2 = [j_s],$$
$$D'_3 = -q^{-j_s-j_{s+1}+l_{s+1}+m_{s+1}+1}[j_{s+1}], \quad D'_4 = q^{-2j_s-2j_{s+1}+l_s+l_{s+1}+2m_{s+1}+2}[j_{s+1}].$$

Proof. Using Proposition 1, one finds that $b_i \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})}$ is a $\mathbf{U}(I_\bullet)$-highest weight vector. By the weight consideration, it should be a linear combination of the following vectors.

$$\boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s}^{(\boldsymbol{l}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{m})},\ \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s}^{(\boldsymbol{l},\boldsymbol{m}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})},\ \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1},\boldsymbol{m})},\ \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l},\boldsymbol{m}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1})}.$$

The four coefficients can be calculated directly. □

Proposition 10. *On $V_l \otimes V_m$, we have*

$$\begin{aligned} b_i f_{i-1} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} = {} & \frac{[l_s+m_s-j_s+1]}{[l_s+m_s-2j_s+1]}(B'_1 \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{m})} + B'_2 \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})}) \\ & + \frac{[j_{s+1}]}{[l_s+m_s-2j_s+1]}(B'_3 \boldsymbol{w}_{\boldsymbol{j}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1},\boldsymbol{m})} + B'_4 \boldsymbol{w}_{\boldsymbol{j}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l},\boldsymbol{m}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1})}) \\ & + \frac{[l_s+m_s-2j_s]}{[l_s+m_s-2j_s+1]}(D'_1 f_{i-1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s}^{(\boldsymbol{l}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{m})} + D'_2 f_{i-1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s}^{(\boldsymbol{l},\boldsymbol{m}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})} \\ & + D'_3 f_{i-1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1},\boldsymbol{m})} + D'_4 f_{i-1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l},\boldsymbol{m}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1})}), \end{aligned}$$

$$\begin{aligned} b_i f_{i+1} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} = {} & \frac{[j_s]}{[l_{s+1}+m_{s+1}-2j_{s+1}+1]}(C'_1 \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1}}^{(\boldsymbol{l}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{m})} + C'_2 \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1}}^{(\boldsymbol{l},\boldsymbol{m}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})}) \\ & + \frac{[l_{s+1}+m_{s+1}-j_{s+1}+1]}{[l_{s+1}+m_{s+1}-2j_{s+1}+1]}(C'_3 \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1},\boldsymbol{m})} + C'_4 \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1})}) \\ & + \frac{[l_{s+1}+m_{s+1}-2j_{s+1}]}{[l_{s+1}+m_{s+1}-2j_{s+1}+1]}(D'_1 f_{i+1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s}^{(\boldsymbol{l}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{m})} + D'_2 f_{i+1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s}^{(\boldsymbol{l},\boldsymbol{m}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})} \\ & + D'_3 f_{i+1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1},\boldsymbol{m})} + D'_4 f_{i+1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l},\boldsymbol{m}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1})}), \end{aligned}$$

where

$$B'_1 = q^{j_s-j_{s+1}-m_s+m_{s+1}}[l_s-j_s], \quad B'_2 = [m_s-j_s],$$
$$B'_3 = -q^{j_s-j_{s+1}-l_s-m_s+l_{s+1}+m_{s+1}}[m_s-j_s], \quad B'_4 = -q^{2j_s-2j_{s+1}-l_s-2m_s+l_{s+1}+2m_{s+1}}[l_s-j_s],$$
$$C'_1 = -q^{-j_s+j_{s+1}+l_s-l_{s+1}}[m_{s+1}-j_{s+1}], \quad C'_2 = -[l_{s+1}-j_{s+1}],$$
$$C'_3 = q^{-j_s+j_{s+1}}[l_{s+1}-j_{s+1}], \quad C'_4 = q^{-2j_s+2j_{s+1}+l_s-l_{s+1}}[m_{s+1}-j_{s+1}],$$

