

Higher Level q -Oscillator Representations for
 $U_q(C_n^{(1)})$, $U_q(C^{(2)}(n+1))$ and
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**HIGHER LEVEL q -OSCILLATOR REPRESENTATIONS FOR
 $U_q(C_n^{(1)}), U_q(C^{(2)}(n+1))$ AND $U_q(B^{(1)}(0, n))$**

JAE-HOON KWON AND MASATO OKADO

ABSTRACT. We introduce higher level q -oscillator representations for the quantum affine (super)algebras of type $C_n^{(1)}, C^{(2)}(n+1)$ and $B^{(1)}(0, n)$. They are constructed from the fusion procedure from the fundamental q -oscillator representations obtained through the studies of the tetrahedron equation. We prove that they are irreducible for type $C_n^{(1)}$ and $C^{(2)}(n+1)$, and give their characters.

1. INTRODUCTION

Let \mathfrak{g} be an affine Lie algebra and $U_q(\mathfrak{g})$ the Drinfeld-Jimbo quantum group (without derivation) associated to it. For a node r of the Dynkin diagram of \mathfrak{g} except 0 and a positive integer s there exists a family of finite-dimensional $U_q(\mathfrak{g})$ -modules $W^{r,s}$ called Kirillov-Reshetikhin modules. They have distinguished properties. One of them is the existence of crystal bases in Kashiwara's sense (see [1, 5, 18] and references therein).

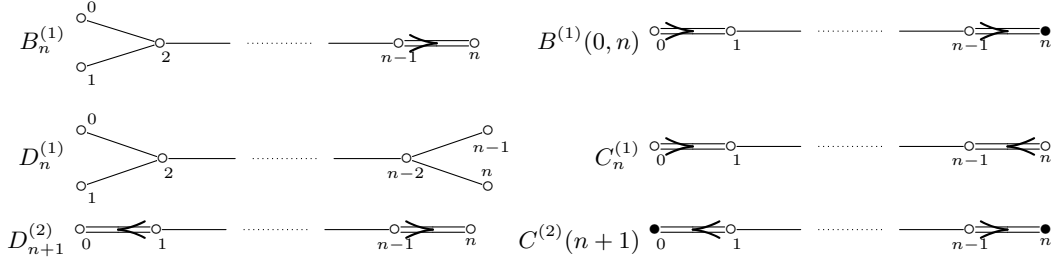


TABLE 1. Dynkin diagrams of $(\mathfrak{g}, \bar{\mathfrak{g}})$

Consider the affine Lie algebras $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}$, whose Dynkin diagrams are given in the left side of Table 1. The Kirillov-Reshetikhin modules corresponding to the node n and the integer 1 have a simple structure. Let V be a two dimensional vector space. The action of $U_q(\mathfrak{g})$ on $W^{n,1}$ has an easy description on $V^{\otimes n}$. It is irreducible when $\mathfrak{g} = B_n^{(1)}, D_{n+1}^{(2)}$, but for $\mathfrak{g} = D_n^{(1)}$ it decomposes into two components; $V^{\otimes n} = W^{n,1} \oplus W^{n-1,1}$. For a quantum

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group $U_q(\mathfrak{g})$ we can consider the quantum R matrix. We introduce a spectral parameter x to the representation $W^{n,1}$, and denote the associated representation by $W^{n,1}(x)$. Let Δ be the coproduct and Δ^{op} its opposite. Then the quantum R matrix $R(x/y)$ is defined as an intertwiner of Δ and Δ^{op} , namely, linear operator satisfying $R(x/y)\Delta(u) = \Delta^{\text{op}}(u)R(x/y)$ for any $u \in U_q(\mathfrak{g})$ on $W^{n,1}(x) \otimes W^{n,1}(y)$. (R is found to depend only on x/y .)

In [15], Kuniba and Sergeev initiated an attempt to obtain quantum R matrices from the solution to the tetrahedron equation, three dimensional analogue of the Yang-Baxter equation. Let \mathcal{L} be a solution of the tetrahedron equation. It is a linear operator on $F \otimes V \otimes V$ where F is an infinite-dimensional vector space spanned by $\{|m\rangle \mid m \in \mathbb{Z}_{\geq 0}\}$. By composing this \mathcal{L} n times and applying suitable boundary vectors in F and F^* , they obtained linear operators on $(V^{\otimes n}) \otimes (V^{\otimes n})$ satisfying the Yang-Baxter equation. The commuting symmetry algebras were found to be $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ or $U_q(D_{n+1}^{(2)})$. The reason they had variations was that there were two choices of boundary vectors in each F and F^* corresponding to the shapes of the Dynkin diagrams at each end.

To the tetrahedron equation, there is yet another solution \mathcal{R} , which is a linear operator on $F^{\otimes 3}$. In [13], Kuniba and the second author performed the same scheme to \mathcal{R} and constructed linear operators on $(F^{\otimes n}) \otimes (F^{\otimes n})$. For the symmetry algebra this time, they found $U_q(C_n^{(1)})$, $U_q(D_{n+1}^{(2)})$ and $U_q(A_{2n}^{(2)})$. They called these representations on $\mathcal{W} = F^{\otimes n}$ q -oscillator ones. To be precise, for type $C_n^{(1)}$ there are two irreducible components \mathcal{W}_+ , \mathcal{W}_- , so one can think \mathcal{W} of either \mathcal{W}_+ or \mathcal{W}_- . By construction, the q -oscillator representation \mathcal{W} is a bosonic analogue of $W^{n,1}$, and it is natural to ask whether we have a higher level q -oscillator representation corresponding to $W^{n,s}$ for $s \geq 1$. However, there is a difficulty in understanding \mathcal{W} since they do not have a suitable classical limit ($q \rightarrow 1$) for type $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}$.

In this paper, we first resolve this difficulty by considering \mathcal{W} for these two types as q -oscillator representations over quantum affine superalgebras $\bar{\mathfrak{g}}$ given in the right side of Table 1 by using the twistor on quantum covering groups [4]. The filled nodes in the Dynkin diagrams signify anisotropic odd simple roots. If they were not filled, the third Dynkin diagram would be $D_{n+1}^{(2)}$ and the first one $A_{2n}^{(2)\dagger}$, where the latter is the same diagram as $A_{2n}^{(2)}$ but the opposite labeling of nodes. We then investigate the quantum R matrices for $\mathcal{W}(x) \otimes \mathcal{W}(y)$ and apply the fusion construction. As a result, we obtain a higher level representation $\mathcal{W}^{(s)}$ for any $s \in \mathbb{Z}_{>0}$ and each $U_q(C_n^{(1)})$, $U_q(C^{(2)}(n+1))$ and $U_q(B^{(1)}(0, n))$.

Our main purpose in this paper is to prove the irreducibility of $\mathcal{W}^{(s)}$ and compute its character for $U_q(C_n^{(1)})$ and $U_q(C^{(2)}(n+1))$. We investigate the crystal base of $\mathcal{W}^{(s)}$ in detail to show this. We further prove that $\mathcal{W}^{(s)}$ is classically irreducible, that is, irreducible as a module over the subalgebra generated by e_i, f_i, k_i for $i \neq 0$. Rather surprisingly, this coincides with the fact that the corresponding $W^{n,s}$ is classically irreducible. We also give conjectures on the irreducibility of $\mathcal{W}^{(s)}$ and its character formula for $B^{(1)}(0, n)$.

We would like to remark that the correspondence between $W^{n,s}$ and $\mathcal{W}^{(s)}$ as representations of finite-dimensional simple Lie (super)algebras after a classical limit, appears in the context of super duality [3]. The theory of super duality is an equivalence between certain

parabolic Bernstein-Gelfand-Gelfand categories of classical Lie (super)algebras of infinite-rank. As a special case, this yields an equivalence between the categories for \mathcal{G}_∞ and $\bar{\mathcal{G}}_\infty$, where $(\mathcal{G}_\infty, \bar{\mathcal{G}}_\infty) = (B_\infty, B(0, \infty)), (D_\infty, C_\infty)$. Their Dynkin diagrams are given in Table 2. Let \mathcal{G}_n and $\bar{\mathcal{G}}_n$ denote the subalgebras of \mathcal{G}_∞ and $\bar{\mathcal{G}}_\infty$ of finite rank n , respectively. Let V_∞ be a given integrable highest weight \mathcal{G}_∞ -module. Under this equivalence, it corresponds to an irreducible highest weight $\bar{\mathcal{G}}_\infty$ -module, say W_∞ , called an oscillator representation. By applying a truncation functor to V_∞ and W_∞ , we also obtain irreducible modules V_n and W_n of \mathcal{G}_n and $\bar{\mathcal{G}}_n$, respectively.

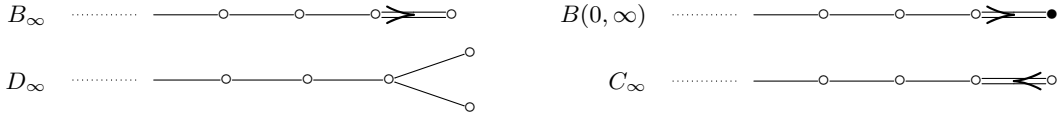


TABLE 2. Dynkin diagrams of $(\mathcal{G}_\infty, \bar{\mathcal{G}}_\infty)$

Let $(\mathfrak{g}, \bar{\mathfrak{g}})$ be one of the pairs of affine Lie (super)algebras $(B_n^{(1)}, B^{(1)}(0, n)), (D_n^{(1)}, C_n^{(1)})$, $(D_{n+1}^{(2)}, C^{(2)}(n+1))$ in Table 1. Let \mathcal{G}_n and $\bar{\mathcal{G}}_n$ be the subalgebra of \mathfrak{g} and $\bar{\mathfrak{g}}$ corresponding to $I \setminus \{0\}$, respectively. Assume that $\mathfrak{g} = D_n^{(1)}, D_{n+1}^{(2)}$. Now we see that if V_n is the classical limit of a classically irreducible Kirillov-Reshetikhin $U_q(\mathfrak{g})$ -module, then W_n corresponds to the classical limit of a higher level q -oscillator $U_q(\bar{\mathfrak{g}})$ -module in Theorems 5.1 and 5.20. The character formula in Conjecture 5.22 is based on this observation in case of $(\mathfrak{g}, \bar{\mathfrak{g}}) = (B_n^{(1)}, B^{(1)}(0, n))$, which is true for $s = 2$. We strongly expect that there is a quantum affine analogue of super duality which relates the category of finite-dimensional $U_q(\mathfrak{g})$ -modules and a suitable category of infinite-dimensional $U_q(\bar{\mathfrak{g}})$ -modules including the q -oscillator modules, and hence explains the correspondence in this paper.

The paper is organized as follows: In Section 2, we briefly review the notion of quantum superalgebras. In Section 3, we construct a level one q -oscillator representation \mathcal{W} of $U_q(\bar{\mathfrak{g}})$ and study some of its properties including the crystal base. In Section 4, we introduce the quantum R matrix on $\mathcal{W}(x) \otimes \mathcal{W}(y)$ and apply fusion construction to define $\mathcal{W}^{(s)}$. In Section 5, we prove the irreducibility of $\mathcal{W}^{(s)}$ and give its character formula when $\bar{\mathfrak{g}} = C_n^{(1)}, C^{(2)}(n+1)$. A conjecture when $\bar{\mathfrak{g}} = B^{(1)}(0, n)$ is also given. In Appendix A, we explain how to construct a level one q -oscillator representation of $U_q(\bar{\mathfrak{g}})$ when $\bar{\mathfrak{g}} = C^{(2)}(n+1)$ and $B^{(1)}(0, n)$ from the one for $D_{n+1}^{(2)}$ and $A_{2n}^{(2)\dagger}$ in [13], respectively, by using the quantum covering groups and twistor [4]. In Appendices B and C, we construct the quantum R matrix on $\mathcal{W}(x) \otimes \mathcal{W}(y)$ for $U_q(\bar{\mathfrak{g}})$ from the one in [13].

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2. QUANTUM SUPERALGEBRAS

2.1. Variant of q -integer. Throughout the paper, we let q be an indeterminate. Following [4], we introduce variants of q -integer, q -factorial and q -binomial coefficient. Let $\epsilon = \pm 1$. For $m \in \mathbb{Z}_{\geq 0}$, we set

$$[m]_{q,\epsilon} = \frac{(\epsilon q)^m - q^{-m}}{\epsilon q - q^{-1}}.$$

For $m \in \mathbb{Z}_{\geq 0}$, set

$$[m]_{q,\epsilon}! = [m]_{q,\epsilon}[m-1]_{q,\epsilon} \cdots [1]_{q,\epsilon} \quad (m \geq 1), \quad [0]_{q,\epsilon}! = 1.$$

For integers m, n such that $0 \leq n \leq m$, we define

$$\begin{bmatrix} m \\ n \end{bmatrix}_{q,\epsilon} = \frac{[m]_{q,\epsilon}!}{[n]_{q,\epsilon}![m-n]_{q,\epsilon}!}.$$

They all belong to $\mathbb{Z}[q, q^{-1}]$. Let A_0 be the subring of $\mathbb{Q}(q)$ consisting of rational functions without a pole at $q = 0$. Then we have

$$[m]_{q,\epsilon} \in q^{1-m}(1 + qA_0), \quad [m]_{q,\epsilon}! \in q^{-m(m-1)/2}(1 + A_0), \quad \begin{bmatrix} m \\ n \end{bmatrix}_{q,\epsilon} \in q^{-n(m-n)}(1 + qA_0).$$

We simply write $[m] = [m]_{q,1}$, $[m]! = [m]_{q,1}!$ and $\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}_{q,1}$.

2.2. Quantum (super)algebra $U_q(sl_2)$ and $U_q(osp_{1|2})$. The quantum (super)algebras $U_q(sl_2)$ ($\epsilon = 1$) and $U_q(osp_{1|2})$ ($\epsilon = -1$) are defined as a $\mathbb{Q}(q)$ -algebra generated by $e, f, k^{\pm 1}$ satisfying the following relations:

$$kk^{-1} = k^{-1}k = 1, \quad kek^{-1} = q^2e, \quad kfk^{-1} = q^{-2}f, \quad ef - \epsilon fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

Set $e^{(m)} = e^m/[m]_{q,\epsilon}!$ and $f^{(m)} = f^m/[m]_{q,\epsilon}!$. We will use the following formula.

Proposition 2.1.

$$e^{(m)}f^{(n)} = \sum_{j \geq 0} \frac{\epsilon^{mn-j(j+1)/2}}{[j]_{q,\epsilon}!} f^{(n-j)} \left(\prod_{l=0}^{j-1} \frac{(\epsilon q)^{2j-m-n-l}k - q^{-2j+m+n+l}k^{-1}}{q - q^{-1}} \right) e^{(m-j)}.$$

Proof. The $U_q(sl_2)$ ($\epsilon = 1$) case is derived easily from (1.1.23) of [12]. The $U_q(osp_{1|2})$ ($\epsilon = -1$) case can be shown by induction. \square

2.3. Quantum affine (super)algebras $U_q(C_n^{(1)})$, $U_q(C^{(2)}(n+1))$, $U_q(B^{(1)}(0, n))$. Set $I = \{0, 1, \dots, n\}$. In this paper, we consider the following three Cartan data $(a_{ij})_{i,j \in I}$, or Dynkin diagrams (cf. [9]), and $(d_i)_{i \in I}$ such that $d_i a_{ij} = d_j a_{ji}$ for $i, j \in I$.

• $C_n^{(1)}$:



$$(a_{ij})_{i,j \in I} = \begin{pmatrix} 2 & -1 & & & \\ -2 & 2 & -1 & & \\ & & \ddots & & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix}$$

$$(d_i)_{i \in I} = (2, 1, \dots, 1, 2)$$

• $C^{(2)}(n+1)$:



$$(a_{ij})_{i,j \in I} = \begin{pmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{pmatrix}$$

$$(d_i)_{i \in I} = \left(\frac{1}{2}, 1, \dots, 1, \frac{1}{2}\right)$$

• $B^{(1)}(0, n)$:



$$(a_{ij})_{i,j \in I} = \begin{pmatrix} 2 & -1 & & & \\ -2 & 2 & -1 & & \\ & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{pmatrix}$$

$$(d_i)_{i \in I} = (2, 1, \dots, 1, \frac{1}{2}).$$

Let $d = \min\{d_i \mid i \in I\}$. For $i \in I$, let $q_i = q^{d_i}$, and let $p(i) = 0, 1$ such that $p(i) \equiv 2d_i \pmod{2}$. Set

$$[m]_i = [m]_{q_i, (-1)^{p(i)}}, \quad [m]_i! = [m]_{q_i, (-1)^{p(i)}}!, \quad \begin{bmatrix} m \\ k \end{bmatrix}_i = \begin{bmatrix} m \\ k \end{bmatrix}_{q_i, (-1)^{p(i)}},$$

for $0 \leq k \leq m$ and $i \in I$.

For a Cartan datum $X = C_n^{(1)}, C^{(2)}(n+1), B^{(1)}(0, n)$, the quantum affine (super)algebra $U_q(X)$ is defined to be the $\mathbb{Q}(q^d)$ -algebra generated by $k_i^{\pm 1}, e_i, f_i$ ($i \in I$) with the following relations:

$$\begin{aligned} k_i k_j &= k_j k_i, & k_i e_j k_i^{-1} &= q_i^{a_{ij}} e_j, & k_i f_j k_i^{-1} &= q_i^{-a_{ij}} f_j, \\ e_i f_j - (-1)^{p(i)p(j)} f_j e_i &= \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \end{aligned}$$

$$\sum_{m=0}^{1-a_{ij}} (-1)^{m+p(i)m(m-1)/2+mp(i)p(j)} e_i^{(1-a_{ij}-m)} e_j e_i^{(m)} = 0 \quad (i \neq j),$$

$$\sum_{m=0}^{1-a_{ij}} (-1)^{m+p(i)m(m-1)/2+mp(i)p(j)} f_i^{(1-a_{ij}-m)} f_j f_i^{(m)} = 0 \quad (i \neq j),$$

where

$$e_i^{(m)} = \frac{e_i^m}{[m]_i!}, \quad f_i^{(m)} = \frac{f_i^m}{[m]_i!}.$$

We define the automorphism τ of $U_q(X)$ for $X = C_n^{(1)}, C^{(2)}(n+1)$ by

$$(2.1) \quad \tau(k_i) = k_{n-i}^{-1}, \quad \tau(e_i) = f_{n-i}, \quad \tau(f_i) = e_{n-i}, \quad \text{if } X = C_n^{(1)},$$

$$(2.2) \quad \tau(k_i) = k_{n-i}^{-1}, \quad \tau(e_i) = (-1)^{\delta_{i0}} f_{n-i}, \quad \tau(f_i) = (-1)^{\delta_{i0}} e_{n-i}, \quad \text{if } X = C^{(2)}(n+1)$$

for $i \in I$ and the anti-automorphism η of $U_q(X)$ by

$$\begin{cases} \eta(k_i) = k_i \\ \eta(e_i) = (-1)^{\delta_{i0}+\delta_{in}} q_i^{-1} k_i^{-1} f_i \\ \eta(f_i) = (-1)^{\delta_{i0}+\delta_{in}} q_i^{-1} k_i e_i \end{cases} \quad \text{if } X = C_n^{(1)}, B^{(1)}(0, n),$$

$$\begin{cases} \eta(k_i) = k_i \\ \eta(e_i) = (-1)^{\delta_{in}} q_i^{-1} k_i^{-1} f_i \\ \eta(f_i) = (-1)^{\delta_{in}} q_i^{-1} k_i e_i \end{cases} \quad \text{if } X = C^{(2)}(n+1)$$

for $i \in I$. Both τ and η are involutions.

