Wavelet characterization of local Muckenhoupt weighted Sobolev spaces with variable exponents

Wavelet characterization of local Muckenhoupt weighted Sobolev spaces with variable exponents

Mitsuo Izuki, Toru Nogayama, Takahiro Noi
and Yoshihiro Sawano

Wavelet characterization of local Muckenhoupt weighted Sobolev spaces with variable exponents

Mitsuo Izuki, Toru Nogayama, Takahiro Noi and Yoshihiro Sawano

May 18, 2021

Abstract

The goal of this paper is to define local weighted variable Sobolev spaces of fractional order and its chracterization by wavelet. We first consider local weighted variable Sobolev spaces by means of weak derivatives and obtain wavelet characterization for these spaces. Using the Bessel potentials, we next define local weighted variable Sobolev spaces of fractional order. We show that Sobolev spaces obtained by weak derivatives and those by the Bessel potentials coincide. We also show that local weighted variable Sobolev spaces are closed under complex interpolation. Some examples are given including the applications to weighted uniformly local Lebesgue spaces with variable exponents and periodic function spaces as a byproduct, although the exponent is constant.

Key words: variable exponent, wavelet, Sobolev spaces, local Muckenhoupt weight
AMS Subject Classification: 42B35, 42C40.

Contents

1 Introduction

Compactly supported wavelets with the proper smoothness characterize various function spaces. In fact, we can obtain norms equivalent to those spaces by using some square functions that involve wavelet coefficients. The first author [22] and Kopaliani [28] have initially and independently obtained the wavelet characterizations of Lebesgue spaces with variable exponents. Later the characterizations were generalized by [25] to the Muckenhoupt weighted setting. One of the purpose of this paper is to enlarge the class of admissible weights to handle the weights $w(x) = \exp(A|x|)$ and $w(x) = (1 + |x|)^A$ with $A \in \mathbb{R}$.

The theory of Lebesgue spaces with variable exponents originates from [38]. After that, Nakano investigated Lebesgue spaces with variable exponents in his Japanese books [35, 36]. The theory of Lebesgue spaces with variable exponents was developed after Kováčik and Rákosník investigated Sobolev spaces with variable exponents in the 1990's [29]. Among others, Diening investigated the boundedness of the Hardy–Littlewood maximal operator in [13], which paved the way to exhaustively investigate of variable exponent Lebesgue spaces. For example, Cruz-Uribe, Fiorenza and Neugebauer further studied the boundedness of the Hardy–Littlewood maximal operator in [8, 9]. We refer to [6, 24] as well as [41, p. 447] for more details. Moreover, the study on generalization of the classical Muckenhoupt weights in terms of variable exponent has been developed [5, 10]. Motivated by

Rychkov [39], the second and fourth authors [37] defined the class of local Muckenhoupt weights and obtained the boundedness of the local Hardy–Littlewood maximal operator.

In this paper, we further develop the theory of wavelets on local weighted Lebesgue spaces with variable exponents, which is a follow-up to [37]. We seek to characterize the spaces in terms of the inhomogeneous wavelet expansion.

In this paper, we use the following notation of a variable exponent. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a variable exponent. That is, $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ is a measurable function. Additionally, let w be a weight. That is, w is a locally integrable function which is positive almost everywhere. Then we define the weighted variable Lebesgue space $L^{p(\cdot)}(w)$ to be the set of all measurable functions f such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} w(x)\, \mathrm{d}x < \infty.$$

The infimum of such λ is called the $L^{p(\cdot)}$-norm of f and is denoted by $\|f\|_{L^{p(\cdot)}}$. When $p(\cdot)$ is the constant function p, $L^{p(\cdot)}(w)$ is simply the weighted Lebesgue space $L^p(w)$ with the coincidence of norms. When we investigate the boundedness of the Hardy–Littlewood maximal operator M defined by (1.8), the following two conditions seem standard:

(1) An exponent $r(\cdot)$ satisfies the local log-Hölder continuity condition if there exists $C > 0$ such that

$$\mathrm{LH}_0 \,:\, |r(x) - r(y)| \le \frac{C}{-\log|x-y|}, \quad x, y \in \mathbb{R}^n, \quad |x-y| \le \frac{1}{2}. \tag{1.1}$$

The set LH_0 collects all exponents $r(\cdot)$ which satisfy (1.1).

(2) An exponent $r(\cdot)$ satisfies the log-Hölder continuity condition at ∞ if there exist $C > 0$ and $r_\infty \in [0, \infty)$ such that

$$\mathrm{LH}_\infty \,:\, |r(x) - r_\infty| \le \frac{C}{\log(e + |x|)}, \quad x \in \mathbb{R}^n. \tag{1.2}$$

The set LH_∞ collects all exponents $r(\cdot)$ which satisfy (1.2).

We also recall the theory of wavelets. We can construct compactly supported wavelets (see [12, 31, 33, 41, 45] for example). Here and below, the wavelets and the scaling functions are assumed to belong to C_{c}, which is the set of all compactly supported continuous functions. For $L \in \mathbb{N} \cup \{0\}$, the set $\mathcal{P}_L^{\perp}$ denotes the set of all the measurable functions f for which $(1 + |\cdot|)^L f \in L^1$ and $\int_{\mathbb{R}^n} x^\alpha f(x)\mathrm{d}x = 0$ for all $\alpha \in (\mathbb{N} \cup \{0\})^n$ with $|\alpha| \le L$. Such a function f satisfies moment condition of order L. In this case, one also writes $f \perp \mathcal{P}_L$.

Choose compactly supported functions

$$\varphi \text{ and } \psi^l \quad (l = 1, 2, \ldots, 2^n - 1) \tag{1.3}$$

so that the following conditions are satisfied:

(1) For any $J \in \mathbb{Z}$, the system

$$\left\{\varphi_{J,k},\, \psi^l_{j,k} \,:\, k \in \mathbb{Z}^n,\, j \geq J,\, l = 1, 2, \ldots, 2^n - 1\right\}$$

is an orthonormal basis of L^2. Here, given a function F defined on $\mathbb{R}^n$, we write

$$F_{j,k} \equiv 2^{\frac{jn}{2}} F(2^j \cdot -k)$$

for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$.

(2) The functions φ and ψ^l ($l = 1, 2, \ldots, 2^n - 1$) belong to $\mathcal{P}^{\perp}_{[s+1]}$. In addition, they are real-valued and compactly supported with

$$\mathrm{supp}(\varphi) = \mathrm{supp}(\psi^l) = [0, 2N - 1]^n \tag{1.4}$$

for some $N \in \mathbb{N}$.

We also define $\chi_{j,k} \equiv 2^{\frac{jn}{2}} \chi_{Q_{j,k}}$ and $\chi^*_{j,k} \equiv 2^{\frac{jn}{2}} \chi_{Q^*_{j,k}}$ for $j \in \mathbb{Z}$ and $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, where $Q_{j,k}$ and $Q^*_{j,k}$ are the dyadic cube and its expansion given by (1.6) and (1.7), respectively. Then using the L^2-inner product $\langle \cdot, \cdot \rangle$, for $f \in L^1_{\mathrm{loc}}$, we define two square functions Vf, $W_s f$ by

$$V f \equiv V^{\varphi} f \equiv \left(\sum_{k\in\mathbb{Z}^n} |\langle f, \varphi_{J,k}\rangle \chi_{J,k}|^2\right)^{\frac{1}{2}},$$

$$W_s f \equiv W_s^{\psi^l} f \equiv \left(\sum_{l=1}^{2^n-1} \sum_{j=J}^{\infty} \sum_{k\in\mathbb{Z}^n} \left|2^{js} \langle f, \psi^l_{j,k}\rangle \chi_{j,k}\right|^2\right)^{\frac{1}{2}}.$$

Here, J is a fixed integer. For the time being, we assume that $s \in \mathbb{N}$. We consider the case $s \in \mathbb{R}$ after Section 4.

Denote by $\mathcal{Q}$ the set of all compact cubes whose edges are parallel to the coordinate axes. We mix the notions considered in [5, 10, 39] to define the local Muckenhoupt class as follows:

Definition 1.1. Given an exponent $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ and a weight w, $w \in A^{\mathrm{loc}}_{p(\cdot)}$ if $[w]_{A^{\mathrm{loc}}_{p(\cdot)}} \equiv \sup\limits_{Q\in\mathcal{Q}, |Q|\leq 1} |Q|^{-1} \|\chi_Q\|_{L^{p(\cdot)}(w)} \|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} < \infty$, where $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$ and the supremum is taken over all cubes $Q \in \mathcal{Q}$ with volumes less than or equal to 1.

Recall that Muckenhoupt and Wheeden considered the class A_p. See [18, 43, 41] for more about this class. Rychkov extended the class A_p to A^{loc}_p in [39], while Cruz-Uribe, Fiorenza, and Neugebauer extended the class A_p to $A_{p(\cdot)}$ in [10]. The present work mixes these two works. A remarkable difference from the classes $A^{\mathrm{loc}}_{p(\cdot)}$ and $A_{p(\cdot)}$ is that our definition restricts the cube sizes. Once we remove the volume restriction in the supremum appearing in the definition of $[w]_{A^{\mathrm{loc}}_{p(\cdot)}}$, we obtain the $A_{p(\cdot)}$-norm.

Unlike the classes A_p and $A_{p(\cdot)}$, we can consider $w(x) = \exp(\alpha|x|)$ for any $\alpha \in \mathbb{R}$. Another typical example is $w(x) = (1 + |x|)^A$ for any $A \in \mathbb{R}$.

We fix $s \in \mathbb{N} \cup \{0\}$ to develop the theory of local weighted variable Sobolev spaces $L^{p(\cdot),s}(w)$. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a variable exponent, and w be a weight. We have defined the local weighted Lebesgue space $L^{p(\cdot)}(w)$ with variable exponents $p(\cdot)$ above. If $w = 1$ almost everywhere, then $L^{p(\cdot)}(w)$ is a non-weighted variable Lebesgue space, and we can write $L^{p(\cdot)} \equiv L^{p(\cdot)}(1)$. Moreover, if $p(\cdot)$ equals to a constant p, then $L^{p(\cdot)} = L^p$, which is the usual L^p space. When we consider non-weighted function spaces defined on $\mathbb{R}^n$, we can simply write $L^2 \equiv L^2(1)$, $L^1_{\rm loc} \equiv L^1_{\rm loc}(1)$, etc. We also note that $L^{p(\cdot)}(w) \subset L^1_{\rm loc}$, provided that $w^{-\frac{1}{p(\cdot)-1}}$ is locally integrable. We define local weighted Sobolev spaces with variable exponents by means of weak derivatives.

Definition 1.2. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a variable exponent, w be a weight and $s \in \mathbb{N}$. Suppose that $w^{-\frac{1}{p(\cdot)-1}}$ is locally integrable. The local weighted Sobolev space $L^{p(\cdot),s}(w)$ is the space of all measurable functions $f \in L^{p(\cdot)}(w)$ satisfying that the weak derivatives $D^\alpha f$ belong to $L^{p(\cdot)}(w)$ for all $\alpha \in (\mathbb{N} \cup \{0\})^n = \{0, 1, \ldots\}^n$ with $|\alpha| \le s$.

The goal of this paper is to establish a wavelet characterization of the local weighted Sobolev space $L^{p(\cdot),s}(w)$ with a variable exponent, which is established in [26] for $s = 0$.

Theorem 1.3. *Suppose that $p(\cdot) \in {\rm LH}_0 \cap {\rm LH}_\infty$ satisfies $1 < p_- \equiv {\rm essinf}_{x\in\mathbb{R}^n} p(x) \le p_+ \equiv {\rm esssup}_{x\in\mathbb{R}^n} p(x) < \infty$. Let $s \in \mathbb{N}$ and $w \in A^{\rm loc}_{p(\cdot)}$. Fix $J \in \mathbb{Z}$ arbitrarily. Then there exists a constant $C > 0$ such that, for all $f \in L^{p(\cdot),s}(w)$,*

$$C^{-1}\|f\|_{L^{p(\cdot),s}(w)} \le \|Vf\|_{L^{p(\cdot)}(w)} + \|W_s f\|_{L^{p(\cdot)}(w)} \le C\,\|f\|_{L^{p(\cdot),s}(w)}. \qquad (1.5)$$

More precisely, for all $f \in L^1_{\rm loc}$, we have the following two assertions:

(1) *If $f \in L^{p(\cdot),s}(w)$, then we have* (1.5)*.*

(2) *If $Vf + W_s f \in L^{p(\cdot)}(w)$, then $f \in L^{p(\cdot),s}(w)$ and* (1.5) *holds.*

We harvest a corollary, which can be obtained from the proof of Theorem 1.3.

Corollary 1.4. *Suppose that $p(\cdot) \in \mathcal{P} \cap {\rm LH}_0 \cap {\rm LH}_\infty$, and let $w \in A^{\rm loc}_{p(\cdot)}$ and $s \in \mathbb{N}$. Then the following two norms*

$$\|f\|_{L^{p(\cdot),s}(w)} \equiv \sum_{|\alpha|\le s} \|D^\alpha f\|_{L^{p(\cdot)}(w)}, \quad \|f\|_{L^{p(\cdot)}(w)} + \sum_{|\alpha|=s} \|D^\alpha f\|_{L^{p(\cdot)}(w)}$$

are equivalent.

It shoudl be noted that Corollary 1.4 extends the results in [20, 23], where Hernández and Weiss considered the case $w = 1$ in [20] and Izuki considered the case where $p(\cdot)$ is a constant exponent. It is noteworthy that these existing results are used for wavelet characterizations in these papers [20, 23]. The crucial point is that the duality result is unnecessary for this characterization.

Our result extends the one in [30] to the variable exponent setting and the one in [25] to the local weight setting. Note that $L^{p(\cdot)}(w)$ is a subset of $L^1_{\rm loc}$ since $w \in A^{\rm loc}_{p(\cdot)}$. See Section 2 for details.

The study of local weighted function spaces is useful since it yields some results for function spaces such as uniformly local Lebesgue spaces and amalgam spaces. Although weighted uniformly local Lebesgue spaces with variable exponents do not fall directly within the scope of the function spaces dealt with in this paper, by using an equivalent characterization, we can obtain the wavelet characterization. Originally, uniformly local Lebesgue spaces were considered as a special case of amalgam spaces handled in [2, 4, 16, 21, 27]. Here, in the context of weighted Lebesgue spaces with variable exponents, we extend the notion of uniformly local Lebesgue spaces to weighted uniformly local Lebesgue spaces with variable exponents. We seek to obtain the wavelet characterization in weighted amalgam spaces with variable exponents, in particular the one in weighted uniformly local Lebesgue spaces with variable exponents. It is noteworthy that the atomic decomposition considered in textbooks [41, 44] cannot be used for our discussion because there is no canonical operator such as Vf and $W_s f$.

The rest of this paper is organized as follows: In Section 2, we collect some preliminary facts. Section 3 proves Theorem 1.3. Section 4 generalizes the definition of $L^{p(\cdot),s}(w)$, and considers the case of $s > 0$ instead of $s \in \mathbb{N}$. As applications, Section 5 considers pointwise multipliers and diffeomorphisms and refines the complex interpolation result obtained in Section 4. We handle local weighted Sobolev spaces with negative smoothness in Section 6. Examples are provided in Section 7. We present two prominent examples of exponential weights and polynomial weights. As another example, we consider weighted uniformly local Lebesgue spaces with variable exponents.

Herein we use the following notation:

(1) The set $\mathbb{N}_0 \equiv \{0, 1, \ldots\}$ consists of all non-negative integers.

(2) Let E be a set. Then we denote its indicator function by χ_E.

(3) A set S is a dyadic cube if

$$S = Q_{j,k} \equiv \prod_{m=1}^{n} \left[2^{-j}k_m, 2^{-j}(k_m+1)\right] \tag{1.6}$$

for some $j \in \mathbb{Z}$ and $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. Likewise, we define

$$Q^*_{j,k} \equiv \prod_{m=1}^{n} \left[2^{-j}k_m, 2^{-j}(k_m+2N-1)\right], \tag{1.7}$$

where N is from (1.4).

(4) Write $\mathcal{D}_j \equiv \{Q_{j,k} \,:\, k \in \mathbb{Z}^n\}$ for each $j \in \mathbb{Z}$.

(5) For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we let

$$\chi^*_{j,k} = 2^{\frac{jn}{2}} \chi_{Q^*_{j,k}} = 2^{\frac{jn}{2}} \chi_{\prod_{l=1}^n [2^{-j}k_l, 2^{-j}(k_l+2N-1)]},$$

where N is from (1.4). Each function $\chi^*_{j,k}$ is called the modified indicator function.

(6) Let $Q_0 \equiv \prod_{m=1}^{n} [a_m, b_m]$ be a cube. A dyadic cube with respect to Q_0 is the set of the form

$$\prod_{m=1}^{n} \left[a_m + \frac{k_m - 1}{2^j}(b_m - a_m), a_m + \frac{k_m}{2^j}(b_m - a_m) \right]$$

for some $j \in \mathbb{N}_0$ and $k = (k_1, k_2, \ldots, k_n) \in \{1, 2, 3, \ldots, 2^j\}^n$. The set $\mathcal{D}(Q_0)$ collects all dyadic cubes with respect to a cube Q_0.

(7) Given a cube Q, we denote by $c(Q)$ the center of Q and by $\ell(Q)$ the side length of Q: $Q = Q(c(Q), \ell(Q))$ and $\ell(Q) \equiv |Q|^{\frac{1}{n}}$, where $|Q|$ denotes the volume of the cube Q. For $r > 0$, we write $Q(r) \equiv Q(0, r)$. In addition, $|E|$ is the Lebesgue measure for general measurable set $E \subset \mathbb{R}^n$.

(8) The letter C denotes positive constants that may change from one occurrence to another. Let $A, B \geq 0$. Then $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq CB$, where C depends only on the parameters of importance. The symbol $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$ occur simultaneously.

(9) Let $\mathcal{M}$ be the set of all complex-valued measurable functions defined on $\mathbb{R}^n$. Likewise, for a measurable set E, let $\mathcal{M}(E)$ be the set of all complex-valued measurable functions defined on E.

(10) The symbol $\langle f, g \rangle$ denotes the L^2-inner product. That is,

$$\langle f, g \rangle \equiv \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, \mathrm{d}x$$

for all complex-valued measurable L^2-functions f, g defined on $\mathbb{R}^n$. Let $L^p(\mathbb{T}^n)$ be the set of all p-locally integrable functions f with period 1 for which

$$\|f\|_{L^p(\mathbb{T}^n)} \equiv \left(\int_{[0,1]^n} |f(x)|^p \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

The symbol $\langle f, g \rangle_{L^2(\mathbb{T}^n)}$ stands for the L^2-inner product on $\mathbb{T}^n$. That is, we write

$$\langle f, g \rangle_{L^2(\mathbb{T}^n)} \equiv \int_{[0,1]^n} f(x)\overline{g(x)}\, \mathrm{d}x$$

for all $f, g \in L^2(\mathbb{T}^n)$. We use these symbols as long as the integral makes sense for any couple (f, g) of measurable functions.

(11) The set $\mathcal{P}$ consists of all $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that $1 < p_- \leq p_+ < \infty$.

(12) Let f be a measurable function. We consider the local maximal operator given by

$$M^{\rm loc}f(x) \equiv \sup_{Q \in \mathcal{Q}, |Q| \le 1} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| {\rm d}y \quad (x \in \mathbb{R}^n).$$

Needless to say, this is an analog of the Hardy–Littlewood maximal operator given by

$$Mf(x) \equiv \sup_{Q \in \mathcal{Q}} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| {\rm d}y \quad (x \in \mathbb{R}^n). \tag{1.8}$$

(13) Let $K \in \mathbb{N}$. The operator $(M^{\rm loc})^K$ is the K-fold composition of $M^{\rm loc}$.

(14) Let E be a measurable set in $\mathbb{R}^n$. For a function $f : E \to \mathbb{C}$, f^* denotes its decreasing rearrangement.

2 Preliminaries

Here, we collect preliminary facts used in this paper. Section 2.1 recalls generalized local Calderón–Zygmund operators. We collect useful inequalities in variable exponent Lebesgue spaces in Section 2.2. Section 2.3 is devoted to relations in a certain pair of compactly supported smooth functions. We summarize the properties of the Gamma functions in Section 2.4. We recall the definition of the complex interpolation functor in Section 2.5. To consider the complex interpolation of weighted Sobolev spaces with variable exponents, we generalize the Calderón product to vector-valued case in Section 2.6.

2.1 Generalized local Calderón–Zygmund operators

The proof of Theorem 1.3 uses the boundedness of generalized local Calderón–Zygmund operators. Given a function space X, $X_{\rm c}$ denotes the set of all functions $f \in X$ with compact support. An L^2-bounded linear operator T is a (generalized) local Calderón–Zygmund operator (with the kernel K), if it satisfies the following conditions:

(1) There exists $K \in L^1_{\rm loc}(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) \,:\, x \in \mathbb{R}^n\})$ such that, for all $f \in L^2_{\rm c}$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) {\rm d}y \text{ for almost all } x \notin {\rm supp}(f). \tag{2.1}$$

(2) There exist constants γ_0, $D_1 = D_1(T)$ and $D_2 = D_2(T)$ such that the two conditions below hold for all $x, y, z \in \mathbb{R}^n$:

(i) Local size condition:

$$|K(x,y)| \le D_1 |x-y|^{-n} \chi_{[-\gamma_0,\gamma_0]^n}(x-y) \tag{2.2}$$

if $x \ne y$,

(*ii*) Hörmander's condition:

$$|K(x,z)-K(y,z)|+|K(z,x)-K(z,y)|\le D_2\frac{|x-y|}{|x-z|^{n+1}} \qquad (2.3)$$

if $0<2|x-y|<|z-x|$.

This is analogous to generalized singular integral operators, which requires

$$|K(x,y)|\le D_1|x-y|^{-n} \qquad (2.4)$$

instead of (2.2) if $x\neq y$. In [26], we showed that all generalized local singular integral operators initially defined on L^2 can be extended to a bounded linear operator on $L^{p(\cdot)}(w)$ for any $p(\cdot)\in\mathcal{P}\cap \mathrm{LH}_0\cap\mathrm{LH}_\infty$ and $w\in A^{\mathrm{loc}}_{p(\cdot)}$.

Proposition 2.1. *Suppose that $p(\cdot)\in\mathcal{P}\cap \mathrm{LH}_0\cap\mathrm{LH}_\infty$. Let T be a generalized local singular integral operator, and $w\in A^{\mathrm{loc}}_{p(\cdot)}$. Then T is bounded on $L^{p(\cdot)}$ with the norm estimate*

$$\|T\|_{L^{p(\cdot)}(w)\to L^{p(\cdot)}(w)}\lesssim\|T\|_{L^2\to L^2}+D_1(T)+D_2(T).$$

In addition to the generalized local singular integral operators considered in [26], we need to consider another type of generalized local singular integral operators. To track the size of the operator norms, we recall [41, Theorem 1.67] with $K=n+2$.

Proposition 2.2 (Hörmander–Mikhlin multiplier theorem)**.** *Choose $\psi\in C^\infty_{\rm c}(Q(4)\setminus Q(1))$ so that $\sum_{j=-\infty}^{\infty}\psi_j\equiv\chi_{\mathbb{R}^n\setminus\{0\}}$. Assume that $m\in C^{n+2}(\mathbb{R}^n\setminus\{0\})\cap L^\infty$ satisfies*

$$M_\alpha\equiv\sup_{\xi\in\mathbb{R}^n\setminus\{0\}}|\xi|^{|\alpha|}|D^\alpha m(\xi)|<\infty \qquad (2.5)$$

for all $|\alpha|\le n+2$.

(1) *The function $\mathcal{K}\equiv\sum_{j=-\infty}^{\infty}\mathcal{F}^{-1}[\psi(2^{-j}\cdot)m]$ is independent of the choice of ψ and all the partial derivatives up to order 1 converge uniformly over any compact set $\mathbb{R}^n\setminus\{0\}$. Furthermore, for $j=1,2,\ldots,n$,*

$$|\mathcal{K}(x)|\lesssim|x|^{-n},\quad |\partial_j\mathcal{K}(x)|\lesssim|x|^{-n-1}\quad (x\in\mathbb{R}^n\setminus\{0\}), \qquad (2.6)$$

where the implicit constants depend on $\{M_\alpha\}_{\alpha\in\mathbb{N}_0{}^n,\,|\alpha|\le n+2}$, and we can write

$$\int_{\mathbb{R}^n}\mathcal{K}(x)\varphi(x)\mathrm{d}x=\int_{\mathbb{R}^n}\mathcal{F}^{-1}m(x)\varphi(x)\mathrm{d}x \qquad (2.7)$$

for all $\varphi\in C^\infty_{\rm c}(\mathbb{R}^n\setminus\{0\})$.

(2) *Let $1<p<\infty$. Then the L^2-bounded operator*

$$f\mapsto m(D)f\equiv\mathcal{F}^{-1}[m\cdot\mathcal{F}f]\quad(f\in L^2)$$

extends to an L^p-bounded linear operator naturally. More precisely, if $f\in L^2$ is compactly supported, then

$$m(D)f(x)=(2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}\mathcal{K}(y)f(x-y)\mathrm{d}y \qquad (2.8)$$

for almost all $x\notin\mathrm{supp}(f)$.

2.2 Observations on local weighted Lebesgue spaces with variable exponents

Here, we prove Hölder's inequality for weighted variable exponent Lebesgue spaces.