Proof. Using Proposition 1, we have

$$e_{i-1} b_i f_{i-1} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} = b_i e_{i-1} f_{i-1} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} = b_i \{k_{i-1}\} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} = [l_s+m_s-2j_s] b_i \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})},$$
$$e_{i+1} b_i f_{i+1} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} = b_i e_{i+1} f_{i+1} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} = b_i \{k_{i+1}\} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} = [l_{s+1}+m_{s+1}-2j_{s+1}] b_i \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})},$$

where $\{k_i\} = \frac{k_i - k_i^{-1}}{q-q^{-1}}$. Thus same in Lemma 9, $e_{i+1} b_i f_{i+1}$ and $e_{i-1} b_i f_{i-1}$ are a linear combination of the following vectors.

$$\boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s}^{(\boldsymbol{l}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{m})},\ \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s}^{(\boldsymbol{l},\boldsymbol{m}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})},\ \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1},\boldsymbol{m})},\ \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l},\boldsymbol{m}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1})}.$$

By considering weight, one find that $b_i f_{i-1}$ and $b_i f_{i+1}$ are a linear combination like a assertion, and coefficients can be calculated directly. □

Corollary 11. *On $V_l \otimes V_m$, we have*

$$b_i f_{i+1} \boldsymbol{w}_{\boldsymbol{o}}^{(\boldsymbol{l},\boldsymbol{m})} = [l_{s+1}] \boldsymbol{w}_{\boldsymbol{o}}^{(\boldsymbol{l}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1},\boldsymbol{m})} + q^{l_s-l_{s+1}}[m_{s+1}] \boldsymbol{w}_{\boldsymbol{o}}^{(\boldsymbol{l},\boldsymbol{m}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1})},$$
$$b_i f_{i-1} \boldsymbol{w}_{\boldsymbol{o}}^{(\boldsymbol{l},\boldsymbol{m})} = q^{m_{s+1}-m_s}[l_s] \boldsymbol{w}_{\boldsymbol{o}}^{(\boldsymbol{l}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{m})} + [m_s] \boldsymbol{w}_{\boldsymbol{o}}^{(\boldsymbol{l},\boldsymbol{m}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})}.$$

Proposition 12. *On $V_l \otimes V_m$, we have*

$$\begin{aligned}
& b_i f_{i-1} f_{i+1} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} \\
&= A_1 \boldsymbol{w}_{\boldsymbol{j}+\boldsymbol{e}_{s+1}}^{(\boldsymbol{l}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{m})} + B_1 f_{i+1} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{m})} + C_1 f_{i-1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1}}^{(\boldsymbol{l}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{m})} + D_1 f_{i-1} f_{i+1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s}^{(\boldsymbol{l}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{m})} \\
&+ A_2 \boldsymbol{w}_{\boldsymbol{j}+\boldsymbol{e}_{s+1}}^{(\boldsymbol{l},\boldsymbol{m}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})} + B_2 f_{i+1} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})} + C_2 f_{i-1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1}}^{(\boldsymbol{l},\boldsymbol{m}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})} + D_2 f_{i-1} f_{i+1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_s}^{(\boldsymbol{l},\boldsymbol{m}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})} \\
&+ A_3 \boldsymbol{w}_{\boldsymbol{j}+\boldsymbol{e}_s}^{(\boldsymbol{l}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1},\boldsymbol{m})} + B_3 f_{i+1} \boldsymbol{w}_{\boldsymbol{j}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1},\boldsymbol{m})} + C_3 f_{i-1} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1},\boldsymbol{m})} + D_3 f_{i-1} f_{i+1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1},\boldsymbol{m})} \\
&+ A_4 \boldsymbol{w}_{\boldsymbol{j}+\boldsymbol{e}_s}^{(\boldsymbol{l},\boldsymbol{m}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1})} + B_4 f_{i+1} \boldsymbol{w}_{\boldsymbol{j}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l},\boldsymbol{m}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1})} + C_4 f_{i-1} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1})} + D_4 f_{i-1} f_{i+1} \boldsymbol{w}_{\boldsymbol{j}-\boldsymbol{e}_{s+1}}^{(\boldsymbol{l},\boldsymbol{m}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1})},
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
A_1 &= q^{j_s+j_{s+1}-l_{s+1}-m_s-1} \frac{[l_s-j_s][m_{s+1}-j_{s+1}][l_s+m_s-j_s+1]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \\
A_2 &= -\frac{[l_{s+1}-j_{s+1}][m_s-j_s][l_s+m_s-j_s+1]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \\
A_3 &= q^{j_s+j_{s+1}-l_s-m_s-1} \frac{[l_{s+1}-j_{s+1}][m_s-j_s][l_{s+1}+m_{s+1}-j_{s+1}+1]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \\
A_4 &= -q^{2j_s+2j_{s+1}-l_s-l_{s+1}-2m_s-2} \frac{[l_s-j_s][m_{s+1}-j_{s+1}][l_{s+1}+m_{s+1}-j_{s+1}+1]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]},
\end{aligned}$$