When $X = C^{(2)}(n+1), B^{(1)}(0, n)$, let

$$U_q(X)^\sigma = U_q(X) \oplus U_q(X)\sigma$$

be the semidirect product of $U_q(X)$ and the group algebra generated by σ , where

$$(2.3) \quad \sigma^2 = 1, \quad \sigma k_i = k_i \sigma, \quad \sigma e_i = (-1)^{p(i)} e_i \sigma, \quad \sigma f_i = (-1)^{p(i)} f_i \sigma \quad (i \in I).$$

τ and η are extended to $U_q(X)^\sigma$ by $\tau(\sigma) = \eta(\sigma) = \sigma$.

The algebras $U_q(C_n^{(1)}), U_q(C^{(2)}(n+1))^\sigma, U_q(B^{(1)}(0, n))^\sigma$ have a Hopf algebra structure.

In particular, the coproduct Δ is given by

$$(2.4) \quad \begin{aligned} \Delta(k_i) &= k_i \otimes k_i, \quad \Delta(\sigma) = \sigma \otimes \sigma, \\ \Delta(e_i) &= e_i \otimes \sigma^{p(i)\delta_{i0}} k_i^{-1} + \sigma^{p(i)\delta_{in}} \otimes e_i, \\ \Delta(f_i) &= f_i \otimes \sigma^{p(i)\delta_{i0}} + \sigma^{p(i)\delta_{in}} k_i \otimes f_i \end{aligned}$$

for $i \in I$.

3. LEVEL ONE q -OSCILLATOR REPRESENTATION

Let \mathcal{W} be an infinite-dimensional vector space over $\mathbb{Q}(q^d)$ defined by

$$\mathcal{W} = \bigoplus_{\mathbf{m}} \mathbb{Q}(q^d) |\mathbf{m}\rangle,$$

where $|\mathbf{m}\rangle$ is a basis vector parametrized by $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n$. Let $|\mathbf{m}| = \sum_{j=1}^n m_j$, and let \mathbf{e}_j be the j -th standard vector in \mathbb{Z}^n for $1 \leq j \leq n$. In this section, we introduce the so-called q -oscillator representation of level one for each algebra.

3.1. Type $C_n^{(1)}$.

3.1.1. $U_q(C_n^{(1)})$ -module \mathcal{W}_{\pm} . Consider the quantum affine algebra $U_q(C_n^{(1)})$. Let $U_q(C_n)$ and $U_q(A_{n-1})$ be the subalgebras generated by k_i, e_i, f_i for $i \in I \setminus \{0\}$ and $i \in I \setminus \{0, n\}$, respectively.

Proposition 3.1. *For a non-zero $x \in \mathbb{Q}(q)$, the space \mathcal{W} admits a $U_q(C_n^{(1)})$ -module structure given as follows:*

$$\begin{aligned} e_0|\mathbf{m}\rangle &= xq^{-1} \frac{[m_1+1][m_1+2]}{[2]} |\mathbf{m} + 2\mathbf{e}_1\rangle, \\ f_0|\mathbf{m}\rangle &= -x^{-1} \frac{q}{[2]} |\mathbf{m} - 2\mathbf{e}_1\rangle, \\ k_0|\mathbf{m}\rangle &= q^{2m_1+1} |\mathbf{m}\rangle, \\ e_j|\mathbf{m}\rangle &= [m_{j+1}+1] |\mathbf{m} - \mathbf{e}_j + \mathbf{e}_{j+1}\rangle, \\ f_j|\mathbf{m}\rangle &= [m_j+1] |\mathbf{m} + \mathbf{e}_j - \mathbf{e}_{j+1}\rangle, \\ k_j|\mathbf{m}\rangle &= q^{-m_j+m_{j+1}} |\mathbf{m}\rangle, \\ e_n|\mathbf{m}\rangle &= -\frac{q}{[2]} |\mathbf{m} - 2\mathbf{e}_n\rangle, \\ f_n|\mathbf{m}\rangle &= q^{-1} \frac{[m_n+1][m_n+2]}{[2]} |\mathbf{m} + 2\mathbf{e}_n\rangle, \\ k_n|\mathbf{m}\rangle &= q^{-2m_n-1} |\mathbf{m}\rangle, \end{aligned}$$

where $1 \leq j \leq n-1$. Here we understand the vector on the right-hand side is zero when any of its components does not belong to $\mathbb{Z}_{\geq 0}$.

Remark 3.2. For $|\mathbf{m}\rangle \in \mathcal{W}$, set $\tau(|\mathbf{m}\rangle) = |m_n, \dots, m_1\rangle$, and extend linearly to any vector of \mathcal{W} . Then, when $x = 1$ we have the following symmetry

$$\tau(u|\mathbf{m}\rangle) = \tau(u)\tau(|\mathbf{m}\rangle),$$

for $u \in U_q(C_n^{(1)})$. Here the automorphism τ on $U_q(C_n^{(1)})$ is given in (2.1).

Remark 3.3. This representation originally appeared in [13, Proposition 3]. The presentation above is obtained from the one in [13] by applying the basis change $|\mathbf{m}\rangle^{\text{new}} = \frac{(q[2])^{|\mathbf{m}|/2}}{\prod_{i=1}^n [m_i]!} |\mathbf{m}\rangle^{\text{old}}$ and the automorphism of $U_q(C_n^{(1)})$ sending $f_0 \mapsto -f_0, e_n \mapsto -e_n, k_i \mapsto -k_i$ for $i = 0, n$ with the other generators fixed.

We assume that ε denotes $+$ or $-$. Set $\varsigma(\varepsilon) = 0$ and 1 , when $\varepsilon = +$ and $-$, respectively. For $m \in \mathbb{Z}_{\geq 0}$, let $\text{sgn}(m)$ be $+$ and $-$ if m is even and odd, respectively.

Define the subspace $\mathcal{W}_{\varepsilon}$ of \mathcal{W} by

$$\mathcal{W}_{\varepsilon} = \bigoplus_{\text{sgn}(|\mathbf{m}|)=\varepsilon} \mathbb{Q}(q)|\mathbf{m}\rangle.$$

Proposition 3.4. *For a non-zero $x \in \mathbb{Q}(q)$, \mathcal{W}_ε is an irreducible $U_q(C_n^{(1)})$ -module.*

We denote this module by $\mathcal{W}_\varepsilon(x)$, and call it a (level one) q -oscillator representation. We simply write $\mathcal{W}_\varepsilon = \mathcal{W}_\varepsilon(1)$ as a $U_q(C_n^{(1)})$ -module.

Let $s_\lambda(x_1, \dots, x_n)$ denote the Schur polynomial in x_1, \dots, x_n corresponding to a partition λ . Then as a $U_q(A_{n-1})$ -module, we have

$$\begin{aligned} \text{ch}\mathcal{W}_+ &= \sum_{l \in 2\mathbb{Z}_{\geq 0}} s_{(l)}(x_1, \dots, x_n) = \frac{1}{\prod_{i=1}^n (1 - x_i^2)}, \\ \text{ch}\mathcal{W}_- &= \sum_{l \in 1+2\mathbb{Z}_{\geq 0}} s_{(l)}(x_1, \dots, x_n) = \frac{\prod_{i=1}^n (1 + x_i) - 1}{\prod_{i=1}^n (1 - x_i^2)}. \end{aligned}$$

Here the weight lattice of $U_q(C_n^{(1)})$ is identified with the \mathbb{Z} -lattice spanned by \mathbf{e}_i for $1 \leq i \leq n$, and hence the variable x_i corresponds to the weight of \mathbf{e}_i .

3.1.2. *Classical limit.* Let A be the localization of $\mathbb{Z}[q, q^{-1}]$ at $[2] = q + q^{-1}$. Let

$$\mathcal{W}_\varepsilon(x)_A = \sum_{\text{sgn}(|\mathbf{m}|) = \varepsilon} A|\mathbf{m}\rangle.$$

Then $\mathcal{W}_\varepsilon(x)_A$ is invariant under e_i, f_i, k_i and $\{k_i\} := \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$ for $i \in I \setminus \{0\}$. Let

$$\overline{\mathcal{W}_\varepsilon(x)} = \mathcal{W}_\varepsilon(x)_A \otimes_A \mathbb{C},$$

where \mathbb{C} is an A -module such that $f(q) \cdot c = f(1)c$ for $f(q) \in A$ and $c \in \mathbb{C}$.

Let E_i, F_i and H_i be the \mathbb{C} -linear endomorphisms on $\overline{\mathcal{W}_\varepsilon(x)}$ induced from e_i, f_i and $\{k_i\}$ for $i \in I \setminus \{0\}$. We can check that they satisfy the defining relations for the universal enveloping algebra $U(C_n)$ of type C_n (cf. [7, Chapter 5]). Hence $\overline{\mathcal{W}_\varepsilon(x)}$ becomes a $U(C_n)$ -module.

Lemma 3.5. *The space $\overline{\mathcal{W}_\varepsilon(x)}$ is isomorphic to the irreducible highest weight $U(C_n)$ -module with highest weight $-(\frac{1}{2} + \varsigma(\varepsilon))\varpi_n$, where ϖ_n is the n -th fundamental weight for C_n .*

Proof. It is clear that $E_i(|\mathbf{0}\rangle \otimes 1) = 0$ for all $i \in I \setminus \{0\}$. Since

$$H_n(|\mathbf{0}\rangle \otimes 1) = \left(\frac{k_n - k_n^{-1}}{q_n - q_n^{-1}} |\mathbf{0}\rangle \right) \otimes 1 = \left(-\frac{1}{q + q^{-1}} |\mathbf{0}\rangle \right) \otimes 1 = -\frac{1}{2} |\mathbf{0}\rangle \otimes 1,$$

and $H_i(|\mathbf{0}\rangle \otimes 1) = 0$ for $1 \leq i \leq n-1$, $\overline{\mathcal{W}_+(x)}$ is a highest weight $U(C_n)$ -module with highest weight $-\frac{1}{2}\varpi_n$. It follows from the actions of E_i for $i \in I \setminus \{0\}$ that any submodule of $\overline{\mathcal{W}_+(x)}$ contains $|\mathbf{0}\rangle \otimes 1$. This implies that $\overline{\mathcal{W}_+(x)}$ is irreducible. The proof for $\mathcal{W}_-(x)$ is similar. \square

3.1.3. *Polarization.* Define a symmetric bilinear form on \mathcal{W}_ε by

$$(3.1) \quad (|\mathbf{m}\rangle, |\mathbf{m}'\rangle) = \delta_{\mathbf{m}, \mathbf{m}'} \frac{q^{-\frac{1}{2} \sum_{i=1}^n m_i(m_i-1)}}{\prod_{i=1}^n [m_i]!},$$

for $|\mathbf{m}\rangle, |\mathbf{m}'\rangle$ with $\mathbf{m} = (m_1, \dots, m_n)$. Note that $(|\mathbf{m}\rangle, |\mathbf{m}\rangle) \in 1 + qA_0$.

Lemma 3.6. *The bilinear form in (3.1) is a polarization on \mathcal{W}_ε , that is,*

$$(uv, v') = (v, \eta(u)v'),$$

for $u \in U_q(C_n^{(1)})$ and $v, v' \in \mathcal{W}_\varepsilon$.

Proof. It suffices to show when u is one of the generators. If $u = k_i$, it is trivial. Let us show that

$$(3.2) \quad (e_i|\mathbf{m}\rangle, |\mathbf{m}'\rangle) = (|\mathbf{m}\rangle, \eta(e_i)|\mathbf{m}'\rangle),$$

for $i \in I$ and $|\mathbf{m}\rangle, |\mathbf{m}'\rangle \in \mathcal{W}_\varepsilon$. The proof for f_i is almost identical since (3.1) is symmetric.

Case 1. Suppose that $1 \leq i \leq n-1$. We may assume $\mathbf{m}' = \mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}$. The right-hand side is

$$(|\mathbf{m}\rangle, \eta(e_i)|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle) = (|\mathbf{m}\rangle, q_i^{-1}k_i^{-1}f_i|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle) = [m_i]q^{-1+m_i-m_{i+1}}(|\mathbf{m}\rangle, |\mathbf{m}\rangle),$$

and the left-hand side is

$$\begin{aligned} (e_i|\mathbf{m}\rangle, |\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle) &= [m_{i+1} + 1](|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle, |\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle) \\ &= \frac{q^A [m_{i+1} + 1]}{[m_i - 1]! [m_{i+1} + 1]! \prod_{j \neq i, i+1} [m_j]!} \\ &= q^{m_i - m_{i+1} - 1} [m_i] (|\mathbf{m}\rangle, |\mathbf{m}\rangle), \end{aligned}$$

since

$$\begin{aligned} A &= -\frac{1}{2} \sum_{j \neq i, i+1} m_j(m_j - 1) - \frac{1}{2}(m_i - 1)(m_i - 2) - \frac{1}{2}(m_{i+1} + 1)m_{i+1} \\ &= -\frac{1}{2} \sum_{1 \leq j \leq n} m_j(m_j - 1) + m_i - m_{i+1} - 1. \end{aligned}$$

Hence (3.2) holds.

Case 2. Suppose that $i = n$. We may assume $\mathbf{m}' = \mathbf{m} - 2\mathbf{e}_n$. The right-hand side is

$$(|\mathbf{m}\rangle, \eta(e_n)|\mathbf{m} - 2\mathbf{e}_n\rangle) = (|\mathbf{m}\rangle, -q_n^{-1}k_n^{-1}f_n|\mathbf{m} - 2\mathbf{e}_n\rangle) = -q^{2m_n-2} \frac{[m_n-1][m_n]}{[2]} (|\mathbf{m}\rangle, |\mathbf{m}\rangle),$$

and the left-hand side is

$$\begin{aligned} (e_n|\mathbf{m}\rangle, |\mathbf{m} - 2\mathbf{e}_n\rangle) &= -\frac{q}{[2]} (|\mathbf{m} - 2\mathbf{e}_n\rangle, |\mathbf{m} - 2\mathbf{e}_n\rangle) = -\frac{q}{[2]} \frac{q^B}{[m_n-2]! \prod_{j \neq n} [m_j]!} (|\mathbf{m}\rangle, |\mathbf{m}\rangle) \\ &= -q^{2m_n-2} \frac{[m_n-1][m_n]}{[2]} (|\mathbf{m}\rangle, |\mathbf{m}\rangle), \end{aligned}$$

since

$$B = -\frac{1}{2} \sum_{j \neq n} m_j(m_j - 1) - \frac{1}{2}(m_n - 2)(m_n - 3) = -\frac{1}{2} \sum_{1 \leq j \leq n} m_j(m_j - 1) + 2m_n - 3.$$

Hence (3.2) holds.

Case 3. Suppose that $i = 0$. We have to show $(e_0v, v') = (v, -q^{-2}k_0^{-1}f_0v')$. By Remark 3.2 and the property $(\tau(|\mathbf{m}\rangle), \tau(|\mathbf{m}'\rangle)) = (|\mathbf{m}\rangle, |\mathbf{m}'\rangle)$, it is equivalent to $(f_n\tau(v), \tau(v')) = (\tau(v), -q^{-2}k_n e_n \tau(v'))$. However, it is equivalent to the one proved in *Case 1*. \square

3.1.4. *Crystal base.* Let M be a $U_q(C_n^{(1)})$ -module. For $1 \leq j \leq n-1$, we assume that e_j and f_j are locally nilpotent on M , and define \tilde{e}_j, \tilde{f}_j to be the usual lower crystal operators [12]. For $i = 0, n$, we introduce new operators \tilde{e}_i and \tilde{f}_i as follows:

Case 1. Let $u \in M$ be a weight vector such that $e_n u = 0$ and $k_n u = q_n^{-l} u$ for some $l > 0$. Put

$$(3.3) \quad u_k := q_n^{\frac{k(k+2l-1)}{2}} f_n^{(k)} u \quad (k \geq 0).$$

Then we define

$$(3.4) \quad \tilde{f}_n u_k = u_{k+1}, \quad \tilde{e}_n u_{k+1} = u_k \quad (k \geq 0).$$

Case 2. Let $u \in M$ be a weight vector such that $f_0 u = 0$ and $k_0 u = q_0^l u$ for some $l > 0$. Put

$$(3.5) \quad u_k := q_0^{\frac{k(k+2l-1)}{2}} e_0^{(k)} u \quad (k \geq 0).$$

Then we define

$$(3.6) \quad \tilde{e}_0 u_k = u_{k+1}, \quad \tilde{f}_0 u_{k+1} = u_k \quad (k \geq 0).$$

Remark 3.7. The definitions of \tilde{e}_i and \tilde{f}_i ($i = 0, n$) are based on the idea that

$$(3.7) \quad (\tilde{f}_n^k u, \tilde{f}_n^k u) \in 1 + qA_0 \quad (\tilde{e}_0^k u', \tilde{e}_0^k u') \in 1 + qA_0 \quad (k \geq 0),$$

for $u, u' \in \mathcal{W}_\varepsilon$ such that $e_n u = 0$ and $f_0 u' = 0$ (use Proposition 2.1).

Let A_0 be the subring of $\mathbb{Q}(q)$ consisting of functions which are regular at $q = 0$. We define A_0 -lattice \mathcal{L}_ε of \mathcal{W}_ε and a \mathbb{Q} -basis \mathcal{B}_ε of $\mathcal{L}_\varepsilon/q\mathcal{L}_\varepsilon$ by

$$\mathcal{L}_\varepsilon = \bigoplus_{\text{sgn}(\mathbf{m})=\varepsilon} A_0|\mathbf{m}\rangle, \quad \mathcal{B}_\varepsilon = \{|\mathbf{m}\rangle \pmod{q\mathcal{L}} \mid \text{sgn}(\mathbf{m}) = \varepsilon\}.$$

It is clear from (3.1) that $(\mathcal{L}_\varepsilon, \mathcal{L}_\varepsilon) \subset A_0$, and \mathcal{B}_ε is an orthonormal basis of $\mathcal{L}_\varepsilon/q\mathcal{L}_\varepsilon$ with respect to $(\ , \)|_{q=0}$.