Lemma 2.3. *Let* $w_0 \in A^{\rm loc}_{p_0(\cdot)}$ *and* $w_1 \in A^{\rm loc}_{p_1(\cdot)}$ *with* $p_0(\cdot), p_1(\cdot) \in \mathcal{P} \cap {\rm LH}_0 \cap {\rm LH}_\infty$. *Define a weight* w *and a variable exponent* $p(\cdot)$ *by*

$$w^{\frac{1}{p(\cdot)}} = w_0{}^{\frac{1-\theta}{p_0(\cdot)}} w_1{}^{\frac{\theta}{p_1(\cdot)}}, \quad \frac{1}{p(\cdot)} = \frac{1-\theta}{p_0(\cdot)} + \frac{\theta}{p_1(\cdot)}.$$

Then $w \in A^{\rm loc}_{p(\cdot)}$ *with the estimate*

$$[w]_{A^{\rm loc}_{p(\cdot)}} \le 2([w_0]_{A^{\rm loc}_{p_0(\cdot)}})^{1-\theta}([w_1]_{A^{\rm loc}_{p_1(\cdot)}})^{\theta}. \tag{2.9}$$

Proof. Let Q be a cube with $|Q| \le 1$. By the definition of w,

$$\|\chi_Q\|_{L^{p(\cdot)}(w)} = \|w^{\frac{1}{p(\cdot)}}\chi_Q\|_{L^{p(\cdot)}} = \|w_0{}^{\frac{1-\theta}{p_0(\cdot)}} w_1{}^{\frac{\theta}{p_1(\cdot)}}\chi_Q\|_{L^{p(\cdot)}}$$

Using Hölder's inequality, we have

$$\|\chi_Q\|_{L^{p(\cdot)}(w)} \le 2(\|w_0{}^{\frac{1}{p_0(\cdot)}}\chi_Q\|_{L^{p_0(\cdot)}})^{1-\theta}(\|w_1{}^{\frac{1}{p_1(\cdot)}}\chi_Q\|_{L^{p_1(\cdot)}})^{\theta}.$$

If we take the supremum over Q, then we obtain $w \in A^{\rm loc}_{p(\cdot)}$ with (2.9). □

We use the localization principle among the other properties of variable exponent Lebesgue spaces.

Proposition 2.4. [19, Theorem 2.4] *Assume that* $p(\cdot) \in \mathcal{P} \cap {\rm LH}_\infty$. *Then for any* $f \in \mathcal{M}$,

$$\left\{\sum_{m \in \mathbb{Z}^n} (\|f\chi_{m+[0,1]^n}\|_{L^{p(\cdot)}})^{p_\infty}\right\}^{\frac{1}{p_\infty}} \sim \|f\|_{L^{p(\cdot)}}.$$

We move on to the maximal inequality. In [37], the maximal inequality (Proposition 2.5) and its vector-valued extension (Proposition 2.7) were obtained.

Proposition 2.5. [37, Theorem 1.2] *Suppose that* $p(\cdot) \in \mathcal{P} \cap {\rm LH}_0 \cap {\rm LH}_\infty$. *Then given any* $w \in A^{\rm loc}_{p(\cdot)}$, *there exists a constant* $D > 0$ *such that* $\|M^{\rm loc}f\|_{L^{p(\cdot)}(w)} \le D\|f\|_{L^{p(\cdot)}(w)}$ *for all* $f \in L^{p(\cdot)}(w)$.

This result corresponds to the ones obtained in [5, 7].

Corollary 2.6. *Let* $B > 2n + 6\log D$ *and write* $K(x) \equiv \exp(-B|x|)$. *Suppose that* $p(\cdot) \in \mathcal{P} \cap {\rm LH}_0 \cap {\rm LH}_\infty$. *Then given any* $w \in A^{\rm loc}_{p(\cdot)}$, *there exists a constant* $C > 0$ *such that* $\|K * f\|_{L^{p(\cdot)}(w)} \le C\|f\|_{L^{p(\cdot)}(w)}$ *for all* $f \in L^{p(\cdot)}(w)$.

We follow the idea of [39, Lemma 2.11] for the proof.

Proof. Fix $x \in \mathbb{R}^n$. We decompose

$$K(x) \le K(x)\chi_{[-1,1]^n}(x) + \sum_{j=1}^{\infty} K(x)\chi_{\left[-\frac{1}{2}j,\frac{1}{2}j\right]^n \setminus \left[-\frac{1}{2}(j-1),\frac{1}{2}(j-1)\right]^n}(x).$$

Since

$$\overbrace{\chi_{[0,1]^n} * \chi_{[0,1]^n} * \cdots * \chi_{[0,1]^n}}^{j \text{ times}}(x) \ge 2^{-n}\chi_{\left[-\frac{1}{2}(j-1),\frac{1}{2}(j-1)\right]^n}(x)$$

for each $j \in \mathbb{N}$, we obtain

$$\begin{aligned} K(x) &\le \chi_{[-1,1]^n}(x) + \sum_{j=1}^{\infty} 2^{jn} \exp\left(-\frac{1}{2}Bj\right) \overbrace{\chi_{[-1,1]^n} * \chi_{[-1,1]^n} * \cdots * \chi_{[-1,1]^n}}^{j \text{ times}}(x) \\ &\le \chi_{[-1,1]^n}(x) + \sum_{j=1}^{\infty} \exp\left(\left(n - \frac{1}{2}B\right) j\right) \overbrace{\chi_{[-1,1]^n} * \chi_{[-1,1]^n} * \cdots * \chi_{[-1,1]^n}}^{j \text{ times}}(x). \end{aligned}$$

As a result,

$$|K * f(x)| \le M^{\rm loc}f(x) + \sum_{j=1}^{\infty} \exp\left(\left(n - \frac{1}{2}B\right) j\right) (M^{\rm loc})^{3j} f(x) \quad (x \in \mathbb{R}^n).$$

Denote by D the constant in Proposition 2.5. Using the triangle inequality for $L^{p(\cdot)}(w)$, we obtain

$$\begin{aligned} \|K * f\|_{L^{p(\cdot)}(w)} &\le \|M^{\rm loc}f\|_{L^{p(\cdot)}(w)} + \sum_{j=1}^{\infty} \exp\left(\left(n - \frac{1}{2}B\right) j\right) \|(M^{\rm loc})^{3j} f\|_{L^{p(\cdot)}(w)} \\ &\le D\|f\|_{L^{p(\cdot)}(w)} + \sum_{j=1}^{\infty} D^{3j} \exp\left(\left(n - \frac{1}{2}B\right) j\right) \|f\|_{L^{p(\cdot)}(w)}. \end{aligned}$$

Since $B > 2n + 6\log D$, the series converges. Thus, we obtain the desired result. □

By adapting the extrapolation result in [11] to our local weight setting, the second and fourth authors obtained the following vector-valued inequality.

Proposition 2.7. [37, Theorem 1.11] *Suppose that $p(\cdot) \in \mathcal{P} \cap {\rm LH}_0 \cap {\rm LH}_\infty$. Additionally, let $w \in A^{\rm loc}_{p(\cdot)}$ and $1 < q \le \infty$. Then for any sequence $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}$,*

$$\left\| \left(\sum_{j=1}^{\infty} \left[M^{\rm loc} f_j\right]^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}(w)} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}(w)}. \tag{2.10}$$

A natural modification is made when $q = \infty$.

As an application of Lemma 2.7, we can prove a vector-valued inequality which extends [40, Theorem 1.3].

Lemma 2.8. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Additionally, let $w \in A_{p(\cdot)}^{\mathrm{loc}}$. If $q \in (\max(2, p_+), \infty)$, then for all $\{f_j\}_{j=1}^\infty \subset L^{p(\cdot)}(w) \cap L^q$ such that each f_j is supported on a cube Q_j,*

$$\left\| \left(\sum_{j=1}^\infty |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)} \lesssim \left\| \left(\sum_{j=1}^\infty \left(\frac{\|f_j\|_{L^q}}{|Q_j|^{\frac{1}{q}}} \right)^2 \chi_{Q_j} \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)}.$$

Proof. We write $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$ as before. We only have to show that

$$\int_{\mathbb{R}^n} \sum_{j=1}^\infty |f_j(x) g_j(x)| \mathrm{d}x \lesssim \left\| \left(\sum_{j=1}^\infty \left(\frac{\|f_j\|_{L^q}}{|Q_j|^{\frac{1}{q}}} \right)^2 \chi_{Q_j} \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)} \left\| \left(\sum_{j=1}^\infty |g_j|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'(\cdot)}(\sigma)}$$

for all $\{g_j\}_{j=1}^\infty \subset L^{p'(\cdot)}(\sigma)$. Since f_j is supported on Q_j,

$$\begin{aligned}
\int_{\mathbb{R}^n} \sum_{j=1}^\infty |f_j(x) g_j(x)| \mathrm{d}x &\le \sum_{j=1}^\infty \|f_j\|_{L^q} \|\chi_{Q_j} g_j\|_{L^{q'}} \\
&\le \int_{\mathbb{R}^n} \sum_{j=1}^\infty \frac{\|f_j\|_{L^q} \|\chi_Q g_j\|_{L^{q'}}}{|Q_j|} \chi_{Q_j}(x) \mathrm{d}x \\
&\le \int_{\mathbb{R}^n} \sum_{j=1}^\infty \frac{\|f_j\|_{L^q}}{|Q_j|^{\frac{1}{q}}} M[|g_j|^{q'}](x)^{\frac{1}{q'}} \chi_{Q_j}(x) \mathrm{d}x.
\end{aligned}$$

By the use of Hölder's inequality and Cauchy–Schwarz's inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{j=1}^\infty |f_j(x) g_j(x)| \mathrm{d}x \\
&\le \left\| \left(\sum_{j=1}^\infty \left(\frac{\|f_j\|_{L^q}}{|Q_j|^{\frac{1}{q}}} \right)^2 \chi_{Q_j} \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)} \left\| \left(\sum_{j=1}^\infty M[|g_j|^{q'}]^{\frac{2}{q'}} \right)^{\frac{1}{2}} \right\|_{L^{p'(\cdot)}(\sigma)}.
\end{aligned}$$

Since $q' < \min(2, (p'(\cdot)_-))$, we can use the Feffereman–Stein vector-valued inequality to give the desired conclusion. □

Let $E \subset \mathbb{R}^n$ be a measurable set. The set C_c^∞ consists of all infinitely differentiable functions defined on $\mathbb{R}^n$ whose support is compact and contained in E. Once we obtain the boundedness of M^{loc}, the density of C_c^∞ as easily be obtained.

Corollary 2.9. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, and let $w \in A_{p(\cdot)}^{\mathrm{loc}}$. Then C_c^∞ is dense in $L^{p(\cdot),s}(w)$. In particular, $L^{p(\cdot),s}(w)$ is separable.*

Remark that this is an analogue of [25, Theorem 2.10] and [34].

Proof. Owing to the Lebesgue convergence theorem obtained in [24], we only have to approximate functions in $L^\infty_{\rm c}$, which is the set of all essentially bounded functions with compact support. Let $f \in L^\infty_{\rm c}$. Choose a non-negative function $\tau \in C^\infty_{\rm c}$ with L^1-norm 1 supported on $[-1,1]^n$ and consider $j^n\tau(j\cdot)*f$ for each $j \in \mathbb{N}$. Then we know that $|j^n\tau(j\cdot)*f| \lesssim M^{\rm loc}f$ for all $j \in \mathbb{N}$. We also know that $j^n\tau(j\cdot)*f(x) \to f(x)$ for almost all $x \in \mathbb{R}^n$ as $j \to \infty$ by the Lebesgue differentiation theorem. Thus, once again we can use the Lebesgue convergence theorem mentioned above to have $j^n\tau(j\cdot)*f \to f$ as $j \to \infty$. □

2.3 Functional equation and its application

We will use the following functional relation obtained in [41, Theorem 1.40].

Lemma 2.10. *Fix $L \in \mathbb{N}$. Then there exist real-valued functions $\Phi^{(L)}, \Psi^{(L)} \in C^\infty_{\rm c}$ such that*

$$\int_{\mathbb{R}^n} \Phi^{(L)}(x){\rm d}x = 1, \quad 2^n\Phi^{(L)}(2\cdot) - \Phi^{(L)} = \Delta^L\Psi^{(L)}, \tag{2.11}$$

where Δ denotes the Laplacian.

It can be arranged that $\Phi^{(L)}$ and $\Psi^{(L)}$ are even since

$$\int_{\mathbb{R}^n} \Phi^{(L)}(-x){\rm d}x = 1, \quad 2^n\Phi^{(L)}(-2\cdot) - \Phi^{(L)}(-\cdot) = \Delta^L\Psi^{(L)}(-\cdot).$$

We set

$$\Theta^{(L)} = \Phi^{(L)} * \Phi^{(L)}, \quad \Gamma^{(L)} = \Phi^{(L)} + 2^n\Phi^{(L)}(2\cdot). \tag{2.12}$$

It is noteworthy that $a^n f * g(a\cdot) = [a^n f(a\cdot)] * [a^n g(a\cdot)]$ for $f, g \in L^1$ and $a > 0$. In fact,

$$\begin{aligned}[a^n f(a\cdot)] * [a^n g(a\cdot)](x) &= a^{2n}\int_{\mathbb{R}^n} f(ax-ay)g(ay)dy \\ &= a^n\int_{\mathbb{R}^n} f(ax-y)g(y)dy\end{aligned}$$

by a change of variables.

Here and below, we suppose that L is a multiple of 4. A direct consequence of Lemma 2.10 is the following equalities:

Lemma 2.11. *Let $L \in \mathbb{N}$. Assume that $\Theta^{(L)}$, $\Gamma^{(L)}$ and $\Psi^{(L)}$ are as above. Then,*

(1) $2^n\Theta^{(L)}(2\cdot) - \Theta^{(L)} = \Delta^{L/2}\Gamma^{(L)} * \Delta^{L/2}\Psi^{(L)}$.

(2) $2^{Jn}\Phi^{(L)}(2^J\cdot) * 2^{Jn}\Phi^{(L)}(2^J\cdot) + \sum_{j=J}^{\infty} 4^{jn}\Gamma^{(L)}(2^j\cdot) * \Delta^L\Psi^{(L)}(2^j\cdot) = \delta$ *in the sense of $\mathcal{D}'$.*

Proof.

(1) We calculate

$$\begin{aligned}2^n\Theta^{(L)}(2\cdot)-\Theta^{(L)}&=2^n\Phi^{(L)}(2\cdot)*2^n\Phi^{(L)}(2\cdot)-\Phi^{(L)}*\Phi^{(L)}\\&=(2^n\Phi^{(L)}(2\cdot)-\Phi^{(L)})*(2^n\Phi^{(L)}(2\cdot)+\Phi^{(L)})\\&=\Gamma^{(L)}*\Delta^L\Psi^{(L)}\\&=\Delta^{L/2}\Gamma^{(L)}*\Delta^{L/2}\Psi^{(L)}.\end{aligned}$$

(2) From (1), we have

$$\begin{aligned}&2^{Jn}\Phi^{(L)}(2^J\cdot)*2^{Jn}\Phi^{(L)}(2^J\cdot)+\sum_{j=J}^{M}4^{jn}\Gamma^{(L)}(2^j\cdot)*\Delta^L\Psi^{(L)}(2^j\cdot)\\&=2^{(M+1)n}\Theta^{(L)}(2^{M+1}\cdot).\end{aligned}$$

Since $\{2^{jn}\Theta^{(L)}(2^j\cdot)\}_{j=1}^{\infty}$ is an approximation to identity in $\mathcal{D}'$,

$$2^{Jn}\Phi^{(L)}(2^J\cdot)*2^{Jn}\Phi^{(L)}(2^J\cdot)+\sum_{j=J}^{\infty}4^{jn}\Gamma^{(L)}(2^j\cdot)*\Delta^L\Psi^{(L)}(2^j\cdot)=\delta$$

in the sense of $\mathcal{D}'$.

□

The next estimate is well known. For example, see [18].

Lemma 2.12 (Grafakos [18, p. 596]). *Let $\mu,\nu\in\mathbb{R}$, $M,N>0$, and $L\in\mathbb{N}_0$ satisfy $\nu\ge\mu$ and $N>M+L+n$. Let x_μ and x_ν be fixed points. Suppose that $\phi_\mu\in C^L(\mathbb{R}^n)$ satisfies*

$$|D^\alpha\phi_\mu(x)|\le A_\alpha\frac{2^{\mu(n+L)}}{(1+2^\mu|x-x_\mu|)^M}\quad \textit{for all } |\alpha|=L.$$

Furthermore, suppose that ϕ_ν is a measurable function satisfying

$$\int_{\mathbb{R}^n}\phi_\nu(x)(x-x_\nu)^\beta\mathrm{d}x=0\quad\textit{for all } |\beta|\le L-1,$$

and

$$|\phi_\nu(x)|\le B\frac{2^{\nu n}}{(1+2^\nu|x-x_\nu|)^N},$$

where the former condition is supposed to be vacuous when $L=0$. Then it holds

$$\left|\int_{\mathbb{R}^n}\phi_\mu(x)\phi_\nu(x)\mathrm{d}x\right|\le C_{A_\alpha,B,L,M,N}\,2^{\mu n-(\nu-\mu)L}(1+2^\mu|x_\mu-x_\nu|)^{-M}$$

with a constant $C_{A_\alpha,B,L,M,N}$ taken as

$$C_{A_\alpha,B,L,M,N}\equiv B\left(\sum_{|\alpha|=L}\frac{A_\alpha}{\alpha!}\right)\frac{\omega_n(N-M-L)}{N-M-L-n},$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

This estimate is useful when estimating the coupling of functions.

Lemma 2.13. *Let $k, k_0 \in \mathbb{Z}$, and $l, l_0 \in \{1, 2, \dots, 2^n - 1\}$. Additionally, let α be a multiindex with $|\alpha| = s$.*

(1) *Unless $Q^*_{j_0,k_0}$ and $Q^*_{J,k}$ intersect, $\langle \varphi_{J,k}, D^\alpha[\psi^{l_0}_{j_0,k_0}]\rangle = 0$.*

(2) *Unless $Q^*_{j_0,k_0}$ and $Q^*_{j,k}$ intersect, $\langle \psi^l_{j,k}, D^\alpha[\psi^{l_0}_{j_0,k_0}]\rangle = 0$.*

(3) *$\left|\langle \varphi_{J,k}, D^\alpha[\psi^{l_0}_{j_0,k_0}]\rangle\right| \lesssim 2^{Js-\frac{(J+j_0)n}{2}+Jn-(j_0-J)}$ for any integers $j_0 \geq J$ such that $|2^{-J}k - 2^{-j_0}k_0| \leq 2^{-J+1}(2N-1)$.*

(4) *$\left|\langle \psi^l_{j,k}, D^\alpha[\psi^{l_0}_{j_0,k_0}]\rangle\right| \lesssim 2^{js-\frac{(j_0+j)n}{2}+\min(j,j_0)n-|j-j_0|}$ for any integers $j_0, j \geq J$ such that $|2^{-j_0}k_0 - 2^{-j}k| \leq 2^{-\min(j,j_0)+1}(2N-1)$.*

(5) *There exists $\kappa > 1$ such that*
$$|2^{j_0 n}\Delta^{L/2}\Psi^{(L)}(2^{j_0}\cdot) * \psi^l_{j,k}| \lesssim 2^{-(s+1)|j-j_0|+\min(j,j_0)n}\chi_{Q(2^{-j}k,\kappa 2^{-\min(j,j_0)n})}.$$

Proof.

(1) This follows from the size of the support. See (1.4).

(2) This once again follows from the size of the support. See (1.4).

(3) Thanks to Lemma 2.12 with $\nu = j_0$, $\mu = J$, $L = s+1$, and
$$\phi_\mu = \phi_J = 2^{\frac{Jn}{2}}\varphi_{J,K}, \quad \phi_\nu = 2^{\frac{j_0 n}{2}}(D^\alpha\psi^{l_0})_{j_0,k_0},$$
we obtain
$$\begin{aligned}\left|\langle \varphi_{J,k}, D^\alpha[\psi^{l_0}_{j_0,k_0}]\rangle\right| &\lesssim 2^{j_0 s-\frac{(J+j_0)n}{2}+Jn-(s+1)(j_0-J)}(1+2^J|2^{-J}k-2^{-j_0}k_0|)^{-N}\\ &\lesssim 2^{j_0 s-\frac{(J+j_0)n}{2}+Jn-(s+1)(j_0-J)}\\ &\leq 2^{Js-\frac{(J+j_0)n}{2}+Jn-(j_0-J)}\end{aligned}$$
as long as $|2^{-J}k - 2^{-j_0}k_0| \leq 2^{-J+1}(2N-1)$.

(4) Likewise, due to Lemma 2.12 with $\nu = j_0$, $\mu = j$, $L = s+1$, and
$$\phi_\mu = \phi_j = 2^{\frac{jn}{2}}\psi^l_{j,k}, \quad \phi_\nu = 2^{\frac{j_0 n}{2}}(D^\alpha\psi^{l_0})_{j_0,k_0}$$
when $j \leq j_0$, and thanks to Lemma 2.12 with $\nu = j$, $\mu = j_0$ $L = s+1$, and
$$\phi_\mu = 2^{\frac{j_0 n}{2}}(D^\alpha\psi^{l_0})_{j_0,k_0}, \quad \phi_\nu = \phi_j = 2^{\frac{jn}{2}}\psi^l_{j,k}$$
when $j \geq j_0$, we obtain
$$\begin{aligned}&\left|\langle \psi^l_{j,k}, D^\alpha[\psi^{l_0}_{j_0,k_0}]\rangle\right|\\ &\lesssim 2^{j_0 s-\frac{(j_0+j)n}{2}+\min(j,j_0)n-(s+1)|j-j_0|}(1+2^{\min(j,j_0)}|2^{-j_0}k_0-2^{-j}k|)^{-N}\\ &\lesssim 2^{j_0 s-\frac{(j_0+j)n}{2}+\min(j,j_0)n-(s+1)|j-j_0|}\\ &\leq 2^{js-\frac{(j_0+j)n}{2}+\min(j,j_0)n-|j-j_0|}\end{aligned}$$
as long as $|2^{-j_0}k_0 - 2^{-j}k| \leq 2^{-\min(j,j_0)+1}(2N-1)$.

(5) Use Lemma 2.12 to argue similarly.

□

2.4 Gamma function and Bessel potential operators

Here, we discuss the boundedness property of $(1-t^2\Delta)^{-a}$ with $0<t\ll 1$ and $a\in\mathbb{C}$ with $\mathrm{Re}(a)\ge 0$. To this end, recall the property of the Gamma function Γ given by

$$\Gamma(z)\equiv\int_0^\infty s^{z-1}\mathrm{e}^{-s}\mathrm{d}s \quad (\mathrm{Re}(z)>0).$$

It is noteworthy that

$$\frac{1}{\Gamma(z)}=z\mathrm{e}^{\gamma z}\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)\mathrm{e}^{-\frac{z}{m}} \quad (\mathrm{Re}(z)>0),$$

where

$$\gamma=\lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right).$$

See [18, Section1.2.2] for more details. Considering that,

$$\lim_{a\to 0}\frac{e^{2a}-(1+a)^2}{a^2}=1,$$

we set

$$\alpha\equiv\sup_{a\in[0,1]}a^{-2}|(1+a)^2\mathrm{e}^{-2a}-1|<\infty.$$

Let $m\in\mathbb{N}$ be fixed. Since

$$(1+a)^2\mathrm{e}^{-2a}\le 1+\alpha a^2, \quad \sqrt{1+a}\le 1+\frac{a}{2}\le\exp\left(\frac{a}{2}\right)$$

for all $a\in[0,1]$,

$$\begin{aligned}\sqrt{\left(1+\frac{s}{m}\right)^2+\frac{t^2}{m^2}}\mathrm{e}^{-\frac{s}{m}}&\le\sqrt{\left(1+\frac{s}{m}\right)^2\mathrm{e}^{-\frac{2s}{m}}+\frac{t^2}{m^2}}\\&\le 1+\alpha\frac{s^2}{2n^2}+\frac{t^2}{2n^2}\\&\le\exp\left(\alpha\frac{s^2}{2n^2}+\frac{t^2}{2n^2}\right)\end{aligned}$$

for all $s\in[0,1]$ and $t\in\mathbb{R}$. Consequently,

$$\frac{1}{|\Gamma(s+it)|}\le(1+|t|)\mathrm{e}^{\gamma}\exp\left(\frac{\alpha\pi^2}{12}+\frac{\pi^2t^2}{12}\right)$$

for all $s\in[0,1]$ and $t\in\mathbb{R}$. Since $|e^{(s+it)^2}|=e^{s^2-t^2}$, we conclude

$$\lim_{t\to\pm\infty}\sup_{s\in[0,1]}\left|\frac{\mathrm{e}^{(s+it)^2}}{\Gamma(s+it)}\right|=0. \tag{2.13}$$

Next, we consider Bessel potential operators. We define the inverse Fourier transform by

$$\mathcal{F}^{-1}f(x) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi.$$

for $f \in L^1$. Recall that

$$(1-t^2\Delta)^{-a}f = \mathcal{F}^{-1}\left[\frac{1}{(1+t^2|\xi|^2)^a}\right] * f$$

for $f \in L^2$. Thus, the integral kernel K_a of $(1-t^2\Delta)^{-a}$ is given formally by the following formula

$$K_a(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(1+t^2|\xi|^2)^{-a}\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi \quad (x \in \mathbb{R}^n). \tag{2.14}$$

We note that

$$(1+t^2|\xi|^2)^{-a} = \frac{1}{\Gamma(a)}\int_0^\infty s^{a-1}\mathrm{e}^{-s(1+t^2|\xi|^2)}\mathrm{d}s.$$

Inserting this expression into the right-hand side above gives

$$K_a(x) = \frac{1}{(2\pi)^{\frac{n}{2}}\Gamma(a)}\int_{\mathbb{R}^n}\left(\int_0^\infty s^{a-1}\mathrm{e}^{-s(1+t^2|\xi|^2)}\mathrm{d}s\right)\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi.$$

Since

$$\int_{\mathbb{R}^n}\mathrm{e}^{-st^2|\xi|^2}\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi = \frac{1}{t^n\sqrt{s^n}}\int_{\mathbb{R}^n}\mathrm{e}^{-|\xi|^2}\mathrm{e}^{i\frac{x}{t\sqrt{s}}\cdot\xi}\mathrm{d}\xi = \frac{\sqrt{\pi^n}}{t^n\sqrt{s^n}}\mathrm{e}^{-\frac{|x|^2}{4t^2s}},$$

we obtain

$$K_a(x) = \frac{1}{2^{\frac{n}{2}}t^n\Gamma(a)}\int_0^\infty s^{a-1-\frac{n}{2}}\mathrm{e}^{-s-\frac{|x|^2}{4t^2s}}\mathrm{d}s. \tag{2.15}$$

Let $|x| \geq 1$. Then

$$\mathrm{e}^{-s-\frac{|x|^2}{4t^2s}} \leq \mathrm{e}^{-\frac{1}{2}s-\frac{1}{2}s-\frac{|x|^2}{8t^2s}-\frac{1}{8t^2s}} \leq \mathrm{e}^{-\frac{1}{2}s-\frac{|x|}{2t}-\frac{1}{8t^2s}}.$$

Consequently,

$$|K_a(x)| \leq \frac{\mathrm{e}^{-\frac{|x|}{2t}}}{2^{\frac{n}{2}}t^n|\Gamma(a)|}\int_0^\infty s^{\mathrm{Re}(a)-1-\frac{n}{2}}\mathrm{e}^{-\frac{1}{2}s-\frac{1}{8t^2s}}\mathrm{d}s \quad (|x| \geq 1) \tag{2.16}$$

and

$$\left|\frac{\partial}{\partial a}K_a(x)\right| \leq C_a\frac{\mathrm{e}^{-\frac{|x|}{2t}}}{2^{\frac{n}{2}}t^n}\int_0^\infty s^{\mathrm{Re}(a)-1-\frac{n}{2}}\mathrm{e}^{-\frac{1}{2}s-\frac{1}{8t^2s}}\log(2+s)\mathrm{d}s. \tag{2.17}$$

Lemma 2.14. *Fix $0 < t \ll 1$.*

1. *Let α be a multi-index such that $|\alpha| > n$. Then $|x^\alpha K_a(x)| \lesssim (1+|a|)^{|\alpha|}$ for all $a \in \mathbb{C}$ with $\mathrm{Re}(a) \geq 0$.*

2. *Let α be a multi-index such that $|\alpha| > n$. Then $|x^\alpha \nabla K_a(x)| \lesssim (1+|a|)^{|\alpha|-1}$ for all $a \in \mathbb{C}$ with $\mathrm{Re}(a) \ge 0$.*

Proof.

