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# SHARP HARDY-LERAY INEQUALITY FOR CURL-FREE FIELDS WITH A REMAINDER TERM

NAOKI HAMAMOTO AND FUTOSHI TAKAHASHI

**ABSTRACT.** In this paper, we give a new and a simpler approach to the result in [7] concerning the best constant of Hardy-Leray inequality for curl-free fields. As a by-product, we obtain an improved inequality with a remainder term. The non-attainability of the best constant is an easy consequence of the new inequality. The proof is based on a decomposition of curl-free fields into radial and spherical parts.

## 1. INTRODUCTION

In this paper, we concern the classical functional inequality called the Hardy-Leray inequality for smooth vector fields and its improvement.

Let  $N \in \mathbb{N}$  be an integer with  $N \geq 2$  and put  $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ . In the following,  $C_c^\infty(\Omega)^N$  denotes the set of smooth vector fields

$$\mathbf{u} = (u_1, u_2, \dots, u_N) : \Omega \ni \mathbf{x} \mapsto \mathbf{u}(\mathbf{x}) \in \mathbb{R}^N$$

having compact supports on an open subset  $\Omega$  of  $\mathbb{R}^N$ .

Let  $\gamma$  be a real number. Then it is well known that

$$\left(\gamma + \frac{N}{2} - 1\right)^2 \int_{\mathbb{R}^N} \frac{|\mathbf{u}|^2}{|\mathbf{x}|^2} |\mathbf{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\nabla \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx$$

holds for any vector field  $\mathbf{u} \in C_c^\infty(\mathbb{R}^N)^N$ , as far as the integral on the left-hand side is finite (or equivalently  $\mathbf{u}(\mathbf{0}) = \mathbf{0}$  for  $\gamma \leq 1 - N/2$ ). This was first proved by J. Leray [10] when the weight  $\gamma = 0$ , see also the book by Ladyzhenskaya [9]. It is also known that the constant  $(\gamma + \frac{N}{2} - 1)^2$  is sharp and never attained by any non-zero vector field.

In [2], Costin and Maz'ya proved that if the smooth vector fields  $\mathbf{u}$  are axisymmetric and subject to the divergence-free constraint  $\operatorname{div} \mathbf{u} \equiv 0$ , then the constant  $(\gamma + \frac{N}{2} - 1)^2$  can be improved and replaced by a larger one. More precisely, they proved the following:

**Theorem A.** (Costin-Maz'ya [2]) *Let  $N \geq 2$ . Let  $\gamma \neq 1 - N/2$  be a real number and  $\mathbf{u} \in C_c^\infty(\mathbb{R}^N)^N$  be an axisymmetric divergence-free vector field. (If  $N = 2$ , the axisymmetric assumption is not needed). Assume that  $\mathbf{u}(\mathbf{0}) = \mathbf{0}$  if  $\gamma < 1 - N/2$ . Then*

$$C_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\mathbf{u}|^2}{|\mathbf{x}|^2} |\mathbf{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\nabla \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx$$

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holds with the optimal constant  $C_{N,\gamma}$  given by

$$C_{N,\gamma} = \begin{cases} \left(\gamma + \frac{N}{2} - 1\right)^2 \frac{N+1 + \left(\gamma - \frac{N}{2}\right)^2}{N-1 + \left(\gamma - \frac{N}{2}\right)^2} & (N \geq 3, \gamma \leq 1), \\ \left(\gamma + \frac{N}{2} - 1\right)^2 + 2 + \min_{\kappa \geq 0} \left( \kappa + \frac{4(N-1)(\gamma-1)}{\kappa + N-1 + \left(\gamma - \frac{N}{2}\right)^2} \right) & (N \geq 4, \gamma > 1), \\ \left(\gamma + \frac{1}{2}\right)^2 + 2 & (N = 3, \gamma > 1), \\ C_{2,\gamma} = \begin{cases} \gamma^2 \frac{3+(\gamma-1)^2}{1+(\gamma-1)^2} & \text{if } |\gamma+1| \leq \sqrt{3}, \\ \gamma^2 + 1 & \text{otherwise.} \end{cases} \end{cases}$$

Note that the expression of the best constant  $C_{N,\gamma}$  is slightly different from that in [2] when  $N \geq 4$ , but a careful checking the proof in [2] leads to the above formula in Theorem A. (See also [3, §2.1].)

Later, the first author of this paper has succeeded in removing the axisymmetric assumption in Theorem A to obtain the best constant [6, 4]. See also [8] for another improvement of [2]. We refer to [3, 5] for the Rellich-Leray inequality for divergence-free vector fields.

For curl-free vector fields, we have recently obtained the following result.