Proposition 3.8. *The pair $(\mathcal{L}_\varepsilon, \mathcal{B}_\varepsilon)$ is a crystal base of \mathcal{W}_ε , that is,*

- (1) \mathcal{L}_ε is invariant under \tilde{e}_i and \tilde{f}_i for $i \in I$,
- (2) $\tilde{e}_i \mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon \cup \{0\}$ and $\tilde{f}_i \mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon \cup \{0\}$ for $i \in I$, where we have

$$\tilde{f}_i |\mathbf{m}\rangle \equiv \begin{cases} |\mathbf{m} + 2\mathbf{e}_n\rangle & \text{if } i = n, \\ |\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle & \text{if } m_{i+1} \geq 1 \text{ and } 1 \leq i \leq n-1, \\ |\mathbf{m} - 2\mathbf{e}_1\rangle & \text{if } m_1 \geq 2 \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases} \pmod{q\mathcal{L}_\varepsilon}.$$

Proof. It is enough to prove (2).

Case 1. Suppose that $1 \leq i \leq n-1$. Let $|\mathbf{m}\rangle = |m_1, \dots, m_n\rangle \in \mathcal{L}_\varepsilon$ be given with $m_{i+1} \geq 1$. Since $e_i |\mathbf{m} - m_i \mathbf{e}_i + m_i \mathbf{e}_{i+1}\rangle = 0$, we have

$$\tilde{f}_i^{m_i} |\mathbf{m} - m_i \mathbf{e}_i + m_i \mathbf{e}_{i+1}\rangle = \frac{f_i^{m_i}}{[m_i]!} |\mathbf{m} - m_i \mathbf{e}_i + m_i \mathbf{e}_{i+1}\rangle = |\mathbf{m}\rangle,$$

and hence $\tilde{f}_i|\mathbf{m}\rangle = \tilde{f}_i^{m_i+1}|\mathbf{m} - m_i\mathbf{e}_i + m_i\mathbf{e}_{i+1}\rangle = |\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle$.

Case 2. Suppose that $i = n$. First, suppose that m_n is even. Since $e_n|\mathbf{m} - m_n\mathbf{e}_n\rangle = 0$ and $k_n|\mathbf{m} - m_n\mathbf{e}_n\rangle = q^{-1}|\mathbf{m} - m_n\mathbf{e}_n\rangle$, we have

$$\tilde{f}_n^{\frac{m_n}{2}}|\mathbf{m} - m_n\mathbf{e}_n\rangle = q^{\binom{m_n}{2}} \frac{f_n^{\frac{m_n}{2}}}{[\frac{m_n}{2}]_n!} |\mathbf{m} - m_n\mathbf{e}_n\rangle = (1+q^2)^{-\frac{m_n}{2}} q^{\binom{m_n}{2}} \frac{[m_n]!}{[\frac{m_n}{2}]_n!} |\mathbf{m}\rangle,$$

and hence

$$\begin{aligned} \tilde{f}_n|\mathbf{m}\rangle &= (1+q^2)^{\frac{m_n}{2}} q^{-\binom{m_n}{2}} \frac{[\frac{m_n}{2}]_n!}{[m_n]!} \tilde{f}_n^{\frac{m_n}{2}+1} |\mathbf{m} - m_n\mathbf{e}_n\rangle \\ &= (1+q^2)^{\frac{m_n}{2}} q^{-\binom{m_n}{2}} \frac{[\frac{m_n}{2}]_n!}{[m_n]!} (1+q^2)^{-\frac{m_n}{2}-1} q^{\binom{m_n}{2}+1} \frac{[m_n+2]!}{[\frac{m_n}{2}+1]_n!} |\mathbf{m} + 2\mathbf{e}_n\rangle \\ &= (1+q^2)^{-1} q^{\binom{m_n}{2}+1} \frac{[m_n+2][m_n+1]}{[\frac{m_n}{2}+1]_n} |\mathbf{m} + 2\mathbf{e}_n\rangle \\ &\equiv |\mathbf{m} + 2\mathbf{e}_n\rangle \pmod{q\mathcal{L}_\varepsilon}, \end{aligned}$$

since

$$q^{\binom{m_n}{2}+1} q^{-\binom{m_n}{2}} \frac{[m_n+2][m_n+1]}{[\frac{m_n}{2}+1]_n} = q^{m_n+1} \frac{[m_n+2][m_n+1]}{[\frac{m_n}{2}+1]_n} \in (1+qA_0).$$

Next, suppose that m_n is odd. Since $e_n|\mathbf{m} - (m_n-1)\mathbf{e}_n\rangle = 0$ and $k_n|\mathbf{m} - (m_n-1)\mathbf{e}_n\rangle = q^{-3}|\mathbf{m} - (m_n-1)\mathbf{e}_n\rangle$, we have

$$\begin{aligned} \tilde{f}_n^{\frac{m_n-1}{2}}|\mathbf{m} - (m_n-1)\mathbf{e}_n\rangle &= q^{\binom{m_n-1}{2}} \frac{f_n^{\frac{m_n-1}{2}}}{[\frac{m_n-1}{2}]_n!} |\mathbf{m} - (m_n-1)\mathbf{e}_n\rangle \\ &= (1+q^2)^{-\frac{m_n-1}{2}} q^{\binom{m_n-1}{2}} \frac{[m_n]!}{[\frac{m_n-1}{2}]_n!} |\mathbf{m}\rangle, \end{aligned}$$

and hence

$$\begin{aligned} \tilde{f}_n|\mathbf{m}\rangle &= (1+q^2)^{\frac{m_n-1}{2}} q^{-\binom{m_n-1}{2}} \frac{[\frac{m_n-1}{2}]_n!}{[m_n]!} \tilde{f}_n^{\frac{m_n-1}{2}+1} |\mathbf{m} - (m_n-1)\mathbf{e}_n\rangle \\ &= (1+q^2)^{\frac{m_n-1}{2}} q^{-\binom{m_n-1}{2}} \frac{[\frac{m_n-1}{2}]_n!}{[m_n]!} (1+q^2)^{-\frac{m_n-1}{2}} q^{\binom{m_n-1}{2}} \frac{[m_n+2]!}{[\frac{m_n-1}{2}+1]_n!} |\mathbf{m} + 2\mathbf{e}_n\rangle \\ &= (1+q^2)^{-1} q^{\binom{m_n-1}{2}} \frac{[m_n+2][m_n+1]}{[\frac{m_n-1}{2}+1]_n} |\mathbf{m} + 2\mathbf{e}_n\rangle \\ &\equiv |\mathbf{m} + 2\mathbf{e}_n\rangle \pmod{q\mathcal{L}_\varepsilon}, \end{aligned}$$

since

$$q^{\binom{m_n-1}{2}} q^{-\binom{m_n-1}{2}} \frac{[m_n+2][m_n+1]}{[\frac{m_n-1}{2}+1]_n} = q^{m_n+1} \frac{[m_n+2][m_n+1]}{[\frac{m_n-1}{2}+1]_n} \in (1+qA_0).$$

Case 3. Suppose that $i = 0$. We can prove this case by the same arguments as in *Case 2* by using the automorphism τ (2.1). \square

3.2. Type $C^{(2)}(n+1)$.

3.2.1. $U_q(C^{(2)}(n+1)$ -module \mathcal{W} . Consider the quantum affine superalgebra of type $C^{(2)}(n+1)$. Let $U_q(B(0, n))$ and $U_q(A_{n-1})$ be the subalgebras of $U_q(C^{(2)}(n+1))$ generated by k_i, e_i, f_i for $i \in I \setminus \{0\}$ and $i \in I \setminus \{0, n\}$, respectively. We also write $U_q(B(0, n)) = U_q(\text{osp}_{1|2n})$, where $\text{osp}_{1|2n}$ is the orthosymplectic Lie superalgebra corresponding to the Dynkin diagram:



Proposition 3.9. *For a non-zero $x \in \mathbb{Q}(q^{\frac{1}{2}})$, the space \mathcal{W} admits an irreducible $U_q(C^{(2)}(n+1))^\sigma$ -module structure given as follows:*

$$\begin{aligned} e_0|\mathbf{m}\rangle &= xq^{-\frac{1}{2}}[m_1 + 1]|\mathbf{m} + \mathbf{e}_1\rangle, \\ f_0|\mathbf{m}\rangle &= x^{-1}q^{\frac{1}{2}}|\mathbf{m} - \mathbf{e}_1\rangle, \\ k_0|\mathbf{m}\rangle &= q^{m_1 + \frac{1}{2}}|\mathbf{m}\rangle, \\ e_j|\mathbf{m}\rangle &= [m_{j+1} + 1]|\mathbf{m} - \mathbf{e}_j + \mathbf{e}_{j+1}\rangle, \\ f_j|\mathbf{m}\rangle &= [m_j + 1]|\mathbf{m} + \mathbf{e}_j - \mathbf{e}_{j+1}\rangle, \\ k_j|\mathbf{m}\rangle &= q^{-m_j + m_{j+1}}|\mathbf{m}\rangle, \\ e_n|\mathbf{m}\rangle &= -q^{\frac{1}{2}}|\mathbf{m} - \mathbf{e}_n\rangle, \\ f_n|\mathbf{m}\rangle &= q^{-\frac{1}{2}}[m_n + 1]|\mathbf{m} + \mathbf{e}_n\rangle, \\ k_n|\mathbf{m}\rangle &= q^{-m_n - \frac{1}{2}}|\mathbf{m}\rangle, \\ \sigma|\mathbf{m}\rangle &= (-1)^{|\mathbf{m}|}|\mathbf{m}\rangle, \end{aligned}$$

where $1 \leq j \leq n-1$.

We denote this module by $\mathcal{W}(x)$ and call it a (level one) q -oscillator representation. We simply write $\mathcal{W} = \mathcal{W}(1)$ as a $U_q(C^{(2)}(n+1))$ -module. Note that as a $U_q(A_{n-1})$ -module, we have

$$\text{ch}\mathcal{W} = \sum_{l \in \mathbb{Z}_{\geq 0}} s_{(l)}(x_1, \dots, x_n) = \frac{1}{\prod_{i=1}^n (1 - x_i)}.$$

Remark 3.10. When $x = 1$ we also have the following symmetry

$$\tau(u|\mathbf{m}\rangle) = \tau(u)\tau(|\mathbf{m}\rangle),$$

for $u \in U_q(C^{(2)}(n+1))$ (cf. Remark 3.2). Here the automorphism τ on $U_q(C^{(2)}(n+1))$ is given in (2.2).

3.2.2. *Classical limit.* Let

$$(3.8) \quad \mathcal{W}(x)_A = \sum_{\mathbf{m}} A|\mathbf{m}\rangle, \quad \overline{\mathcal{W}(x)} = \mathcal{W}(x)_A \otimes_A \mathbb{C},$$

where $A = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ and \mathbb{C} is an A -module such that $f(q^{\frac{1}{2}}) \cdot c = f(1)c$ for $f(q^{\frac{1}{2}}) \in A$ and $c \in \mathbb{C}$.

One can check directly that $\mathcal{W}(x)_A$ is invariant under e_i, f_i , and $\{k_i\}$ for $i \in I \setminus \{0\}$, and the induced operators E_i, F_i , and H_i on $\overline{\mathcal{W}(x)}$, respectively, satisfy the defining relations of $U(\mathfrak{osp}_{1|2n})$.

Lemma 3.11. *The space $\overline{\mathcal{W}(x)}$ is isomorphic to the irreducible highest weight $U(\mathfrak{osp}_{1|2n})$ -module with highest weight $-\varpi_n$, where ϖ_n is the n -th fundamental weight for $\mathfrak{osp}_{1|2n}$.*

Proof. We have

$$H_n(|\mathbf{0}\rangle \otimes 1) = \left(\frac{k_n - k_n^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} |\mathbf{0}\rangle \right) \otimes 1 = \left(\frac{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} |\mathbf{0}\rangle \right) \otimes 1 = -|\mathbf{0}\rangle \otimes 1,$$

and $H_i(|\mathbf{0}\rangle \otimes 1) = 0$ for $1 \leq i \leq n-1$. By the same argument as in Lemma 3.5, $\overline{\mathcal{W}(x)}$ is an irreducible highest weight $U_q(\mathfrak{osp}_{1|2n})$ -module with highest weight $-\varpi_n$. \square

3.2.3. *Polarization.* Define a symmetric bilinear form on \mathcal{W} by (3.1).

Lemma 3.12. *The bilinear form in (3.1) is a polarization on \mathcal{W} , that is,*

$$(uv, v') = (v, \eta(u)v'),$$

for $u \in U_q(C^{(2)}(n+1))$ and $v, v' \in \mathcal{W}$.

Proof. Let us show $(e_n|\mathbf{m}\rangle, |\mathbf{m}'\rangle) = (|\mathbf{m}\rangle, \eta(e_n)|\mathbf{m}'\rangle)$ for $|\mathbf{m}\rangle, |\mathbf{m}'\rangle \in \mathcal{W}$ only. The proof for e_i ($1 \leq i \leq n-1$) is identical to Lemma 3.1, and the proof for e_0 is obtained by using τ . We may assume $\mathbf{m}' = \mathbf{m} - \mathbf{e}_n$. The right-hand side is

$$(|\mathbf{m}\rangle, \eta(e_n)|\mathbf{m} - \mathbf{e}_n\rangle) = (|\mathbf{m}\rangle, -q_n^{-1}k_n^{-1}f_n|\mathbf{m} - \mathbf{e}_n\rangle) = -q_n^{2m_n-1}[m_n](|\mathbf{m}\rangle, |\mathbf{m}\rangle),$$

and the left-hand side is

$$\begin{aligned} (e_n|\mathbf{m}\rangle, |\mathbf{m} - \mathbf{e}_n\rangle) &= -q^{\frac{1}{2}}(|\mathbf{m} - \mathbf{e}_n\rangle, |\mathbf{m} - \mathbf{e}_n\rangle) \\ &= -q^{\frac{1}{2}} \frac{q^{-\frac{1}{2}} \sum m_i(m_i-1)}{\prod_{i=1}^n [m_i]!} [m_n] q^{m_n-1} = -q^{m_n-\frac{1}{2}} [m_n](|\mathbf{m}\rangle, |\mathbf{m}\rangle). \end{aligned}$$

Hence the equality holds. \square

3.2.4. *Crystal base.* Let M be a $U_q(C^{(2)}(n+1))$ -module. For $1 \leq j \leq n-1$, we assume that e_j and f_j are locally nilpotent on M , and define \tilde{e}_j, \tilde{f}_j to be the usual lower crystal operators. For $i = 0, n$, we consider the operators \tilde{e}_i and \tilde{f}_i defined in the same way as in $U_q(C_n^{(1)})$ (3.3)–(3.6), which also satisfy (3.7).

Let A_0 be the subring of $\mathbb{Q}(q^{\frac{1}{2}})$ consisting of functions which are regular at $q^{\frac{1}{2}} = 0$. We define the A_0 -lattice \mathcal{L} of \mathcal{W} and a \mathbb{Q} -basis \mathcal{B} of $\mathcal{L}/q^{\frac{1}{2}}\mathcal{L}$ by

$$(3.9) \quad \mathcal{L} = \bigoplus_{\mathbf{m}} A_0|\mathbf{m}\rangle, \quad \mathcal{B} = \{ |\mathbf{m}\rangle \pmod{q^{\frac{1}{2}}\mathcal{L}} \}.$$

It is clear from (3.1) that $(\mathcal{L}, \mathcal{L}) \subset A_0$, and \mathcal{B} is an orthonormal basis of $\mathcal{L}/q^{\frac{1}{2}}\mathcal{L}$ with respect to $(\ , \)|_{q^{\frac{1}{2}}=0}$.

Proposition 3.13. *The pair $(\mathcal{L}, \mathcal{B})$ is a crystal base of \mathcal{W} in the sense of Proposition 3.8, where*

$$\tilde{f}_i |\mathbf{m}\rangle \equiv \begin{cases} |\mathbf{m} + \mathbf{e}_n\rangle & \text{if } i = n, \\ |\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle & \text{if } m_{i+1} \geq 1 \text{ and } 1 \leq i \leq n-1, \\ |\mathbf{m} - \mathbf{e}_1\rangle & \text{if } m_1 \geq 1 \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{mod } q^{\frac{1}{2}}\mathcal{L}).$$

Proof. It suffices to prove (2) when $i = 0, n$ since the other cases are proved in Proposition 3.8. Let us prove the case of \tilde{f}_n only. Recall that $[m]_n = [m]_{q^{\frac{1}{2}}, -1}$ for $m \in \mathbb{Z}_{\geq 0}$.

Let $|\mathbf{m}\rangle$ be given. Since $e_n |\mathbf{m} - m_n \mathbf{e}_n\rangle = 0$ and $k_n |\mathbf{m} - m_n \mathbf{e}_n\rangle = q^{-\frac{1}{2}} |\mathbf{m} - m_n \mathbf{e}_n\rangle$, we have

$$\tilde{f}_n^{m_n} |\mathbf{m} - m_n \mathbf{e}_n\rangle = q_n^{\frac{m_n(m_n+1)}{2}} \frac{f_n^{m_n}}{[m_n]_{q_n, -1}!} |\mathbf{m} - m_n \mathbf{e}_n\rangle = q_n^{\frac{m_n(m_n-1)}{2}} \frac{[m_n]!}{[m_n]_{q_n, -1}!} |\mathbf{m}\rangle,$$

and hence

$$\begin{aligned} \tilde{f}_n |\mathbf{m}\rangle &= q_n^{-\frac{m_n(m_n-1)}{2}} \frac{[m_n]_{q_n, -1}!}{[m_n]!} \tilde{f}_n^{m_n+1} |\mathbf{m} - m_n \mathbf{e}_n\rangle \\ &= q_n^{-\frac{m_n(m_n-1)}{2}} \frac{[m_n]_{q_n, -1}!}{[m_n]!} q_n^{\frac{m_n(m_n+1)}{2}} \frac{[m_n+1]!}{[m_n+1]_{q_n, -1}!} |\mathbf{m} + \mathbf{e}_n\rangle \\ &\equiv q_n^{m_n} \frac{[m_n+1]}{[m_n+1]_{q_n, -1}} |\mathbf{m} + \mathbf{e}_n\rangle = |\mathbf{m} + \mathbf{e}_n\rangle \quad (\text{mod } q^{\frac{1}{2}}\mathcal{L}). \end{aligned}$$

□

3.3. Type $B^{(1)}(0, n)$.

3.3.1. $U_q(B^{(1)}(0, n))$ -module \mathcal{W} . Consider the quantum affine superalgebra of type $B^{(1)}(0, n)$. Let $U_q(B(0, n))$ (or $U_q(\mathfrak{osp}_{1|2n})$) and $U_q(A_{n-1})$ be the subalgebras of $U_q(B^{(1)}(0, n))$ generated by k_i, e_i, f_i for $i \in I \setminus \{0\}$ and $i \in I \setminus \{0, n\}$, respectively.