1. Thanks to (2.14), we have

$$(ix)^\alpha K_a(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (1+t^2|\xi|^2)^{-a} \partial_\xi^\alpha [\mathrm{e}^{ix\cdot\xi}] \mathrm{d}\xi.$$

   If we integrate by parts α times, then we obtain the desired result.

2. Similar to above.

□

Considering the pointwise estimates (2.16) and (2.17), we prove the boundedness of the operator $(1-t^2\Delta)^{-a}$ in $L^{p(\cdot)}(w)$.

Theorem 2.15. *Fix $0 < t \ll 1$. Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, and let $w \in A_{p(\cdot)}^{\mathrm{loc}}$. Let $a \in \mathbb{C}$ satisfy $\mathrm{Re}(a) \ge 0$. Then the operator*

$$f \in L^2 \mapsto (1-t^2\Delta)^{-a} f \in L^2$$

extends to a bounded linear operator on $L^{p(\cdot)}(w)$ once we restrict it to $L^{p(\cdot)}(w) \cap L^2$. More precisely,

$$\|(1-t^2\Delta)^{-a} f\|_{L^{p(\cdot)}(w)} \lesssim \left((1+|a|)^{n+2} + \frac{1}{|\Gamma(a)|} \right) \|f\|_{L^{p(\cdot)}(w)}$$

for all $f \in L^{p(\cdot)}(w)$ with the implicit constant independent of a and f.

Proof. Thanks to Corollary 2.9, $L^{p(\cdot)}(w) \cap L^2$ is dense in $L^{p(\cdot)}(w)$. Thus, we can assume that $f \in L^{p(\cdot)}(w) \cap L^2$. Then

$$\|(1-t^2\Delta)^{-a} f\|_{L^{p(\cdot)}(w)} \lesssim \left(\sum_{m \in \mathbb{Z}^n} (\|\chi_{Q_{0,m}} (1-t^2\Delta)^{-a} f\|_{L^{p(\cdot)}(w)})^{p_\infty} \right)^{\frac{1}{p_\infty}}$$

by the localization principle, Proposition 2.4. We fix m for the time being and estimate

$$\begin{aligned} &\|\chi_{Q_{0,m}} (1-t^2\Delta)^{-a} f\|_{L^{p(\cdot)}(w)} \\ &\le \|\chi_{Q_{0,m}} (1-t^2\Delta)^{-a} [\chi_{3Q_{0,m}} f]\|_{L^{p(\cdot)}(w)} + \|\chi_{Q_{0,m}} (1-t^2\Delta)^{-a} [\chi_{\mathbb{R}^n \setminus 3Q_{0,m}} f]\|_{L^{p(\cdot)}(w)}. \end{aligned} \tag{2.18}$$

The first term in (2.18) can be controlled by boundedness of the generalized local singular integral operators after truncating the integral kernel suitably. If we let

$$L_a f(x) = (1-t^2\Delta)^{-a} [\tau(x-\cdot) f](x) \tag{2.19}$$

for some bump function $\tau \in C^\infty$ satisfying $\chi_{Q(4n)} \le \tau \le \chi_{Q(5n)}$, then

$$\begin{aligned}\|\chi_{Q_{0,m}}(1-t^2\Delta)^{-a}[\chi_{3Q_{0,m}}f]\|_{L^{p(\cdot)}(w)} &= \|\chi_{Q_{0,m}}L_a[\chi_{3Q_{0,m}}f]\|_{L^{p(\cdot)}(w)}\\ &\lesssim (1+|a|)^{n+2}\|\chi_{3Q_{0,m}}f\|_{L^{p(\cdot)}(w)},\end{aligned}$$

where we use the estimate $D_1(L_a)+D_2(L_a) \lesssim (1+|a|)^{n+2}$. See Lemma 2.14. For the second term, we employ the size estimate of the kernel K_a to give

$$\|\chi_{Q_{0,m}}(1-t^2\Delta)^{-a}[\chi_{\mathbb{R}^n\setminus 3Q_{0,m}}f]\|_{L^{p(\cdot)}(w)} \lesssim \frac{1}{|\Gamma(a)|}\left\|\chi_{Q_{0,m}}\exp\left(-\frac{|\cdot|}{2t}\right)*f\right\|_{L^{p(\cdot)}(w)}. \tag{2.20}$$

Thanks to Corollary 2.6, the operator $f \mapsto \exp\left(-\frac{|\cdot|}{2t}\right)*f$ is bounded on $L^{p(\cdot)}(w)$ as long as $0<t\ll 1$. Combining this fact with Proposition 2.4, (2.16) and (2.20), we can control the second term in (2.18). Thus, $(1-t^2\Delta)^{-a}$ can be extended to a bounded linear operator in $L^{p(\cdot)}(w)$. □

We harvest two corollaries.

Corollary 2.16. *Let $s>n+2$. Then the mapping $z \in S \mapsto (1-t^2\Delta)^{-z-s} \in B(L^{p(\cdot)}(w))$ is holomorphic.*

Proof. This follows from Proposition 2.1, (2.15), (2.16) and (2.17). □

Corollary 2.17. *Let $f \in L^{p(\cdot)}(w)$ and let $a>n$. Then $\lim_{N\to\infty}(1-N^{-1}\Delta)^{-a}f = f$ in the topology of $L^{p(\cdot)}(w)$.*

Proof. Since $a>n$, K_a is a bounded function. Thus, Corollary 2.6 can be used. □

2.5 Complex interpolation functors and Hirschman's lemma

Here we will review the definition of the complex interpolation functor together with Hirschman's lemma. Write $S \equiv \{z\in\mathbb{C} : 0<\mathrm{Re}\,(z)<1\}$ and $\overline{S} \equiv \{z\in\mathbb{C} : 0\le \mathrm{Re}\,(z)\le 1\}$. For $j=0,1$, set $j+i\mathbb{R} \equiv \{z\in\mathbb{C} : \mathrm{Re}(z)=j\}$. We define the complex interpolation functors as follows:

Definition 2.18 (Calderón's first complex interpolation space)**.** Suppose that $\overline{X}=(X_0,X_1)$ is a compatible couple of complex Banach spaces.

(1) The space $\mathcal{F}(X_0,X_1)$ is defined as the set of all functions $F:\bar{S}\to X_0+X_1$ such that

  (*i*) F is continuous on $\bar{S}$ and $\sup_{z\in\bar{S}}\|F(z)\|_{X_0+X_1}<\infty$,

  (*ii*) F is holomorphic on S,

  (*iii*) the function $t\in\mathbb{R}\mapsto F(j+it)\in Xj$ is bounded and continuous on $\mathbb{R}$ for each $j=0,1$.

The space $\mathcal{F}(X_0, X_1)$ is equipped with the norm

$$\|F\|_{\mathcal{F}(X_0,X_1)} \equiv \max_{j=0,1} \left(\sup_{z \in j+i\mathbb{R}} \|F(z)\|_{X_j} \right).$$

(2) Let $\theta \in (0,1)$. The *first complex interpolation space* $[X_0, X_1]_\theta$ with respect to $\overline{X} = (X_0, X_1)$ is defined as the set of all functions $f \in X_0 + X_1$ such that $f = F(\theta)$ for some $F \in \mathcal{F}(X_0, X_1)$. The norm on $[X_0, X_1]_\theta$ is defined by

$$\|f\|_{[X_0,X_1]_\theta} \equiv \inf\{\|F\|_{\mathcal{F}(X_0,X_1)} : f = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\}.$$

The space $[X_0, X_1]_\theta$ is also called the *Calderón's first complex interpolation space*, or the *lower complex interpolation space* of (X_0, X_1).

Example 2.19. Let $X_0 = X_1 = X$. It is useful to note that $\mathcal{F}(X_0, X_0)$ as the set of all the bounded and continuous functions $F : \bar{S} \to X$ such that F is holomorphic on S.

Recall vector-valued Hirschman's lemma.

Lemma 2.20 (Hirschman, [47])**.** *Let $\mathcal{F}$ be a Banach space. Let $F : S \to \mathcal{F}$ be analytic on the open strip S and continuous on its closure such that*

$$\sup_{z \in \overline{S}} \mathrm{e}^{-a|\mathrm{Im}(z)|} \log \|F(z)\|_{\mathcal{F}} \le A < \infty \tag{2.21}$$

for some fixed A and $a < \pi$. Then, for all $0 < \theta < 1$,

$$\log \|F(\theta)\|_{\mathcal{F}} \le \int_{\mathbb{R}} \{\log \|F(it)\|_{\mathcal{F}} \mu_0(t) + \log \|F(1+it)\|_{\mathcal{F}} \mu_1(t)\} \, \mathrm{d}t, \tag{2.22}$$

where μ_j $(j = 0, 1)$ are functions satisfying $\|\mu_0\|_{L^1(\mathbb{R})} = 1 - \theta$ and $\|\mu_1\|_{L^1(\mathbb{R})} = \theta$, respectively.

Let (X, Σ, μ) be a measure space. A Banach lattice $E(\mu)$ is a linear space of $\mathcal{M}$ which carries the structure of a Banach space such that the lattice property: $f \in E(\mu)$, $g \in \mathcal{M}$, $|g| \le |f| \implies g \in E(\mu)$ and $\|g\|_{E(\mu)} \le \|f\|_{E(\mu)}$ holds. The following general lemma seems known. We can find the statement in the case of $E(\mu) = L^p(\mu)$ can be found in [1]. Here for the sake of convenience for readers we give its proof.

Lemma 2.21. *Let $E(\mu)$, $E_0(\mu)$, and $E_1(\mu)$ be Banach lattices such that $E(\mu) \hookrightarrow E_0(\mu) \cap E_1(\mu)$. Let $g^0, g^1 \in E(\mu)$. Define $G(z) = |g^0|^{1-z}|g^1|^z$ for $z \in \bar{S}$. Then $G : \bar{S} \to E(\mu)$ is continuous and $G|S$ is analytic.*

Proof. We content ourselves with the proof of continuity. The proof of analyticity is similar. We write

$$S_+ = \left\{ z \in S \,:\, 0 \le \mathrm{Re}(z) < \frac{2}{3} \right\}, \quad S_- = \left\{ z \in S \,:\, 0 \le 1 - \mathrm{Re}(z) < \frac{2}{3} \right\}.$$

Suppose that $0 \le \mathrm{Re}(z), \mathrm{Re}(w) < \frac{2}{3}$. Fix $N \gg 1$. We observe

$$\begin{aligned} &G(z) - G(w) \\ &= (G(z) - G(w))\chi_{[0,N^{-1}]}\left(\frac{|g^0|}{|g^1|}\right) + (G(z) - G(w))\chi_{[0,N)}\left(\frac{|g^1|}{|g^0|}\right), \end{aligned}$$

By the triangle inequality,

$$\left\|(G(z) - G(w))\chi_{[0,N^{-1}]}\left(\frac{|g^0|}{|g^1|}\right)\right\|_{E(\mu)} \le \frac{2}{\sqrt[3]{N}}\|g_1\|_{E(\mu)}.$$

Meanwhile,

$$\begin{aligned} &(G(z) - G(w))\chi_{[0,N)}\left(\frac{|g^1|}{|g^0|}\right) \\ &= \int_w^z G(\tau)\log\frac{|g^1|}{|g^0|}\chi_{[0,N)}\left(\frac{|g^1|}{|g^0|}\right)\mathrm{d}\tau \\ &= \int_w^z G(\tau)\left(\frac{|g^1|}{|g^0|}\right)^{-\frac{1}{6}}\left(\frac{|g^1|}{|g^0|}\right)^{\frac{1}{6}}\log\frac{|g^1|}{|g^0|}\chi_{[0,N)}\left(\frac{|g^1|}{|g^0|}\right)\mathrm{d}\tau. \end{aligned}$$

Observe that

$$\sup_{t \in (0,N)} t^{\frac{1}{6}}|\log t| = N^{\frac{1}{6}}\log N.$$

Thus,

$$|G(z) - G(w))|\chi_{[0,N)}\left(\frac{|g^1|}{|g^0|}\right) \le |w - z|N^{\frac{1}{6}}\log N \times \int_0^1 \left|G\left((z - w)u - \frac{1}{6}\right)\right|\mathrm{d}u.$$

Thus,

$$|G(z) - G(w))|\chi_{[0,N)}\left(\frac{|g^1|}{|g^0|}\right) \le |w - z|N^{\frac{1}{6}}\log N \times (|g^0| + |g^1|).$$

Consequently,

$$\|G(z) - G(w)\|_{E(\mu)} \le \frac{2}{\sqrt[3]{N}}\|g_1\|_{E(\mu)} + |w - z|N^{\frac{1}{6}}\log N \times (\|g^0\|_{E(\mu)} + \|g^1\|_{E(\mu)})$$

for all $N \gg 1$. As a result, $G : S_+ \to E(\mu)$ is continuous. Likewise we can show that $G : S_- \to E(\mu)$ is continuous. Thus, $G : \bar{S} = S_+ \cup S_- \to E(\mu)$ is continuous. □

2.6 Banach lattices and complex interpolation

We will define vector-valued Banach lattices. Let E and $\mathcal{A}$ be Banach lattices over measure spaces (X, Σ, μ) and $(Y, \mathcal{T}, \nu)$, respectively. Define the $\mathcal{A}$-valued Banach lattice $E(\mathcal{A}) = E(\mathcal{A}, \mu)$ by the set of all weakly measurable $\mathcal{A}$-valued functions over X such that

$$\|f\|_{E(\mathcal{A},\mu)} = \|\|f(\cdot)\|_{\mathcal{A}}\|_{E(\mu)} < \infty.$$

Recall that an $\mathcal{A}$-valued function f on a measure space (X, Σ, μ) is said to be weakly measurable if $x^*(f)$ is measurable for all $x^* \in \mathcal{A}^*$.

Definition 2.22. Let $E_0(\mathcal{A},\mu)$ and $E_1(\mathcal{A},\mu)$ be $\mathcal{A}$-valued Banach lattices. Define the Calderón product $E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta$ by

$$E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta \equiv \{|f_0|^{1-\theta}|f_1|^\theta g \,:\, g\in L^\infty(\mathcal{A},\mu), f_0\in E_0(\mu), f_1\in E_1(\mu)\}.$$

For $f\in E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta$, define

$$\begin{aligned}&\|f\|_{E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta}\\ &\equiv \inf\{\|f_0\|_{E_0(\mu)}{}^{1-\theta}\|f_1\|_{E_1(\mu)}{}^{\theta} \,:\, f_0\in E_0(\mu), f_1\in E_1(\mu), \|f(\cdot)\|_{\mathcal{A}}\le |f_0|^{1-\theta}|f_1|^\theta\}\\ &= \inf\{\|f_0\|_{E_0(\mu)}{}^{1-\theta}\|f_1\|_{E_1(\mu)}{}^{\theta} \,:\, f_0\in E_0(\mu), f_1\in E_1(\mu), \|f(\cdot)\|_{\mathcal{A}}= |f_0|^{1-\theta}|f_1|^\theta\}.\end{aligned}$$

Remark 2.23. A weakly measurable function $f: X\to\mathcal{A}$ belongs to the Calderón product $E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta$ if and only if there exist $h_0\in E_0(\mathcal{A},\mu)$ and $h_1\in E_1(\mathcal{A},\mu)$ such that $\|f(\cdot)\|_{\mathcal{A}}\le(\|h_0(\cdot)\|_{\mathcal{A}})^{1-\theta}(\|h_1(\cdot)\|_{\mathcal{A}})^\theta$. Furthermore,

$$\|f\|_{E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta} = \inf \|h_0\|_{E_0(\mathcal{A},\mu)}^{1-\theta}\|h_1\|_{E_1(\mathcal{A},\mu)}^{\theta},$$

where h_0 and h_1 move over the above conditions.

In fact, if there exist $h_0\in E_0(\mathcal{A},\mu)$ and $h_1\in E_1(\mathcal{A},\mu)$ such that $\|f(\cdot)\|_{\mathcal{A}}\le(\|h_0(\cdot)\|_{\mathcal{A}})^{1-\theta}(\|h_1(\cdot)\|_{\mathcal{A}})^\theta$, then set $g\equiv\frac{1}{(\|h_0(\cdot)\|_{\mathcal{A}})^{1-\theta}(\|h_1(\cdot)\|_{\mathcal{A}})^\theta}f$, $f_0\equiv\|h_0(\cdot)\|_{\mathcal{A}}$ and $f_1\equiv\|h_1(\cdot)\|_{\mathcal{A}}$. Then $f=|f_0|^{1-\theta}|f_1|^\theta g$, $g\in L^\infty(\mathcal{A},\mu)$, $f_0\in E_0(\mu)$ and $f_1\in E_1(\mu)$. Hence $f\in E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta$. Thus, the "if" part as well as the estimate

$$\|f\|_{E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta} \le \inf \|h_0\|_{E_0(\mathcal{A},\mu)}^{1-\theta}\|h_1\|_{E_1(\mathcal{A},\mu)}^{\theta},$$

is proved.

Conversely, if f has an expression: $f=|f_0|^{1-\theta}|f_1|^\theta g$, where $g\in L^\infty(\mathcal{A},\mu)$, $f_0\in E_0(\mu)$ and $f_1\in E_1(\mu)$, then set $h_0=f_0g$, $h_1=f_1g$ to have $h_0\in E_0(\mathcal{A},\mu)$, $h_1\in E_1(\mathcal{A},\mu)$ and

$$\|f(\cdot)\|_{\mathcal{A}}=|f_0|^{1-\theta}|f_1|^\theta\|g(\cdot)\|_{\mathcal{A}}=(\|h_0(\cdot)\|_{\mathcal{A}})^{1-\theta}(\|h_1(\cdot)\|_{\mathcal{A}})^\theta.$$

Thus, the "only if" part as well as the estimate

$$\|f\|_{E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta} \ge \inf \|h_0\|_{E_0(\mathcal{A},\mu)}^{1-\theta}\|h_1\|_{E_1(\mathcal{A},\mu)}^{\theta},$$

is proved.

Note that this definition boils down to the original Calderón product by Calderón [3] in the case where $\mathcal{A}=\mathbb{C}$. In other words, using the original Calderón product we can say that

$$E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta=[E_0(\mu)^{1-\theta}E_1(\mu)^\theta](\mathcal{A}).$$

Here we make the following observation:

Lemma 2.24. *Let $E_0(\mu)$ and $E_1(\mu)$ be Banach lattices. Let $f\in E_0(\mathcal{A},\mu)\cap E_1(\mathcal{A},\mu)$. Then there exists a decomposition*

$$f=|f^0|^{1-\theta}|f^1|^\theta g$$

for some $g\in L^\infty(\mathcal{A},\mu)$ and $f^0,f^1\in E_0(\mu)\cap E_1(\mu)$ such that

$$\|g\|_{L^\infty(\mathcal{A},\mu)}=1,\quad \|f^0\|_{E_0(\mu)}{}^{1-\theta}\|f^1\|_{E_1(\mu)}{}^{\theta}\le 4\|f\|_{E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta}.$$

Proof. We know that there exists a decomposition

$$f = |k^0|^{1-\theta}|k^1|^{\theta}k$$

for some $k \in L^\infty(\mathcal{A}, \mu)$ and $(k^0, k^1) \in E_0(\mu) \times E_1(\mu)$ such that

$$\|k\|_{L^\infty(\mathcal{A},\mu)} = 1, \quad \|k^0\|_{E_0(\mu)}{}^{1-\theta}\|k^1\|_{E_1(\mu)}{}^{\theta} \le 2\|f\|_{E_0(\mu)^{1-\theta}E_1(\mu)^{\theta}}.$$

Let

$$h^0 = |k^0| + 2^{-j}\|f(\cdot)\|_{\mathcal{A}}, \quad h^1 = \min(|k^1|, 2^j\|f(\cdot)\|_{\mathcal{A}}),$$

where $j \gg 1$. Then $h^0 \in E_0(\mu)$ and $h^1 \in E_0(\mu) \cap E_1(\mu)$ with

$$\|h^0\|_{E_0(\mu)} \le 2\|k^0\|_{E_0(\mu)}, \quad \|h^1\|_{E_1(\mu)} \le \|k^1\|_{E_1(\mu)}.$$

Likewise, for $\ell \gg 1$, let

$$f^0 = \min(h^0, 2^{\ell}\|f(\cdot)\|_{\mathcal{A}}), \quad f^1 = h^1 + 2^{-\ell}\|f(\cdot)\|_{\mathcal{A}}.$$

Then $f^0, f^1 \in E_0(\mu) \cap E_1(\mu)$ with $\|f^0\|_{E_0(\mu)} \le 2\|k^0\|_{E_0(\mu)}$, $\|f^1\|_{E_0(\mu)} \le 2\|k^1\|_{E_0(\mu)}$ and $\|f(\cdot)\|_{\mathcal{A}} \le |f^0|^{1-\theta}|f^1|^{\theta}$. If $g = \frac{1}{|f^0|^{1-\theta}|f^1|^{\theta}}f$, then we obtain the desired decomposition. □

Let (X, Σ, μ) be a measure space, and let $E_1(\mu)$ and $E_2(\mu)$ be Banach lattices. Shestakov [42] showed

$$(E_0(\mu)^{1-\theta}E_1(\mu)^{\theta})^{\circ}(= \overline{E_0(\mu) \cap E_1(\mu)}^{E_0(\mu)^{1-\theta}E_1(\mu)^{\theta}}) = [E_0(\mu), E_1(\mu)]_{\theta}. \tag{2.23}$$

We generalize this to the vector-valued case.

Theorem 2.25. *Let (X, Σ, μ) be a measure space, $\mathcal{A}(\mu)$ be a Banach space, and $E_1(\mathcal{A}, \mu)$ and $E_2(\mathcal{A}, \mu)$ be Banach lattices. Then*

$$\begin{aligned}(E_0(\mathcal{A}, \mu)^{1-\theta}E_1(\mathcal{A}, \mu)^{\theta})^{\circ}(&\equiv \overline{E_0(\mathcal{A}, \mu) \cap E_1(\mathcal{A}, \mu)}^{E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^{\theta}}) \\ &= [E_0(\mathcal{A}, \mu), E_1(\mathcal{A}, \mu)]_{\theta}.\end{aligned}$$

Proof. We show that

$$(E_0(\mathcal{A}, \mu)^{1-\theta}E_1(\mathcal{A}, \mu)^{\theta})^{\circ} \hookleftarrow [E_0(\mathcal{A}, \mu), E_1(\mathcal{A}, \mu)]_{\theta}.$$

Let $f \in [E_0(\mathcal{A}, \mu), E_1(\mathcal{A}, \mu)]_{\theta}$. Let $F \in \mathcal{F}(E_0(\mathcal{A}, \mu), E_1(\mathcal{A}, \mu))$ be such that $f = F(\theta)$. Then by Lemma 2.20

$$\|f\|_{\mathcal{A}} = \|F(\theta)\|_{\mathcal{A}} \le \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|F(it)\|_{\mathcal{A}}\mu_0(t)\mathrm{d}t\right)^{1-\theta} \left(\frac{1}{\theta}\int_{\mathbb{R}} \|F(1+it)\|_{\mathcal{A}}\mu_1(t)\mathrm{d}t\right)^{\theta}.$$

Setting

$$f_0 \equiv \frac{1}{1-\theta}\int_{\mathbb{R}} \|F(it)\|_{\mathcal{A}}\mu_0(t)\mathrm{d}t, \quad f_1 \equiv \frac{1}{\theta}\int_{\mathbb{R}} \|F(1+it)\|_{\mathcal{A}}\mu_1(t)\mathrm{d}t,$$

we have $f \in E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta$ and $\|f\|_{E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta} \le \|f\|_{[E_0(\mathcal{A},\mu),E_1(\mathcal{A},\mu)]_\theta}$.

We next show that

$$(E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta)^\circ \hookrightarrow [E_0(\mathcal{A},\mu),E_1(\mathcal{A},\mu)]_\theta.$$

Let $f \in E_0(\mathcal{A},\mu)\cap E_1(\mathcal{A},\mu)$. We claim that there exists $F \in \mathcal{F}(E_0(\mathcal{A},\mu),E_1(\mathcal{A},\mu))$ such that $F(\theta)=f$ and

$$\|F\|_{\mathcal{F}(E_0(\mathcal{A},\mu),E_1(\mathcal{A},\mu))} \le 4\|f\|_{E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta}. \tag{2.24}$$

Once (2.24) is proved, then this is valid for all $f \in (E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta)^\circ$, including the assertion that $[E_0(\mathcal{A},\mu),E_1(\mathcal{A},\mu)]_\theta \hookleftarrow (E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta)^\circ$.