**Theorem B.** ([7]) *Let  $N \geq 2$ . Let  $\gamma \neq 1 - N/2$  be a real number and let  $\mathbf{u} \in C_c^\infty(\mathbb{R}^N)^N$  be a curl-free vector field. We assume that  $\mathbf{u}(\mathbf{0}) = \mathbf{0}$  if  $\gamma < 1 - N/2$ . Then*

$$H_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\mathbf{u}|^2}{|\mathbf{x}|^2} |\mathbf{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\nabla \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx$$

with the optimal constant  $H_{N,\gamma}$  given by

$$(1) \quad H_{N,\gamma} = \begin{cases} \left(\gamma + \frac{N}{2} - 1\right)^2 \frac{3(N-1) + \left(\gamma + \frac{N}{2} - 2\right)^2}{N-1 + \left(\gamma + \frac{N}{2} - 2\right)^2} & \text{if } \left|\gamma + \frac{N}{2}\right| \leq \sqrt{N+1}, \\ \left(\gamma + \frac{N}{2} - 1\right)^2 + N - 1 & \text{otherwise.} \end{cases}$$

The method of the proof of Theorem B, which followed from that of Costin-Maz'ya [2], consists of the following items: A representation of curl-free vector fields in the spherical polar coordinates, a transformation of vector fields called Brezis-Vázquez-Maz'ya, the one-dimensional Fourier transform in the radial direction, and the eigenvector expansion for the Laplace-Beltrami operator in  $L^2(\mathbb{S}^{N-1})$ .

A main purpose of this paper is to give another and a simpler approach to Theorem B. We avoid the use of Fourier transform, in the hope of being helpful for the possible extension of the result to  $L^p$ -setting or to domains other than the whole space. As a by-product, we obtain the sharp Hardy-Leray inequality for curl-free vector fields with a remainder term, which is the main result of this paper:

**Theorem 1.** *Let  $N \geq 2$ . Let  $H_{N,\gamma}$  be defined in (1) and let  $\mathbf{u} \in C_c^\infty(\mathbb{R}^N)^N$  be a curl-free field such that  $\mathbf{u}(\mathbf{0}) = \mathbf{0}$  if  $\gamma \leq 1 - \frac{N}{2}$ . Then the inequality*

$$(2) \quad \int_{\mathbb{R}^N} |\nabla \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx \geq H_{N,\gamma} \int_{\mathbb{R}^N} |\mathbf{u}|^2 |\mathbf{x}|^{2\gamma-2} dx \\ + \int_{\mathbb{R}^N} \left( (N-1) \mathcal{E}_{N,\gamma}[\mathbf{u}] + \left| \mathbf{x} \cdot \nabla (|\mathbf{x}|^{\gamma+\frac{N}{2}-1} \mathbf{u}) \right|^2 \right) |\mathbf{x}|^{-N} dx$$

holds with the nonnegative function  $\mathcal{E}_{N,\gamma}[\mathbf{u}]$  given by

$$\mathcal{E}_{N,\gamma}[\mathbf{u}](\mathbf{x}) = \begin{cases} \left( \left( \gamma + \frac{N}{2} \right)^2 - N - 1 \right) \varphi^2 + (\mathbf{x} \cdot \nabla \varphi)^2 & \text{for } \left| \gamma + \frac{N}{2} \right| \geq \sqrt{N+1}, \\ \frac{\left( N + 1 - \left( \gamma + \frac{N}{2} \right)^2 \right) f^2 + 4(1-\gamma)(\mathbf{x} \cdot \nabla \varphi)^2}{\left( \gamma + \frac{N}{2} - 2 \right)^2 + N - 1} & \text{for } \left| \gamma + \frac{N}{2} \right| < \sqrt{N+1}. \end{cases}$$

Here  $f$  and  $\varphi$  are scalar fields defined by

$$f(\mathbf{x}) = \omega_{N-1}^{-1} |\mathbf{x}|^{\gamma + \frac{N}{2} - 1} \int_{\mathbb{S}^{N-1}} \boldsymbol{\sigma} \cdot \mathbf{u}(|\mathbf{x}| \boldsymbol{\sigma}) d\boldsymbol{\sigma},$$

$$\varphi(\mathbf{x}) = |\mathbf{x}|^{\gamma + \frac{N}{2} - 2} \left( \phi(\mathbf{x}) - \omega_{N-1}^{-1} \int_{\mathbb{S}^{N-1}} \phi(|\mathbf{x}| \boldsymbol{\sigma}) d\boldsymbol{\sigma} \right),$$

in terms of the scalar potential  $\phi$  of  $\mathbf{u}$  (that is,  $\mathbf{u} = \nabla \phi$ ), and  $\omega_{N-1}$  denotes the surface measure of the unit sphere  $\mathbb{S}^{N-1}$  in  $\mathbb{R}^N$ . Moreover, the equality in (2) is realized if and only if the equation

$$-\Delta_{\boldsymbol{\sigma}} \varphi(r \boldsymbol{\sigma}) = (N-1) \varphi(r \boldsymbol{\sigma})$$

holds for all  $r > 0$  and  $\boldsymbol{\sigma} \in \mathbb{S}^{N-1}$ , where  $\Delta_{\boldsymbol{\sigma}}$  denotes the Laplace-Beltrami operator on  $\mathbb{S}^{N-1}$ .