Proposition 3.14. *For a non-zero $x \in \mathbb{Q}(q^{\frac{1}{2}})$, the space \mathcal{W} admits an irreducible $U_q(B^{(1)}(0, n))^\sigma$ -module structure given as follows:*

$$\begin{aligned} e_0 |\mathbf{m}\rangle &= xq^{-1} \frac{[m_1+1][m_1+2]}{[2]} |\mathbf{m} + 2\mathbf{e}_1\rangle, \\ f_0 |\mathbf{m}\rangle &= -x^{-1} \frac{q}{[2]} |\mathbf{m} - 2\mathbf{e}_1\rangle, \\ k_0 |\mathbf{m}\rangle &= q^{2m_1+1} |\mathbf{m}\rangle, \\ e_j |\mathbf{m}\rangle &= [m_{j+1}+1] |\mathbf{m} - \mathbf{e}_j + \mathbf{e}_{j+1}\rangle, \\ f_j |\mathbf{m}\rangle &= [m_j+1] |\mathbf{m} + \mathbf{e}_j - \mathbf{e}_{j+1}\rangle, \\ k_j |\mathbf{m}\rangle &= q^{-m_j+m_{j+1}} |\mathbf{m}\rangle, \\ e_n |\mathbf{m}\rangle &= -q^{\frac{1}{2}} |\mathbf{m} - \mathbf{e}_n\rangle, \\ f_n |\mathbf{m}\rangle &= q^{-\frac{1}{2}} [m_n+1] |\mathbf{m} + \mathbf{e}_n\rangle, \end{aligned}$$

$$\begin{aligned} k_n |\mathbf{m}\rangle &= q^{-m_n - \frac{1}{2}} |\mathbf{m}\rangle, \\ \sigma |\mathbf{m}\rangle &= (-1)^{|\mathbf{m}|} |\mathbf{m}\rangle, \end{aligned}$$

where $1 \leq j \leq n-1$.

We also denote this module by $\mathcal{W}(x)$ and call it a (level one) q -oscillator representation. Note that the classical limit of $\mathcal{W}(x)$ as a $U_q(\mathfrak{osp}_{1|2n})$ -module is the same as in Lemma 3.11.

3.3.2. Polarization and crystal base.

Lemma 3.15. *The bilinear form in (3.1) is a polarization on \mathcal{W} , that is,*

$$(uv, v') = (v, \eta(u)v'),$$

for $u \in U_q(B^{(1)}(0, n))$ and $v, v' \in \mathcal{W}$.

Proof. All the cases are already shown in Lemmas 3.6 and 3.12 since the action of e_i for $0 \leq i < n$ (resp. $i = n$) is the same as the one for $C_n^{(1)}$ (resp. $C^{(2)}(n+1)$). \square

We define the A_0 -lattice \mathcal{L} of \mathcal{W} and a \mathbb{Q} -basis \mathcal{B} of $\mathcal{L}/q^{\frac{1}{2}}\mathcal{L}$ as in (3.9). We also define the operators \tilde{e}_i and \tilde{f}_i in the same way as in $U_q(C_n^{(1)})$ and $U_q(C^{(2)}(n+1))$.

Proposition 3.16. *The pair $(\mathcal{L}, \mathcal{B})$ is a crystal base of \mathcal{W} in the sense of Proposition 3.8, where*

$$\tilde{f}_i |\mathbf{m}\rangle \equiv \begin{cases} |\mathbf{m} + \mathbf{e}_n\rangle & \text{if } i = n, \\ |\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle & \text{if } m_{i+1} \geq 1 \text{ and } 1 \leq i \leq n-1, \\ |\mathbf{m} - 2\mathbf{e}_1\rangle & \text{if } m_1 \geq 2 \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases} \pmod{q^{\frac{1}{2}}\mathcal{L}}.$$

Proof. It follows from Propositions 3.8 and 3.13. \square

4. QUANTUM R -MATRIX AND FUSION CONSTRUCTION

In this section, we review the quantum R -matrix and its spectral decomposition for each quantum affine (super)algebra and explain how to construct higher level q -oscillator representations by so-called fusion construction.

Let $x, y \in \mathbb{Q}(q^d)$ be generic, and let \mathcal{W} be a level one q -oscillator representation of $U_q(X)$ including \mathcal{W}_ε ($\varepsilon = \pm$) for type $C_n^{(1)}$. The quantum R -matrix $R(x, y)$ on $\mathcal{W}(x) \otimes \mathcal{W}(y)$ is defined as a linear operator satisfying

$$R(x, y)\Delta(a) = \Delta^{\text{op}}(a)R(x, y)$$

for $a \in U_q(X)$, where Δ^{op} denotes the opposite coproduct, namely, the coproduct obtained by interchanging the first and second components in Δ . If $\mathcal{W}(x) \otimes \mathcal{W}(y)$ is irreducible, then $R(x, y)$ is unique up to a scalar function of x, y and depends only on $z = x/y$. Let P be the linear operator on $\mathcal{W}(x) \otimes \mathcal{W}(y)$ such that $P(u \otimes v) = v \otimes u$ and set $\check{R}(x, y) = PR(x, y)$. Then $\check{R}(x, y)$ maps $\mathcal{W}(x) \otimes \mathcal{W}(y)$ to $\mathcal{W}(y) \otimes \mathcal{W}(x)$.

We also need to care about the difference between the coproduct (2.4) and that of [13] and Appendix A. Let $\bar{\Delta}$ be the coproduct of the latter. Let ς be the automorphism given by $\varsigma(e_i) = e_i k_i^{-1}$, $\varsigma(f_i) = k_i f_i$, $\varsigma(k_i) = k_i$. Then we have $\Delta(a) = \bar{\Delta}^{\text{op}}(\varsigma(a))$. Hence, to translate the results in [13] and Appendix A, we replace $\check{R}(x, y)$ with $\check{R}(y, x)$. The component V_l or V_l^ε appearing in the spectral decomposition should be replaced with PV_l . Thus, we obtain the spectral decomposition of $\check{R}(x, y)$ as follows. Note that $z = x/y$.

For type $C_n^{(1)}$, we have

$$(4.1) \quad \check{R}_\varepsilon(x, y) = \sum_{l \in 2\mathbb{Z}_{\geq 0}} \prod_{j=1}^{l/2} \frac{1 - q^{4j-2}z}{z - q^{4j-2}} P_l^\varepsilon$$

where $\check{R}_\varepsilon(x, y) : \mathcal{W}_\varepsilon(x) \otimes \mathcal{W}_\varepsilon(y) \rightarrow \mathcal{W}_\varepsilon(y) \otimes \mathcal{W}_\varepsilon(x)$ for $\varepsilon = +, -$ and P_l^ε is the projection onto V_l^ε .

For $C^{(2)}(n+1)$, from Proposition C.4 and the spectral decomposition for $U_q(D_{n+1}^{(2)})$ in [13, Proposition 7], we have

$$(4.2) \quad \check{R}(x, y) = \sum_{l \in \mathbb{Z}_{\geq 0}} \prod_{j=1}^l \frac{1 + (-q)^j z}{z + (-q)^j} P_l,$$

where P_l is the projection onto V_l .

Finally for $B^{(1)}(0, n)$, from Proposition C.4 and the spectral decomposition for $U_q(A_{2n}^{(2)\dagger})$ in Appendix B, we have

$$(4.3) \quad \check{R}(x, y) = \sum_{l \in 2\mathbb{Z}_{\geq 0}} \prod_{j=1}^{l/2} \frac{1 - q^{4j-1}z}{z - q^{4j-1}} P_l + \sum_{l \in 1+2\mathbb{Z}_{\geq 0}} \prod_{j=0}^{(l-1)/2} \frac{1 - q^{4j+1}z}{z - q^{4j+1}} P_l.$$

Next, we explain the fusion construction. For $s \geq 2$, let \mathfrak{S}_s denote the group of permutations on s letters generated by $s_i = (i \ i+1)$ for $1 \leq i \leq s-1$. We have $U_q(X)$ -linear maps

$$\check{R}_w(x_1, \dots, x_s) : \mathcal{W}(x_1) \otimes \dots \otimes \mathcal{W}(x_s) \longrightarrow \mathcal{W}(x_{w(1)}) \otimes \dots \otimes \mathcal{W}(x_{w(s)}),$$

for $w \in \mathfrak{S}_s$ and generic $x_1, \dots, x_s \in \mathbb{Q}(q)$ satisfying

$$\begin{aligned} \check{R}_1(x_1, \dots, x_s) &= \text{id}_{\mathcal{W}(x_1) \otimes \dots \otimes \mathcal{W}(x_s)}, \\ \check{R}_{s_i}(x_1, \dots, x_s) &= (\otimes_{j < i} \text{id}_{\mathcal{W}(x_j)}) \otimes \check{R}(x_i/x_{i+1}) \otimes (\otimes_{j > i+1} \text{id}_{\mathcal{W}(x_j)}), \\ \check{R}_{ww'}(x_1, \dots, x_s) &= \check{R}_{w'}(x_{w(1)}, \dots, x_{w(s)}) \check{R}_w(x_1, \dots, x_s), \end{aligned}$$

for $w, w' \in \mathfrak{S}_s$ with $\ell(ww') = \ell(w) + \ell(w')$ where $\ell(w)$ denotes the length of w . Hence we have a $U_q(X)$ -linear map $\check{R}_s = \check{R}_{w_0}(x_1, \dots, x_s)$ with $x_i = q^{d(2i-s-1)}$:

$$\check{R}_s : \mathcal{W}(q^{d(1-s)}) \otimes \dots \otimes \mathcal{W}(q^{d(s-1)}) \longrightarrow \mathcal{W}(q^{d(s-1)}) \otimes \dots \otimes \mathcal{W}(q^{d(1-s)}).$$

Here w_0 is the longest element in \mathfrak{S}_s and $d = \min\{d_i \mid i \in I\}$. Now we define a $U_q(X)$ -module

$$(4.4) \quad \mathcal{W}^{(s)} = \text{Im} \check{R}_s.$$

Remark 4.1. Let R^{univ} be the universal R matrix for the quantum affine (super)algebra $U_q(X)$ [6]. Suppose that \mathcal{W} is a finite-dimensional irreducible $U_q(X)$ -module. Then R^{univ} is rationally renormalizable in the sense of [11], that is, there exists $c \in \mathbb{Q}(q^d)((y/x))$ such that we have a well-defined map

$$(4.5) \quad cR^{\text{univ}} : \mathcal{W}(x) \otimes \mathcal{W}(y) \longrightarrow \mathcal{W}(y) \otimes \mathcal{W}(x),$$

for x, y . Then we may apply [11, Theorem 3.12] to prove that $\mathcal{W}^{(s)}$ is irreducible. However, the q -oscillator module \mathcal{W} is infinite dimensional and R^{univ} on $\mathcal{W}(x) \otimes \mathcal{W}(y)$ is not rationally renormalizable. We expect that (4.5) still has a meaning, but do not know how to justify it.

5. HIGHER LEVEL q -OSCILLATOR REPRESENTATION

5.1. Type $C_n^{(1)}$. For $s \geq 2$ and $\varepsilon = \pm$, let $\mathcal{W}_\varepsilon^{(s)}$ denote the higher level q -oscillator module in (4.4) corresponding to \mathcal{W}_ε . The following is the main result in this section.

Theorem 5.1. *For $s \geq 2$, $\mathcal{W}_\varepsilon^{(s)}$ is an irreducible $U_q(C_n^{(1)})$ -module, which is also irreducible as a $U_q(C_n)$ -module. Moreover, its character is given by*

$$\text{ch}\mathcal{W}_\varepsilon^{(s)} = \sum_{\substack{\lambda \in \mathcal{P}_\varepsilon \\ \ell(\lambda) \leq s}} s_\lambda(x_1, \dots, x_n),$$

where \mathcal{P}_ε is the set of partitions $\lambda = (\lambda_i)_{i \geq 1}$ with $\text{sgn}(\lambda_i) = \varepsilon$ for all i with $\lambda_i \neq 0$, and $\ell(\lambda)$ denotes the length of λ .

Corollary 5.2. *The character of $\mathcal{W}_\varepsilon^{(s)}$ has a stable limit for $s \geq n$ as follows:*

$$\begin{aligned} \text{ch}\mathcal{W}_\varepsilon^{(s)} &= \sum_{\substack{\lambda \in \mathcal{P}_\varepsilon \\ \ell(\lambda) \leq n}} s_\lambda(x_1, \dots, x_n) \\ &= \frac{1}{\prod_{1 \leq i \leq j \leq n} (1 - x_i x_j)} \quad (\varepsilon = +). \end{aligned}$$

Let us construct a certain $\mathbb{Q}(q)$ -basis of $\mathcal{W}_\varepsilon^{(2)}$, which is compatible with the action of $\check{R}(z)$, and plays an important role in the proof of Theorem 5.1. We note from (4.1) that

$$\mathcal{W}_\varepsilon^{(2)} = V_0^\varepsilon = U_q(C_n)(|\zeta(\varepsilon)\mathbf{e}_n\rangle \otimes |\zeta(\varepsilon)\mathbf{e}_n\rangle),$$

and hence it is irreducible. Moreover, we have the following character formula for $\mathcal{W}_\varepsilon^{(2)}$.

Proposition 5.3. *We have*

$$\text{ch}\mathcal{W}_\varepsilon^{(2)} = \text{ch}V_0^\varepsilon = \sum_{\substack{\lambda \in \mathcal{P}_\varepsilon \\ \ell(\lambda) \leq 2}} s_\lambda(x_1, \dots, x_n).$$

Proof. Write $\mathcal{W}_\varepsilon = \mathcal{W}_\varepsilon(q^{\pm 1})$ for short since we may consider the action of $U_q(C_n)$ only. Let $(\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon)_A$ be the A -span of $|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle$ in $\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon$. Then $(\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon)_A$ is also invariant under e_i, f_i, k_i and $\{k_i\}$ for $i \in I \setminus \{0\}$. This yields its classical limit $\overline{\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon} := (\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon)_A \otimes_A \mathbb{C}$, which is a $U(C_n)$ -module. Also, we have as a $U(C_n)$ -module

$$\overline{\mathcal{W}_\varepsilon \otimes \mathcal{W}_\varepsilon} \cong \overline{\mathcal{W}_\varepsilon} \otimes \overline{\mathcal{W}_\varepsilon}.$$

By Lemma 3.5, $\overline{\mathcal{W}_\varepsilon}$ is an irreducible highest weight module. By the theory of super duality [3], it belongs to a semisimple category of $U(C_n)$ -module which is closed under tensor product (see [16, Section 5.4] for more details, where we put $m = 0$ there). Hence $\overline{\mathcal{W}_\varepsilon} \otimes \overline{\mathcal{W}_\varepsilon}$ is semisimple, and the classical limit $\overline{V_0^\varepsilon}$, the submodule generated by $(|\varsigma(\varepsilon)\mathbf{e}_n\rangle \otimes |\varsigma(\varepsilon)\mathbf{e}_n\rangle) \otimes 1$, is an irreducible highest weight $U(C_n)$ -module with highest weight $-(1 + 2\varsigma(\varepsilon))\varpi_n$. The character of $\overline{V_0^\varepsilon}$ and hence V_0^ε follows from [17, Theorem 6.1]. \square

We construct a $\mathbb{Q}(q)$ -basis of $\mathcal{W}_\varepsilon^{(2)}$ which is compatible with its $U_q(A_{n-1})$ -crystal base. For this, we find all the $U_q(A_{n-1})$ -highest weight vectors in $\mathcal{W}_\varepsilon^{(2)}$.

For $l \in \mathbb{Z}_{\geq 0}$, let

$$(5.1) \quad \mathbf{v}_l = \sum_{k=0}^l (-1)^k q^{k(k-l+1)} \begin{bmatrix} l \\ k \end{bmatrix}^{-1} |k\mathbf{e}_{n-1} + (l-k)\mathbf{e}_n\rangle \otimes |(l-k)\mathbf{e}_{n-1} + k\mathbf{e}_n\rangle.$$

Lemma 5.4. *For $l \in \mathbb{Z}_{\geq 0}$, \mathbf{v}_l is a $U_q(A_{n-1})$ -highest weight vector in $\mathcal{W}_\varepsilon^{(2)}$, and*

$$\mathbf{v}_l \equiv |l\mathbf{e}_n\rangle \otimes |l\mathbf{e}_{n-1}\rangle \pmod{q\mathcal{L}_\varepsilon^{\otimes 2}},$$

where $\text{sgn}(l) = \varepsilon$.

Proof. It is straightforward to check that $e_i \mathbf{v}_l = 0$ for $1 \leq i \leq n-1$. Next we claim that $\mathbf{v}_l \in \mathcal{W}_\varepsilon^{(2)}$. Note that

$$\text{ch}\mathcal{W}_\varepsilon = \sum_{l \in \varsigma(\varepsilon) + 2\mathbb{Z}_{>0}} s_{(l)}(x_1, \dots, x_n),$$

and hence

$$(5.2) \quad \text{ch}\mathcal{W}_\varepsilon^{\otimes 2} = (\text{ch}\mathcal{W}_\varepsilon)^2 = \sum_{\substack{\text{sgn}(|\lambda|)=+ \\ \ell(\lambda) \leq 2}} m_\lambda s_\lambda(x_1, \dots, x_n),$$

where for $\lambda = (\lambda_1, \lambda_2)$,

$$m_\lambda = \begin{cases} \frac{\lambda_1 - \lambda_2}{2} - \varsigma(\varepsilon) & \text{if } \lambda_1 > \lambda_2, \\ 1 & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Let S_l be the $U_q(A_{n-1})$ -submodule of $\mathcal{W}_\varepsilon^{\otimes 2}$ generated by \mathbf{v}_l . Since the character of S_l is $s_{(l^2)}(x_1, \dots, x_n)$, and the multiplicity of $s_{(l^2)}(x_1, \dots, x_n)$ in (5.2) is one, it follows from Proposition 5.3 that $S_l \subset \mathcal{W}_\varepsilon^{(2)}$. This shows that $\mathbf{v}_l \in \mathcal{W}_\varepsilon^{(2)}$. The lemma follows from $q^{k(k-l+1)} \begin{bmatrix} l \\ k \end{bmatrix}^{-1} \in q^k(1 + qA_0)$. \square

One can prove more directly that $\mathbf{v}_l \in \mathcal{W}_\varepsilon^{(2)}$ using the following lemma.