Recall that f admits a decomposition

$$f = |g^0|^{1-\theta}|g^1|^\theta h$$

for some $h \in L^\infty(\mathcal{A},\mu)$, $g^0, g^1 \in E_0(\mu)\cap E_1(\mu)$ such that $\|h\|_{L^\infty(\mathcal{A},\mu)} \le 1$ and $\|g^0\|_{E_0(\mu)} = \|g^1\|_{E_1(\mu)} \le 4\|f\|_{E_0(\mathcal{A},\mu)^{1-\theta}E_1(\mathcal{A},\mu)^\theta}$ due to Lemma 2.24. Let $F(z) \equiv |g^0|^{1-z}|g^1|^z h$. Then $F \in \mathcal{F}(E_0(\mathcal{A},\mu),E_1(\mathcal{A},\mu))$ by Lemma 2.21, $F(\theta)=f$ and

$$\|F\|_{\mathcal{F}(E_0(\mathcal{A},\mu),E_1(\mathcal{A},\mu))} \le 4(\|f_0\|_{E_0(\mu)})^{1-\theta}(\|f_1\|_{E_1(\mu)})^\theta.$$

As a result, (2.24) holds. □

3 Proof of Theorem 1.3 including Corollary 1.4

Theorem 1.3 is proved by decomposing its proof into three parts. First, Section 3.1 proves the right inequality in (1.5) for $f \in L^{p(\cdot),s}(w)$. The key inequality is a pointwise estimate (3.3). Section 3.2 concentrates on establishing the existence of derivatives of $f \in L^{p(\cdot)}(w)$ such that $Vf+W_s f \in L^{p(\cdot)}(w)$. In Section 3.2, we reduce matters to the case where f is sufficiently smooth and compactly supported. At this point, matters are reduced to establishing the left inequality for any $f \in C_c \cap \mathcal{P}_{s+1}^\perp$ such that $Vf+W_s f \in L^{p(\cdot)}(w)$. Section 3.3 actually considers the left inequality in (1.5) for such f via key pointwise estimate (3.7). Finally, by reexamining the argument in Section 3.1, we prove Corollary 1.4 in Section 3.4.

3.1 Proof of the right inequality in (1.5) for $f \in L^{p(\cdot),s}(w)$

Let $f \in L^{p(\cdot),s}(w)\cap L^2$. Choose even functions $\Phi^{(L)}$ and $\Psi^{(L)}$ as described in Lemma 2.10, where L is a multiple of 4 with $L \gg s$. We can write

$$\Delta^{L/2} = \sum_{m=1}^n P_m(D)D_m^s$$

with some polynomials $P_1, P_2, \dots, P_n$ of the order $L/2-s$. Accordingly, we write

$$\Gamma_m^{(L)} = P_m(D)\Gamma^{(L)},$$

where $\Gamma^{(L)}$ and $\Theta^{(L)}$ are as in (2.12). Then

$$
\begin{aligned}
f &= 2^{Jn}\Theta^{(L)}(2^J\cdot) * f + \sum_{j=J}^{\infty} 4^{jn}\Delta^{L/2}\Gamma^{(L)}(2^j\cdot) * \Delta^{L/2}\Psi^{(L)}(2^j\cdot) * f \\
&= 2^{Jn}\Theta^{(L)}(2^J\cdot) * f + (-1)^s \sum_{m=1}^{n}\sum_{j=J}^{\infty} 2^{2jn-js}\Gamma^{(L)}_m(2^j\cdot) * \Delta^{L/2}\Psi^{(L)}(2^j\cdot) * D^s_m f.
\end{aligned}
$$

thanks to Lemma 2.11. Fix $j_0, j \in \mathbb{Z}\cap[J,\infty)$, $k \in \mathbb{Z}^n$, and $l \in \{1,2,\ldots,2^n-1\}$. Then we estimate

$$
2^{(j-j_0)s+2j_0n}\left|\left\langle \Gamma^{(L)}_m(2^{j_0}\cdot) * \Delta^{L/2}\Psi^{(L)}(2^{j_0}\cdot) * D^s_m f, \psi^l_{j,k}\right\rangle\right| \chi_{j,k}.
$$

Recall that $\Psi^{(L)}$ is even. Thus, we obtain

$$
\begin{aligned}
&2^{(j-j_0)s+2j_0n}\left|\left\langle \Gamma^{(L)}_m(2^{j_0}\cdot) * \Delta^{L/2}\Psi^{(L)}(2^{j_0}\cdot) * D^s_m f, \psi^l_{j,k}\right\rangle\right| \chi_{j,k} \\
&= 2^{(j-j_0)s+2j_0n}\left|\left\langle \Gamma^{(L)}_m(2^{j_0}\cdot) * D^s_m f, \Delta^{L/2}\Psi^{(L)}(2^{j_0}\cdot) * \psi^l_{j,k}\right\rangle\right| \chi_{j,k}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&2^{(j-j_0)s+2j_0n}\left|\left\langle \Gamma^{(L)}_m(2^{j_0}\cdot) * \Delta^{L/2}\Psi^{(L)}(2^{j_0}\cdot) * D^s_m f, \psi^l_{j,k}\right\rangle\right| \chi_{j,k} \\
&\lesssim 2^{-|j-j_0|+j_0n} M^{\rm loc}\left[\Gamma^{(L)}_m(2^{j_0}\cdot) * D^s_m f\right] \chi_{j,k}
\end{aligned}
\tag{3.1}
$$

due to Lemma 2.13 (5). A similar estimate is available for $2^{Jn}\Theta^{(L)}(2^J\cdot) * f$:

$$
2^{js+Jn}\left|\left\langle \Theta^{(L)}(2^J\cdot) * f, \psi^l_{j,k}\right\rangle\right| \chi_{j,k} \lesssim 2^{-|j-J|+Jn} M^{\rm loc}\left[\Phi^{(L)} * f\right] \chi_{j,k}. \tag{3.2}
$$

We add estimates (3.1) and (3.2) over $j_0 \in \mathbb{Z}\cap[J,\infty)$. The result is:

$$
\begin{aligned}
2^{js}\left|\left\langle f, \psi^l_{j,k}\right\rangle\right| \chi_{j,k} &\lesssim 2^{-|j-J|+Jn} M^{\rm loc}\left[\Phi^{(L)} * f\right] \chi_{j,k} \\
&\quad + \sum_{m=1}^{n}\sum_{j_0=J}^{\infty} 2^{-|j-j_0|+j_0n} M^{\rm loc}\left[\Gamma^{(L)}_m(2^{j_0}\cdot) * D^s_m f\right] \chi_{j,k}.
\end{aligned}
$$

Due to the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
\left(2^{js}\left|\left\langle f, \psi^l_{j,k}\right\rangle\right| \chi_{j,k}\right)^2 &\lesssim 2^{-2(j-J)+2Jn} M^{\rm loc}\left[\Phi^{(L)} * f\right]^2 \chi_{j,k} \\
&\quad + \left(\sum_{m=1}^{n}\sum_{j_0=J}^{\infty} 2^{-|j-j_0|+j_0n} M^{\rm loc}\left[\Gamma^{(L)}_m(2^{j_0}\cdot) * D^s_m f\right] \chi_{j,k}\right)^2 \\
&\lesssim 2^{-(j-J)+2Jn} M^{\rm loc}\left[\Phi^{(L)} * f\right]^2 \chi_{j,k} \\
&\quad + \sum_{m=1}^{n}\sum_{j_0=J}^{\infty} 2^{-|j-j_0|+2j_0n} M^{\rm loc}\left[\Gamma^{(L)}_m(2^{j_0}\cdot) * D^s_m f\right]^2 \chi_{j,k}.
\end{aligned}
$$

If we add this inequality over $j \in \mathbb{Z}\cap[J,\infty)$, $k \in \mathbb{Z}^n$, $l \in \{1,2,\ldots,2^n-1\}$, and subsequently replace j_0 in the right-hand side with j, then we obtain

$$
(W_s f)^2 \lesssim 2^{2Jn} M^{\rm loc}\left[\Phi^{(L)}(2^J\cdot) * f\right]^2 + \sum_{m=1}^{n}\sum_{j=J}^{\infty} 2^{2jn} M^{\rm loc}\left[\Gamma^{(L)}_m(2^j\cdot) * D^s_m f\right]^2.
$$

If we argue similarly, we obtain

$$(Vf)^2 \lesssim 2^{2Jn} M^{\rm loc} \left[\Phi^{(L)}(2^J \cdot) * f\right]^2 + \sum_{m=1}^{n} \sum_{j=J}^{\infty} 2^{2jn} M^{\rm loc} \left[\Gamma_m^{(L)}(2^j \cdot) * D_m^s f\right]^2.$$

In total, we obtain

$$\begin{aligned} & Vf + W_s f \qquad (3.3)\\ & \lesssim 2^{Jn} M^{\rm loc} \left[\Phi^{(L)}(2^J \cdot) * f\right] + \sum_{m=1}^{n} \left(\sum_{j=J}^{\infty} 2^{2jn} M^{\rm loc} \left[\Gamma_m^{(L)}(2^j \cdot) * D_m^s f\right]^2\right)^{\frac{1}{2}}. \end{aligned}$$

By the vector-valued inequality for $M^{\rm loc}$ (see Proposition 2.7), we have

$$\begin{aligned} & \|Vf\|_{L^{p(\cdot)}(w)} + \|W_s f\|_{L^{p(\cdot)}(w)} \\ & \lesssim \|2^{Jn} \Phi^{(L)}(2^J \cdot) * f\|_{L^{p(\cdot)}(w)} + \sum_{m=1}^{n} \left\| \left(\sum_{j=J}^{\infty} 2^{2jn} \left|\Gamma_m^{(L)}(2^j \cdot) * D_m^s f\right|^2\right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)}. \end{aligned} \qquad (3.4)$$

The first term in the right-hand side can be controlled by $M^{\rm loc}$ (see Proposition 2.5), while we can control the second term using the Rademacher sequence similar to [26, §4] and the boundedness of generalized local singular integral operators. The result is

$$\|Vf\|_{L^{p(\cdot)}(w)} + \|W_s f\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{L^{p(\cdot)}(w)} + \sum_{m=1}^{n} \|D_m^s f\|_{L^{p(\cdot)}(w)},$$

as required.

3.2 Reduction for the proof of (2)

We claim that we have only to prove Theorem 1.3 for $f \in (C_{\rm c} \cap \mathcal{P}_{s+1}^{\perp} \subset) L^{p(\cdot),s}(w) \cap L^2$.

Assume that $f \in L^1_{\rm loc}$ satisfies $Vf + W_s f \in L^{p(\cdot)}(w)$. Then $f \in L^{p(\cdot)}(w)$. Hence, we have the wavelet decomposition for $s = 0$ since Theorem 1.3 with $s = 0$ is proved in [26].

We set

$$f_N \equiv \sum_{k \in \mathbb{Z}^n} \chi_{[0,N]}(|k|) \langle f, \varphi_{J,k} \rangle \varphi_{J,k} + \sum_{l=1}^{2^n-1} \sum_{j=J}^{N} \sum_{k \in \mathbb{Z}^n} \chi_{[0,N]}(|k|) \langle f, \psi^l_{j,k} \rangle \psi^l_{j,k}.$$

Since f_N is compactly supported and $s+1$-times continuously differentiable, and each wavelet $\varphi_{J,k}$ and scaling function $\psi^l_{J,k}$ belong to $C_{\rm c} \cap \mathcal{P}_{s+1}^{\perp}$, we see that $f_N \in C_{\rm c} \cap \mathcal{P}_{s+1}^{\perp} \subset L^{p(\cdot),s}(w) \cap L^2$.

Assume that the two-sided inequality (1.5) holds for any $f \in L^{p(\cdot),s}(w) \cap L^2$, that is,

$$\|g\|_{L^{p(\cdot),s}(w)} \lesssim \|Vg + W_s g\|_{L^{p(\cdot)}(w)} \qquad (3.5)$$

for any $g \in L^{p(\cdot),s}(w) \cap L^2$.

Then for any $f \in L^{p(\cdot)}(w)$ satisfying $Vf + W_s f \in L^{p(\cdot)}(w)$, we have

$$\|f_N\|_{L^{p(\cdot),s}(w)} \lesssim \|Vf_N + W_s f_N\|_{L^{p(\cdot)}(w)} \le \|Vf + W_s f\|_{L^{p(\cdot)}(w)}$$

where the implicit constant is independent of N. This means that $\{D^\alpha f_N\}_{N=1}^\infty$ forms a bounded set in $L^{p(\cdot)}(w)$ for any α with length less than or equal to s. According to the Banach–Alaoglu theorem, there exists a subsequence $\{N_m\}_{m=1}^\infty$ such that $\{D^\alpha f_{N_m}\}_{m=1}^\infty$ converges weakly to a function $f_{(\alpha)} \in L^{p(\cdot)}(w)$ for such α. In particular, since we know that f_{N_m} converges strongly to f thanks to Theorem 1.3 with $s = 0$, which is proved in [26], we see that $f_{(0)} = f \in L^{p(\cdot)}(w)$. From the definition of the weak topology of $L^{p(\cdot)}(w)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} f_{(\alpha)}(x)\theta(x)\mathrm{d}x &= \lim_{m\to\infty}\int_{\mathbb{R}^n} D^\alpha f_{N_m}(x)\theta(x)\mathrm{d}x\\ &= \lim_{m\to\infty}(-1)^{|\alpha|}\int_{\mathbb{R}^n} f_{N_m}(x)D^\alpha\theta(x)\mathrm{d}x\\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^n} f(x)D^\alpha\theta(x)\mathrm{d}x\end{aligned}$$

for all test functions $\theta \in C_{\mathrm{c}}^\infty$.

Thus, $D^\alpha f = f_{(\alpha)}$ in the weak sense and consequently, $D^\alpha f$ belongs to $L^{p(\cdot)}(w)$. Due to the Fatou lemma, we obtain the left inequality of (1.5):

$$\begin{aligned}\|f\|_{L^{p(\cdot),s}(w)} &\le \liminf_{N\to\infty}\|f_N\|_{L^{p(\cdot),s}(w)}\\ &\lesssim \lim_{N\to\infty}\|Vf_N + W_s f_N\|_{L^{p(\cdot)}(w)}\\ &\le \|Vf + W_s f\|_{L^{p(\cdot)}(w)} < \infty.\end{aligned}$$

This implies that $f \in L^{p(\cdot),s}(w)$ and (1.5) holds.

Here and below, we suppose that $f \in L^{p(\cdot),s}(w) \cap L^2$ to prove the left inequality of (1.5).

3.3 Proof of the left inequality in (1.5)

We suppose that $f \in L^{p(\cdot),s}(w) \cap L^2$. According to the previous section, we have $Vf + W_s f \in L^{p(\cdot)}(w)$. To see that $\|f\|_{L^{p(\cdot),s}(w)} \lesssim \|Vf + W_s f\|_{L^{p(\cdot)}(w)}$, it suffices to show that $\|V[D^\alpha f]\|_{L^{p(\cdot)}(w)} + \|W_0[D^\alpha f]\|_{L^{p(\cdot)}(w)} \lesssim \|Vf + W_s f\|_{L^{p(\cdot)}(w)}$ for any α with $|\alpha| \le s$, since we proved that $\|V[D^\alpha f]\|_{L^{p(\cdot)}(w)} + \|W_0[D^\alpha f]\|_{L^{p(\cdot)}(w)} \sim \|D^\alpha f\|_{L^{p(\cdot)}(w)}$ in [26].

Let α be a multiindex with length s. Fix $j_0 \in \mathbb{Z} \cap [J, \infty)$, $k_0 \in \mathbb{Z}^n$, and $l_0 \in \{1, 2, \ldots, 2^n - 1\}$. We observe

$$\begin{aligned}&\left|\langle D^\alpha f, \psi^{l_0}_{j_0,k_0}\rangle\right|\\ &= \left|\langle f, D^\alpha[\psi^{l_0}_{j_0,k_0}]\rangle\right|\\ &\le \sum_{k\in\mathbb{Z}^n}\left|\langle f, \varphi_{J,k}\rangle\langle\varphi_{J,k}, D^\alpha[\psi^{l_0}_{j_0,k_0}]\rangle\right| + \sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|\langle f, \psi^l_{j,k}\rangle\langle\psi^l_{j,k}, D^\alpha[\psi^{l_0}_{j_0,k_0}]\rangle\right|.\end{aligned}$$

Consequently, denoting by $\chi^*_{j,k}$, the modified indicator function of $\chi_{j,k}$, we obtain

$$\begin{aligned}
&\left|\langle D^\alpha f, \psi^{l_0}_{j_0,k_0}\rangle\right| \chi_{j_0,k_0} \\
&\quad\lesssim \sum_{k\in\mathbb{Z}^n} |\langle f, \varphi_{J,k}\rangle|\, 2^{Js-Jn-\frac{j_0 n}{2}+Jn-(j_0-J)} \chi^*_{j_0,k_0}\chi^*_{J,k} \\
&\qquad + \sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi^l_{j,k}\rangle\right| 2^{js-jn-\frac{j_0 n}{2}+\min(j,j_0)n-|j-j_0|} \chi^*_{j_0,k_0}\chi^*_{j,k},
\end{aligned} \tag{3.6}$$

due to Lemma 2.13 (3) and (4). If we add (3.6) over j_0, k_0, and l_0, we obtain

$$\begin{aligned}
&\sum_{l_0=1}^{2^n-1}\sum_{j_0=J}^{\infty}\sum_{k_0\in\mathbb{Z}^n} \left(\left|\langle D^\alpha f, \psi^{l_0}_{j_0,k_0}\rangle\right| \chi_{j_0,k_0}\right)^2 \\
&\lesssim \sum_{l_0=1}^{2^n-1}\sum_{j_0=J}^{\infty}\sum_{k_0\in\mathbb{Z}^n} \left(\sum_{k\in\mathbb{Z}^n} |\langle f, \varphi_{J,k}\rangle|\, 2^{Js-Jn-\frac{j_0 n}{2}+Jn-(j_0-J)} \chi^*_{j_0,k_0}\chi^*_{J,k}\right)^2 \\
&\quad + \sum_{l_0=1}^{2^n-1}\sum_{j_0=J}^{\infty}\sum_{k_0\in\mathbb{Z}^n} \left(\sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi^l_{j,k}\rangle\right| 2^{js-jn-\frac{j_0 n}{2}+\min(j,j_0)n-|j-j_0|} \chi^*_{j_0,k_0}\chi^*_{j,k}\right)^2 .
\end{aligned}$$

Since l_0 moves over a finite set, we have

$$\begin{aligned}
&\sum_{l_0=1}^{2^n-1}\sum_{j_0=J}^{\infty}\sum_{k_0\in\mathbb{Z}^n} \left(\left|\langle D^\alpha f, \psi^{l_0}_{j_0,k_0}\rangle\right| \chi_{j_0,k_0}\right)^2 \\
&\lesssim \sum_{j_0=J}^{\infty}\sum_{k_0\in\mathbb{Z}^n} \left(\sum_{k\in\mathbb{Z}^n} |\langle f, \varphi_{J,k}\rangle|\, 2^{Js-(j_0-J)} (2^{-\frac{j_0 n}{2}}\chi^*_{j_0,k_0})\chi^*_{J,k}\right)^2 \\
&\quad + \sum_{j_0=J}^{\infty}\sum_{k_0\in\mathbb{Z}^n} \left(\sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi^l_{j,k}\rangle\right| 2^{js-jn+\min(j,j_0)n-|j-j_0|} (2^{-\frac{j_0 n}{2}}\chi^*_{j_0,k_0})\chi^*_{j,k}\right)^2 .
\end{aligned}$$

Using the Cauchy-Schwarz inequality and

$$\sum_{j=J}^{\infty} 2^{-jn+\min(j,j_0)n-|j-j_0|} \le \sum_{j=-\infty}^{\infty} 2^{-jn+\min(j,j_0)n-|j-j_0|} < \infty,$$

we obtain

$$\begin{aligned}
&\sum_{l_0=1}^{2^n-1}\sum_{j_0=J}^{\infty}\sum_{k_0\in\mathbb{Z}^n} \left(\left|\langle D^\alpha f, \psi^{l_0}_{j_0,k_0}\rangle\right| \chi_{j_0,k_0}\right)^2 \\
&\lesssim \sum_{j_0=J}^{\infty}\sum_{k_0\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n} \left(|\langle f, \varphi_{J,k}\rangle|\, 2^{Js-(j_0-J)} (2^{-\frac{j_0 n}{2}}\chi^*_{j_0,k_0})\chi^*_{J,k}\right)^2 \\
&\quad + \sum_{j_0=J}^{\infty}\sum_{k_0\in\mathbb{Z}^n}\sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} \left(\left|\langle f, \psi^l_{j,k}\rangle\right| 2^{js-jn+\min(j,j_0)n-|j-j_0|} (2^{-\frac{j_0 n}{2}}\chi^*_{j_0,k_0})\chi^*_{j,k}\right)^2 \\
&\lesssim \sum_{k\in\mathbb{Z}^n} \left(|\langle f, \varphi_{J,k}\rangle|\, \chi^*_{J,k}\right)^2 + \sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} \left(2^{js}\left|\langle f, \psi^l_{j,k}\rangle\right| \chi^*_{j,k}\right)^2 .
\end{aligned}$$

We can incorporate the term for $\langle D^\alpha f, \varphi_{J,k}\rangle$. The result is

$$\sum_{k\in\mathbb{Z}^n}\left(|\langle D^\alpha f, \varphi_{J,k}\rangle|\,\chi^*_{J,k}\right)^2 + \sum_{l_0=1}^{2^n-1}\sum_{j_0=J}^{\infty}\sum_{k_0\in\mathbb{Z}^n}\left(|\langle D^\alpha f, \psi^{l_0}_{j_0,k_0}\rangle|\,\chi_{j_0,k_0}\right)^2$$
$$\lesssim \sum_{k\in\mathbb{Z}^n}\left(|\langle f, \varphi_{J,k}\rangle|\,\chi^*_{J,k}\right)^2 + \sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n}\left(2^{js}\,|\langle f, \psi^{l}_{j,k}\rangle|\,\chi^*_{j,k}\right)^2.$$

Consequently, we have a pointwise estimate

$$V[D^\alpha f] + W_0[D^\alpha f] \lesssim Vf + W_s f. \tag{3.7}$$

Thus, by Theorem 1.3 with $s = 0$, we obtain that $f \in L^{p(\cdot),s}(w)$.

3.4 Proof of Corollary 1.4

It is trivial that

$$\|f\|_{L^{p(\cdot),s}(w)} \geq \|f\|_{L^{p(\cdot)}(w)} + \sum_{|\alpha|=s}\|D^\alpha f\|_{L^{p(\cdot)}(w)}.$$

For the opposite inequality, simply combine the left inequality in (1.5) with (3.4).

4 Fractional local weighted Sobolev spaces with variable exponents

This section considers the local weighted Sobolev space $L^{p(\cdot),s}(w)$ with a variable exponent for $s > 0$. As a preliminary step, Section 4.1 defines $L^{p(\cdot),s}(w)$. Section 4.2 investigates the complex interpolation to the minimum. By using this key tool, we extend Theorem 1.3 for $s > 0$ in Section 4.3. Sections 4.4 and 4.5 refine the wavelet decomposition obtained above. To loosen the postulate on φ and ψ Section 4.4 investigates the atomic decomposition. We try to decrease the order of the smoothness and the moment. Section 4.5 provides further refinement based on [17, 32] and the Haar function, which is a fundamental function that is useful to characterize function spaces.

4.1 Bessel potential operator $(1-t^2\Delta)^{-a}$ with $0 < t \ll 1$ and $\mathrm{Re}(a) \geq 0$

First, we define the Sobolev space $L^{p(\cdot),s}(w)$ with a positive fractional order s.

Definition 4.1 (Sobolev spaces with variable exponents defined by the Bessel potential)**.** Fix $0 < t \ll 1$. Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, and let $w \in A^{\mathrm{loc}}_{p(\cdot)}$. For $s \geq 0$, define

$$L^{p(\cdot),s}(w) \equiv \{g \in L^{p(\cdot)}(w) \,:\, g = (1-t^2\Delta)^{-\frac{s}{2}} f \text{ for some } f \in L^{p(\cdot)}(w)\}.$$

If $g = (1-t^2\Delta)^{-\frac{s}{2}} f \in L^{p(\cdot),s}(w)$ for some $f \in L^{p(\cdot)}(w)$, then define

$$\|g\|_{L^{p(\cdot),s}(w)} \equiv \|f\|_{L^{p(\cdot)}(w)}.$$

If $0 < t_1, t_2 < \infty$ and $a > 0$, then $(1 - {t_1}^2\Delta)^a(1 - {t_2}^2\Delta)^{-a}$ is a generalized local singular integral operator. Hence, the definition of $L^{p(\cdot),s}(w)$ is independent of t.

Thanks to Theorem 2.15, $L^{p(\cdot),s}(w)$ is nested: $L^{p(\cdot),s_0}(w) \subset L^{p(\cdot),s_1}(w)$ for $s_0 > s_1 > 0$.

Note that the space $L^{p(\cdot),s}(w)$ with $s \in \mathbb{N}$ has two different definitions. One is from the weak derivative in $\mathcal{D}'$, and the other is from the Bessel potential. We show that they coincide with the equivalence of norms.

Theorem 4.2. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, and let $w \in A^{\rm loc}_{p(\cdot)}$. Then the two definitions of $L^{p(\cdot),s}(w)$ coincide for all $s \in \mathbb{N}$.*

Proof. Similar to Theorem 2.15, we can show that $D^\alpha(1 - t^2\Delta)^{-\frac{s}{2}}$ is bounded on $L^{p(\cdot)}(w)$ as long as $|\alpha| \le s$. This means that, as long as $|\alpha| \le s$, $D^\alpha g \in L^{p(\cdot)}(w)$ is defined by the Bessel potential for any $g \in L^{p(\cdot),s}(w)$. Thus, the space $L^{p(\cdot),s}(w)$ which is defined by the Bessel potential is included in the one defined by the partial derivative.