$$\begin{aligned}
B_j &= B'_j \frac{[l_s+m_s-j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \quad (j=1,2), \\
&= B'_j \frac{[j_{s+1}][l_{s+1}+m_{s+1}-2j_{s+1}]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \quad (j=3,4), \\
C_j &= C'_j \frac{[j_s][l_s+m_s-2j_s]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \quad (j=1,2), \\
&= C'_j \frac{[l_s+m_s-2j_s][l_{s+1}+m_{s+1}-j_{s+1}+1]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \quad (j=3,4), \\
D_j &= D'_j \frac{[l_s+m_s-2j_s][l_{s+1}+m_{s+1}-2j_{s+1}]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \quad (j=1,2,3,4).
\end{aligned}$$

Proof. Similar to Proposition9 and 10, $b_i f_{i-1} f_{i+1} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})}$ can be expressed with suitable scalars A_j, B_j, C_j, D_j $(1 \le j \le 4)$ as (26). By applying $e_{i-1}e_{i+1}$ on both sides, the first to third terms in each line of the right hand side vanish. So by Proposition 9, D_j $(1 \le j \le 4)$ is determined. Then, by applying e_{i+1} on both sides of (26), B_j $(1 \le j \le 4)$ is determined, and by applying e_{i-1}, C_j $(1 \le j \le 4)$ is done by Proposition 10. Finally, A_j $(1 \le j \le 4)$ is determined by a direct calculation. □

Corollary 13. *On $V_l \otimes V_m$, we have*

$$\begin{aligned}
b_i f_{i-1} f_{i+1} \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} =& A_1 \boldsymbol{w}_{\boldsymbol{j}+\boldsymbol{e}_{s+1}}^{(\boldsymbol{l}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{m})} + A_2 \boldsymbol{w}_{\boldsymbol{j}+\boldsymbol{e}_{s+1}}^{(\boldsymbol{l},\boldsymbol{m}-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})} + A_3 \boldsymbol{w}_{\boldsymbol{j}+\boldsymbol{e}_s}^{(\boldsymbol{l}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1},\boldsymbol{m})} + A_4 \boldsymbol{w}_{\boldsymbol{j}+\boldsymbol{e}_s}^{(\boldsymbol{l},\boldsymbol{m}+\boldsymbol{e}_s-\boldsymbol{e}_{s+1})} \\
&+ (\textit{other terms}),
\end{aligned}$$

where A_j $(j = 1, 2, 3, 4)$ is given in Proposition 12 and (other terms) stands for the linear combination of vectors of the form $\boldsymbol{w}_{\boldsymbol{j}'}^{(\boldsymbol{l}',\boldsymbol{m}')}$ possibly applied by f_{i-1}, f_{i+1} with $(\boldsymbol{l}', \boldsymbol{m}')$ appearing in the right hand side and $j'_k \le j_k$ for $1 \le k \le n$.

4.3. **Proof of Theorem 7.** We prove Theorem 7 when $\varepsilon = 0$. Suppose W is a nonzero $\mathbf{U}^\imath$-invariant subspace of $V_l \otimes V_m$. Note that $\mathbf{U}^\imath$ contains $\mathbf{U}(I_\bullet)$. In view of Proposition 8, one can assume that W contains a vector of the form

$$\sum_{\boldsymbol{l},\boldsymbol{m},\boldsymbol{j}} c(\boldsymbol{l},\boldsymbol{m},\boldsymbol{j}) \boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})} \tag{27}$$

where $c(\boldsymbol{l},\boldsymbol{m},\boldsymbol{j})\in\mathbb{Q}(q)$ and $\boldsymbol{l},\boldsymbol{m},\boldsymbol{j}$ run over all possible integer vectors such that $l_s+m_s-2j_s$ is constant for any $s=1,\dots,n$. By applying b_i $(i\in I_\circ)$ in a suitable order, from Proposition 9 one can assume $\boldsymbol{j}=\boldsymbol{o}$ in (27). Then by Corollary 11, one can eventually assume $\boldsymbol{l}=l\boldsymbol{e}_1,\boldsymbol{m}=m\boldsymbol{e}_1$ where $l=|\boldsymbol{l}|,m=|\boldsymbol{m}|$. Hence, we have $\boldsymbol{w}_{\boldsymbol{o}}^{(l\boldsymbol{e}_1,m\boldsymbol{e}_1)}\in W$.