Lemma 5.5. *Set $\mathcal{E} = e_{n-2}^{(2)} \cdots e_1^{(2)} e_0$, where it should be understood as e_0 when $n = 2$. Then for $l \in \mathbb{Z}_{\geq 0}$ we have*

$$(\mathcal{E} e_1^{(2)} \mathcal{E} - \frac{1}{[3]!} (e_1 \mathcal{E})^2) \mathbf{v}_l = q^{-2} \frac{[2]}{[3]} ([l+1][l+2])^2 \mathbf{v}_{l+2}.$$

Proof. Denote the module \mathcal{W}_ε by $\mathcal{W}_{\varepsilon,n}$ to signify the rank n and let $\mathcal{W}_{\varepsilon,n}^0$ be a linear subspace of $\mathcal{W}_{\varepsilon,n}$ spanned by the vectors $|0^{n-2}, m_{n-1}, m_n\rangle$. Let $\pi : \mathcal{W}_{\varepsilon,n} \rightarrow \mathcal{W}_{\varepsilon,2}$ be a linear map defined by $\pi(|\mathbf{m}\rangle) = |m_{n-1}, m_n\rangle$, where $\mathbf{m} = (m_1, \dots, m_n)$. Then we can show by direct calculation that the following diagram commutes.

$$\begin{array}{ccc} (\mathcal{W}_{\varepsilon,n}^0)^{\otimes 2} & \xrightarrow{\pi^{\otimes 2}} & \mathcal{W}_{\varepsilon,2}^{\otimes 2} \\ \mathcal{E} \downarrow & & \downarrow e_0 \\ (\mathcal{W}_{\varepsilon,n}^0)^{\otimes 2} & \xrightarrow{\pi^{\otimes 2}} & \mathcal{W}_{\varepsilon,2}^{\otimes 2} \end{array}$$

This fact reduces the proof of the lemma to the case of $n = 2$.

When $n = 2$, one calculates

$$\begin{aligned} e_0 e_1^{(2)} e_0 \mathbf{v}_l &= \sum_k c_k [l-k+1][l-k+2] \\ &\quad \times \{q^{-2l+2k-4}[k+1][k+2]|k+2, l-k+2\rangle \otimes |l-k, k\rangle \\ &\quad + (q^{-1}[l-k+1][l-k+2] + q^{-2l-7}[k-1][k])|k, l-k+2\rangle \otimes |l-k+2, k\rangle \\ &\quad + q^{-2k}[l-k+3][l-k+4]|k-2, l-k+2\rangle \otimes |l-k+4, k\rangle\}. \end{aligned}$$

Here $c_k = (-1)^k q^{k(k-l+1)} \begin{bmatrix} l \\ k \end{bmatrix}^{-1}$ and we have used the relation $q^{l-2k-2}[l-k]c_{k+1} + [k+1]c_k = 0$. On the other hand, we also get

$$\begin{aligned} (e_1 e_0)^2 \mathbf{v}_l &= [2] \sum_k c_k [l-k+1][l-k+2] \\ &\quad \times \{[3]q^{-2l+2k-4}[k+1][k+2]|k+2, l-k+2\rangle \otimes |l-k, k\rangle \\ &\quad + A_k |k, l-k+2\rangle \otimes |l-k+2, k\rangle \\ &\quad + [3]q^{-2k}[l-k+3][l-k+4]|k-2, l-k+2\rangle \otimes |l-k+4, k\rangle\}, \end{aligned}$$

where

$$\begin{aligned} A_k &= \frac{q^{l-2k}}{q-q^{-1}} \{(1+q^{-2l-6})(q^2[k+1][l-k+2] - [k][l-k+3]) \\ &\quad - q^{-2l+2k}(1+q^{-4})([k+1][l-k+2] - q^{-4}[k][l-k+3])\}. \end{aligned}$$

Combining these results, we obtain

$$\begin{aligned} &(e_0 e_1^{(2)} e_0 - \frac{1}{[3]!} (e_1 e_0)^2) \mathbf{v}_l \\ &= \frac{[2]}{[3]} [l+1][l+2] \sum_k c_k q^{-2k-2} [l-k+1][l-k+2] |k, l-k+2\rangle \otimes |l-k+2, k\rangle \\ &= q^{-2} \frac{[2]}{[3]} ([l+1][l+2])^2 \mathbf{v}_{l+2}. \end{aligned}$$

□

For $l \in \mathbb{Z}_{\geq 0}$ and $l' = 2m \in 2\mathbb{Z}_{\geq 0}$, set

$$\mathbf{v}_{l,l'} = q_n^{\frac{m(m+2l+1)}{2}} f_n^{(m)} \mathbf{v}_l.$$

Note that $\mathbf{v}_{l,l'}$ may not be equal to $\tilde{f}_n^m \mathbf{v}_l$ in the sense of (3.4) since $e_n \mathbf{v}_{l,l'} \neq 0$ in general.

Lemma 5.6. *For $l \in \mathbb{Z}_{\geq 0}$ and $l' \in 2\mathbb{Z}_{\geq 0}$ with $\text{sgn}(l) = \varepsilon$, $\mathbf{v}_{l,l'}$ is a $U_q(A_{n-1})$ -highest weight vector in $\mathcal{W}_\varepsilon^{(2)}$, and*

$$\mathbf{v}_{l,l'} \equiv |l\mathbf{e}_n\rangle \otimes |l\mathbf{e}_{n-1} + l'\mathbf{e}_n\rangle \pmod{q\mathcal{L}_\varepsilon^{\otimes 2}}.$$

Proof. Let us assume that l is even, and hence $\varepsilon = +$, since the proof for odd l is almost identical. Since e_j ($1 \leq j \leq n-1$) commutes with f_n , it is clear that $\mathbf{v}_{l,l'}$ is a $U_q(A_{n-1})$ -highest weight vector in $\mathcal{W}_\varepsilon^{(2)}$.

Let $l' = 2m$. For $0 \leq c \leq l$, we have

$$|c\mathbf{e}_{n-1} + (l-c)\mathbf{e}_n\rangle \equiv \begin{cases} \tilde{f}_n^{\lfloor \frac{l-c}{2} \rfloor} |c\mathbf{e}_{n-1}\rangle & \text{if } c \text{ is even,} \\ \tilde{f}_n^{\lfloor \frac{l-c}{2} \rfloor} |c\mathbf{e}_{n-1} + \mathbf{e}_n\rangle & \text{if } c \text{ is odd,} \end{cases} \pmod{q\mathcal{L}_+}.$$

Put $a = \lfloor \frac{l-c}{2} \rfloor$ and $b = \lfloor \frac{c}{2} \rfloor$.

Case 1. Suppose that c is even. Let

$$u_1 = |c\mathbf{e}_{n-1}\rangle, \quad u_2 = |(l-c)\mathbf{e}_{n-1}\rangle.$$

We have

$$\begin{aligned} & \Delta(f_n^{(m)})(\tilde{f}_n^a u_1 \otimes \tilde{f}_n^b u_2) \\ &= \sum_{k=0}^m q_n^{-k(m-k)} f_n^{(m-k)} k_n^k (\tilde{f}_n^a u_1) \otimes f_n^{(k)} (\tilde{f}_n^b u_2) \\ &= \sum_{k=0}^m q_n^{-k(m-k) - (\frac{1}{2} + 2a)k} f_n^{(m-k)} \tilde{f}_n^a u_1 \otimes f_n^{(k)} \tilde{f}_n^b u_2 \\ &= \sum_{k=0}^m q_n^{-k(m-k) - (\frac{1}{2} + 2a)k + \frac{a^2}{2} + \frac{b^2}{2}} f_n^{(m-k)} f_n^{(a)} u_1 \otimes f_n^{(k)} f_n^{(b)} u_2 \\ &= \sum_{k=0}^m q_n^{-k(m-k) - (\frac{1}{2} + 2a)k + \frac{a^2}{2} + \frac{b^2}{2}} \begin{bmatrix} m-k+a \\ a \end{bmatrix}_n \begin{bmatrix} k+b \\ b \end{bmatrix}_n f_n^{(m-k+a)} u_1 \otimes f_n^{(k+b)} u_2 \\ &= \sum_{k=0}^m q_n^{-k(m-k) - (\frac{1}{2} + 2a)k + \frac{a^2}{2} + \frac{b^2}{2} - \frac{(m-k+a)^2}{2} - \frac{(k+b)^2}{2}} \\ & \quad \begin{bmatrix} m-k+a \\ a \end{bmatrix}_n \begin{bmatrix} k+b \\ b \end{bmatrix}_n \tilde{f}_n^{m-k+a} u_1 \otimes \tilde{f}_n^{k+b} u_2 \\ &= \sum_{k=0}^m f_{a,b}(q) \tilde{f}_n^{m-k+a} u_1 \otimes \tilde{f}_n^{k+b} u_2. \end{aligned}$$

Multiplying $q_n^{\frac{m(m+2l+1)}{2}}$ on both sides, we have $q_n^{\frac{m(m+2l+1)}{2}} f_{a,b}(q) \in q^d(1+qA_0)$, where

$$\begin{aligned}
d &= m(m+2l+1) - 2k(m-k) - 2\left(\frac{1}{2} + 2a\right)k \\
&\quad + a^2 + b^2 - (m-k+a)^2 - (k+b)^2 - 2(m-k)a - 2kb \\
(5.3) \quad &= 2lm + (m-k) - 4ma - 4kb = 2lm + (m-k) - 4m\left(\frac{l-c}{2}\right) - 4k\left(\frac{c}{2}\right) \\
&= (m-k) + 2c(m-k) = (2c+1)(m-k)
\end{aligned}$$

since $a = \frac{l-c}{2}$ and $b = \frac{c}{2}$.

Case 2. Suppose that c is odd. Let

$$u_1 = |c\mathbf{e}_{n-1} + \mathbf{e}_n\rangle, \quad u_2 = |(l-c)\mathbf{e}_{n-1} + \mathbf{e}_n\rangle.$$

We have

$$\begin{aligned}
&\Delta(f_n^{(m)})(\tilde{f}_n^a u_1 \otimes \tilde{f}_n^b u_2) \\
&= \sum_{k=0}^m q_n^{-k(m-k)} f_n^{(m-k)} k_n^k(\tilde{f}_n^a u_1) \otimes f_n^{(k)}(\tilde{f}_n^b u_2) \\
&= \sum_{k=0}^m q_n^{-k(m-k) - (\frac{3}{2} + 2a)} f_n^{(m-k)} \tilde{f}_n^{(a)} u_1 \otimes f_n^{(k)} \tilde{f}_n^{(b)} u_2 \\
&= \sum_{k=0}^m q_n^{-k(m-k) - (\frac{3}{2} + 2a)k + \frac{a(a+2)}{2} + \frac{b(b+2)}{2}} f_n^{(m-k)} f_n^{(a)} u_1 \otimes f_n^{(k)} f_n^{(b)} u_2 \\
&= \sum_{k=0}^m q_n^{-k(m-k) - (\frac{3}{2} + 2a)k + \frac{a(a+2)}{2} + \frac{b(b+2)}{2}} \begin{bmatrix} m-k+a \\ a \end{bmatrix}_n \begin{bmatrix} k+b \\ b \end{bmatrix}_n f_n^{(m-k+a)} u_1 \otimes f_n^{(k+b)} u_2 \\
&= \sum_{k=0}^m q_n^{-k(m-k) - (\frac{3}{2} + 2a)k + \frac{a(a+2)}{2} + \frac{b(b+2)}{2} - \frac{(m-k+a)(m-k+a+2)}{2} - \frac{(k+b)(k+b+2)}{2}} \\
&\quad \begin{bmatrix} m-k+a \\ a \end{bmatrix}_n \begin{bmatrix} k+b \\ b \end{bmatrix}_n \tilde{f}_n^{m-k+a} u_1 \otimes \tilde{f}_n^{k+b} u_2 \\
&= \sum_{k=0}^m g_{a,b}(q) \tilde{f}_n^{m-k+a} u_1 \otimes \tilde{f}_n^{k+b} u_2.
\end{aligned}$$

Multiplying $q_n^{\frac{m(m+2l+1)}{2}}$ on both sides, we have $q_n^{\frac{m(m+2l+1)}{2}} g_{a,b}(q) \in q^{d'}(1+qA_0)$, where

$$\begin{aligned}
d' &= d - 2k + 2a + 2b - 2(m-k+a) - 2(k+b) \\
&= d - 2k - 2m \\
(5.4) \quad &= 2lm + (m-k) - 4ma - 4kb - 2k - 2m \\
&= 2lm + (m-k) - 4m\left(\frac{l-c-1}{2}\right) - 4k\left(\frac{c-1}{2}\right) - 2k - 2m \\
&= (m-k) + 2c(m-k) = (2c+1)(m-k)
\end{aligned}$$

by putting $a = \frac{l-c-1}{2}$ and $b = \frac{c-1}{2}$. By (5.3), (5.4), and Lemma 5.4, we have

$$q_n^{\frac{m(m+2l+1)}{2}} f_n^{(m)} \mathbf{v}_l \equiv |l\mathbf{e}_n\rangle \otimes |l\mathbf{e}_{n-1} + 2m\mathbf{e}_n\rangle \pmod{q\mathcal{L}_+^{\otimes 2}}.$$

□

Corollary 5.7. *The set $\{\mathbf{v}_{l,l'} \mid l \in \mathbb{Z}_{\geq 0}, l' \in 2\mathbb{Z}_{\geq 0}, \text{sgn}(l) = \varepsilon\}$ is the set of $U_q(A_{n-1})$ -highest weight vectors in $\mathcal{W}_\varepsilon^{(2)}$.*

Proof. The character of the $U_q(A_{n-1})$ -submodule of \mathcal{W}_ε generated by $\mathbf{v}_{l,l'}$ is $s_\lambda(x_1, \dots, x_n)$ where $\lambda = (l' + l, l)$. Hence it follows from Proposition 5.3 that there is no other $U_q(A_{n-1})$ -highest weight vectors in $\mathcal{W}_\varepsilon^{(2)}$. □

Now we define the pair $(\mathcal{L}_\varepsilon^{(2)}, \mathcal{B}_\varepsilon^{(2)})$ by

$$\mathcal{L}_\varepsilon^{(2)} = \sum_{\substack{l_1 \in \mathbb{Z}_{\geq 0} \\ \text{sgn}(l_1) = \varepsilon}} \sum_{l_2 \in 2\mathbb{Z}_{\geq 0}} \sum_{\substack{r \geq 0 \\ 1 \leq i_1, \dots, i_r \leq n-1}} A_0 \tilde{f}_{i_1} \dots \tilde{f}_{i_r} \mathbf{v}_{l_1, l_2},$$

$$\mathcal{B}_\varepsilon^{(2)} = \left\{ \tilde{f}_{i_1} \dots \tilde{f}_{i_r} \mathbf{v}_{l_1, l_2} \pmod{q\mathcal{L}_\varepsilon^{(2)}} \mid l_1 \in \mathbb{Z}_{\geq 0}, \text{sgn}(l_1) = \varepsilon, l_2 \in 2\mathbb{Z}_{\geq 0}, r \geq 0, 1 \leq i_1, \dots, i_r \leq n-1 \right\} \setminus \{0\}.$$

Proposition 5.8. *We have*

- (1) $\mathcal{L}_\varepsilon^{(2)} \subset \mathcal{L}_\varepsilon^{\otimes 2}$ and $\mathcal{B}_\varepsilon^{(2)} \subset \mathcal{B}_\varepsilon^{\otimes 2}$,
- (2) $(\mathcal{L}_\varepsilon^{(2)}, \mathcal{B}_\varepsilon^{(2)})$ is a $U_q(A_{n-1})$ -crystal base of $\mathcal{W}_\varepsilon^{(2)}$.

Proof. (1) By Proposition 3.8, $\mathcal{L}_\varepsilon^{\otimes 2}$ is a crystal base of $\mathcal{W}_\varepsilon^{\otimes 2}$ as a $U_q(A_{n-1})$ -module, hence it is invariant under \tilde{f}_i for $1 \leq i \leq n-1$. By Lemma 5.6, we have $\tilde{f}_{i_1} \dots \tilde{f}_{i_r} \mathbf{v}_{l_1, l_2} \in \mathcal{L}_\varepsilon^{\otimes 2}$ and hence $\tilde{f}_{i_1} \dots \tilde{f}_{i_r} \mathbf{v}_{l_1, l_2} \in \mathcal{B}_\varepsilon^{\otimes 2} \pmod{q\mathcal{L}_\varepsilon^{\otimes 2}}$.

(2) By definition of $(\mathcal{L}_\varepsilon^{(2)}, \mathcal{B}_\varepsilon^{(2)})$ and Lemma 5.6, $(\mathcal{L}_\varepsilon^{(2)}, \mathcal{B}_\varepsilon^{(2)})$ is a $U_q(A_{n-1})$ -crystal base of the submodule V of $\mathcal{W}_\varepsilon^{(2)}$ generated by \mathbf{v}_{l_1, l_2} for l_1, l_2 . On the other hand, we have $V = \mathcal{W}_\varepsilon^{(2)}$ by Proposition 5.3. Hence $(\mathcal{L}_\varepsilon^{(2)}, \mathcal{B}_\varepsilon^{(2)})$ is a $U_q(A_{n-1})$ -crystal base of $\mathcal{W}_\varepsilon^{(2)}$. □

For $|\mathbf{m}\rangle = |m_1, \dots, m_n\rangle \in \mathcal{W}$, let $T(\mathbf{m})$ denote the semistandard tableau of shape $(|\mathbf{m}|)$, a single row of length $|\mathbf{m}|$, with letters in $\{\bar{n} < \dots < \bar{1}\}$ such that the number of occurrences of \bar{i} is m_i for $1 \leq i \leq n$.

Suppose that $|\mathbf{m}_1\rangle, \dots, |\mathbf{m}_s\rangle$ are given such that $|\mathbf{m}_1| \leq \dots \leq |\mathbf{m}_s|$. Let $\lambda = (|\mathbf{m}_s| \geq \dots \geq |\mathbf{m}_1|)$, which is a partition or its Young diagram, and λ^π denote the Young diagram obtained by 180° -rotation of λ . We denote by $T(\mathbf{m}_1, \dots, \mathbf{m}_s)$ the row-semistandard tableau of shape λ^π , whose j -th row from the top is equal to $T(\mathbf{m}_j)$ for $1 \leq j \leq s$.