Conversely, let $f \in L^{p(\cdot)}(w)$ be such that $D^\alpha f \in L^{p(\cdot)}(w)$ for any multiindex α with $|\alpha| \le s$. We choose a finite collection of polynomials $\{P_k\}_{k\in K}$ of degree less than or equal to s so that

$$(1 + t^2|\xi|^2)^s = \sum_{k\in K} P_k(\xi)^2.$$

Then

$$f \in L^{p(\cdot)}(w) \mapsto P_k\left(\frac{1}{i}D\right)(1 - t^2\Delta)^{-\frac{s}{2}} f \in L^{p(\cdot)}(w)$$

is a bounded linear operator, and we have a decomposition

$$f = (1 - t^2\Delta)^{-\frac{s}{2}} \sum_{k\in K} P_k\left(\frac{1}{i}D\right)(1 - t^2\Delta)^{-\frac{s}{2}}\left[P_k\left(\frac{1}{i}D\right) f\right],$$

meaning that f is a member in the Sobolev space $L^{p(\cdot),s}(w)$ by the Bessel potential. □

We investigate the lifting property of weighted Sobolev spaces with variable exponents.

Proposition 4.3. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Let $w \in A^{\rm loc}_{p(\cdot)}$ and $s_1 > s_2 \ge 0$. Then $(1 - t^2\Delta)^{\frac{s_2 - s_1}{2}}$ is an isomorphism from $L^{p(\cdot),s_2}(w)$ to $L^{p(\cdot),s_1}(w)$.*

Proof. It is clear that $(1 - t^2\Delta)^{\frac{s_2-s_1}{2}}$ is a bijection from $L^{p(\cdot),s_2}(w)$ to $L^{p(\cdot),s_1}(w)$ in view of the definitions of $L^{p(\cdot),s_2}(w)$ and $L^{p(\cdot),s_1}(w)$. The operator $(1 - t^2\Delta)^{\frac{s_2-s_1}{2}}$ is also injective, since $(1-t^2\Delta)^{[s_1+1]}(1-t^2\Delta)^{-[s_1+1]-\frac{s_2-s_1}{2}}(1-t^2\Delta)^{\frac{s_2-s_1}{2}} = \mathrm{id}_{L^{p(\cdot),s_2}(w)}$. □

4.2 Complex interpolation I

We show that $L^{p(\cdot),s}(w)$ with any $s>0$ has a wavelet characterization.

We start by calculating the first complex interpolation.

Proposition 4.4. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, and let $w \in A_{p(\cdot)}^{\rm loc}$. Additionaly, let s_0, s_1, s and θ satisfy $s_0, s_1, s \ge 0$, $0<\theta<1$ and $s=(1-\theta)s_0+\theta s_1$. Then*

$$[L^{p(\cdot),s_0}(w), L^{p(\cdot),s_1}(w)]_\theta = L^{p(\cdot),s}(w).$$

Proof. We may assume $s_0 \ne s_1$. Otherwise, there is nothing to prove since $s=s_0=s_1$. We may also assume $s_0>s_1$ by symmetry. Let $F \in \mathcal{F}(L^{p(\cdot),s_0}(w), L^{p(\cdot),s_1}(w))$. We aim to show that $F(\theta) \in L^{p(\cdot),s}(w)$.

Fix t as in Theorem 2.15. We define

$$G_N(z)={\rm e}^{(z-\theta)^2}(1-t^2\Delta)^{\frac{s_1-s_0}{2}z}[(1-N^{-1}\Delta)^{-s_0-s_1-n-2}[F(z)] \quad (z \in \bar{S}, N \in \mathbb{N}).$$

Fix $N \gg 1$ and $(a,b) \in [0,1] \times \mathbb{R}$. We write out $\|G_N(a+bi)\|_{L^{p(\cdot),s_0}(w)}$ in full:

$$\begin{aligned}&\|G_N(a+bi)\|_{L^{p(\cdot),s_0}(w)}\\ &={\rm e}^{(a-\theta)^2-b^2}\left\|(1-t^2\Delta)^{\frac{s_1-s_0}{2}(a+bi)}(1-N^{-1}\Delta)^{-s_0-s_1-n-2}[F(a+bi)]\right\|_{L^{p(\cdot),s_0}(w)}.\end{aligned}$$

If we employ Theorem 2.15, then

$$\begin{aligned}&\|G_N(a+bi)\|_{L^{p(\cdot),s_0}(w)}\\ &\lesssim {\rm e}^{-b^2}\left((1+|b|)^{n+2}+\frac{1}{|\Gamma(a+bi)|}\right)\left\|(1-N^{-1}\Delta)^{-s_0-s_1-n-2}[F(a+bi)]\right\|_{L^{p(\cdot),s_0}(w)}.\end{aligned}$$

From (2.13),

$$\begin{aligned}\|G_N(a+bi)\|_{L^{p(\cdot),s_0}(w)} &\lesssim \left\|(1-N^{-1}\Delta)^{-s_0-s_1-n-2}[F(a+bi)]\right\|_{L^{p(\cdot),s_0}(w)}\\ &=\left\|(1-N^{-1}\Delta)^{-s_0-s_1-n-2}(1-t^2\Delta)^{\frac{s_0}{2}}[F(a+bi)]\right\|_{L^{p(\cdot)}(w)}.\end{aligned}$$

Then thanks to Theorem 2.15

$$\begin{aligned}\|G_N(a+bi)\|_{L^{p(\cdot),s_0}(w)} &\lesssim \left\|(1-t^2\Delta)^{\frac{s_0}{2}}[F(a+bi)]\right\|_{L^{p(\cdot)}(w)}\\ &=\|F(a+bi)\|_{L^{p(\cdot),s_0}(w)}\end{aligned}$$

with the implicit constant independent of N. Meanwhile

$$z \in \bar{S} \mapsto {\rm e}^{(z-\theta)^2}(1-t^2\Delta)^{\frac{s_1-s_0}{2}z}(1-N^{-1}\Delta)^{-s_0-s_1-n-2} \in B(L^{p(\cdot)}(w))$$

is bounded and continuous on $\bar{S}$ due to Theorem 2.15 and holomorphic in S due to Corollary 2.16. Consequently, it follows from Theorem 2.15 that

$$G_N \in \mathcal{F}(L^{p(\cdot),s_0}(w), L^{p(\cdot),s_0}(w)).$$

From Example 2.19

$$G_N(\theta) = (1-t^2\Delta)^{\frac{s_1-s_0}{2}\theta}[(1-N^{-1}\Delta)^{-s_0-s_1-n-2}F(\theta)] \in L^{p(\cdot),s_0}(w),$$

and

$$\|(1-t^2\Delta)^{\frac{s_1-s_0}{2}\theta}[(1-N^{-1}\Delta)^{-s_0-s_1-n-2}F(\theta)]\|_{L^{p(\cdot),s_0}(w)} \lesssim \|f\|_{[\mathcal{F}(L^{p(\cdot),s_0}(w),L^{p(\cdot),s_0}(w))]_\theta}$$

with the implicit constant independent of N. Hence

$$(1-N^{-1}\Delta)^{-s_0-s_1-n-2}F(\theta) \in L^{p(\cdot),s}(w)$$

and

$$\|(1-N^{-1}\Delta)^{-s_0-s_1-n-2}F(\theta)\|_{L^{p(\cdot),s}(w)} \lesssim \|f\|_{[\mathcal{F}(L^{p(\cdot),s_0}(w),L^{p(\cdot),s_0}(w))]_\theta}$$

with the implicit constant independent of N. Meanwhile, Corollary 2.17 yields $(1-N^{-1}\Delta)^{-s_0-s_1-n-2}F(\theta) \to F(\theta)$ as $N \to \infty$. Similar to Section 3.2, we conclude $F(\theta) \in L^{p(\cdot),s}(w)$ using the Banach–Alaoglu theorem. Consequently $[L^{p(\cdot),s_0}(w), L^{p(\cdot),s_1}(w)]_\theta \subset L^{p(\cdot),s}(w)$.

Conversely, we let $f \in L^{p(\cdot),s}(w)$, so that $f = (1-t^2\Delta)^{-\frac{s}{2}}g$ for some $g \in L^{p(\cdot)}(w)$. We have a decomposition

$$g_0 = (1-{N_0}^{-1}\Delta)^{-4n}g, \quad g_k = (1-{N_k}^{-1}\Delta)^{-4n}g-(1-{N_{k-1}}^{-1}\Delta)^{-4n}g, \quad g = \sum_{k=0}^{\infty} g_k,$$

where $\{N_k\}_{k=0}^{\infty}$ is an increasing sequence of integers such that

$$\sum_{k=1}^{\infty} \|g_k\|_{L^{p(\cdot)}(w)} \le 2\|g\|_{L^{p(\cdot)}(w)}.$$

Define

$$F_k(z) = \mathrm{e}^{(z-\theta)^2}(1-t^2\Delta)^{\frac{s_0-s_1}{2}(z-\theta)-\frac{s}{2}}g_k$$

for each $k \in \mathbb{N}$ and $z \in \bar{S}$. Then $F_k \in \mathcal{F}(L^{p(\cdot),s_0}(w), L^{p(\cdot),s_1}(w))$ with

$$\|F_k\|_{\mathcal{F}(L^{p(\cdot),s_0}(w),L^{p(\cdot),s_1}(w))} \lesssim \|g_k\|_{L^{p(\cdot)}(w)}$$

due to Theorem 2.15 and Corollary 2.16. Consequently

$$f = \sum_{k=0}^{\infty} F_k(\theta) \in [L^{p(\cdot),s_0}(w), L^{p(\cdot),s_1}(w)]_\theta.$$

□

4.3 Wavelet characterization of $L^{p(\cdot),s}(w)$ with $s > 0$

We characterize $L^{p(\cdot),s}(w)$ in terms of the wavelet coefficients for any $s > 0$ by means of the complex interpolation and Theorem 1.3, which considers the case of $s \in \mathbb{N}$.

According to Theorem 1.3, it is useful to define the space $\mathbf{f}_2^{p(\cdot),s}(w)$ for $s \in \mathbb{R}$ which is the set of all sequences $\{\lambda_{j,k}\}_{j\in\mathbb{Z}\cap[J,\infty),k\in\mathbb{Z}^n}$ satisfying

$$\|\{\lambda_{j,k}\}_{j\in\mathbb{Z}\cap[J,\infty),k\in\mathbb{Z}^n}\|_{\mathbf{f}_2^{p(\cdot),s}(w)} = \left\| \left(\sum_{j=J}^{\infty} \left| 2^{js} \sum_{k\in\mathbb{Z}^n} \lambda_{j,k}\chi_{j,k} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)} < \infty.$$

We define

$$\begin{aligned}\widetilde{\mathbf{f}}_2^{p(\cdot),s}(w) = \{\{\lambda_k\}_{k\in\mathbb{Z}^n} \cup \{\lambda_{j,k}^l\}_{j\in\mathbb{Z}\cap[J,\infty),k\in\mathbb{Z}^n,l\in\{1,2,\ldots,2^n-1\}} :& \\ \{\lambda_{j,k}^l\}_{j\in\mathbb{Z}\cap[J,\infty),k\in\mathbb{Z}^n} \in \mathbf{f}_2^{p(\cdot),s}(w) \text{ for all } l = 0,1,\ldots,2^n-1\},&\end{aligned}$$

where it is understood that

$$\lambda_{j,k}^0 = \begin{cases} \lambda_k & (j = J), \\ 0 & (\text{otherwise}). \end{cases}$$

Theorem 1.3 asserts that

$$\begin{aligned}\Psi = \Psi^{\varphi,\psi^l} &: f \in L^{p(\cdot),s}(w) \\ &\mapsto \{\langle \varphi_{J,k}, f\rangle\}_{k\in\mathbb{Z}^n} \cup \{\langle \psi_{j,k}^l, f\rangle\}_{j\in\mathbb{Z}\cap[J,\infty),k\in\mathbb{Z}^n,l=1,2,\ldots,2^n-1} \in \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)\end{aligned}$$

and

$$\begin{aligned}\Phi = \Phi^{\varphi,\psi^l} &: \{\lambda_k\}_{k\in\mathbb{Z}^n} \cup \{\lambda_{j,k}^l\}_{j\in\mathbb{Z}\cap[J,\infty),k\in\mathbb{Z}^n,l\in\{1,2,\ldots,2^n-1\}} \in \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w) \\ &\mapsto \sum_{k\in\mathbb{Z}^n} \lambda_k\varphi_{J,k} + \sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} \lambda_{j,k}^l\psi_{j,k}^l \in L^{p(\cdot),s}(w)\end{aligned}$$

are bounded linear operators for $s \in \mathbb{N}$. Furthermore, by the property of wavelets

$$\Phi(\Psi f) = f \tag{4.1}$$

for all $f \in L^{p(\cdot),s}(w)$.

To show that the space $L^{p(\cdot),s}(w)$ with $s \in (0,\infty)\setminus\mathbb{N}$ is realized as a complex interpolation, we need the following lemma.

Lemma 4.5. *Let $s > 0$ and suppose that we have two couples (φ,ψ^l) and $(\widetilde{\varphi},\widetilde{\psi}^l)$, $l = 1,2,\ldots,2^n-1$ of $C^{[s+1]}$-functions as in (1.3). Then for all $f \in L^1_{\rm loc}$,*

$$\|V^{\widetilde{\varphi}}f\|_{L^{p(\cdot)}(w)} + \|W_s^{\widetilde{\psi}^l}f\|_{L^{p(\cdot)}(w)} \sim \|V^{\varphi}f\|_{L^{p(\cdot)}(w)} + \|W_s^{\psi^l}f\|_{L^{p(\cdot)}(w)}.$$

Proof. It suffices to show

$$\|V^{\widetilde{\varphi}}f\|_{L^{p(\cdot)}(w)} + \|W_s^{\widetilde{\psi}^l}f\|_{L^{p(\cdot)}(w)} \lesssim \|V^{\varphi}f\|_{L^{p(\cdot)}(w)} + \|W_s^{\psi^l}f\|_{L^{p(\cdot)}(w)}$$

by symmetry. We content ourselves with the proof of

$$\|W_s^{\widetilde{\psi}^l}f\|_{L^{p(\cdot)}(w)} \lesssim \|V^{\varphi}f\|_{L^{p(\cdot)}(w)} + \|W_s^{\psi^l}f\|_{L^{p(\cdot)}(w)},$$

since the proof of $\|V^{\widetilde{\varphi}} f\|_{L^{p(\cdot)}(w)} \lesssim \|V^{\varphi} f\|_{L^{p(\cdot)}(w)} + \|W_s^{\psi^l} f\|_{L^{p(\cdot)}(w)}$ is similar.

Let $j_0 \geq J$, $k_0 \in \mathbb{Z}^n$, and $l_0 = 1, 2, \ldots, 2^n - 1$ be fixed. We calculate

$$\begin{aligned}
&|\langle \widetilde{\psi}^{l_0}_{j_0,k_0}, f\rangle| \\
&\leq \sum_{k\in\mathbb{Z}^n} |\langle \widetilde{\psi}^{l_0}_{j_0,k_0}, \varphi_{J,k}\rangle| \cdot |\langle f, \varphi_{J,k}\rangle| + \sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} |\langle \widetilde{\psi}^{l_0}_{j_0,k_0}, \psi^l_{j,k}\rangle| \cdot |\langle f, \psi^l_{j,k}\rangle|.
\end{aligned}$$

The moment condition yields

$$\begin{aligned}
&|\langle \widetilde{\psi}^{l_0}_{j_0,k_0}, f\rangle| \\
&\lesssim \sum_{k\in\mathbb{Z}^n} \chi_{[0,2^{-J+1}(2N-1)]}(|2^{-J}k - 2^{-j_0}k_0|) 2^{-\frac{(J+j_0)n}{2} + Jn - (j_0-J)[s+1]} |\langle f, \varphi_{J,k}\rangle| \\
&\quad + \sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} \chi_{[0,2^{1-\min(j,j_0)}(2N-1)]}(|2^{-j}k - 2^{-j_0}k_0|) \\
&\qquad \times 2^{-\frac{(j+j_0)n}{2} + \min(j,j_0)n - |j_0-j|[s+1]} |\langle f, \psi^l_{j,k}\rangle|,
\end{aligned}$$

where we also employed Lemma 2.13 (3) and (4). Consequently

$$\begin{aligned}
&2^{j_0 s} |\langle \widetilde{\psi}^{l_0}_{j_0,k_0}, f\rangle| \chi_{j_0,k_0} \\
&\lesssim \sum_{k\in\mathbb{Z}^n} 2^{j_0 s - \frac{(J+j_0)n}{2} + Jn - (j_0-J)[s+1]} |\langle f, \varphi_{J,k}\rangle| \chi^*_{j_0,k_0} \chi^*_{J,k} \\
&\quad + \sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} 2^{j_0 s - \frac{(j+j_0)n}{2} + \min(j,j_0)n - |j_0-j|[s+1]} |\langle f, \psi^l_{j,k}\rangle| \chi^*_{j_0,k_0} \chi^*_{j,k} \\
&= \sum_{k\in\mathbb{Z}^n} 2^{-(j_0-J)([s+1]-s)} 2^{Jn + Js - \frac{(J+j_0)n}{2}} |\langle f, \varphi_{J,k}\rangle| \chi^*_{j_0,k_0} \chi^*_{J,k} \\
&\quad + \sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} 2^{-|j_0-j|([s+1]-s)} 2^{js - \frac{(j+j_0)n}{2} + \min(j,j_0)n} |\langle f, \psi^l_{j,k}\rangle| \chi^*_{j_0,k_0} \chi^*_{j,k}.
\end{aligned}$$

If we add this estimate over $k_0 \in \mathbb{Z}^n$, then

$$\begin{aligned}
&\sum_{k_0\in\mathbb{Z}^n} 2^{j_0 s} |\langle \widetilde{\psi}^{l_0}_{j_0,k_0}, f\rangle| \chi_{j_0,k_0} \\
&\lesssim \sum_{k\in\mathbb{Z}^n} 2^{-(j_0-J)([s+1]-s)} 2^{Jn + Js - \frac{(J+j_0)n}{2}} |\langle f, \varphi_{J,k}\rangle| \chi^*_{J,k} \\
&\quad + \sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} 2^{-|j_0-j|([s+1]-s)} 2^{js - \frac{(j+j_0)n}{2} + \min(j,j_0)n} |\langle f, \psi^l_{j,k}\rangle| \chi^*_{j,k} \\
&\lesssim \sum_{k\in\mathbb{Z}^n} 2^{-(j_0-J)([s+1]-s)} 2^{Jn + Js - \frac{(J+j_0)n}{2}} |\langle f, \varphi_{J,k}\rangle| \chi^*_{J,k} \\
&\quad + \left(\sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} 2^{-2|j_0-j|([s+1]-s)} 2^{2js} |\langle f, \psi^l_{j,k}\rangle|^2 \chi^*_{j,k} \right)^{\frac{1}{2}}.
\end{aligned}$$

The last inequality uses the Cauchy–Schwartz inequality. Next, we take the ℓ^2-norm over $j_0 \geq J$, which gives

$$\sum_{j_0=J}^{\infty}\left(\sum_{k_0\in\mathbb{Z}^n} 2^{j_0 s}|\langle\widetilde{\psi}^{l_0}_{j_0,k_0}, f\rangle|\chi_{j_0,k_0}\right)^2$$

$$\lesssim \sum_{k\in\mathbb{Z}^n}|\langle f,\varphi_{J,k}\rangle|\chi^*_{J,k} + \left(\sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} 2^{2js}|\langle f,\psi^l_{j,k}\rangle|^2\chi^*_{j,k}\right)^{\frac{1}{2}}.$$

Thus, we obtain $W^{\widetilde{\psi}^l}_s f \lesssim V^{\varphi}f + W^{\psi^l}_s f$, which yields (3.7) and the desired result. □

Theorem 4.6. *Suppose that $p(\cdot) \in \mathcal{P}\cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, and $w \in A^{\mathrm{loc}}_{p(\cdot)}$. Let $s > 0$ and $\Psi = \Psi^{\varphi,\psi^l} : f \in L^{p(\cdot)}(w) \mapsto \Psi(f) = \Psi^{\varphi,\psi^l}(f) \in \widetilde{\mathbf{f}}^{p(\cdot),0}_2(w)$ be the wavelet coefficient operator with $\varphi, \psi^l \in C^{[s+1]}$. Then $f \in L^{p(\cdot)}(w)$ belongs to $L^{p(\cdot),s}(w)$ if and only if $W_s f = W^{\psi^l}_s f \in L^{p(\cdot)}(w)$.*

It should be noted that $Vf \in L^{p(\cdot)}(w)$ since the case of $s = 0$ is already covered in [26].

Proof. We may assume that $s \in (0,\infty)\setminus\mathbb{N}$. Otherwise, Theorem 4.6 is covered by Theorem 1.3. First, assume that the collection $\widetilde{\varphi}, \widetilde{\psi}^l, l = 1, 2, \ldots, 2^n - 1$ satisfies the requirements in (1.3) with $[s+1]$ replaced by $[s+2]$. Here, we keep in mind that (1.3) requires that $\varphi, \psi^l \in C^{m+1}$ for the function space $L^{p(\cdot),m}$ with $m \in \mathbb{N}$. Since we know that $\Psi^{\widetilde{\varphi},\widetilde{\psi}^l}$ is an isomorphism from $L^{p(\cdot)}(w)$ to $\widetilde{\mathbf{f}}^{p(\cdot),0}_2(w)$ and from $L^{p(\cdot),[s+1]}(w)$ to $\widetilde{\mathbf{f}}^{p(\cdot),[s+1]}_2(w)$ by the complex interpolation, we see that $\Psi^{\widetilde{\varphi},\widetilde{\psi}^l}$ is an isomorphism from $L^{p(\cdot),s}(w)$ to $\widetilde{\mathbf{f}}^{p(\cdot),s}_2(w)$.

Since the expressions

$$\|V^{\widetilde{\varphi}}f\|_{L^{p(\cdot)}(w)} + \|W^{\widetilde{\psi}^l}_s f\|_{L^{p(\cdot)}(w)}, \|V^{\varphi}f\|_{L^{p(\cdot)}(w)} + \|W^{\psi^l}_s f\|_{L^{p(\cdot)}(w)}$$

are equivalent due to Lemma 4.5 as long as the collection φ, ψ^l satisfies the requirements in (1.3), it follows that $f \in L^{p(\cdot)}(w)$ belongs to $L^{p(\cdot),s}(w)$ if and only if $W^{\widetilde{\psi}^l}_s f \in L^{p(\cdot)}(w)$, or equivalently, $W^{\psi^l}_s f \in L^{p(\cdot)}(w)$.

□

4.4 Atomic decomposition

Fix $J \in \mathbb{Z}$.

Definition 4.7. Let $s \in \mathbb{R}$, $j \in \mathbb{Z}\cap(J,\infty)$, and $k \in \mathbb{Z}^n$. Suppose that the integers $K, L \in \mathbb{Z}$ satisfy $K \geq 0$ and $L \geq -1$.

1. A function $a \in C^K$ is an *atom centered at $Q_{J,k}$* if the following differential inequality holds:

$$|\partial^\alpha a| \leq \chi_{3Q_{J,k}}, \quad |\alpha| \leq K. \tag{4.2}$$

2. A function $a \in C^K \cap \mathcal{P}_L^\perp$ is an *atom centered at* $Q_{j,k}$ if it satisfies the following differential inequality:

$$|\partial^\alpha a| \le 2^{\frac{jn}{2}+j|\alpha|}\chi_{3Q_{j,k}}, \quad |\alpha| \le K. \tag{4.3}$$

Theorem 4.8. *Suppose that* $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$*, and* $w \in A^{\mathrm{loc}}_{p(\cdot)}$*. Let* $s > 0$*. Suppose that we have atoms* $a_{\nu,m}$ *centered at* $Q_{\nu,m}$ *for each* $\nu \in \mathbb{Z} \cap [J,\infty)$ *and* $m \in \mathbb{Z}^n$*. Additionally, let*

$$\lambda \equiv \{\lambda_{\nu,m}\}_{\nu \in \mathbb{Z} \cap [J,\infty), m \in \mathbb{Z}^n} \subset \mathbb{C}$$

satisfy

$$\left\| \left(\sum_{\nu=J}^\infty \left| 2^{\nu s} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m}\chi_{\nu,m} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)} < \infty.$$

Then

$$f \equiv \sum_{\nu=J}^\infty \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}$$

converges in $L^{p(\cdot),s}(w)$ *and satisfies*

$$\|f\|_{L^{p(\cdot),s}(w)} \lesssim \left\| \left(\sum_{\nu=J}^\infty \left| 2^{\nu s} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m}\chi_{\nu,m} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)}.$$

Proof. We can assume that $\lambda_{\nu,m} = 0$ if $\nu + |m| \gg 1$ by the truncation argument. Thus, we can concentrate on the norm estimate. The proof of the convergence of the series defining f follows immediately if this norm estimate is valid at least for the type above of sequences.

We prove

$$\|Vf + W_s f\|_{L^{p(\cdot)}(w)} \lesssim \left\| \left(\sum_{\nu=J}^\infty \left| 2^{\nu s} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m}\chi_{\nu,m} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(w)}.$$

Since Vf is easier to handle than $W_s f$, we concentrate on $W_s f$. Let $l = 1, 2, \ldots, 2^n - 1$ be fixed. Write

$$\delta(j,\nu,k,m) \equiv \chi_{[0,2^{-\min(j,\nu)}D]}(|2^{-\nu}m - 2^{-j}k|)$$

for some large constant $D > 0$.