Next show $\boldsymbol{w}_{\boldsymbol{o}}^{(l_1\boldsymbol{e}_1+l_2\boldsymbol{e}_2,m_1\boldsymbol{e}_1+m_2\boldsymbol{e}_2)}\in W$ for any l_1,l_2,m_1,m_2 such that $l_1+l_2=l,m_1+m_2=m$. We do it by induction on $k=l_2+m_2$. The $k=0$ case is done. Assume $\boldsymbol{w}_{\boldsymbol{o}}^{(l_1\boldsymbol{e}_1+l_2\boldsymbol{e}_2,m_1\boldsymbol{e}_1+m_2\boldsymbol{e}_2)}\in W$ for $l_2+m_2=k$. By Corollary 11, we have

$$b_2f_1\boldsymbol{w}_{\boldsymbol{o}}^{(l_1\boldsymbol{e}_1+l_2\boldsymbol{e}_2,m_1\boldsymbol{e}_1+m_2\boldsymbol{e}_2)}=q^{m_2-m_1}[l_1]\boldsymbol{w}_{\boldsymbol{o}}^{((l_1-1)\boldsymbol{e}_1+(l_2+1)\boldsymbol{e}_2,m_1\boldsymbol{e}_1+m_2\boldsymbol{e}_2)}+[m_1]\boldsymbol{w}_{\boldsymbol{o}}^{(l_1\boldsymbol{e}_1+l_2\boldsymbol{e}_2,(m_1-1)\boldsymbol{e}_1+(m_2+1)\boldsymbol{e}_2)},$$

$$\begin{aligned}(b_4f_5)\cdots(b_{2n-2}f_{2n-1})(b_0f_1)&\boldsymbol{w}_{\boldsymbol{o}}^{(l_1\boldsymbol{e}_1+l_2\boldsymbol{e}_2,m_1\boldsymbol{e}_1+m_2\boldsymbol{e}_2)}\\&=[l_1]\boldsymbol{w}_{\boldsymbol{o}}^{((l_1-1)\boldsymbol{e}_1+(l_2+1)\boldsymbol{e}_2,m_1\boldsymbol{e}_1+m_2\boldsymbol{e}_2)}+q^{l_2-l_1}[m_1]\boldsymbol{w}_{\boldsymbol{o}}^{(l_1\boldsymbol{e}_1+l_2\boldsymbol{e}_2,(m_1-1)\boldsymbol{e}_1+(m_2+1)\boldsymbol{e}_2)}.\end{aligned}$$

If $l_1+m_1\neq l_2+m_2$, these two vectors are linearly independent. Hence the induction proceeds up to $k\le l_1+m_1$. When $l_2+m_2\ge l_1+m_1$, we first recognize that $\boldsymbol{w}_{\boldsymbol{o}}^{(l\boldsymbol{e}_2,m\boldsymbol{e}_2)}\in W$ by applying $(b_2f_1)^{l+m}$ to $\boldsymbol{w}_{\boldsymbol{o}}^{(l\boldsymbol{e}_1,m\boldsymbol{e}_1)}$. We then do the same exercise as before.

Let us now show W contains $\boldsymbol{w}_{\boldsymbol{o}}^{(\boldsymbol{l},\boldsymbol{m})}$ for any possible $\boldsymbol{l}$ and $\boldsymbol{m}$. From the previous paragraph, we know $\boldsymbol{w}_{\boldsymbol{o}}^{(l_1\boldsymbol{e}_1+l_2\boldsymbol{e}_2,m\boldsymbol{e}_1)}\in W$. Applying b_if_{i-1} $(i=4,\dots,2n-2)$ suitable times, we know $\boldsymbol{w}_{\boldsymbol{o}}^{(\boldsymbol{l},m\boldsymbol{e}_1)}\in W$ for any $\boldsymbol{l}$. Then by doing similarly including $i=2$, we know $\boldsymbol{w}_{\boldsymbol{o}}^{(\boldsymbol{l},\boldsymbol{m})}\in W$ for any $\boldsymbol{l},\boldsymbol{m}$.