Example 5.9. Suppose that $n = 5$. If $|\mathbf{m}_1\rangle = |2, 1, 0, 0, 2\rangle$ and $|\mathbf{m}_2\rangle = |0, 1, 2, 3, 1\rangle$, then

$$T(\mathbf{m}_1, \mathbf{m}_2) = \begin{array}{|c|c|c|c|c|} \hline \bar{5} & \bar{5} & \bar{2} & \bar{1} & \bar{1} \\ \hline \bar{5} & \bar{4} & \bar{4} & \bar{4} & \bar{3} \\ \hline \bar{4} & \bar{4} & \bar{3} & \bar{3} & \bar{2} \\ \hline \end{array}.$$

Proposition 5.10. *We have*

$$\mathcal{B}_\varepsilon^{(2)} = \left\{ |\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle \pmod{q\mathcal{L}_\varepsilon^{(2)}} \mid |\mathbf{m}_1| \leq |\mathbf{m}_2|, T(\mathbf{m}_1, \mathbf{m}_2) \text{ is semistandard} \right\}.$$

Proof. For $l_1 \in \mathbb{Z}_{\geq 0}$ and $l_2 \in 2\mathbb{Z}_{\geq 0}$ with $\text{sgn}(l_1) = \varepsilon$, let us identify $\mathbf{v}_{l_1, l_2} = |l_1 \mathbf{e}_n\rangle \otimes |l_1 \mathbf{e}_{n-1} + l_2 \mathbf{e}_n\rangle$ in $\mathcal{B}_\varepsilon^{\otimes 2}$ with the pair $(l_1 \mathbf{e}_n, l_1 \mathbf{e}_{n-1} + l_2 \mathbf{e}_n)$ and the connected component of \mathbf{v}_{l_1, l_2} as a $U_q(A_{n-1})$ -crystal with the set of corresponding set of pairs $(\mathbf{m}_1, \mathbf{m}_2)'$'s. Then $T(\mathbf{v}_{l_1, l_2})$ is the semistandard tableau of shape $(l_1 + l_2, l_1)^\pi$. Since $\tilde{e}_j \mathbf{v}_{l_1, l_2} = 0$ for $1 \leq j \leq n-1$, $T(\mathbf{v}_{l_1, l_2})$ is the tableau of highest weight and the set

$$(5.5) \quad \left\{ T \left(\tilde{f}_{i_1} \dots \tilde{f}_{i_r} \mathbf{v}_{l_1, l_2} \right) \mid r \geq 0, 1 \leq i_1, \dots, i_r \leq n-1 \right\} \setminus \{0\}$$

is equal to the set of semistandard tableau of shape $(l_1 + l_2, l_1)^\pi$ with letters $\{\bar{n} < \dots < \bar{1}\}$. \square

Let $|\mathbf{m}_1\rangle, |\mathbf{m}_2\rangle \in \mathcal{B}_\varepsilon$ be given with $|\mathbf{m}_1| = d_1$ and $|\mathbf{m}_2| = d_2$, let $P(\mathbf{m}_1, \mathbf{m}_2)$ denote a unique semistandard tableau of shape μ^π for some partition μ , which is equivalent to $|\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle$ as an element of $U_q(A_{n-1})$ -crystals. Indeed, if we read the row word of $T(\mathbf{m}_1)$ from left to right, and then apply the Schensted's column insertion to $T(\mathbf{m}_2)$ in a reverse way starting from the right-most column, then the resulting tableau is $P(\mathbf{m}_1, \mathbf{m}_2)$. So $P(\mathbf{m}_1, \mathbf{m}_2)$ is of shape $(d'_2, d'_1)^\pi$ for some $d'_1 \leq d'_2$ with $d'_1 \leq d_1$, $d'_2 \geq d_2$, and $d'_1 + d'_2 = d_1 + d_2$. In particular, $P(\mathbf{m}_1, \mathbf{m}_2) = T(\mathbf{m}_1, \mathbf{m}_2)$ if $d_1 \leq d_2$ and $|\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle \in \mathcal{B}_\varepsilon^{(2)}$.

Example 5.11. Let $|\mathbf{m}_1\rangle, |\mathbf{m}_2\rangle$ be as in Example 5.9. Then

$$P(\mathbf{m}_1, \mathbf{m}_2) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline & & & & & & & \bar{5} & \bar{5} \\ \hline \bar{5} & \bar{4} & \bar{4} & \bar{4} & \bar{3} & \bar{3} & \bar{2} & \bar{2} & \bar{1} & \bar{1} \\ \hline \end{array}.$$

Let $l_1 \in \mathbb{Z}_{\geq 0}$ and $l_2 \in 2\mathbb{Z}_{\geq 0}$ be given with $\text{sgn}(l_1) = \varepsilon$. Put $\lambda = (\lambda_1, \lambda_2) = (l_1 + l_2, l_1)$. Let $SST(\lambda^\pi)$ be the set of semistandard tableaux of shape λ^π with letters in $\{\bar{n} < \dots < \bar{1}\}$. For each $T \in SST(\lambda^\pi)$, we choose $i_1, \dots, i_r \in I \setminus \{0, n\}$ such that $T = T(\tilde{f}_{i_1} \dots \tilde{f}_{i_r} \mathbf{v}_{l_1, l_2})$ (see (5.5)), and define

$$(5.6) \quad \mathbf{v}_T = \tilde{f}_{i_1} \dots \tilde{f}_{i_r} \mathbf{v}_{l_1, l_2} \in \mathcal{L}_\varepsilon^{(2)}.$$

By Proposition 5.10, we have a $\mathbb{Q}(q)$ -basis of $\mathcal{W}_\varepsilon^{(2)}$

$$(5.7) \quad \bigsqcup_{\substack{\lambda \in \mathcal{P}_\varepsilon \\ \ell(\lambda) \leq 2}} \{ \mathbf{v}_T \mid T \in SST(\lambda^\pi) \}.$$

Lemma 5.12. For $T \in SST(\lambda^\pi)$, we have

$$\mathbf{v}_T = |\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle + \sum_{\mathbf{m}'_1, \mathbf{m}'_2} c_{\mathbf{m}'_1, \mathbf{m}'_2} |\mathbf{m}'_1\rangle \otimes |\mathbf{m}'_2\rangle,$$

where $P(\mathbf{m}_1, \mathbf{m}_2) = T$, $P(\mathbf{m}'_1, \mathbf{m}'_2)$ is of shape μ^π with $\mu \triangleright \lambda$ and $\mu \neq \lambda$, and $c_{\mathbf{m}'_1, \mathbf{m}'_2} \in qA_0$. Here \triangleright denotes a dominance order on partitions, that is, $\mu_1 > \lambda_1$, and $\mu_1 + \mu_2 = \lambda_1 + \lambda_2$.

Proof. By Lemmas 5.4 and 5.6 (see also their proofs), we observe that

$$(5.8) \quad \mathbf{v}_{l_1, l_2} = |l_1 \mathbf{e}_n\rangle \otimes |l_1 \mathbf{e}_{n-1} + l_2 \mathbf{e}_n\rangle + \sum c_{x, y, z, w} |x \mathbf{e}_{n-1} + y \mathbf{e}_n\rangle \otimes |z \mathbf{e}_{n-1} + w \mathbf{e}_n\rangle,$$

where the sum is over (x, y, z, w) such that

- (1) $0 < x \leq l_1$ with $x + z = l_1$,

- (2) $y \geq z, w \geq x$ with $y + w = l_1 + l_2$,
(3) $c_{x,y,z,w} \in qA_0$.

We may regard $|l_1\mathbf{e}_n\rangle \otimes |l_1\mathbf{e}_{n-1} + l_2\mathbf{e}_n\rangle$ as the case when $(x, y, z, w) = (0, l_1, l_1, l_2)$. Then it is not difficult to see that if the shape of $P(x\mathbf{e}_{n-1} + y\mathbf{e}_n, z\mathbf{e}_{n-1} + w\mathbf{e}_n)$ is $\mu^\pi = (\mu_1, \mu_2)^\pi$, then $\mu_2 = z = l_1 - x \leq l_1$ and hence $\mu \triangleright \lambda$, and $\mu \neq \lambda$ when $x > 0$.

Let $i_1, \dots, i_r \in I \setminus \{0, n\}$ be the sequence in (5.6). By the tensor product rule of crystals, we have

$$(5.9) \quad \tilde{f}_{i_1} \dots \tilde{f}_{i_r}(|x\mathbf{e}_{n-1} + y\mathbf{e}_n\rangle \otimes |z\mathbf{e}_{n-1} + w\mathbf{e}_n\rangle) = \sum_{\mathbf{m}_1, \mathbf{m}_2} c_{\mathbf{m}_1, \mathbf{m}_2} |\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle,$$

where the sum is over $\mathbf{m}_1, \mathbf{m}_2$ such that

- (1) $c_{\mathbf{m}_1, \mathbf{m}_2}(q) \in A_0$ such that

$$c_{\mathbf{m}_1, \mathbf{m}_2}(0) = \begin{cases} 1 & \text{if } |\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle = \tilde{f}_{i_1} \dots \tilde{f}_{i_r}(|x\mathbf{e}_{n-1} + y\mathbf{e}_n\rangle \otimes |z\mathbf{e}_{n-1} + w\mathbf{e}_n\rangle), \\ 0 & \text{otherwise.} \end{cases}$$

- (2) $\nu \triangleright \lambda$ and $\nu \neq \lambda$, where ν^π is the shape of $P(\mathbf{m}_1, \mathbf{m}_2)$.

Therefore, we obtain the result by (5.8) and (5.9). \square

Corollary 5.13. *We have $\mathcal{L}_\varepsilon^{(2)} = \mathcal{L}_\varepsilon^{\otimes 2} \cap \mathcal{W}_\varepsilon^{(2)}$.*

Proof. It is clear that $\mathcal{L}_\varepsilon^{(2)} \subset \mathcal{L}_\varepsilon^{\otimes 2} \cap \mathcal{W}_\varepsilon^{(2)}$ by Proposition 5.8. Conversely, suppose that $v \in \mathcal{L}_\varepsilon^{\otimes 2} \cap \mathcal{W}_\varepsilon^{(2)}$ is given. By (5.7), we have

$$(5.10) \quad v = \sum_T c_T \mathbf{v}_T,$$

for some $c_T \in \mathbb{Q}(q)$. We may assume that all the shape of T in (5.10) is the same. Fix T with $c_T \neq 0$. Let $|\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle$ be such that $|\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle$ appears in (5.10) with non-zero coefficient, and $P(\mathbf{m}_1, \mathbf{m}_2) = T$. By Lemma 5.12, the coefficient of $|\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle$ is c_T . Hence $c_T \in A_0$, and $v \in \mathcal{L}_\varepsilon^{(2)}$. \square

Proof of Theorem 5.1. Let $\mathcal{W}_\varepsilon^{\otimes 2} = \mathcal{W}_\varepsilon^{(2)} \oplus W$, where W is the complement of $\mathcal{W}_\varepsilon^{(2)}$ in $\mathcal{W}_\varepsilon^{\otimes 2}$ as a $U_q(A_{n-1})$ -module since it is completely reducible. By Corollary 5.13, we have

$$(5.11) \quad \mathcal{L}_\varepsilon^{\otimes 2} = \mathcal{L}_\varepsilon^{(2)} \oplus \mathcal{M}^{(2)},$$

where $\mathcal{M}^{(2)} = \mathcal{L}_\varepsilon^{\otimes 2} \cap W$ is the crystal lattice of W as a $U_q(A_{n-1})$ -module. Then we have

$$(5.12) \quad \check{R}_2(\mathcal{L}_\varepsilon^{\otimes 2}) \subset \mathcal{L}_\varepsilon^{(2)}, \quad \check{R}_2|_{q=0}(\mathcal{B}_\varepsilon^{\otimes 2}) \subset \mathcal{B}_\varepsilon^{(2)}.$$

More generally, by (4.1) and (5.11), we have for $a \in \mathbb{Z}_{>0}$

$$(5.13) \quad \check{R}(q^{-2a})(\mathcal{L}_\varepsilon^{\otimes 2}) \subset \mathcal{L}_\varepsilon^{\otimes 2}.$$

For each $1 \leq i \leq s-1$, we have

$$\check{R}_s = \check{R}_{s_i}(\dots, \underbrace{q^{s-2i-1}}_i, \underbrace{q^{s-2i+1}}_{i+1}, \dots) \check{R}_{w_0 s_i}(q^{1-s}, \dots, q^{s-1}).$$

We have $\check{R}_{w_0 s_i}(q^{1-s}, \dots, q^{s-1})(\mathcal{L}_\varepsilon^{\otimes s}) \subset \mathcal{L}_\varepsilon^{\otimes s}$ by (5.13), and hence by (5.12)

$$\begin{aligned}\check{R}_s(\mathcal{L}_\varepsilon^{\otimes s}) &\subset \mathcal{L}_\varepsilon^{\otimes i-1} \otimes \mathcal{L}_\varepsilon^{(2)} \otimes \mathcal{L}_\varepsilon^{\otimes s-i-1}, \\ \check{R}_s(\mathcal{B}_\varepsilon^{\otimes s}) &\subset \mathcal{B}_\varepsilon^{\otimes i-1} \otimes \mathcal{B}_\varepsilon^{(2)} \otimes \mathcal{B}_\varepsilon^{\otimes s-i-1}.\end{aligned}$$

Therefore $\check{R}_s(\mathcal{B}_\varepsilon^{\otimes s})$ is spanned by $\mathcal{B}_\varepsilon^{(s)}$, where

$$\mathcal{B}_\varepsilon^{(s)} = \left\{ |\mathbf{m}_1\rangle \otimes \dots \otimes |\mathbf{m}_s\rangle \pmod{q\mathcal{L}_\varepsilon^{\otimes s}} \mid |\mathbf{m}_j\rangle \otimes |\mathbf{m}_{j+1}\rangle \in \mathcal{B}_\varepsilon^{(2)} \ (1 \leq j \leq s-2) \right\}.$$

By Proposition 5.10, the set

$$\left\{ T(\mathbf{m}_1, \dots, \mathbf{m}_s) \mid |\mathbf{m}_1\rangle \otimes \dots \otimes |\mathbf{m}_s\rangle \in \mathcal{B}_\varepsilon^{(s)} \right\}$$

is equal to the set of semistandard tableau of shape λ^π where $\lambda = (|\mathbf{m}_s| \geq \dots \geq |\mathbf{m}_1|)$.

Hence

$$(5.14) \quad \text{ch}\mathcal{W}_\varepsilon^{(s)} = \sum_{\substack{\lambda \in \mathcal{P}_\varepsilon \\ \ell(\lambda) \leq s}} s_\lambda(x_1, \dots, x_n).$$

Let $V_0^{(s)}$ be the $U_q(C_n)$ -submodule of $\mathcal{W}_\varepsilon^{(s)}$ generated $|\varsigma(\varepsilon)\mathbf{e}_n\rangle^{\otimes s}$. The classical limit $\overline{V_0^{(s)}}$ of $V_0^{(s)}$ is a highest weight $U(C_n)$ -module with highest weight

$$\Lambda^{(s)} := -s\left(\frac{1}{2} + \varsigma(\varepsilon)\right)\varpi_n.$$

On the other hand, by [17, Theorem 6.1] the character of the irreducible highest weight $U(C_n)$ -module with highest weight $\Lambda^{(s)}$, say $V(\Lambda^{(s)})$, is also equal to (5.14). Since $V(\Lambda^{(s)})$ is a quotient of $\overline{V_0^{(s)}}$, we conclude that

$$\text{ch}\mathcal{W}_\varepsilon^{(s)} = \text{ch}V_0^{(s)} = \text{ch}\overline{V_0^{(s)}} = \text{ch}V(\Lambda^{(s)}).$$

In particular, $V_0^{(s)}$ is an irreducible $U_q(C_n)$ -module and hence $\mathcal{W}_\varepsilon^{(s)} = V_0^{(s)}$ is an irreducible $U_q(C_n^{(1)})$ -module. This completes the proof. \square

5.2. Type $C^{(2)}(n+1)$. Let us prove that $\mathcal{W}^{(s)}$ is an irreducible $U_q(C^{(2)}(n+1))$ -module. The proof is similar to that of Theorem 5.1 for $U_q(C_n^{(1)})$. So we give a sketch of the proof and leave the details to the reader.

We first consider $\mathcal{W}^{(2)}$. By (4.2), we have

$$(5.15) \quad \mathcal{W}^{(2)} = V_0 = U_q(\mathfrak{osp}_{1|2n})|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle,$$

which is an irreducible representation of $U_q(\mathfrak{osp}_{1|2n})$ and hence of $U_q(C^{(1)}(n+1))$. By similar arguments as in Proposition 5.3, we have the following.

Proposition 5.14. *We have*

$$\text{ch}\mathcal{W}^{(2)} = \text{ch}V_0 = \sum_{\substack{\lambda \in \mathcal{P} \\ \ell(\lambda) \leq 2}} s_\lambda(x_1, \dots, x_n).$$

Lemma 5.15. *For $l \in \mathbb{Z}_{\geq 0}$, let \mathbf{v}_l be the vector in (5.1). Then \mathbf{v}_l is a $U_q(A_{n-1})$ -highest weight vector in $\mathcal{W}^{(2)}$, and $\mathbf{v}_l \equiv |l\mathbf{e}_n\rangle \otimes |l\mathbf{e}_{n-1}\rangle \pmod{q^{\frac{1}{2}}\mathcal{L}^{\otimes 2}}$.*

Proof. Since the actions of Chevalley generators for $1 \leq i \leq n-1$ is the same as in the case of $C_n^{(1)}$, it follows from Lemma 5.4 that \mathbf{v}_l is a $U_q(A_{n-1})$ -highest weight vector. Note that

$$(5.16) \quad \text{ch}\mathcal{W}^{\otimes 2} = (\text{ch}\mathcal{W})^2 = \sum_{\ell(\lambda) \leq 2} m_\lambda s_\lambda(x_1, \dots, x_n),$$

where $m_\lambda = \lambda_1 - \lambda_2$. Then we have $\mathbf{v}_l \in \mathcal{W}^{(2)}$ by the same argument as in Lemma 5.4. \square

We have an analogue of Lemma 5.5, which also proves that $\mathbf{v}_l \in \mathcal{W}^{(2)}$.

Lemma 5.16. *Set $\mathcal{E} = e_{n-2} \cdots e_1 e_0$, where it is understood as e_0 when $n = 2$. Then for $l \geq 0$ we have*

$$(\mathcal{E}e_{n-1}\mathcal{E} - \frac{1}{[2]}e_{n-1}\mathcal{E}^2)\mathbf{v}_l = (-1)^l q^{-5/2} \frac{(1+q)}{[2]} [l+1]^2 \mathbf{v}_{l+1}.$$

Lemma 5.17. *For $l, m \in \mathbb{Z}_{\geq 0}$, let*

$$\mathbf{v}_{l,m} = q_n^{\frac{m(m+4l+3)}{2}} f_n^{(m)} \mathbf{v}_l.$$

Then $\mathbf{v}_{l,m}$ is a $U_q(A_{n-1})$ -highest weight vector in $\mathcal{W}^{(2)}$, and

$$\mathbf{v}_{l,m} \equiv |\mathbf{e}_n\rangle \otimes |\mathbf{e}_{n-1} + m\mathbf{e}_n\rangle \pmod{q^{\frac{1}{2}}\mathcal{L}^{\otimes 2}}.$$

Proof. Since e_j for $1 \leq j \leq n-1$ commutes with f_n , $\mathbf{v}_{l,m}$ is a $U_q(A_{n-1})$ -highest weight vector in $\mathcal{W}_\epsilon^{(2)}$.