As long as D is sufficiently large, we have

$$|\langle a_{\nu,m}, \psi^l_{j,k}\rangle| \lesssim 2^{-\frac{\nu+j}{2}n - |\nu-j|[s+1] + \min(j,\nu)n}\delta(j,\nu,k,m).$$

Thus

$$
\begin{aligned}
&\sum_{j=J}^{\infty} 2^{2js} \sum_{k\in\mathbb{Z}^n} |\langle f, \psi^l_{j,k}\rangle|^2 \chi_{j,k}\\
&= \sum_{j=J}^{\infty} 2^{2js} \sum_{k\in\mathbb{Z}^n} \left| \sum_{\nu=J}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} \langle a_{\nu,m}, \psi^l_{j,k}\rangle \right|^2 \chi_{j,k}\\
&\lesssim \sum_{j=J}^{\infty} 2^{2js} \sum_{k\in\mathbb{Z}^n} \left(\sum_{\nu=J}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}| \left|\langle a_{\nu,m}, \psi^l_{j,k}\rangle\right| \right)^2 \chi_{j,k}\\
&\lesssim \sum_{j=J}^{\infty} 2^{2js} \sum_{k\in\mathbb{Z}^n} \left(\sum_{\nu=J}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}| 2^{-\frac{\nu+j}{2}n - |\nu-j|[s+1] + \min(j,\nu)n} \delta(j,\nu,k,m) \right)^2 \chi_{j,k}\\
&\lesssim \sum_{j=J}^{\infty} \sum_{k\in\mathbb{Z}^n} \left(\sum_{\nu=J}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}| 2^{-\frac{\nu+j}{2}n - |\nu-j|([s+1]-s) - s|\nu-j| + sj + \min(j,\nu)n} \delta(j,\nu,k,m) \right)^2 \chi_{j,k}.
\end{aligned}
$$

Arithmetic shows

$$
\begin{aligned}
&-\frac{\nu+j}{2}n - |\nu-j|([s+1]-s) - s|\nu-j| + sj + \min(j,\nu)n\\
&\le \begin{cases} s\nu + \dfrac{\nu}{2}n - (\nu-j)([s+1]+n) - \dfrac{j}{2}n & (\text{if } \nu \ge j),\\ s\nu + \dfrac{\nu}{2}n - (j-\nu)([s+1]-s) - \dfrac{j}{2}n & (\text{if } \nu \le j). \end{cases}
\end{aligned}
$$

We also note that

$$
\sum_{m\in\mathbb{Z}^n} 2^{\nu s} 2^{\frac{\nu}{2}n} |\lambda_{\nu,m}| \delta(j,\nu,k,m) \chi_{Q_{j,k}} \lesssim M^{\mathrm{loc}} \left[\sum_{m\in\mathbb{Z}^n} 2^{\nu s} \lambda_{\nu,m} \chi_{\nu,m} \right] \chi_{Q_{j,k}}.
$$

By virtue of these observations, we obtain

$$
\begin{aligned}
&\sum_{j=J}^{\infty} 2^{2js} \sum_{k\in\mathbb{Z}^n} |\langle f, \psi^l_{j,k}\rangle|^2 \chi_{j,k}\\
&\lesssim \sum_{j=J}^{\infty} \sum_{k\in\mathbb{Z}^n} \left(\sum_{\nu=J}^{\infty} 2^{-|j-\nu|([s+1]-s)} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}| 2^{s\nu + \frac{\nu}{2}n} \delta(j,\nu,k,m) \chi_{Q_{j,k}} \right)^2\\
&\lesssim \sum_{j=J}^{\infty} \sum_{k\in\mathbb{Z}^n} \left(\sum_{\nu=J}^{\infty} 2^{-|j-\nu|([s+1]-s)} M^{\mathrm{loc}} \left[\sum_{m\in\mathbb{Z}^n} 2^{\nu s} \lambda_{\nu,m} \chi_{\nu,m} \right] \chi_{Qj,k} \right)^2.
\end{aligned}
$$

A geometric observation shows that

$$
\sum_{k\in\mathbb{Z}^n} \chi_{Q_{j,k}} = 1
$$

for all $j \in \mathbb{Z}$. By the Cauchy–Schwartz inequality for the summation over ν and j, we obtain

$$\begin{aligned}
&\sum_{j=J}^{\infty} 2^{2js} \sum_{k\in\mathbb{Z}^n} |\langle f, \psi^l_{j,k}\rangle|^2 \chi_{j,k} \\
&\lesssim \sum_{j=J}^{\infty} \left(\sum_{\nu=J}^{\infty} 2^{-|j-\nu|([s+1]-s)} M^{\rm loc}\left[\sum_{m\in\mathbb{Z}^n} 2^{\nu s} \lambda_{\nu,m}\chi_{\nu,m} \right] \right)^2 \\
&\lesssim \sum_{j=J}^{\infty} \left(\sum_{\nu=J}^{\infty} 2^{-|j-\nu|([s+1]-s)} \right) \left(\sum_{\nu=J}^{\infty} 2^{-|j-\nu|([s+1]-s)} \left(M^{\rm loc}\left[\sum_{m\in\mathbb{Z}^n} 2^{\nu s} \lambda_{\nu,m}\chi_{\nu,m} \right] \right)^2 \right) \\
&\sim \sum_{j=J}^{\infty} \sum_{\nu=J}^{\infty} 2^{-|j-\nu|([s+1]-s)} \left(M^{\rm loc}\left[\sum_{m\in\mathbb{Z}^n} 2^{\nu s} \lambda_{\nu,m}\chi_{\nu,m} \right] \right)^2 \\
&\sim \sum_{\nu=J}^{\infty} \left(M^{\rm loc}\left[\sum_{m\in\mathbb{Z}^n} 2^{\nu s} \lambda_{\nu,m}\chi_{\nu,m} \right] \right)^2,
\end{aligned}$$

where we change the order of the summation in the last inequality. Finally, taking the $L^{p(\cdot)}(w)$-norm, we obtain the desired result. □

4.5 Characterization by Haar wavelets

Here, we demonstrate the Haar-wavelet characterization. See [17, 32] for more information on some recent research. We define

$$h^M = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}, \quad h^F = \chi_{[0,1)}.$$

The functions h^M and h^F are called the Haar wavelet and the Haar scaling function, respectively.

We set

$$\{M,F\}^{n*} \equiv \{M,F\}^n \setminus \{(F,F,\dots,F)\}.$$

Haar wavelets on $\mathbb{R}^n$ are defined by the usual tensor product procedure:

$$H^G_{j,k} \equiv \otimes_{r=1}^n h^{G_r}_{j,k_r}$$

for $j \in \mathbb{Z} \cap [J,\infty)$,

$$G = (G_1,\dots,G_n) \in G^j = \begin{cases} \{M,F\}^n & (j = J), \\ \{M,F\}^{n*} & (j > J) \end{cases}$$

and

$$k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n.$$

In our next theorem, we show that $H^G_{j,k}$ can be substituted for $\psi^l_{j,k}$ and $\varphi_{J,k}$.

Theorem 4.9. *Let $s \in [0, \frac{1}{\max(2,p_+)})$. Then, for all $f \in L^{p(\cdot),s}(w)$,*

$$\left\| \left(\sum_{j=J}^{\infty} \sum_{G \in G^j} \left| 2^{js} \sum_{k \in \mathbb{Z}^n} \langle f, H_{j,k}^G \rangle H_{j,k}^G \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot),s}(w)} \lesssim \|f\|_{L^{p(\cdot),s}(w)}. \tag{4.4}$$

Proof. We will justify that we may assume $f \in L^{p(\cdot),s}(w) \cap L^2$. In fact, thanks to Theorem 4.8, the space $L^{p(\cdot),s}(w) \cap L^2$ is dense in $L^{p(\cdot),s}(w)$. Thus, for $f \in L^{p(\cdot),s}(w)$, we can choose $\{f_l\}_{l=1}^{\infty} \subset L^{p(\cdot),s}(w) \cap L^2$ such that $f_l \to f$ in $L^{p(\cdot),s}(w)$. Suppose that we have proved (4.4) for any $f \in L^{p(\cdot),s}(w) \cap L^2$. Then we have

$$\left\| \left(\sum_{j=J}^{\infty} \sum_{G \in G^j} \left| 2^{js} \sum_{k \in \mathbb{Z}^n} \langle f_l, H_{j,k}^G \rangle H_{j,k}^G \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot),s}(w)} \lesssim \|f_l\|_{L^{p(\cdot),s}(w)}.$$

Then by Fatou's lemma and $\lim_{l \to \infty} \langle f_l, H_{j,k}^G \rangle = \langle f, H_{j,k}^G \rangle$, we have

$$\begin{aligned} &\left\| \left(\sum_{j=J}^{\infty} \sum_{G \in G^j} \left| 2^{js} \sum_{k \in \mathbb{Z}^n} \langle f, H_{j,k}^G \rangle H_{j,k}^G \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot),s}(w)} \\ &\le \liminf_{k \to \infty} \left\| \left(\sum_{j=J}^{\infty} \sum_{G \in G^j} \left| 2^{js} \sum_{k \in \mathbb{Z}^n} \langle f_l, H_{j,k}^G \rangle H_{j,k}^G \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot),s}(w)}. \end{aligned}$$

Since $\|f_l\|_{L^{p(\cdot),s}(w)} \to \|f\|_{L^{p(\cdot),s}(w)}$ as $l \to \infty$, (4.4) holds for any $f \in L^{p(\cdot),s}(w)$.

Let $f \in L^{p(\cdot),s}(w) \cap L^2$ here and below. Let

$$f = \sum_{j=J}^{\infty} \sum_{k \in \mathbb{Z}^n} \sum_{G \in G^j} \langle f, H_{j,k}^G \rangle H_{j,k}^G$$

be the wavelet decomposition. We may assume that the sum is finite using the simple truncation procedure. Then

$$\begin{aligned} f = &\sum_{j=J}^{\infty} \sum_{k,m \in \mathbb{Z}^n} \sum_{G \in G^j} \langle f, H_{j,k}^G \rangle \cdot \langle H_{j,k}^G, \varphi_{J,m} \rangle \varphi_{J,m} \\ &+ \sum_{j,\nu=J}^{\infty} \sum_{k,m \in \mathbb{Z}^n} \sum_{G \in G^j} \sum_{l=1}^{2^n-1} \langle f, H_{j,k}^G \rangle \cdot \langle H_{j,k}^G, \psi_{\nu,m}^l \rangle \psi_{\nu,m}^l. \end{aligned}$$

Write

$$\begin{aligned} \mu_{\nu,m}^{l,G,1} &\equiv \sum_{j=\nu}^{\infty} \sum_{k \in \mathbb{Z}^n} \langle f, H_{j,k}^G \rangle \cdot \langle H_{j,k}^G, \psi_{\nu,m}^l \rangle, \\ \mu_{\nu,m}^{l,G,2} &\equiv \sum_{j=J}^{\nu-1} \sum_{k \in \mathbb{Z}^n} \langle f, H_{j,k}^G \rangle \cdot \langle H_{j,k}^G, \psi_{\nu,m}^l \rangle, \\ \mu_{\nu,m}^{l,G} &\equiv \mu_{\nu,m}^{l,G,1} + \mu_{\nu,m}^{l,G,2}. \end{aligned}$$

We estimate
$$|\mu^{l,G,1}_{\nu,m}| \lesssim \sum_{j=\nu}^{\infty} \sum_{\substack{k\in\mathbb{Z}^n,\\ \langle H^G_{j,k},\psi^l_{\nu,m}\rangle\neq 0}} 2^{\frac{n}{2}(\nu-j)}|\langle f, H^G_{j,k}\rangle|.$$
We fix a small constant ε that is specified in the end of the proof. Note that $\langle H^G_{j,k}, \psi^l_{\nu,m}\rangle = \mathrm{O}(2^{\frac{n}{2}(\nu-j)})$. As a result,
$$\begin{aligned}
(|\mu^{l,G,1}_{\nu,m}|\chi_{\nu,m})^2 &\lesssim \left(\sum_{j=\nu}^{\infty} \sum_{\substack{k\in\mathbb{Z}^n,\\ \langle H^G_{j,k},\psi^l_{\nu,m}\rangle\neq 0}} 2^{\frac{n}{2}(\nu-j)}|\langle f, H^G_{j,k}\rangle|\chi_{\nu,m}\right)^2\\
&\lesssim \sum_{j=\nu}^{\infty} 2^{(n-\varepsilon)(\nu-j)}\left(\sum_{\substack{k\in\mathbb{Z}^n,\\ \langle H^G_{j,k},\psi^l_{\nu,m}\rangle\neq 0}} |\langle f, H^G_{j,k}\rangle|\chi_{\nu,m}\right)^2\\
&\lesssim \sum_{j=\nu}^{\infty} 2^{-5\varepsilon(\nu-j)}\left(\sum_{\substack{k\in\mathbb{Z}^n,\\ \langle H^G_{j,k},\psi^l_{\nu,m}\rangle\neq 0}} \frac{2^{\frac{jn}{2}}|\langle f, H^G_{j,k}\rangle|}{(1+|2^j\cdot-k|)^{n+2\varepsilon}}\chi_{Q_{\nu,m}}\right)^2\\
&\lesssim \sum_{j=\nu}^{\infty} 2^{-5\varepsilon(\nu-j)}\left(\sum_{\substack{k\in\mathbb{Z}^n,\\ \langle H^G_{j,k},\psi^l_{\nu,m}\rangle\neq 0}} 2^{\frac{jn}{2}}|\langle f, H^G_{j,k}\rangle| M^{\mathrm{loc}}[\chi_{Q_{j,k}}](x)^{\frac{n+\varepsilon}{n}}\right)^2 \chi_{Q_{\nu,m}}.
\end{aligned}$$
Thus, by the vector-valued maximal inequality for M^{loc},
$$\begin{aligned}
&\sum_{\nu=J}^{\infty}\sum_{m\in\mathbb{Z}^n}\sum_{G\in G^j}\sum_{l=1}^{2^n-1} \mu^{l,G,1}_{\nu,m}\psi^l_{\nu,m}\\
&\lesssim \sum_{\nu=J}^{\infty}\sum_{m\in\mathbb{Z}^n}\sum_{j=\nu}^{\infty} 2^{-5\varepsilon(\nu-j)}\left(\sum_{\substack{k\in\mathbb{Z}^n,\\ \langle H^G_{j,k},\psi^l_{\nu,m}\rangle\neq 0}} 2^{\frac{jn}{2}}|\langle f, H^G_{j,k}\rangle| M^{\mathrm{loc}}[\chi_{Q_{j,k}}](x)^{\frac{n+\varepsilon}{n}}\right)^2 \chi_{Q_{\nu,m}}.
\end{aligned}$$
we can control the first term.

Next, we consider the second term. We calculate

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}^n}(|\mu^{l,G,2}_{\nu,m}|\chi_{\nu,m})^2 &\lesssim \sum_{m\in\mathbb{Z}^n}\sum_{j=J}^{\nu-1}2^{2(\nu-j)\varepsilon-(\nu-j)n}\sum_{\substack{k\in\mathbb{Z}^n,\\ \langle H^G_{j,k},\psi^l_{\nu,m}\rangle\neq 0}}(|\langle f,H^G_{j,k}\rangle|\chi_{\nu,m})^2\\
&=\sum_{k\in\mathbb{Z}^n}\sum_{j=J}^{\nu-1}2^{2(\nu-j)\varepsilon-(\nu-j)n}\sum_{\substack{m\in\mathbb{Z}^n,\\ \langle H^G_{j,k},\psi^l_{\nu,m}\rangle\neq 0}}(|\langle f,H^G_{j,k}\rangle|\chi_{\nu,m})^2\\
&\le\sum_{k\in\mathbb{Z}^n}\sum_{j=J}^{\nu-1}2^{2\varepsilon(\nu-j)}\sum_{\substack{m\in\mathbb{Z}^n,\\ \langle H^G_{j,k},\psi^l_{\nu,m}\rangle\neq 0}}(2^{\frac{jn}{2}}|\langle f,H^G_{j,k}\rangle|\chi_{Q_{\nu,m}})^2.
\end{aligned}
$$

Thus, if we write $a^G_{j,k}\equiv 2^{js+\frac{jn}{2}}\langle f,H^G_{j,k}\rangle$, then we obtain

$$
\begin{aligned}
&\left\{\sum_{\nu=J}^{\infty}\sum_{m\in\mathbb{Z}^n}(2^{\nu s}|\mu^{l,G,2}_{\nu,m}|\chi_{\mu,m})^2\right\}^{\frac12}\\
&\lesssim\left\{\sum_{\nu=J}^{\infty}\sum_{k\in\mathbb{Z}^n}\sum_{j=J}^{\nu-1}2^{2(\nu-j)(s+\varepsilon)}\sum_{\substack{m\in\mathbb{Z}^n,\\ \langle H^G_{j,k},\psi^l_{\nu,m}\rangle\neq 0}}(|a^G_{j,k}|\chi_{Q_{\nu,m}})^2\right\}^{\frac12}\\
&=\left\{\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n}|a^G_{j,k}|^2\sum_{\nu=j+1}^{\infty}2^{2(\nu-j)(s+\varepsilon)}\sum_{\substack{m\in\mathbb{Z}^n,\\ \langle H^G_{j,k},\psi^l_{\nu,m}\rangle\neq 0}}\chi_{Q_{\nu,m}}\right\}^{\frac12}.
\end{aligned}
$$

Taking the $L^{p(\cdot)}(w)$-norm, we obtain

$$
\begin{aligned}
&\left\|\left\{\sum_{\nu=J}^{\infty}\sum_{m\in\mathbb{Z}^n}(2^{\nu s}|\mu^{l,G,2}_{\nu,m}|\chi_{\nu,m})^2\right\}^{\frac12}\right\|_{L^{p(\cdot)}(w)}\\
&\lesssim\left\|\left\{\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n}|a^G_{j,k}|^2\sum_{\nu=j+1}^{\infty}2^{2(\nu-j)(s+\varepsilon)}\sum_{\substack{m\in\mathbb{Z}^n,\\ \langle H^G_{j,k},\psi^l_{\nu,m}\rangle\neq 0}}\chi_{Q_{\nu,m}}\right\}^{\frac12}\right\|_{L^{p(\cdot)}(w)}.
\end{aligned}
$$

Given a measurable function F, we write

$$
m^{(q)}_Q(F)=\frac{\|F\|_{L^q(Q)}}{|Q|^{\frac1q}}.
$$

Using Lemma 2.8, we deduce

$$\begin{aligned}
&\left\| \left\{ \sum_{\nu=J}^{\infty} \sum_{m\in\mathbb{Z}^n} (2^{\nu s}|\mu^{l,G,2}_{\nu,m}|\chi_{\nu,m})^2 \right\}^{\frac12} \right\|_{L^{p(\cdot)}(w)} \\
&\lesssim \left\| \left\{ \sum_{j=J}^{\infty} \sum_{k\in\mathbb{Z}^n} |a^G_{j,k}|^2 \sum_{\nu=j+1}^{\infty} 2^{2(\nu-j)(s+\varepsilon)} m^{(q)}_{D\,Q_{j,k}} \left(\sum_{\substack{m\in\mathbb{Z}^n,\\ \langle H^G_{j,k},\psi^l_{\nu,m}\rangle\neq 0}} \chi_{Q_{\nu,m}} \right) \chi_{D\,Q_{j,k}} \right\}^{\frac12} \right\|_{L^{p(\cdot)}(w)} \\
&\lesssim \left\| \left\{ \sum_{j=J}^{\infty} \sum_{k\in\mathbb{Z}^n} |a^G_{j,k}|^2 \sum_{\nu=j+1}^{\infty} 2^{(\nu-j)(2s+2\varepsilon-1/q)} \chi_{D\,Q_{j,k}} \right\}^{\frac12} \right\|_{L^{p(\cdot)}(w)}.
\end{aligned}$$

Since $s < \frac{1}{\max(2,p_+)}$, we can choose $q > \frac{\max(2,p_+)}{2}$ and $\varepsilon > 0$ such that $s+\varepsilon < \frac{1}{2q}$. Thus

$$\left\| \left\{ \sum_{\nu=J}^{\infty} \sum_{m\in\mathbb{Z}^n} (2^{\nu s}|\mu^{l,G,2}_{\nu,m}|\chi_{\nu,m})^2 \right\}^{\frac12} \right\|_{L^{p(\cdot)}(w)} \lesssim \left\| \left\{ \sum_{j=J}^{\infty} \sum_{k\in\mathbb{Z}^n} |a^G_{j,k}|^2 \chi_{D\,Q_{j,k}} \right\}^{\frac12} \right\|_{L^{p(\cdot)}(w)},$$

proving Theorem 4.9. □

5 Key theorems to more applications

Here, we are interested in applications of wavelet decomposition and complex interpolation. Section 5.1 considers pointwise multiplication. As an application of Section 5.1 and wavelet characterization, Section 5.2 considers a variant of the compact embedding. Section 5.3 deals with diffeomorphisms. Section 5.4 generalizes Proposition 4.4.

5.1 Pointwise multipliers

For $m = 0, 1, 2, \ldots$, define $\mathcal{B}^m$ to be the set of all $f \in C^m$ with bounded partial derivatives up to order m. If $s \in \mathbb{N}_0$, then any $h \in \mathcal{B}^s$ induces the bounded pointwise multiplication M_h by the Leibnitz rule. We generalize this fact to $s > 0$.

Theorem 5.1. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, and let $(s,w) \in ((0,\infty)\setminus\mathbb{N}) \times A^{\mathrm{loc}}_{p(\cdot)}$. Then for all $h \in \mathcal{B}^{[s+1]}$, mapping $M_h : f \in L^{p(\cdot),s}(w) \mapsto h\cdot f \in L^{p(\cdot),s}(w)$ is bounded.*

Proof. As mentioned above, M_h is bounded on $L^{p(\cdot),[s]}(w)$ and $L^{p(\cdot),[s+1]}(w)$. Hence, we can interpolate these estimates to have the desired result using Proposition 4.4. □

5.2 Compact embedding

We consider the Rellich–Kondrachev-type compact embedding. Let $s_0 > s_1 > 0$. Then $L^{p(\cdot),s_0}(w) \subset L^{p(\cdot),s_1}(w)$. Thus, it makes sense to consider the embedding operator $\iota : L^{p(\cdot),s_0}(w) \hookrightarrow L^{p(\cdot),s_1}(w)$. After suitably truncating the functions, we can prove the Rellich–Kondrachev-type compact embedding resuit.

Theorem 5.2. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, and let $w \in A_{p(\cdot)}^{\mathrm{loc}}$. Then for $s_0 > s_1 > 0$ and $h \in C^{[s_0+1]}$ with compact support, the operator $M_h \circ \iota$ is a compact operator.*

Proof. Let Φ, Ψ be as in Section 4.3. For $j_0 \geq J$, we define $T_{j_0} : \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w) \to \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)$ to be the truncation operator given by

$$\begin{aligned} &T_{j_0}(\{\lambda_k\}_{k \in \mathbb{Z}^n} \cup \{\lambda_{j,k}^l\}_{j \in \mathbb{Z} \cap [J,\infty), k \in \mathbb{Z}^n, l \in \{1,2,\ldots,2^n-1\}}) \\ &\quad = \{\lambda_k\}_{k \in \mathbb{Z}^n} \cup \{\chi_{[J,j_0]}(j)\lambda_{j,k}^l\}_{j \in \mathbb{Z} \cap [J,\infty), k \in \mathbb{Z}^n, l \in \{1,2,\ldots,2^n-1\}}. \end{aligned}$$

Denote by $\iota^{\mathbf{f}} : \widetilde{\mathbf{f}}_2^{p(\cdot),s_0}(w) \to \widetilde{\mathbf{f}}_2^{p(\cdot),s_1}(w)$ the natural embedding. Set $P_{j_0} \equiv \Phi \circ T_{j_0} \circ \Psi$. Then thanks to Theorem 4.6, $\|\iota - P_{j_0}\|_{L^{p(\cdot),s_0}(w) \to L^{p(\cdot),s_1}(w)} \lesssim 2^{-j_0(s_0-s_1)}$ since $\|\iota^{\mathbf{f}} - T_{j_0}\|_{\widetilde{\mathbf{f}}_2^{p(\cdot),s_0}(w) \to \widetilde{\mathbf{f}}_2^{p(\cdot),s_1}(w)} \lesssim 2^{-j_0(s_0-s_1)}$. Hence, it follows from Theorem 5.1 that

$$\|M_h \circ \iota - M_h \circ P_{j_0}\|_{L^{p(\cdot),s_0}(w) \to L^{p(\cdot),s_1}(w)} \lesssim 2^{-j_0(s_0-s_1)}.$$

Thus, $M_h \circ \iota$ is realized as the norm limit of the operators $\{M_h \circ P_{j_0}\}_{j_0=J}^\infty$ with a finite rank. Hence $M_h \circ \iota$ is a compact operator. □

5.3 Diffeomorphism

A C^M-diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *regular* if ψ and its inverse belong to $\mathcal{B}^M$.

Theorem 5.3. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, and let $(s,w) \in ((0,\infty) \setminus \mathbb{Z}) \times A_{p(\cdot)}^{\mathrm{loc}}$. Then for all regular $C^{[s+1]}$-diffeomorphisms $\psi : \mathbb{R}^n \to \mathbb{R}^n$ that preserves $L^{p(\cdot)}(w)$, composition mapping $f \in L^{p(\cdot),s}(w) \mapsto f \circ \psi \in L^{p(\cdot),s}(w)$ is bounded.*

If $s \in \mathbb{N}_0$, then any $h \in \mathcal{B}^s$ induces composition mapping.

Proof. It is clear that $f \in L^{p(\cdot),[s]}(w) \mapsto f \circ \psi \in L^{p(\cdot),[s]}(w)$ and $f \in L^{p(\cdot),[s+1]}(w) \mapsto f \circ \psi \in L^{p(\cdot),[s+1]}(w)$ are bounded. Hence, we can interpolate these estimates to have the desired result. □

5.4 Complex interpolation II

We obtain the complex interpolation as a direct consequence of Theorem 2.25.

Proposition 5.4. *Suppose that $p_0(\cdot), p_1(\cdot), p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Let $(w_0, w_1, w) \in A_{p_0(\cdot)}^{\mathrm{loc}} \times A_{p_1(\cdot)}^{\mathrm{loc}} \times A_{p(\cdot)}^{\mathrm{loc}}$, s_0, s_1, s and θ satisfy $s_0, s_1, s \geq 0$, $0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$, respectively. In addition, assume that*

$$w^{\frac{1}{p(\cdot)}} = w_0^{\frac{1-\theta}{p_0(\cdot)}} w_1^{\frac{\theta}{p_1(\cdot)}}, \quad \frac{1}{p(\cdot)} = \frac{1-\theta}{p_0(\cdot)} + \frac{\theta}{p_1(\cdot)}.$$

Then $[\widetilde{\mathbf{f}}_2^{p_0(\cdot),s_0}(w_0), \widetilde{\mathbf{f}}_2^{p_1(\cdot),s_1}(w_1)]_\theta = \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)$.