By Proposition 8, it is enough to show W contains $\boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})}$ for any possible $\boldsymbol{l},\boldsymbol{m},\boldsymbol{j}$. From the considerations so far, it is true when $|\boldsymbol{j}|=0$. The following proposition makes the induction on $|\boldsymbol{j}|$ work and finishes the proof of Theorem 7.

Proposition 14. *Consider the following matrix C depending on l,m,j. Its row index runs over all $(i,\boldsymbol{l},\boldsymbol{m},\boldsymbol{j})$ with $i=0,2,\dots,2n-2$ and $|\boldsymbol{l}|=l,|\boldsymbol{m}|=m,|\boldsymbol{j}|=j$, and its column index runs over all $(\boldsymbol{l}',\boldsymbol{m}',\boldsymbol{j}')$ with $|\boldsymbol{l}'|=l,|\boldsymbol{m}'|=m,|\boldsymbol{j}'|=j+1$. The entry for the pair $((i,\boldsymbol{l},\boldsymbol{m},\boldsymbol{j}),(\boldsymbol{l}',\boldsymbol{m}',\boldsymbol{j}'))$ is given by the coefficient of $\boldsymbol{w}_{\boldsymbol{j}'}^{(\boldsymbol{l}',\boldsymbol{m}')}$ in $b_if_{i-1}f_{i+1}\boldsymbol{w}_{\boldsymbol{j}}^{(\boldsymbol{l},\boldsymbol{m})}$ in the previous proposition. Then C is of full rank. Note that the rank does not depend on the orders of the index sets.*

Proof. Let A be the subring of $\mathbb{Q}(q)$ defined by $A=\{f(q)\in\mathbb{Q}(q)\,|\,f(q)\text{ is regular at }q=0\}$. Let α_t $(t=1,2,3,4)$ be the largest integer such that A_t in Corollary 13 belongs to $q^{\alpha_t}A$. We have

$$\begin{aligned}\alpha_1-\alpha_2=\alpha_4-\alpha_3&=j_s+j_{s+1}-l_s-m_{s+1}-1<0,\\\alpha_4-\alpha_1&=2j_{s+1}-l_{s+1}-m_{s+1}-1<0,\end{aligned}$$

since $j_t\le\min(l_t,m_t)$ $(t=s,s+1)$. Therefore, α_4 is minimal and the others are strictly larger.

For $\boldsymbol{w}_{\boldsymbol{j}'}^{(\boldsymbol{l}',\boldsymbol{m}')}$ such that $|\boldsymbol{l}'|=l,|\boldsymbol{m}'|=m,|\boldsymbol{j}'|=j+1$, choose the minimal s such $j'_s>0$ and consider $b_if_{i-1}f_{i+1}\boldsymbol{w}_{\boldsymbol{j}'-\boldsymbol{e}_s}^{(\boldsymbol{l}',\boldsymbol{m}'-\boldsymbol{e}_s+\boldsymbol{e}_{s+1})}$ with $i=2s$. By Proposition 12 the fourth term of the above is nonzero. Consider the row of C corresponding to the index $(i,\boldsymbol{l}',\boldsymbol{m}'-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{j}'-\boldsymbol{e}_s)$. By multiplying a suitable scalar to this row, one can make the $((i,\boldsymbol{l}',\boldsymbol{m}'-\boldsymbol{e}_s+\boldsymbol{e}_{s+1},\boldsymbol{j}'-\boldsymbol{e}_s),(\boldsymbol{l}',\boldsymbol{m}',\boldsymbol{j}'))$-entry of C be 1, and the other three nonzero entries in the same row belong to qA. Consider the square matrix C' obtained by varying all possible $(\boldsymbol{l}',\boldsymbol{m}',\boldsymbol{j}')$ and picking the corresponding renormalized rows. Then from the construction, $\det C'$ belongs to $\{\pm1\}+qA$. Hence the assertion is confirmed. □

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