For $0 \leq c \leq l$, put $a = l - c$ and $b = c$. Let

$$u_1 = |c\mathbf{e}_{n-1}\rangle, \quad u_2 = |(l-c)\mathbf{e}_{n-1}\rangle.$$

By (2.4), we have

$$\begin{aligned} & \Delta(f_n^{(m)})(\tilde{f}_n^a u_1 \otimes \tilde{f}_n^b u_2) \\ &= \sum_{k=0}^m \sigma^k q_n^{-k(m-k)} f_n^{(m-k)} k_n^k (\tilde{f}_n^a u_1) \otimes f_n^{(k)} (\tilde{f}_n^b u_2) \\ &= \sum_{k=0}^m \sigma^k q_n^{-k(m-k) - (1+2a)k} f_n^{(m-k)} \tilde{f}_n^a u_1 \otimes f_n^{(k)} \tilde{f}_n^b u_2 \\ &= \sum_{k=0}^m \sigma^k q_n^{-k(m-k) - (1+2a)k + \frac{a(a+1)}{2} + \frac{b(b+1)}{2}} f_n^{(m-k)} f_n^{(a)} u_1 \otimes f_n^{(k)} f_n^{(b)} u_2 \\ &= \sum_{k=0}^m \sigma^k q_n^{-k(m-k) - (1+2a)k + \frac{a(a+1)}{2} + \frac{b(b+1)}{2}} \begin{bmatrix} m-k+a \\ a \end{bmatrix}_n \begin{bmatrix} k+b \\ b \end{bmatrix}_n f_n^{(m-k+a)} u_1 \otimes f_n^{(k+b)} u_2 \\ &= \sum_{k=0}^m \sigma^k q_n^{-k(m-k) - (1+2a)k + \frac{a(a+1)}{2} + \frac{b(b+1)}{2} - \frac{(m-k+a)(m-k+a+1)}{2} - \frac{(k+b)(k+b+1)}{2}} \\ & \quad \begin{bmatrix} m-k+a \\ a \end{bmatrix}_n \begin{bmatrix} k+b \\ b \end{bmatrix}_n \tilde{f}_n^{m-k+a} u_1 \otimes \tilde{f}_n^{k+b} u_2 \\ &= \sum_{k=0}^m \sigma^k f_{a,b}(q) \tilde{f}_n^{m-k+a} u_1 \otimes \tilde{f}_n^{k+b} u_2. \end{aligned}$$

Multiplying $q_n^{\frac{m(m+4l+3)}{2}}$ on both sides, it is straightforward to see that

$$q_n^{\frac{m(m+4l+3)}{2}} f_{a,b}(q) \in q_n^{(2c+1)(m-k)} (1 + q^{\frac{1}{2}} A_0).$$

This implies that $\mathbf{v}_{l,m} \equiv |l\mathbf{e}_n\rangle \otimes |l\mathbf{e}_{n-1} + m\mathbf{e}_n\rangle \pmod{q^{\frac{1}{2}}\mathcal{L}^{\otimes 2}}$. \square

Now we define the pair $(\mathcal{L}^{(2)}, \mathcal{B}^{(2)})$ by

$$\mathcal{L}^{(2)} = \sum_{l_1, l_2 \in \mathbb{Z}_{\geq 0}} \sum_{\substack{r \geq 0 \\ 1 \leq i_1, \dots, i_r \leq n-1}} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{v}_{l_1, l_2},$$

$$\mathcal{B}^{(2)} = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{v}_{l_1, l_2} \pmod{q^{\frac{1}{2}}\mathcal{L}^{(2)}} \mid l_1, l_2 \in \mathbb{Z}_{\geq 0}, r \geq 0, 1 \leq i_1, \dots, i_r \leq n-1 \right\} \setminus \{0\}.$$

Proposition 5.18. *We have*

- (1) $\mathcal{L}^{(2)} \subset \mathcal{L}^{\otimes 2}$ and $\mathcal{B}^{(2)} \subset \mathcal{B}^{\otimes 2}$,
- (2) $(\mathcal{L}^{(2)}, \mathcal{B}^{(2)})$ is a $U_q(A_{n-1})$ -crystal base of $\mathcal{W}^{(2)}$, where

$$\mathcal{B}^{(2)} = \left\{ |\mathbf{m}_1\rangle \otimes |\mathbf{m}_2\rangle \pmod{q^{\frac{1}{2}}\mathcal{L}^{(2)}} \mid |\mathbf{m}_1| \leq |\mathbf{m}_2|, T(\mathbf{m}_1, \mathbf{m}_2) \text{ is semistandard} \right\}.$$

Proof. It follows from the same arguments as in Propositions 5.8 and 5.10. \square

Corollary 5.19. *We have $\mathcal{L}^{(2)} = \mathcal{L}^{\otimes 2} \cap \mathcal{W}^{(2)}$.*

Proof. By Proposition 5.18, one can check that Lemma 5.12 also holds for $\mathcal{W}^{(2)}$, which implies $\mathcal{L}^{(2)} = \mathcal{L}^{\otimes 2} \cap \mathcal{W}^{(2)}$. \square

Theorem 5.20. *For $s \geq 2$, $\mathcal{W}^{(s)}$ is an irreducible $U_q(C^{(2)}(n+1))$ -module, which is also irreducible as a $U_q(\mathfrak{osp}_{1|2n})$ -module. Moreover, its character is given by*

$$\text{ch}\mathcal{W}^{(s)} = \sum_{\substack{\lambda \in \mathcal{P} \\ \ell(\lambda) \leq s}} s_\lambda(x_1, \dots, x_n).$$

Proof of Theorem 5.20. We may apply the same arguments as in Theorem 5.1 and the result in [17, Theorem 6.1] by using Proposition 5.18 and Corollary 5.19. \square

Corollary 5.21. *The character of $\mathcal{W}_\varepsilon^{(s)}$ has a stable limit for $s \geq n$ as follows:*

$$\text{ch}\mathcal{W}^{(s)} = \sum_{\substack{\lambda \in \mathcal{P} \\ \ell(\lambda) \leq n}} s_\lambda(x_1, \dots, x_n) = \frac{1}{\prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq i < j \leq n} (1 - x_i x_j)}.$$

5.3. Type $B^{(1)}(0, n)$. As usual, we identify the weight lattice for $U_q(B(0, n))$ with $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}\mathbf{e}_i$ equipped with the standard symmetric bilinear form such that $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$. Then the simple roots α_i ($i \in I \setminus \{0\}$) are given by $\alpha_i = \mathbf{e}_{i+1} - \mathbf{e}_i$ for $1 \leq i \leq n-1$ and $\alpha_n = -\mathbf{e}_n$, and $\varpi_n = -\frac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n)$.

For $\lambda \in \mathcal{P}_+$ with $\ell(\lambda) \leq \min\{n, s/2\}$, we put

$$\Lambda_\lambda^{(s)} = -s\varpi_n + \sum_{i=1}^n \lambda_i \mathbf{e}_{n-i+1}.$$

Let $V(\Lambda_\lambda^{(s)})$ be the irreducible highest weight $U(\mathfrak{osp}_{1|2n})$ -module with highest weight $\Lambda_\lambda^{(s)}$. Note that $\Lambda_{(l)}^{(2)}$ is the weight of the maximal vector v_l and $V(\Lambda_{(l)}^{(2)}) = V_l$ for $l \geq 0$. Generalizing

the decomposition of $\mathcal{W}^{(2)}$ into $U_q(\mathfrak{osp}_{1|2n})$ -modules, we have the following conjecture on $\mathcal{W}^{(s)}$.

Conjecture 5.22. *For $s \geq 2$, $\mathcal{W}^{(s)}$ is an irreducible $U_q(B^{(1)}(0, n))$ -module and its character is given by*

$$\mathrm{ch}\mathcal{W}^{(s)} = \sum_{\substack{\lambda \in \mathcal{P}_+ \\ \ell(\lambda) \leq \min\{n, s/2\}}} \mathrm{ch}V(\Lambda_\lambda^{(s)}).$$

Remark 5.23. The family of infinite-dimensional $U(\mathfrak{osp}_{1|2n})$ -modules $V(\Lambda_\lambda^{(s)})$ have been introduced in [2] in connection with Howe duality. They are unitarizable and form a semisimple tensor category. The Weyl-Kac type character formula for $V(\Lambda_\lambda^{(s)})$ can be found in [2, Theorem 6.13].

Corollary 5.24. *For $s \geq 2n$, we have*

$$\begin{aligned} \mathrm{ch}\mathcal{W}^{(s)} &= \frac{\sum_{\lambda \in \mathcal{P}_+} s_\lambda(x_1, \dots, x_n)}{\prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq i < j \leq n} (1 - x_i x_j)} \\ &= \frac{1}{\prod_{1 \leq i \leq n} (1 - x_i)(1 - x_i^2) \prod_{1 \leq i < j \leq n} (1 - x_i x_j)^2}. \end{aligned}$$

Proof. The first equation follows from the fact [17, Corollary 6.6] that if $\lambda \in \mathcal{P}_+$ with $\ell(\lambda) \leq n$, then

$$\mathrm{ch}V(\Lambda_\lambda^{(s)}) = \frac{s_\lambda(x_1, \dots, x_n)}{\prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq i < j \leq n} (1 - x_i x_j)}.$$

The second one follows from the well-known Littlewood identity. \square

APPENDIX A. TWISTOR

In this appendix, we review the twistor introduced in [4] that relate quantum groups to quantum supergroups. We use it to relate the q -oscillator representation of $U_q(D_{n+1}^{(2)})$ in [13] to a representation of $U_q(C^{(2)}(n+1))$. An advantage to do so is that in the latter we can take a classical limit $q \rightarrow 1$. We also obtain a representation of $U_q(B^{(1)}(0, n))$ from the q -oscillator representation of $U_q(A_{2n}^{(2)\dagger})$, where $A_{2n}^{(2)\dagger}$ is the same Dynkin diagram as $A_{2n}^{(2)}$ in [8] but the labeling of nodes are opposite.

A.1. The twistor of the covering quantum group. We review the covering quantum group and the twistor map introduced in [4]. Our notations for a Cartan datum is closer to Kac's book [8]. Let I be the index set of the Dynkin diagram, $\{\alpha_i\}_{i \in I}$ the set of simple roots, $(a_{ij})_{i, j \in I}$ the Cartan matrix. The symmetric bilinear form (\cdot, \cdot) on the weight lattice is normalized so that it satisfies $d_i = (\alpha_i, \alpha_i)/2 \in \mathbb{Z}$ for any $i \in I$. It is also assumed that $a_{ij} \in 2\mathbb{Z}$ if $d_i \equiv 1 \pmod{2}$ and $j \in I$. The parity function $p(i)$ taking values in $\{0, 1\}$ is consistent with d_i , namely, $p(i) \equiv d_i \pmod{2}$. We set $q_i = q^{d_i}$, $\pi_i = \pi^{d_i}$.

Let q, π be indeterminates and $\mathbf{i} = \sqrt{-1}$. For a ring R with 1, we set $R^\pi = R[\pi]/(\pi^2 - 1)$. The covering quantum group \mathbf{U} associated to a Cartan datum is the $\mathbb{Q}^\pi(q, \mathbf{i})$ -algebra with

generators $E_i, F_i, K_i^{\pm 1}, J_i^{\pm 1}$ for $i \in I$ subject to the following relations.

$$\begin{aligned}
J_i J_j &= J_j J_i, & K_i K_j &= K_j K_i, & J_i K_j &= K_j J_i, \\
J_i E_j &= \pi^{a_{ij}} E_j J_i, & J_i F_j &= \pi^{a_{ij}} F_j J_i, \\
K_i E_j &= q^{a_{ij}} E_j K_i, & K_i F_j &= q^{a_{ij}} F_j K_i, \\
E_i F_j - \pi^{p(i)p(j)} F_j E_i &= \delta_{ij} \frac{J_i K_i - K_i^{-1}}{\pi_i q_i - q_i^{-1}}, \\
\sum_{l=0}^{1-a_{ij}} (-1)^l \pi^{l(l-1)p(i)/2 + lp(i)p(j)} \begin{bmatrix} 1 - a_{ij} \\ l \end{bmatrix}_{q_i, \pi_i} E_i^{1-a_{ij}-l} E_j E_i^l &= 0 \quad (i \neq j), \\
\sum_{l=0}^{1-a_{ij}} (-1)^l \pi^{l(l-1)p(i)/2 + lp(i)p(j)} \begin{bmatrix} 1 - a_{ij} \\ l \end{bmatrix}_{q_i, \pi_i} F_i^{1-a_{ij}-l} F_j F_i^l &= 0 \quad (i \neq j).
\end{aligned}$$

Remark A.1. We changed the notations from [4]. We replaced v with q , \mathbf{t} with \mathbf{i} , $J_{d_i i}$ and $K_{d_i i}$ with J_i and K_i .

We extend \mathbf{U} by introducing generators T_i, Υ_i for $i \in I$. They commute with each other and with J_i, K_i . They also have the commutation relations with E_i, F_i as

$$T_i E_j = \mathbf{i}^{a_{ij}} E_j T_i, \quad T_i F_j = \mathbf{i}^{-a_{ij}} F_j T_i, \quad \Upsilon_i E_j = \mathbf{i}^{\phi_{ij}} E_j \Upsilon_i, \quad \Upsilon_i F_j = \mathbf{i}^{-\phi_{ij}} F_j \Upsilon_i$$

where

$$\phi_{ij} = \begin{cases} d_i a_{ij} & \text{if } i > j, \\ d_i & \text{if } i = j, \\ -2p(i)p(j) & \text{if } i < j. \end{cases}$$

We denote this extended algebra by $\widehat{\mathbf{U}}$.

Theorem A.2 ([4]). *There is a $\mathbb{Q}(\mathbf{i})$ -algebra automorphism $\widehat{\Psi}$ on $\widehat{\mathbf{U}}$ such that*

$$\begin{aligned}
E_i &\mapsto \mathbf{i}^{-d_i} \Upsilon_i^{-1} T_i E_i, & F_i &\mapsto F_i \Upsilon_i, & K_i &\mapsto T_i K_i, \\
J_i &\mapsto T_i^2 J_i, & T_i &\mapsto T_i, & \Upsilon_i &\mapsto \Upsilon_i, \\
q &\mapsto \mathbf{i}^{-1} q, & \pi &\mapsto -\pi.
\end{aligned}$$

A.2. Image of the twistor $\widehat{\Psi}$. We apply the twistor $\widehat{\Psi}$ given in the previous subsection for the Cartan datum corresponding to B_n , namely, $I = \{1, 2, \dots, n\}$ and the Cartan matrix is given by

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & & & \\ & & \ddots & & \\ & & & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}.$$

Through it, we are to regard the q -oscillator representation $\mathcal{W} = \bigoplus_{\mathbf{m}} \mathbb{Q}(q^{\frac{1}{2}}) | \mathbf{m} \rangle$ of $U_q(B_n)$ (subalgebra of $U_q(D_{n+1}^{(2)})$ generated by e_i, f_i, k_i for $1 \leq i \leq n$) given in Proposition 1 of [13] as a representation of $U_q(\mathfrak{osp}_{1|2n})$. Although we normalized the symmetric bilinear

form on the weight lattice so that $(\alpha_i, \alpha_i) \in \mathbb{Z}$ for any $i \in I$ in the previous subsection, we renormalize it so that $(\alpha_n, \alpha_n) = \frac{1}{2}$ to adjust it to the notations in [13]. The generators T_i, Υ_i are represented on \mathcal{W} as

$$T_i|\mathbf{m}\rangle = \begin{cases} \mathbf{i}^{m_{i+1}-m_i}|\mathbf{m}\rangle & (1 \leq i < n) \\ \mathbf{i}^{-2m_n}|\mathbf{m}\rangle & (i = n) \end{cases}, \quad \Upsilon_i|\mathbf{m}\rangle = \begin{cases} \mathbf{i}^{-2m_i}|\mathbf{m}\rangle & (1 \leq i < n) \\ \mathbf{i}^{|\mathbf{m}|-2m_n}|\mathbf{m}\rangle & (i = n) \end{cases}.$$

Let u_i ($i \in I$, $u = e, f, k$) be the generators of $U_q(B_n)$ ($\pi = 1$) and $\bar{u}_i = \hat{\Psi}(u_i)$ be the image ($\pi = -1$) of the twistor $\hat{\Psi}$. Then \bar{u}_i satisfy the relations for $U_{\bar{q}}(\text{osp}_{1|2n})$ where $\bar{q}^{\frac{1}{2}} = \mathbf{i}^{-1}q^{\frac{1}{2}}$. On the space \mathcal{W} , they act as follows.

$$\begin{aligned} \bar{e}_i|\mathbf{m}\rangle &= \mathbf{i}^{2m_{i+1}}[m_i]|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle, \\ \bar{f}_i|\mathbf{m}\rangle &= \mathbf{i}^{-2m_i}[m_{i+1}]|\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle, \\ \bar{k}_i|\mathbf{m}\rangle &= \mathbf{i}^{2m_i-2m_{i+1}}q^{-m_i+m_{i+1}}|\mathbf{m}\rangle, \\ \bar{e}_n|\mathbf{m}\rangle &= \kappa \mathbf{i}^{1-|\mathbf{m}|}[m_n]|\mathbf{m} - \mathbf{e}_n\rangle, \\ \bar{f}_n|\mathbf{m}\rangle &= \mathbf{i}^{|\mathbf{m}|-2m_n}|\mathbf{m} + \mathbf{e}_n\rangle, \\ \bar{k}_n|\mathbf{m}\rangle &= \mathbf{i}^{2m_n+1}q^{-m_n-\frac{1}{2}}|\mathbf{m}\rangle, \end{aligned}$$

where $1 \leq i < n$, $\kappa = (q+1)/(q-1)$.

By introducing the actions of $\bar{e}_0, \bar{f}_0, \bar{k}_0$, we want to make \mathcal{W} a quantum group module in Section A.1 associated to the affine Dynkin datum $C^{(2)}(n+1)$ or $B^{(1)}(0, n)$. For the former we set

$$\begin{aligned} \bar{e}_0|\mathbf{m}\rangle &= x \mathbf{i}^{2m_1-|\mathbf{m}|}|\mathbf{m} + \mathbf{e}_1\rangle, \\ \bar{f}_0|\mathbf{m}\rangle &= x^{-1} \kappa \mathbf{i}^{|\mathbf{m}|+1}[m_1]|\mathbf{m} - \mathbf{e}_1\rangle, \\ \bar{k}_0|\mathbf{m}\rangle &= \mathbf{i}^{-2m_1-1}q^{m_1+\frac{1}{2}}|\mathbf{m}\rangle, \end{aligned}$$

and for the latter

$$\begin{aligned} \bar{e}_0|\mathbf{m}\rangle &= x(-1)^{|\mathbf{m}|}|\mathbf{m} + 2\mathbf{e}_1\rangle, \\ \bar{f}_0|\mathbf{m}\rangle &= x^{-1}(-1)^{|\mathbf{m}|} \frac{[m_1][m_1-1]}{[2]^2}|\mathbf{m} - 2\mathbf{e}_1\rangle, \\ \bar{k}_0|\mathbf{m}\rangle &= -q^{2m_1+1}|\mathbf{m}\rangle, \end{aligned}$$

where x is the so-called spectral parameter. We also note that the quantum parameter is still $\bar{q} = \mathbf{i}^{-1}q^{\frac{1}{2}}$.