Proof. According to [15, Theorem 7.1.2], $(L^{p_0(\cdot)})^{1-\theta}(L^{p_1(\cdot)})^\theta = L^{p(\cdot)}$. Since $f \in L^{p(\cdot)}(w)$ if and only if $f w^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}$, $(L^{p_0(\cdot)}(w_0))^{1-\theta}(L^{p_1(\cdot)}(w_1))^\theta = L^{p(\cdot)}(w)$. This still needs to be combined with Theorem 2.25. □

Using Proposition 5.4 and the mappings Φ and Ψ, we obtain the following conclusion:

Theorem 5.5. *Suppose that* $p_0(\cdot), p_1(\cdot), p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. *Let* $(w_0, w_1, w) \in A_{p_0(\cdot)}^{\mathrm{loc}} \times A_{p_1(\cdot)}^{\mathrm{loc}} \times A_{p(\cdot)}^{\mathrm{loc}}$. *Additionally, let* s_0, s_1, s *and* θ *satisfy* $s_0, s_1, s \geq 0$, $0 < \theta < 1$ *and* $s = (1-\theta)s_0 + \theta s_1$, *respectively. Assume that*

$$w^{\frac{1}{p(\cdot)}} = {w_0}^{\frac{1-\theta}{p_0(\cdot)}} {w_1}^{\frac{\theta}{p_1(\cdot)}}, \quad \frac{1}{p(\cdot)} = \frac{1-\theta}{p_0(\cdot)} + \frac{\theta}{p_1(\cdot)}.$$

Then $[L^{p(\cdot),s_0}(w_0), L^{p(\cdot),s_1}(w_1)]_\theta = L^{p(\cdot),s}(w)$.

Proof. Let $f \in [L^{p(\cdot),s_0}(w_0), L^{p(\cdot),s_1}(w_1)]_\theta$. Then $\Psi(f) \in [\widetilde{\mathbf{f}}_2^{p_0(\cdot),s_0}(w_0), \widetilde{\mathbf{f}}_2^{p_1(\cdot),s_1}(w_1)]_\theta = \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)$. Thus, $f = \Phi \circ \Psi(f) \in L^{p(\cdot),s}(w)$. Conversely, we let $f \in L^{p(\cdot),s}(w)$. Then $\Psi(f) \in \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w) = [\widetilde{\mathbf{f}}_2^{p_0(\cdot),s_0}(w_0), \widetilde{\mathbf{f}}_2^{p_1(\cdot),s_1}(w_1)]_\theta$. Thus, it follows that $f = \Phi \circ \Psi(f) \in [L^{p(\cdot),s_0}(w_0), L^{p(\cdot),s_1}(w_1)]_\theta$. □

6 Sobolev space $L^{p(\cdot),s}(w)$ with $s < 0$

By virture of the duality, we define $L^{p(\cdot),s}(w)$ for $s \in \mathbb{R}$, especially for $s < 0$. Section 6.1, gives its definition and investigate the duality. Section 6.2 expands Section 5. We mimic the proof of the corresponding theorems or use the results in Section 5. Recall that we write $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$.

6.1 Definition of the Sobolev space $L^{p(\cdot),s}(w)$ with $s < 0$

Definition 6.1. Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Let $w \in A_{p(\cdot)}^{\mathrm{loc}}$ and $s < 0$. Then define $L^{p(\cdot),s}(w)$ to be the dual space of $L^{p'(\cdot),-s}(\sigma)$.

Let $s < 0$. Since $L^{p'(\cdot),-s}(\sigma)$ is embedded into $L^{p'(\cdot)}(\sigma)$,

$$\iota : f \in L^{p(\cdot)}(w) \mapsto \left[g \in L^{p'(\cdot),-s}(\sigma) \mapsto \int_{\mathbb{R}^n} g(x) f(x) \mathrm{d}x \in \mathbb{C} \right]$$

is a well-defined injective linear mapping. Thus, ι embeds $L^{p(\cdot)}(w)$ into $L^{p(\cdot),s}(w)$. The next proposition shows that $L^{p(\cdot),s}(w)$ is embedded into $\mathcal{D}'$.

Proposition 6.2. *Suppose that* $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, *and let* $w \in A_{p(\cdot)}^{\mathrm{loc}}$ *and* $s \in \mathbb{R}$. *Then* $L^{p(\cdot),s}(w) \hookrightarrow \mathcal{D}'$ *for* $s \in \mathbb{R}$.

Proof. For $s < 0$, since $\mathcal{D} \subset L^{p'(\cdot),-s}(\sigma)$, we only take a duality. Otherwise, the proof is simpler. □

Next, we shall consider the some characterizations of $L^{p(\cdot),s}(w)$ for $s<0$. To this end, we recall the sequence space $\widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)$ (Section 4.3) and investigate its duality.

Proposition 6.3. *Suppose that $p(\cdot)\in\mathcal{P}\cap\mathrm{LH}_0\cap\mathrm{LH}_\infty$. Let $w\in A_{p(\cdot)}^{\mathrm{loc}}$ and $s<0$. We define the coupling*

$$\langle\lambda,\lambda^*\rangle=\sum_{k\in\mathbb{Z}^n}\lambda_k\lambda_k^*+\sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n}\lambda_{j,k}^l\lambda_{j,k}^{l*}$$

for

$$\lambda=\{\lambda_k\}_{k\in\mathbb{Z}^n}\cup\{\lambda_{j,k}^l\}_{j\in\mathbb{Z}\cap[J,\infty),k\in\mathbb{Z}^n,l\in\{1,2,\ldots,2^n-1\}}\in\widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)$$

and

$$\lambda^*=\{\lambda_k^*\}_{k\in\mathbb{Z}^n}\cup\{\lambda_{j,k}^{l*}\}_{j\in\mathbb{Z}\cap[J,\infty),k\in\mathbb{Z}^n,l\in\{1,2,\ldots,2^n-1\}}\in\widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma).$$

Then $\widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma)$ is canonically embedded into the dual space of $\widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)$ via the coupling

$$\lambda^*\in\widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma)\mapsto[\lambda\in\widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)\mapsto\langle\lambda,\lambda^*\rangle\in\mathbb{C}].$$

Proof. First, we check the well-definedness of the coupling for $\lambda\in\widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)$ and $\lambda\in\widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma)$: We need to check that two series defining $\langle\lambda,\lambda^*\rangle$ converge absolutely. We concentrate on the second term. The first term can be handled by the same way. Moreover, thanks to the triangle inequality and finiteness of l, we only estimate for fixed l. By the orthonomality of $\chi_{j,k}$,

$$\begin{aligned}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n}|\lambda_{j,k}^l\lambda_{j,k}^{l*}|&=\int_{\mathbb{R}^n}\sum_{j=J}^{\infty}2^{\frac{jn}{2}}\sum_{k\in\mathbb{Z}^n}|\lambda_{j,k}^l\lambda_{j,k}^{l*}\chi_{j,k}(x)|\,\mathrm{d}x\\&=\int_{\mathbb{R}^n}\sum_{j=J}^{\infty}2^{js}\sum_{k\in\mathbb{Z}^n}|\lambda_{j,k}^l\chi_{j,k}(x)|\cdot2^{-js}\sum_{k'\in\mathbb{Z}^n}|\lambda_{j,k'}^{l*}\chi_{j,k'}(x)|\,\mathrm{d}x.\end{aligned}$$

Using the Cauchy–Schwartz inequality and the Hölder inequality, we have

$$\begin{aligned}&\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n}|\lambda_{j,k}^l\lambda_{j,k}^{l*}|\\&\le\int_{\mathbb{R}^n}\left(\sum_{j=J}^{\infty}\left|2^{js}\sum_{k\in\mathbb{Z}^n}\lambda_{j,k}^l\chi_{j,k}(x)\right|^2\right)^{\frac12}\left(\sum_{j=J}^{\infty}\left|2^{-js}\sum_{k'\in\mathbb{Z}^n}\lambda_{j,k'}^{l*}\chi_{j,k'}(x)\right|^2\right)^{\frac12}\mathrm{d}x\\&\le\left\|\left(\sum_{j=J}^{\infty}\left|2^{js}\sum_{k\in\mathbb{Z}^n}\lambda_{j,k}^l\chi_{j,k}\right|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}\left\|\left(\sum_{j=J}^{\infty}\left|2^{-js}\sum_{k'\in\mathbb{Z}^n}\lambda_{j,k'}^{l*}\chi_{j,k'}\right|^2\right)^{\frac12}\right\|_{L^{p'(\cdot)}(\sigma)}\\&=\|\lambda\|_{\mathbf{f}_2^{p(\cdot),s}(w)}\|\lambda^*\|_{\mathbf{f}_2^{p'(\cdot),-s}(\sigma)}.\end{aligned}\tag{6.1}$$

Thus, the coupling is well defined.

Finally, by the inequality (6.1), $\widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma)$ is embedded into the dual space of $\widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)$. □

Moreover, we can show that the dual space of $\widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)$ is embeded into $\widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma)$. The case $p(\cdot)$ is constant is considered in [32, 46].

Proposition 6.4. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Let $w \in A^{\rm loc}_{p(\cdot)}$ and $s < 0$. For all $\widetilde{\lambda} \in \left(\widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)\right)^*$, we can represent it uniquely as*

$$\widetilde{\lambda}(\lambda) = \sum_{k\in\mathbb{Z}^n} \lambda_k \lambda_k^* + \sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n} \lambda_{jk}^l \lambda_{jk}^{l*} \ \ (= \langle \lambda, \lambda^* \rangle)$$

for all

$$\lambda = \{\lambda_k\}_{k\in\mathbb{Z}^n} \cup \{\lambda_{j,k}^l\}_{j\in\mathbb{Z}\cap[J,\infty),k\in\mathbb{Z}^n,l\in\{1,2,\ldots,2^n-1\}} \in \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w),$$

where

$$\lambda^* = \{\lambda_k^*\}_{k\in\mathbb{Z}^n} \cup \{\lambda_{j,k}^{l*}\}_{j\in\mathbb{Z}\cap[J,\infty),k\in\mathbb{Z}^n,l\in\{1,2,\ldots,2^n-1\}} \in \widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma).$$

Moreover, we obtain the norm equivalence

$$\|\lambda^*\|_{\widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma)} \sim \|\widetilde{\lambda}\|.$$

Before we move on the proof, we prepare a notation. The space $L^{p(\cdot)}(w,\ell^2)$ is the set of all sequences of measurable functions $\{f_j^l\}_{j\in\mathbb{Z}\cap[J,\infty),l=0,\ldots,2^n-1}$ satisfying

$$\left\|\{f_j^l\}\right\|_{L^{p(\cdot)}(w,\ell^2)} = \left\|\left(\sum_{l=0}^{2^n-1}\sum_{j=J}^{\infty}|f_j^l|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)} < \infty.$$

The proof of this proposition follows the method in [32, 46].

Proof. Note that for simplisity, we denote $\lambda = \{\lambda_{jk}^l\}_{j\in\mathbb{Z}\cap[J,\infty),k\in\mathbb{Z}^n,l=0,\ldots,2^n-1}$, where it is understood that

$$\lambda_{j,k}^0 = \begin{cases} \lambda_k & (j=J), \\ 0 & (\text{otherwise}). \end{cases}$$

Let $\widetilde{\lambda} \in \left(\widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)\right)^*$. Define the map $T : \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w) \to L^{p(\cdot)}(w,\ell^2)$, which assigns $\lambda = \{\lambda_{jk}^l\}_{j,k,l} \mapsto \{f_j^l\}_{j,l}$, where

$$f_j^l \equiv 2^{js}\sum_{k\in\mathbb{Z}^n} \lambda_{jk}^l \chi_{j,k}.$$

Here and below, we omit the range where j,k and l move in the elements such as $\{\lambda_{jk}^l\}_{j,k,l}$. Then, we calculate

$$\begin{aligned}\left\|\{f_j^l\}_{j,l}\right\|_{L^{p(\cdot)}(w,\ell^2)} &= \left\|\left(\sum_{l=0}^{2^n-1}\sum_{j=J}^{\infty}|f_j^l|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)} \\ &= \left\|\left(\sum_{l=0}^{2^n-1}\sum_{j=J}^{\infty}\left|2^{js}\sum_{k\in\mathbb{Z}^n}\lambda_{jk}^l\chi_{j,k}\right|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}.\end{aligned}$$

Thus, T is an isometry. By the Hahn–Banach Theorem, there exists $\Lambda \in \left(L^{p(\cdot)}(w,\ell^2)\right)^*$ such that $\Lambda \circ T = \widetilde{\lambda}$ and $\|\Lambda\| = \|\widetilde{\lambda}\|$. Since we know that $\left(L^{p(\cdot)}(w,\ell^2)\right)^* = L^{p'(\cdot)}(\sigma,\ell^2)$, there exists $g = \{g_j^l\}_{j,l} \in L^{p'(\cdot)}(\sigma,\ell^2)$ such that

$$\Lambda(f) = \langle f, g\rangle, \quad \|\Lambda\| = \|g\|_{L^{p'(\cdot)}(\sigma,\ell^2)}$$

for any $f \equiv \{f_j^l\}_{j,l} \in L^{p(\cdot)}(w,\ell^2)$, where

$$\langle f, g\rangle = \sum_{l=0}^{2^n-1} \sum_{j=J}^{\infty} \int_{\mathbb{R}^n} f_j^l(x) g_j^l(x) dx$$

We define the projection $P : L^{p(\cdot)}(w,\ell^2) \to \widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma)$ by

$$P\left(\{h_j^l\}_{j,l}\right) = \{\lambda_{jk}^{l\,*}\}_{j,k,l} = \left\{\int_{\mathbb{R}^n} 2^{js} h_j^l(y)\chi_{j,k}(y)dy\right\}_{j,k,l}$$

for some $\{h_j^l\}_{j,l} \in L^{p(\cdot)}(w,\ell^2)$. Then, there is a positive number N such that

$$\begin{aligned}
&\left\|P\left(\{h_j^l\}_{j,l}\right)\right\|_{\widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma)} \\
&= \left\|\left(\sum_{l=0}^{2^n-1}\sum_{j=J}^{\infty}\left|2^{-js}\sum_{k\in\mathbb{Z}^n}\left(\int_{\mathbb{R}^n} 2^{js}h_j^l(y)\chi_{j,k}(y)dy\right)\chi_{j,k}\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p'(\cdot)}(\sigma)} \\
&\lesssim \left\|\left(\sum_{l=0}^{2^n-1}\sum_{j=J}^{\infty}\left|\left(M^{\mathrm{loc}}\right)^N\left[h_j^l\right]\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p'(\cdot)}(\sigma)} \\
&\lesssim \left\|\left(\sum_{l=0}^{2^n-1}\sum_{j=J}^{\infty}\left|h_j^l\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p'(\cdot)}(\sigma)}.
\end{aligned}$$

Thus, P is continuous.

Combining these ovservations, we can justify the change of the order of the summation and integration to have

$$\begin{aligned}
\widetilde{\lambda}(\lambda) = \Lambda(T(\lambda)) &= \sum_{l=0}^{2^n-1}\sum_{j=J}^{\infty}\int_{\mathbb{R}^n} f_j^l(x)g_j^l(x)dx \\
&= \sum_{l=0}^{2^n-1}\sum_{j=J}^{\infty}\int_{\mathbb{R}^n} g_j^l(x)\sum_{k\in\mathbb{Z}^n}\lambda_{jk}^l 2^{js}\chi_{j,k}(x)dx \\
&= \sum_{l=0}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n}\lambda_{jk}^l 2^{js}\int_{\mathbb{R}^n} g_j^\ell(x)\chi_{j,k}(x)dx \\
&= \sum_{l=0}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n}\lambda_{jk}^l\lambda_{jk}^{l\,*} = \langle\lambda,\lambda^*\rangle.
\end{aligned}$$

Finally, we conclude the proof of the norm equivalence keeping in mind Proposition 6.3 as follows:

$$\|\lambda^*\|_{\widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma)} = \|P\left(g_j^l\right)\|_{\widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma)} \lesssim \|g_j^l\|_{L^{p(\cdot)}(w,\ell^2)} = \|\Lambda\| = \|\widetilde{\lambda}\|.$$

□

Theorem 6.5. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, and let $w \in A_{p(\cdot)}^{\mathrm{loc}}$. Let $s < 0$. Then $\Psi : f \in L^{p(\cdot),s}(w) \mapsto \Psi(f) \in \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)$ and $\Phi : \lambda \in \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w) \mapsto \Phi(\lambda) \in L^{p(\cdot),s}(w)$ are continuous linear operators satisfying $\Phi \circ \Psi(f) = f$ for all $f \in L^{p(\cdot),s}(w)$.*

Proof. Let $\lambda^* \in \widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma)$. Then $\langle \Psi(f), \lambda^* \rangle = \langle f, \Phi(\lambda^*) \rangle$ due to the orthogonality of wavelets. Since $\Phi : \widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma) \to L^{p'(\cdot),-s}(\sigma)$ is a bounded linear operator, it follows that $\Psi : f \in L^{p(\cdot),s}(w) \to \Psi(f) \in \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)$ is a continuous linear operator. Similarly, $\Phi : \lambda \in \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w) \to \Phi(\lambda) \in L^{p(\cdot),s}(w)$ is a continuous linear operator.

Finally, we verify the equality. Let $g \in L^{p'(\cdot),-s}(\sigma)$ be arbitrary. Then

$$\langle \Phi \circ \Psi(f), g \rangle = \langle \Psi(f), \Psi(g) \rangle = \langle f, \Phi \circ \Psi(g) \rangle = \langle f, g \rangle,$$

proving that $f = \Phi \circ \Psi(f)$. □

Corollary 6.6. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $w \in A_{p(\cdot)}^{\mathrm{loc}}$. Let $s < 0$. Then C_{c}^∞ is dense in $L^{p(\cdot),s}(w)$.*

Proof. Thanks to Theorem 6.5, we have only to approximate $\varphi_{J,k}$ and $\psi_{j,k}^l$. This is achieved by a routine mollification procedure since $L^{p(\cdot)}(w)$ is continuously embedded into $L^{p(\cdot),s}(w)$. □

Next, we investigate the duality.

Theorem 6.7. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Let $w \in A_{p(\cdot)}^{\mathrm{loc}}$ and $s < 0$. Then $L^{p'(\cdot),-s}(\sigma)$ is the dual space of $L^{p(\cdot),s}(w)$ via the coupling*

$$f \in L^{p'(\cdot),-s}(\sigma) \mapsto [g \in L^{p(\cdot),s}(w) \mapsto g(f) = \langle g, f \rangle \in \mathbb{C}].$$

Proof. By the definition of the duality $L^{p(\cdot),s}(w)$-$L^{p'(\cdot),-s}(\sigma)$, $L^{p'(\cdot),-s}(\sigma)$ is embedded into the dual space of $L^{p(\cdot),s}(w)$. Thus, we have to show the reverse inclusion. Let $L : L^{p(\cdot),s}(w) \to \mathbb{C}$ be a continuous linear mapping. Then $L \circ \Phi : \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w) \mapsto \mathbb{C}$ is a continuous linear mapping. Thus, by Propositions 6.3 and 6.4, there exists $\lambda^* \in \widetilde{\mathbf{f}}_2^{p'(\cdot),-s}(\sigma)$ such that $L \circ \Phi(\lambda) = \langle \lambda, \lambda^* \rangle$ for all $\lambda \in \widetilde{\mathbf{f}}_2^{p(\cdot),s}(w)$. Since

$$L(f) = L \circ \Phi \circ \Psi(f) = \langle \Psi(f), \lambda^* \rangle = \langle f, \Phi(\lambda^*) \rangle \quad (f \in L^{p(\cdot),s}(w)),$$

it follows that $\Phi(\lambda^*) \in L^{p'(\cdot),-s}(\sigma)$ is the element, which realizes L. □

6.2 Applications

We can upgrade Theorems 5.1, 5.2, 5.3 and 5.5 to the case where $s \in \mathbb{R}$. The case of $s > 0$ is already covered in Theorems 5.1, 5.2, 5.3 and 5.5. Hence, we are mainly interested in the case when $s < 0$.

Theorem 6.8. *Suppose that* $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, *and let* $(s, w) \in \mathbb{R} \times A_{p(\cdot)}^{\mathrm{loc}}$. *Then for all* $h \in \mathcal{B}^{[|s|+1]}$, *the pointwise multiplication* $M_h : f \in L^{p(\cdot),s}(w) \mapsto h \cdot f \in L^{p(\cdot),s}(w)$ *is bounded.*

Proof. Simply employ $\langle h \cdot f, g \rangle = \langle f, h \cdot g \rangle$ for $f \in L^{p(\cdot),s}(w)$ and $g \in L^{p'(\cdot),-s}(\sigma)$. □

Theorem 6.9. *Suppose that* $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, *and let* $w \in A_{p(\cdot)}^{\mathrm{loc}}$. *Then for* $0 > s_0 > s_1$ *and* $h \in C^{[|s_1|+1]}$ *with compact support, the operator* $M_h \circ \iota : L^{p(\cdot),s_0}(w) \to L^{p(\cdot),s_1}(w)$ *is a compact operator.*

Proof. Simply go through the same argument as Theorem 5.2. □

Theorem 6.10. *Suppose that* $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, *and let* $(s, w) \in \mathbb{R} \times A_{p(\cdot)}^{\mathrm{loc}}$. *Then for all regular* $C^{[|s|+2]}$*-diffeomorphisms* $\psi : \mathbb{R}^n \to \mathbb{R}^n$ *that preserves* $L^{p(\cdot)}(w)$, *the composition mapping* $f \in L^{p(\cdot),s}(w) \mapsto f \circ \psi \in L^{p(\cdot),s}(w)$ *is bounded.*

Proof. Let $s < 0$. Let $f \in \mathcal{D}$. Observe that

$$\int_{\mathbb{R}^n} f(\psi(x))g(x)\mathrm{d}x = \int_{\mathbb{R}^n} f(x)g(\psi^{-1}(x))|\det(D\psi)|^{-1}\mathrm{d}x.$$

Since $|\det(D\psi)|^{-1} = \pm\det(D\psi)^{-1}$ is a $\mathcal{B}^{[-s+1]}$-function, we can use Theorems 5.3 and 6.8. □

If we reexamine the argument in the proof of Theorem 5.5, we obtain the complex interpolation.

Theorem 6.11. *The conclusion of Theorem 5.5 remains valid if we merely assume* $s_0, s_1 \in \mathbb{R}$.

7 Examples

Finally, we will provide some examples. In addition to the examples of $w \in A_p$ or more generally $w \in A_p^{\mathrm{loc}}$, we can consider $w(x) = \exp(A|x|)$ and $w(x) = (1 + |x|)^A$ with $A \in \mathbb{R}$ as we mentioned in Section 1. Section 7.1 investigates the former, while Section 7.2 considers latter. Section 7.3 is oriented to an application rather than an example. We define weighted uniformly local Lebesgue spaces with variable exponents and easily develop the theory of wavelet decomposition as an application of the results in this paper. Although we consider the case where $p(\cdot)$ is a constant exponent, Section 7.4 shows the periodic case as an application of the class A_p^{loc}.

7.1 $w_n(x) = \mathrm{e}^{A|x|}, A \in \mathbb{R}$

Since we need to work on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$ at the same time, we add subscript n to w and write w_n instead of w. We prove a trace theorem. We define the trace operator Tr by

$$\mathrm{Tr} : f \in C_{\mathrm{c}}^\infty = C_{\mathrm{c}}^\infty(\mathbb{R}^n) \mapsto f(\cdot', 0_n) \in C_{\mathrm{c}}^\infty(\mathbb{R}^{n-1}).$$

Theorem 7.1. *Let $w_n(x) = \mathrm{e}^{A|x|}, x \in \mathbb{R}^n$ for $A > 0$. Let $s > \frac{1}{2}$. Then* Tr : $L^{2,s}(w_n) \mapsto L^{2,s-\frac{1}{2}}(w_{n-1})$ *is extended to a bounded linear surjective operator.*

Proof. It is sufficient to consider the corresponding trace result in the level of sequences. Consider $\mathrm{Tr} : \widetilde{\mathbf{f}}_2^{2,s}(w_n) \mapsto \widetilde{\mathbf{f}}_2^{2,s-\frac{1}{2}}(w_{n-1})$ given by

$$\begin{aligned}&\mathrm{Tr}(\{\lambda_k\}_{k\in\mathbb{Z}^n} \cup \{\lambda_{j,k}^l\}_{j\in\mathbb{Z}\cap[J,\infty),k\in\mathbb{Z}^n,l\in\{1,2,\ldots,2^n-1\}})\\ &\quad= \{\lambda_{(k',0_n)}\}_{k'\in\mathbb{Z}^{n-1}} \cup \{\lambda_{j,(k',0_n)}^l\}_{j\in\mathbb{Z}\cap[J,\infty),k'\in\mathbb{Z}^{n-1},l\in\{1,2,\ldots,2^n-1\}}.\end{aligned}$$

We claim that this is a well-defined and bounded linear operator by proving the corresponding inequality for sequences. We set

$$\mathrm{I} = \left\| \left(\sum_{j=J}^\infty \left| 2^{j\left(s-\frac{1}{2}\right)} \sum_{k'\in\mathbb{Z}^{n-1}} \lambda_{j,(k',0_n)}\chi_{j,k'} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(w_{n-1})}$$

for $\{\lambda_{j,k}\}_{j\in\mathbb{Z}\cap[J,\infty),k\in\mathbb{Z}^n}$. We observe

$$\mathrm{I} \sim \left\| \left(\sum_{j=J}^\infty \left| 2^{js} \sum_{k'\in\mathbb{Z}^{n-1}} \lambda_{j,(k',0_n)}\chi_{j,(k',0)} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(w_n)}$$

since $w_{n-1}(x') \sim w_n(x', a)$ for all $a \in [0, 2^{-J}]$ and $x' \in \mathbb{R}^{n-1}$. Since

$$\left\| \left(\sum_{j=J}^\infty \left| 2^{js} \sum_{k'\in\mathbb{Z}^{n-1}} \lambda_{j,(k',0_n)}\chi_{j,(k',0)} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(w_n)} \le \left\| \left(\sum_{j=J}^\infty \left| 2^{js} \sum_{k\in\mathbb{Z}^n} \lambda_{j,k}\chi_{j,k} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(w_n)},$$

we obtain

$$\mathrm{I} \lesssim \left\| \left(\sum_{j=J}^\infty \left| 2^{js} \sum_{k\in\mathbb{Z}^n} \lambda_{j,k}\chi_{j,k} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(w_n)}.$$

If we employ this inequality, Tr is obtained as a bounded linear operator.