To obtain the representation for the quantum parameter q , we need to we switch $q^{\frac{1}{2}}$ to $\mathbf{i}q^{\frac{1}{2}}$ ($\bar{q}^{\frac{1}{2}}$ to $q^{\frac{1}{2}}$). Also, the relations in Section A.1 and those in Section 2.3 are different. For the node i that is signified as \bullet in the Dynkin diagram, there is a relation

$$e_i f_i + f_i e_i = \frac{k_i - k_i^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

in Section 2.3 rather than

$$e_i f_i + f_i e_i = \frac{k_i - k_i^{-1}}{-q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

in Section A.1. The former relation is realized by deleting κ from the action of \bar{e}_i or \bar{f}_i in the formulas of the q -oscillator representation above. By doing so, we obtain

$$\begin{aligned}\bar{e}_0|\mathbf{m}\rangle &= \begin{cases} x \mathbf{i}^{2m_1-|\mathbf{m}|}|\mathbf{m} + \mathbf{e}_1\rangle & \text{for } U_q(C^{(2)}(n+1)) \\ x(-1)^{|\mathbf{m}|}|\mathbf{m} + 2\mathbf{e}_1\rangle & \text{for } U_q(B^{(1)}(1, n)) \end{cases}, \\ \bar{f}_0|\mathbf{m}\rangle &= \begin{cases} x^{-1}\mathbf{i}^{|\mathbf{m}|+2m_1+1}[m_1]|\mathbf{m} - \mathbf{e}_1\rangle & \text{for } U_q(C^{(2)}(n+1)) \\ x^{-1}(-1)^{|\mathbf{m}|+1}\frac{[m_1][m_1-1]}{[2]^2}|\mathbf{m} - 2\mathbf{e}_1\rangle & \text{for } U_q(B^{(1)}(0, n)) \end{cases}, \\ \bar{k}_0|\mathbf{m}\rangle &= \begin{cases} q^{m_1+\frac{1}{2}}|\mathbf{m}\rangle & \text{for } U_q(C^{(2)}(n+1)) \\ q^{2m_1+1}|\mathbf{m}\rangle & \text{for } U_q(B^{(1)}(0, n)) \end{cases}, \\ \bar{e}_i|\mathbf{m}\rangle &= (-1)^{-m_i+m_{i+1}+1}[m_i]|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle, \\ \bar{f}_i|\mathbf{m}\rangle &= (-1)^{-m_i+m_{i+1}+1}[m_{i+1}]|\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle, \\ \bar{k}_i|\mathbf{m}\rangle &= q^{-m_i+m_{i+1}}|\mathbf{m}\rangle, \\ \bar{e}_n|\mathbf{m}\rangle &= \mathbf{i}^{1-|\mathbf{m}|+2m_n}[m_n]|\mathbf{m} - \mathbf{e}_n\rangle, \\ \bar{f}_n|\mathbf{m}\rangle &= \mathbf{i}^{|\mathbf{m}|-2m_n}|\mathbf{m} + \mathbf{e}_n\rangle, \\ \bar{k}_n|\mathbf{m}\rangle &= q^{-m_n-\frac{1}{2}}|\mathbf{m}\rangle,\end{aligned}$$

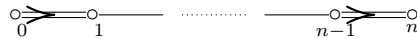
for $1 \leq i \leq n-1$.

To obtain the actions of $U_q(C^{(2)}(n+1))$ (resp. $U_q(B^{(1)}(0, n))$) in Proposition 3.9 (resp. 3.14), we perform the basis change $|\mathbf{m}\rangle$ to $\mathbf{i}^{s(\mathbf{m})}q^{-|\mathbf{m}|/2} \prod_{j=1}^n [m_j]!|\mathbf{m}\rangle$ where $s(\mathbf{m}) = -|\mathbf{m}|(|\mathbf{m}|+1)/2 - \sum_j m_j^2$. Next we apply the algebra automorphism sending $e_n \mapsto -e_n, f_n \mapsto -f_n$ and the other generators fixed. For $U_q(C^{(2)}(n+1))^\sigma$, we also apply $e_0 \mapsto \sigma e_0, f_0 \mapsto f_0 \sigma$. Accordingly, the coproduct also changes. For $U_q(B^{(1)}(0, n))$, we alternatively apply $e_0 \mapsto \mathbf{i}[2]e_0, f_0 \mapsto \frac{1}{\mathbf{i}[2]}f_0$.

APPENDIX B. QUANTUM R MATRIX FOR $U_q(A_{2n}^{(2)\dagger})$

In this appendix, we consider the quantum R matrix for the q -oscillator representation of $U_q(A_{2n}^{(2)\dagger})$ where $A_{2n}^{(2)\dagger}$ is the Dynkin diagram whose nodes have the opposite labelings to $A_{2n}^{(2)}$. Next we identify it as the one for $U_q(B^{(1)}(0, n))$.

B.1. q -oscillator representation for $U_q(A_{2n}^{(2)\dagger})$. By $A_{2n}^{(2)\dagger}$ we denote the following Dynkin diagram.



Although we did not deal with the q -oscillator representation for $U_q(A_{2n}^{(2)\dagger})$ in [13], it is easy to guess from other cases given there. On the space \mathcal{W} , the actions are given as follows.

$$\begin{aligned}
e_0|\mathbf{m}\rangle &= x|\mathbf{m} + 2\mathbf{e}_1\rangle, \\
f_0|\mathbf{m}\rangle &= x^{-1} \frac{[m_1][m_1 - 1]}{[2]^2} |\mathbf{m} - 2\mathbf{e}_1\rangle, \\
k_0|\mathbf{m}\rangle &= -q^{2m_1+1} |\mathbf{m}\rangle, \\
e_i|\mathbf{m}\rangle &= [m_i] |\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle, \\
f_i|\mathbf{m}\rangle &= [m_{i+1}] |\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle, \\
k_i|\mathbf{m}\rangle &= q^{-2m_i+2m_{i+1}} |\mathbf{m}\rangle, \\
e_n|\mathbf{m}\rangle &= \mathbf{i}\kappa[m_n] |\mathbf{m} - \mathbf{e}_n\rangle, \\
f_n|\mathbf{m}\rangle &= |\mathbf{m} + \mathbf{e}_n\rangle, \\
k_n|\mathbf{m}\rangle &= \mathbf{i}q^{-m_n-1/2} |\mathbf{m}\rangle,
\end{aligned}$$

where $0 < i < n$, $\kappa = (q+1)/(q-1)$. Denote this representation map by π_x .

$U_q(B_n)$ -highest weight vectors $\{v_l \mid l \in \mathbb{Z}_{\geq 0}\}$ are calculated in [13, Prop 4]. We take the coproduct (C.1) with $\pi = 1$.

Lemma B.1. *For $x, y \in \mathbb{Q}(q)$ we have*

$$\begin{aligned}
(1) \quad & (\pi_x \otimes \pi_y) \Delta(f_0 f_1^{(2)} \cdots f_{n-1}^{(2)}) v_l = -\frac{[l][l-1]}{[2]^2} (q^{2l-2} x^{-1} + q^{-1} y^{-1}) v_{l-2} \quad (l \geq 2), \\
(2) \quad & (\pi_x \otimes \pi_y) \Delta(e_n e_{n-1}^{(2)} \cdots e_1^{(2)} e_0) v_0 = \frac{\mathbf{i}\kappa[2]}{1-q} ((y+qx)v_1 - q(y+x)\Delta(f_n)v_0).
\end{aligned}$$

Define $\check{R}_{KO}(z, q)$ as in Proposition C.4 for $U_q(B_n^{(1)})$. The existence of such $\check{R}_{KO}(z, q)$ is essentially given in [13, Theorem 13]. Namely, although $A_{2n}^{(2)\dagger}$ is not listed there, the corresponding gauge transformed quantum R matrix is $S^{2,1}(z)$ and the proof has been done as the cases (i),(iv) and (v).

Proposition B.2. *We have the following spectral decomposition*

$$\check{R}_{KO}(z) = \sum_{l \in 2\mathbb{Z}_+} \prod_{j=1}^{l/2} \frac{z + q^{4j-1}}{1 + q^{4j-1}z} P_l + \sum_{l \in 1+2\mathbb{Z}_+} \prod_{j=0}^{(l-1)/2} \frac{z + q^{4j+1}}{1 + q^{4j+1}z} P_l,$$

where P_l is the projector on the subspace generated by the $U_q(B_n)$ -highest weight vector v_l ($l \geq 0$).

APPENDIX C. QUANTUM R MATRIX FOR $U_q(C^{(2)}(n+1))$ AND $U_q(B^{(1)}(0, n))$

In this appendix, we compare the quantum R matrix for the q -oscillator representation for $U_q(C^{(2)}(n+1))$ with the one for $U_q(D_{n+1}^{(2)})$ given in [13]. We also consider the quantum R matrix for $U_q(B^{(1)}(0, n))$ based on the results in [13].

C.1. Gauge transformation. We take the following coproduct

$$\begin{aligned}
(C.1) \quad & \Delta(k_i) = k_i \otimes k_i, \\
& \Delta(e_i) = 1 \otimes e_i + e_i \otimes \sigma^{\frac{1-\pi}{2}p(i)} k_i, \\
& \Delta(f_i) = f_i \otimes \sigma^{\frac{1-\pi}{2}p(i)} + k_i^{-1} \otimes f_i,
\end{aligned}$$

for $i \in I$, where σ satisfies (2.3). We also take the same coproduct (C.1) for \bar{u}_i . Let Γ be an operator acting on $\mathcal{W}^{\otimes 2}$ by

$$(C.2) \quad \Gamma|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle = \mathbf{i}^{\sum_{k,l} \varphi_{kl} m_k m'_l} |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle,$$

for $\mathbf{m} = (m_1, \dots, m_n)$ and $\mathbf{m}' = (m'_1, \dots, m'_n)$. Here we have the constraint $\varphi_{kl} + \varphi_{lk} = 0$. Then by [20] (see also [19]),

$$\Delta^\Gamma(u) = \Gamma^{-1} \Delta(u) \Gamma$$

gives another coproduct of $U_q(B_n)$ acting on $\mathcal{W}^{\otimes 2}$. Take φ_{kl} to be 1 for $k < l$. We also set

$$(C.3) \quad K|\mathbf{m}\rangle = \mathbf{i}^{c(\mathbf{m})} |\mathbf{m}\rangle,$$

where

$$c(\mathbf{m}) = -\frac{1}{2} \sum_k m_k^2 + \sum_k \left(k - n - \frac{1}{2}\right) m_k.$$

Set

$$\gamma_i(\mathbf{m}) = \begin{cases} -|\mathbf{m}| + m_1 & (i = 0 \text{ and for } U_q(C^{(2)}(n+1))) \\ -2|\mathbf{m}| + 2m_1 & (i = 0 \text{ and for } U_q(B^{(1)}(0, n))) \\ m_i + m_{i+1} & (0 < i < n) \\ -|\mathbf{m}| + m_n & (i = n) \end{cases},$$

$$\beta_i(\mathbf{m}) = \begin{cases} m_1 + n & (i = 0 \text{ and } U_q(C^{(2)}(n+1))) \\ 2m_1 + 2n + 1 & (i = 0 \text{ and } U_q(B^{(1)}(0, n))) \\ -m_i + m_{i+1} & (0 < i < n) \\ -m_n & (i = n) \end{cases}.$$

Let $\alpha_0 = \mathbf{e}_1$ for $U_q(C^{(2)}(n+1))$, $2\mathbf{e}_1$ for $U_q(B^{(1)}(0, n))$, $\alpha_i = -\mathbf{e}_i + \mathbf{e}_{i+1}$ ($0 < i < n$), and $\alpha_n = -\mathbf{e}_n$.

Lemma C.1. *The following formulas hold for \mathbf{m} , \mathbf{m}' , and $i \in I$;*

- (1) $\Gamma^{-1}(1 \otimes e_i) \Gamma |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle = \mathbf{i}^{-\gamma_i(\mathbf{m})} |\mathbf{m}\rangle \otimes e_i |\mathbf{m}'\rangle$,
- (2) $\Gamma^{-1}(e_i \otimes 1) \Gamma |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle = \mathbf{i}^{\gamma_i(\mathbf{m}')} e_i |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle$,
- (3) $\Gamma^{-1}(1 \otimes f_i) \Gamma |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle = \mathbf{i}^{\gamma_i(\mathbf{m} - \alpha_i)} |\mathbf{m}\rangle \otimes f_i |\mathbf{m}'\rangle$,
- (4) $\Gamma^{-1}(f_i \otimes 1) \Gamma |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle = \mathbf{i}^{-\gamma_i(\mathbf{m}' - \alpha_i)} f_i |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle$.

Lemma C.2. *The following formulas hold for \mathbf{m} and $i \in I$;*

- (1) $K^{-1} e_i K |\mathbf{m}\rangle = \mathbf{i}^{\beta_i(\mathbf{m})} e_i |\mathbf{m}\rangle$,
- (2) $K^{-1} f_i K |\mathbf{m}\rangle = \mathbf{i}^{-\beta_i(\mathbf{m} - \alpha_i)} f_i |\mathbf{m}\rangle$.

Proposition C.3. *For u_i ($i \in I$, $u = e, f, k$), we have*

$$\Delta(\bar{u}_i) |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle = \mathbf{i}^{\Lambda_i(\mathbf{m} + \mathbf{m}')} (K \otimes K)^{-1} \Delta^\Gamma(u_i) (K \otimes K) |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle,$$

on $\mathcal{W}^{\otimes 2}$. Here

$$\Lambda_i(\mathbf{m}) = \begin{cases} m_i + m_{i+1} - (\delta_{i0} + \delta_{in})|\mathbf{m}| - n\delta_{i0} & (u = e) \\ m_i + m_{i+1} + (\delta_{i0} + \delta_{in})(|\mathbf{m}| + 1) - 2 & (u = f), \\ 2m_i - 2m_{i+1} & (u = k) \end{cases}$$

except when $i = 0$ and for $U_q(B^{(1)}(0, n))$, where

$$\Lambda_0(\mathbf{m}) = \begin{cases} 2m_1 - 2|\mathbf{m}| - 2n + 1 & (u = e) \\ 2m_1 - 2|\mathbf{m}| - 2n + 3 & (u = f). \\ 0 & (u = k) \end{cases}$$

Here we should understand $m_0 = m_{n+1} = 0$.

Proof. It follows from Lemmas C.1 and C.2, and the following calculations. For instance, for $i = n$

$$\begin{aligned} \Delta(\bar{e}_n)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle &= (1 \otimes \bar{e}_n + \bar{e}_n \otimes \sigma \bar{k}_n)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle \\ &= \kappa(\mathbf{i}^{1-|\mathbf{m}'|}[m'_n]|\mathbf{m}\rangle \otimes |\mathbf{m}' - \mathbf{e}_n\rangle \\ &\quad + (-1)^{|\mathbf{m}'|} \mathbf{i}^{2-|\mathbf{m}|+2m'_n} q^{-2m'_n-1} [m_n]|\mathbf{m} - \mathbf{e}_n\rangle \otimes |\mathbf{m}'\rangle), \\ \Delta^\Gamma(e_n)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle &= (\Gamma^{-1}(1 \otimes e_n)\Gamma + \Gamma^{-1}(e_n \otimes 1)\Gamma \cdot (1 \otimes k_n))|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle \\ &= \kappa(\mathbf{i}^{|\mathbf{m}|-m_n+1}[m'_n]|\mathbf{m}\rangle \otimes |\mathbf{m}' - \mathbf{e}_n\rangle \\ &\quad + \mathbf{i}^{-|\mathbf{m}'|+m'_n+2} q^{-2m'_n-1} [m_n]|\mathbf{m} - \mathbf{e}_n\rangle \otimes |\mathbf{m}'\rangle), \end{aligned}$$

and for $i \neq n$

$$\begin{aligned} \Delta(\bar{e}_i)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle &= (1 \otimes \bar{e}_i + \bar{e}_i \otimes \bar{k}_i)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle \\ &= \mathbf{i}^{2m'_i+1}[m'_i]|\mathbf{m}\rangle \otimes |\mathbf{m}' - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle \\ &\quad + \mathbf{i}^{2m_{i+1}+2m'_i-2m'_{i+1}} q^{-2m'_i+2m'_{i+1}} [m_i]|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle \otimes |\mathbf{m}'\rangle, \\ \Delta^\Gamma(e_i)|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle &= (\Gamma^{-1}(1 \otimes e_i)\Gamma + \Gamma^{-1}(e_i \otimes 1)\Gamma \cdot (1 \otimes k_i))|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle \\ &= \mathbf{i}^{-m_i-m_{i+1}} [m'_i]|\mathbf{m}\rangle \otimes |\mathbf{m}' - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle \\ &\quad + \mathbf{i}^{m'_i+m'_{i+1}} q^{-2m'_i+2m'_{i+1}} [m_i]|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle \otimes |\mathbf{m}'\rangle. \end{aligned}$$

□

For a quantum group such as $U = U_q(D_{n+1}^{(2)})$, $U_q(A_{2n}^{(2)\dagger})$, $U_q(C^{(2)}(n+1))$, $U_q(B^{(1)}(0, n))$ a quantum R matrix $R(z)$ is defined, if it exists, as an intertwiner satisfying

$$\check{R}(z)(\pi_x \otimes \pi_y)\Delta(u) = (\pi_y \otimes \pi_x)\Delta(u)\check{R}(z),$$

where $\check{R}(z) = PR(z)$, P is the transposition of the tensor components and $z = x/y$. We also note that the coproduct we use here is (C.1). For $U = U_q(D_{n+1}^{(2)})$ or $U_q(A_{2n}^{(2)\dagger})$, the existence of quantum R matrices are proved in [13] or Appendix B. We denote them by $\check{R}_{KO}(z)$. Let $\check{R}_{new}(z)$ be the quantum R matrices for the quantum groups $U = U_q(C^{(2)}(n+1))$ or $U_q(B^{(1)}(0, n))$. From the previous proposition, we have

Proposition C.4. For generic $x, y \in \mathbb{Q}(q)$, $\check{R}_{new}(z)$ and $\check{R}_{KO}(z)$ have the following relation:

$$\check{R}_{new}(z, -q) = (K \otimes K)^{-1} \Gamma^{-1} \check{R}_{KO}(z, q) \Gamma(K \otimes K).$$

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