Similarly, by examining the sequence space, we can show that Tr is a surjective operator. □

We can also consider the case where $w(x) = \exp(Ap(x)|x|)$ for some $A > 0$. In addition to the analogies to Proposition 6.2 and Theorem 7.1, we can show the lifting property.

Proposition 7.2. *Suppose that* $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ *and* $A, B, s \in \mathbb{R}$. *Let* $w(x) \equiv \exp(Ap(x)|x|)$ *and* $v(x) \equiv \exp(Bp(x)|x|)$, $x \in \mathbb{R}^n$. *Set* $u(x) \equiv \exp((A - B)\sqrt{1+|x|^2})$, $x \in \mathbb{R}^n$. *Then* $f \in L^{p(\cdot),s}(w) \mapsto u \cdot f \in L^{p(\cdot),s}(v)$ *is a bounded linear operator.*

We can also consider the case where $w(x) = \exp(Ap(x)|x|)$ for some $A > 0$. In addition to the analogies to Proposition 6.2 and Theorem 7.1, we can show the lifting property.

Proof. The proof is similar to Theorem 5.3. We may assume that $s \in \mathbb{N}$. In this case, this assertion follows from the Leibniz rule. □

7.2 $w(x) = (1+|x|)^A, A \in \mathbb{R}$

Next, we consider the case of polynomials, that is, $w(x) = (1+|x|)^A$. Remark that $w \in A_\infty$ if and only if $A > -n$. However, if we work within the framework of local weights, any value of $A \in \mathbb{R}$ is available.

Proposition 7.3. *Suppose that* $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, *and let* $A, s \in \mathbb{R}$. *We write* $w(x) \equiv (1+|x|)^A$, $x \in \mathbb{R}^n$. *Then* $L^{p(\cdot),s}(w) \hookrightarrow \mathcal{S}'$.

Proof. Simply observe, for $f \in L^{p(\cdot),s}(w)$ and $\varphi \in \mathcal{S}$, when $s \geq 0$,

$$\|f\varphi\|_{L^1} \lesssim \|f\|_{L^{p(\cdot)}(w)} \cdot \|\varphi\|_{L^{p'(\cdot)}(\sigma)} \lesssim \|f\|_{L^{p(\cdot),s}(w)} \sup_{x \in \mathbb{R}^n} (1+|x|)^{\frac{A}{p_- - 1}+n+1} |\varphi(x)|.$$

Meanwhile, when $s < 0$,

$$|\langle f, \varphi \rangle| \leq \|f\|_{L^{p(\cdot),s}(w)} \cdot \|\varphi\|_{L^{p'(\cdot),-s}(\sigma)} \lesssim \|f\|_{L^{p(\cdot),s}(w)} \sup_{x \in \mathbb{R}^n} (1+|x|)^{\frac{A}{p_- - 1}+n+1} |\varphi(x)|.$$

Thus, we obtain the desired result. □

Analogous to Proposition 7.2, we can show the lifting property.

Proposition 7.4. *Suppose that* $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ *and* $s, A, B \in \mathbb{R}$. *Let* $w(x) = (1+|x|)^A$, $v(x) = (1+|x|)^B$ *and* $m(x) = (\sqrt{1+|x|^2})^{\frac{A-B}{p_\infty}}$ *for* $x \in \mathbb{R}^n$. *Then* $M_m : f \in L^{p(\cdot),s}(w) \mapsto m \cdot f \in L^{p(\cdot),s}(v)$ *is a bounded linear operator.*

Proof. The proof is similar to Theorem 5.3 and Proposition 7.2, since $m(x) \sim (\sqrt{1+|x|^2})^{\frac{A-B}{p(x)}}$. We may assume that $s \in \mathbb{N}$. In this case, this assertion follows from the Leibniz rule. □

Although the Fefferman–Stein vector-valued inequality for this function space is not available if $A \leq -n$, it is useful to obtain the vector-valued inequality for the η-function as in [14]. To this end, we prove a simple estimate.

Lemma 7.5. *Let* $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, $w \in A_{p(\cdot)}^{\mathrm{loc}}$, *and* $A, B \in \mathbb{R}$. *Write* $w(x) = (1+|x|)^A$. *Assume* $B > \frac{|A|}{p_\infty} + n$. *Then the convolution operator*

$$f \in L^{p(\cdot)}(w) \mapsto (1+|\cdot|)^{-B} * f \in L^{p(\cdot)}(w)$$

is bounded.

Proof. We may assume that f is non-negative. Simply observe

$$\begin{aligned}(1+|\cdot|)^{-B}*f\cdot(1+|\cdot|)^{\frac{A}{p_\infty}} &\lesssim (1+|\cdot|)^{-B+\frac{|A|}{p_\infty}}*[(1+|\cdot|)^{\frac{A}{p_\infty}}f]\\ &\lesssim M[(1+|\cdot|)^{\frac{A}{p_\infty}}f].\end{aligned}$$

Thus, we can use the $L^{p(\cdot)}$-boundedness of M. □

Corollary 7.6. *Let $p(\cdot)\in\mathcal{P}\cap\mathrm{LH}_0\cap\mathrm{LH}_\infty$, $w\in A^{\mathrm{loc}}_{p(\cdot)}$, and $A,B\in\mathbb{R}$. Write $w(x)\equiv(1+|x|)^A$, $x\in\mathbb{R}^n$. Assume $B>\frac{|A|}{p_\infty}+n$. Then*

$$\left\|\left(\sum_{j=1}^\infty 2^{2jn}|(1+2^j|\cdot|)^{-B}*f_j|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}\lesssim\left\|\left(\sum_{j=1}^\infty|f_j|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}. \tag{7.1}$$

Proof. We devide the left-hand side of (7.1) into two parts:

$$\mathrm{I}=\left\|\left(\sum_{j=1}^\infty 2^{2jn}|(1+\chi_{[-1,1]^n}2^j|\cdot|)^{-B}*f_j|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}$$

$$\mathrm{II}=\left\|\left(\sum_{j=1}^\infty 2^{2jn}|(1+\chi_{\mathbb{R}^n\setminus[-1,1]^n}2^j|\cdot|)^{-B}*f_j|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}$$

For I, using the Feffereman–Stein type vector-valued inequality [37, Theorem 1.11]

$$\left\|\left(\sum_{j=1}^\infty M^{\mathrm{loc}}{f_j}^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}\lesssim\left\|\left(\sum_{j=1}^\infty|f_j|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)},$$

we can obtain desired estimate. Thus, we only have to show

$$\mathrm{II}\lesssim\left\|\left(\sum_{j=1}^\infty|f_j|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)},$$

or equivalently,

$$\left\|\left(\sum_{j=1}^\infty 2^{2j(n-B)}|(1+|\cdot|)^{-B}*f_j|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}\lesssim\left\|\left(\sum_{j=1}^\infty|f_j|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}. \tag{7.2}$$

Estimate (7.2) is a consequence of Lemma 7.5 and the triangle inequality:

$$\left\|\left(\sum_{j=1}^\infty 2^{2j(n-B)}|(1+|\cdot|)^{-B}*f_j|^2\right)^{\frac12}\right\|_{L^{p(\cdot)}(w)}\le\sum_{j=1}^\infty 2^{2j(n-B)}\left\|(1+|\cdot|)^{-B}*f_j\right\|_{L^{p(\cdot)}(w)}.$$

□

As mentioned, once Corollary 7.6 is proved, we can argue similar to [14, Theorem 3.4] to demonstrate various results.

7.3 Weighted uniformly local Lebesgue spaces with variable exponents

Let $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $w \in A^{\rm loc}_{p(\cdot)}$. Then the weighted uniformly local Lebesgue space $L^{p(\cdot)}_{\rm uloc}(w)$ with a variable exponent is defined to be all $f \in L^1_{\rm loc}$ for which the norm $\|f\|_{L^{p(\cdot)}_{\rm uloc}(w)} = \sup\limits_{m \in \mathbb{Z}^n} \|\chi_{Q_{0,m}} f\|_{L^{p(\cdot)}(w)}$ is finite. This is a natural extension of the uniformly local Lebesgue space $L^p_{\rm uloc}$, which considers a substitute of L^∞. If we replace the supremum by the ℓ^r-norm, then the weighted amalgam space $(\ell^r, L^{p(\cdot)}_{\rm uloc}(w))$ with a variable exponent is obtained as an extension of the amalgam space (ℓ^r, L^p) considered in [2, 4, 16, 21, 27]. Although our results are applicable to amalgam spaces, to simplify the argument, we consider uniformly local Lebesgue spaces with variable exponents.

For $w \in A^{\rm loc}_{p(\cdot)}$, we write $w_m(x) = w(x)(1+|x-m|)^{-p_+(1+n)}$. Then by the triangle inequality, we can check that

$$\|f\|_{L^{p(\cdot)}_{\rm uloc}(w)} \sim \sup_{m \in \mathbb{Z}^n} \|f\|_{L^{p(\cdot)}(w_m)}. \tag{7.3}$$

Likewise, for $s \in \mathbb{N}$, the weighted uniformly local Sobolev space $L^{p(\cdot),s}_{\rm uloc}(w)$ with a variable exponent is defined to be all $f \in L^1_{\rm loc}$ for which $\partial^\alpha f \in L^{p(\cdot)}_{\rm uloc}(w)$ for all $|\alpha| \le s$. The norm of $f \in L^{p(\cdot),s}_{\rm uloc}(w)$ is defined by $\|f\|_{L^{p(\cdot),s}_{\rm uloc}(w)} = \sum\limits_{|\alpha| \le s} \|\partial^\alpha f\|_{L^{p(\cdot)}_{\rm uloc}(w)}$. We present an application of the wavelet characterization.

Theorem 7.7. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and $w \in A^{\rm loc}_{p(\cdot)}$. Let $s \in \mathbb{N}_0$. Then the function $f \in L^1_{\rm loc}$ belongs to $L^{p(\cdot),s}_{\rm uloc}(w)$ if and only if $Vf + W_s f \in L^{p(\cdot)}_{\rm uloc}(w)$.*

Proof. We note that $f \in L^{p(\cdot),s}_{\rm uloc}(w)$ if and only if $\partial^\alpha f \in L^{p(\cdot)}_{\rm uloc}(w)$ for all $|\alpha| \le s$ by definition. This is equivalent to $\sup\limits_{|\alpha| \le s} \sup\limits_{m \in \mathbb{Z}^n} \|\partial^\alpha f\|_{L^{p(\cdot)}(w_m)} < \infty$ by (7.3), which is also equivalent to $\sup\limits_{m \in \mathbb{Z}^n} \|Vf + W_s f\|_{L^{p(\cdot)}(w_m)} < \infty$ thanks to Theorem 1.3. If we use (7.3) once again, we obtain the desired conclusion. □

Note that using the interpolation and duality results, we have same conclusion for $s \in \mathbb{R}$.

As an application of the weights considered in this paper, we can characterize various function spaces. Below is an example of Morrey spaces with constant exponent.

Example 7.8. Let $1 \le q \le p \le \infty$. For an $L^q_{\rm loc}$-function f, its Morrey norm is defined by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_{(x,r) \in \mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q {\rm d}y \right)^{\frac{1}{q}}. \tag{7.4}$$

The Morrey space $\mathcal{M}^p_q$ is the set of all $f \in L^q_{\rm loc}$ for which $\|f\|_{\mathcal{M}^p_q} < \infty$. Note that

$$\|f\|_{\mathcal{M}^p_q} \sim \sup_{(x,r) \in \mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q((M\chi_{B(x,r)})^\theta)}$$

as long as $1-\frac{1}{p}<\theta<1$. Because of this fact, a similar argument can be used to obtain the wavelet characterization for Morrey spaces.

A passage to Morrey spaces with variable exponents can be performed. We omit further details.

7.4 Periodic function spaces

Although the exponent $p(\cdot)$ must be constant in this subsection, it seems useful to discuss periodic function spaces. Let $L^p(\mathbb{T}^n)$ be the set of all p-locally integrable functions f with period 1 for which

$$\|f\|_{L^p(\mathbb{T}^n)}=\left(\int_{[0,1]^n}|f(x)|^p{\rm d}x\right)^{\frac{1}{p}}<\infty.$$

Similarly, for $s\in\mathbb{N}$, $L^{p,s}(\mathbb{T}^n)$ is the set of all functions $f\in L^p(\mathbb{T}^n)$ for which the weak derivative $\partial^\alpha f$ exists and belongs to $L^p(\mathbb{T}^n)$ as long as the multiindex α satisfies $|\alpha|\le s$. The norm is given by

$$\|f\|_{L^{p,s}(\mathbb{T}^n)}=\sum_{|\alpha|\le s}\|\partial^\alpha f\|_{L^p(\mathbb{T}^n)}.$$

If a constant $p(\cdot)$ is periodic and satisfies the global log-Hölder condition, then $p(\cdot)$ must be constant. Thus, we assume that $p(\cdot)$ is a constant here. The following observation is the starting point of Section 7.4.

Lemma 7.9. *Let $w(x)=(1+|x|)^{-n-1}\sim M\chi_{Q_{0,0}}(x)^{\frac{n+1}{n}}$ for $x\in\mathbb{R}^n$. Then for any $1\le p\le\infty$, $L^p(\mathbb{T}^n)\hookrightarrow L^p(w)$ and*

$$\|f\|_{L^p(\mathbb{T}^n)}\sim\|f\|_{L^p(w)}.$$

Proof. Simply use $\sum_{m\in\mathbb{Z}^n}(1+|m|)^{-n-1}<\infty$. □

Let $s\in\mathbb{N}\cup\{0\}$. In view of Lemma 7.9, for $f\in L^{p,s}(\mathbb{T}^n)$, we have the wavelet expansion

$$f=\sum_{k\in\mathbb{Z}^n}\langle f,\varphi_{J,k}\rangle\varphi_{J,k}+\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n}\sum_{l=1}^{2^n-1}\langle f,\psi^l_{j,k}\rangle\psi^l_{j,k},$$

where the convergence occurs in $L^p(w)$ and consequently, in $L^p(\mathbb{T}^n)$. Assume that J is so large that the support of each $\varphi_{J,k}$ is contained in a cube with side-length 1/3. Fix such J. We set

$$\Phi_{J,k}=\sum_{k'\equiv k \pmod{2^J\mathbb{Z}^n}}\varphi_{J,k}$$

and

$$\Psi^l_{j,k}=\sum_{k'\equiv k \pmod{2^j\mathbb{Z}^n}}\psi^l_{j,k}$$

for $k \in \mathbb{Z}$, $j = J, J+1, \ldots$ and $l = 1, 2, \ldots, 2^n - 1$. Then, as in [45, Proposition 2.21], we have an orthogonal decomposition

$$f = \sum_{k \in \mathbb{Z}^n \cap [0,2^J)^n} \langle f, \Phi_{J,k} \rangle_{L^2(\mathbb{T}^n)} \Phi_{J,k} + \sum_{j=J}^{\infty} \sum_{k \in \mathbb{Z}^n \cap [0,2^j)^n} \sum_{l=1}^{2^n-1} \langle f, \Psi^l_{j,k} \rangle_{L^2(\mathbb{T}^n)} \Psi^l_{j,k}.$$

Hence, we can easily obtain the wavelet decomposition for periodic functions.

Corollary 7.10. *Let $1 < p < \infty$, $s \in \mathbb{N} \cup \{0\}$ and let $f \in L^p(\mathbb{T}^n)$. Set*

$$V_{\mathbb{T}^n} f = V^{\varphi}_{\mathbb{T}^n} f \equiv \left(\sum_{k \in \mathbb{Z}^n \cap [0,2^J)^n} \left| \langle f, \Phi_{J,k} \rangle_{L^2(\mathbb{T}^n)} \chi_{J,k} \right|^2 \right)^{\frac{1}{2}}$$

$$W_{s,\mathbb{T}^n} f = W^{\psi^l}_{s,\mathbb{T}^n} f \equiv \left(\sum_{l=1}^{2^n-1} \sum_{j=J}^{\infty} \sum_{k \in \mathbb{Z}^n \cap [0,2^j)^n} \left| 2^{js} \langle f, \Psi^l_{j,k} \rangle_{L^2(\mathbb{T}^n)} \chi_{j,k} \right|^2 \right)^{\frac{1}{2}}.$$

Then $f \in L^{p,s}(\mathbb{T}^n)$ if and only if $V_{\mathbb{T}^n} f + W_{s,\mathbb{T}^n} f \in L^p(\mathbb{T}^n)$. If this is the case, then $\|f\|_{L^{p,s}(\mathbb{T}^n)} \sim \|V_{\mathbb{T}^n} f\|_{L^p(\mathbb{T}^n)} + \|W_{s,\mathbb{T}^n} f\|_{L^p(\mathbb{T}^n)}$.

Acknowledgements

Mitsuo Izuki was partially supported by Grand-in-Aid for Scientific Research (C), (Grant No. 15K04928), for Japan Society for the Promotion of Science. Toru Nogayama was partially supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists (Grant No. 20J10403). Takahiro Noi was partially supported by Grand-in-Aid for Young Scientists (B), (Grant No. 17K14207) for Japan Society for the Promotion of Science. Yoshihiro Sawano was partially supported by Grand-in-Aid for Scientific Research (C), (Grant No. 19K03546), for Japan Society for the Promotion of Science.

This work was partly supported by Osaka City University Advanced Mathematical Institute: MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849.

References

[1] J. Bergh and J. Löfström, Interpolation spaces. An introduction. Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976. [20]

[2] J. Bertrandias, C. Datry and C. Dupuis, Unions et intersections d'espaces L^p invariantes par translation ou convolution, Ann. Inst. Fourier (Grenoble) **28** (2) (1978) 53–84. [6, 53]

[3] A.P. Calderón, Intermediate spaces and interpolation, the complex method, Studia Math. **24** (1964), 113–190. no. 1, 46–56. [22]

[4] M. Cowling, S. Meda and R. Pasquale, Riesz potentials and amalgams, Ann. Inst. Fourier (Grenoble) **49** (4) (1999) 1345–1367. [6, 53]

[5] D. Cruz-Uribe, L. Diening and P. Hästö, The maximal operator on weighted variable Lebesgue spaces, Fract. Calc. Appl. Anal. **14** (2011), 361–374. [2, 4, 10]

[6] D. V. Cruz-Uribe and A. Fiorenza, Variable Lebesgue Spaces, Foundations and Harmonic Analysis, Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Heidelberg, 2013. [2]

[7] D. Cruz-Uribe, A. Fiorenza, J. M. Martell and C. Pérez, The boundedness of classical operators on variable L^p spaces, Ann. Acad. Sci. Fenn. Math. **31** (2006), 239–264. [10]

[8] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, The maximal function on variable L^p spaces, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238. [2]

[9] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, Corrections to: "The maximal function on variable L^p spaces" [Ann. Acad. Sci. Fenn. Math. **28** (1) (2003), 223–238.], Ann. Acad. Sci. Fenn. Math. **29** (2004), 247–249. [2]

[10] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, Weighted norm inequalities for the maximal operator on variable Lebesgue spaces, J. Math. Anal. Appl. **394** (2012), 744–760. [2, 4]

[11] D. V. Cruz-Uribe, A. Fiorenza, J. M. Martell and C. Pérez, Weights, Extrapolation and the Theory of Rubio de Francia, Operator Theory: Advances and Applications, **215**. Birkhäuser/Springer Basel AG, Basel, 2011. [11]

[12] I. Daubechies, Orthonormal basis of compactly supported wavelets, Comm. Pure Appl. Math. **41** (1988), 909–996. [3]

[13] L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, Math. Inequal. Appl. **7** (2004), 245–253. [2]

[14] L. Diening, P. Hästö and S. Roudenko, Spaces of variable integrability and differentiability. J. Funct. Anal. **256**, 1731–1768 (2009) [51, 52]

[15] L. Diening, P. Harjulehto and P. Hästö, M. Růžička, Lebesgue and Sobolev Spaces with variable exponents, Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Berlin, 2011. [44]

[16] J. Fournier and J. Stewart, Amalgams of L^p and l^q, Bull. Amer. Math. Soc. (N.S.) **13** (1) (1985) 1–21. [6, 53]

[17] G. Garrigós, A Seeger and T. Ullrich, The Haar System as a Schauder Basis in Spaces of Hardy–Sobolev Type, J. Four. Anal Appl (2018) **24**, 1319–1339 [29, 38]

[18] L. Grafakos, Modern Fourier Analysis, Graduate texts in mathematics ; 250, New York, Springer, 2014. [4, 14, 16]

[19] P. Hästö, Local-to-global results in variable exponent spaces, Math. Res. Letters **16** (2009), no. 2, 263–278. [10]

[20] E. Hernández and G. Weiss, A First Course on Wavelets, CRC Press, Boca Raton, FL.,1996. [5]

[21] F. Holland, Harmonic analysis on amalgams of L^p and ℓ^q, J. London Math. Soc. (2) **10** (1975) 295–305. [6, 53]

[22] M. Izuki, Wavelets and modular inequalities in variable L^p spaces, Georgian Math. J. **15** (2008), 281–293. [2]

[23] M. Izuki, The characterizations of weighted Sobolev spaces by wavelets and scaling functions, Taiwanese J. Math. **13** (2009), 467–492. [5]

[24] M. Izuki, E. Nakai and Y. Sawano, Function spaces with variable exponents –an introduction–, Sci. Math. Jpn. **77** (2014), 187–315. [2, 13]

[25] M. Izuki, E. Nakai and Y. Sawano, Wavelet characterization and modular inequalities for weighted Lebesgue spaces with variable exponents, Ann. Acad. Sci. Fenn. Math. **40** (2015), 551–571. [2, 6, 12]

[26] M. Izuki, T. Nogayama, T. Noi and Y. Sawano, Wavelet characterization of local Muckenhoupt weighted Lebesgue spaces with variable exponents, Nonlinear Anal., **198** (2020), 111930, 14 pp. [5, 9, 26, 27, 35]

[27] N. Kikuchi, E. Nakai, N. Tomita, K. Yabuta and T. Yoneda, Calderón–Zygmund operators on amalgam spaces and in the discrete case, J. Math. Anal. Appl. **335** (2007) 198–212 [6, 53]

[28] T. S. Kopaliani, Greediness of the wavelet system in $L^{p(t)}(\mathbb{R})$ spaces, East J. Approx. **14** (2008), 59–67. [2]

[29] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. **41** (1991), 592–618. [2]

[30] P. Lemarié-Rieusset, Ondelettes et poids de Muckenhoupt, Studia Math. **108** (1994), 127–147. [6]

[31] P. G. Lemarié-Rieusset and G. Malgouyres, Support des fonctions de base dans une analyse multi-résolution, C. R. Acad. Sci. Paris **313** (1991), 377–380. [3]

[32] A. Małecka, Haar functions in weighted Besov and Triebel–Lizorkin spaces, J. Approx. Theory **200** (2015) 1–27. [29, 38, 46]

[33] Y. Meyer, Wavelets and Operators, Cambridge Univ. Press, Cambridge, 1992. [3]

[34] E. Nakai, N. Tomita and K. Yabuta, Density of the set of all infinitely differentiable functions with compact support in weighted Sobolev spaces, Sci. Math. Jpn. **60** (2004), 121–127. [12]

[35] H. Nakano, Modulared Semi-Ordered Linear Spaces, Maruzen Co., Ltd., Tokyo, 1950. [2]

[36] H. Nakano, Topology of Linear Topological Spaces, Maruzen Co., Ltd., Tokyo, 1951. [2]

[37] T. Nogayama and Y. Sawano, Local Muckenhoupt class for variable exponents, J. Inequal. Appl. online (2021), https://doi.org/10.1186/s13660-021-02601-2. [3, 10, 11, 52]

[38] W. Orlicz, Über konjugierte Exponentenfolgen. Studia Math. **3** (1931), 200–212. [2]

[39] V. S. Rychkov, Littlewood–Paley theory and function spaces with $A_p^{\rm loc}$ weights, Math. Nachr. **224** (2001), 145–180. [3, 4, 10]

[40] Y. Sawano, Atomic decompositions of Hardy spaces with variable exponents and its application to bounded linear operators, Integr. Equ. Oper. Theory, **77**, no. 1, 123–148 (2013) [11]

[41] Y. Sawano, Theory of Besov Spaces, Development in Mathematics **56**, Springer, Singapore, 2018. [2, 3, 4, 6, 9, 13]

[42] V.A. Shestakov, On complex interpolation of Banach spaces of measurable functions, Vestnik Leningrad. Univ. **19** (1974), 64–68. [23]

[43] E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton Univ. Press, 1993. [4]

[44] H. Triebel, Theory of function spaces II, Birkhäuser (1992). [6]

[45] P. Wojtaszczyk, A Mathematical Introduction to Wavelets, Cambridge Univ. Press, Cambridge, 1997. [3, 55]

[46] A. Wojciechowska, Multidimensional wavelet bases in Besov and Triebel–Lizorkin spaces (Ph.D. thesis), Adam Mickiewicz University Poznań, Poznań, 2012. [46]

[47] K. Zhu, Operator theory in function spaces. No. 138. American Mathematical Society, Providence (2007). [20]

Mitsuo Izuki,
Faculty of Liberal Arts and Sciences,
Tokyo City University,
1-28-1, Tamadutsumi Setagaya-ku Tokyo 158-8557, Japan.
E-mail: izuki@tcu.ac.jp

Toru Nogayama,
Department of Mathematical Science,
Tokyo Metropolitan University,
Hachioji, 192-0397, Japan.
E-mail: toru.nogayama@gmail.com

Takahiro Noi,
Department of Mathematical Science,
Tokyo Metropolitan University,
Hachioji, 192-0397, Japan.
E-mail: taka.noi.hiro@gmail.com

Yoshihiro Sawano (Corresponding author),
Department of Mathematical Science,
Chuo University,
Kasuga,, Bunkyo-ku 112-8551, Tokyo, Japan.
+
People's Friendship University of Russia.
E-mail: yoshihiro-sawano@celery.ocn.ne.jp