**Remark 2.** We directly see from (2) that the equation

$$\int_{\mathbb{R}^N} |\nabla \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx = H_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\mathbf{u}|^2}{|\mathbf{x}|^2} |\mathbf{x}|^{2\gamma} dx$$

does not hold for any  $\mathbf{u} \in C_c^\infty(\mathbb{R}^N)^N \setminus \{\mathbf{0}\}$  as far as the integral on the right-hand side is finite. Indeed, this equation together with (2) implies that the function

$$\mathbb{R}_+ \times \mathbb{S}^{N-1} \ni (r, \boldsymbol{\sigma}) \mapsto r^{\gamma + \frac{N}{2} - 1} \mathbf{u}(r \boldsymbol{\sigma})$$

is independent of  $r$ , which violates the finiteness of the integral unless  $\mathbf{u} \equiv \mathbf{0}$ .

The remaining content of this paper is organized as follows: In §2, we give a quick review of some differential formulae with respect to radial-spherical variables and derive an equivalent condition to the curl-free condition for vector-fields on  $\mathbb{R}^N$  (the Poincaré lemma); there Proposition 4 gives a characterization of curl-free fields, which serves as a key tool for the proof of our main theorem. In §3 we prove Theorem 1 by making full use of Proposition 4. In §4 we prove the sharp Rellich-Leray inequality for curl-free vector fields with a remainder term, as another application of the method described in §2–§3.

## 2. REPRESENTATION OF CURL-FREE FIELDS IN TERMS OF RADIAL-SPHERICAL VARIABLES

In this section, we recall the Poincaré lemma, which gives a scalar-potential representation of smooth curl-free fields on  $\mathbb{R}^N$ . By deforming this potential via Brezis-Vázquez-Maz'ya transformation, we derive another equivalent condition for test vector fields to be curl-free.

**2.1. Radial-spherical variables and the Poincaré lemma.** First of all, we introduce the transformation

$$\mathbb{R}_+ \times \mathbb{S}^{N-1} \rightarrow \mathbb{R}^N \setminus \{\mathbf{0}\}, \quad (r, \boldsymbol{\sigma}) \mapsto \mathbf{x} = r \boldsymbol{\sigma}$$

together with its inverse

$$\mathbb{R}^N \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}_+ \times \mathbb{S}^{N-1}, \quad \mathbf{x} \mapsto (r, \boldsymbol{\sigma}) = \left( |\mathbf{x}|, \frac{\mathbf{x}}{|\mathbf{x}|} \right) \in \mathbb{R}_+ \times \mathbb{S}^{N-1}.$$

Let  $\mathbf{u} = (u_1, u_2, \dots, u_N) : \mathbb{R}^N \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}^N$  be a vector field. Then its radial scalar component  $u_R = u_R(\mathbf{x})$  and spherical vector part  $\mathbf{u}_S = \mathbf{u}_S(\mathbf{x})$  are defined by the formulae

$$\mathbf{u} = \boldsymbol{\sigma} u_R + \mathbf{u}_S, \quad \boldsymbol{\sigma} \cdot \mathbf{u}_S = 0$$

for all  $\mathbf{x} = r\boldsymbol{\sigma} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ . In a similar way, we denote by  $\partial_r$  and  $\nabla_\sigma$  the radial derivative and the spherical gradient, respectively:

$$\partial_r f = \boldsymbol{\sigma} \cdot \nabla f, \quad \nabla_\sigma f = r(\nabla f)_S$$

for all  $f = f(\mathbf{x}) \in C^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})$ , or equivalently

$$(3) \quad \nabla = \boldsymbol{\sigma} \partial_r + \frac{1}{r} \nabla_\sigma, \quad \boldsymbol{\sigma} \cdot \nabla_\sigma = 0.$$

The Laplace operator  $\Delta = \sum_{k=1}^N \partial^2 / \partial x_k^2$  is known to be represented in terms of radial-spherical variables by the formula

$$(4) \quad \Delta = \frac{1}{r^{N-1}} \partial_r (r^{N-1} \partial_r) + \frac{1}{r^2} \Delta_\sigma,$$

where  $\Delta_\sigma$  denotes the Laplace-Beltrami operator on  $\mathbb{S}^{N-1}$ . In the following, we use a convention with some ambiguity that for smooth scalar fields and vector fields on  $\mathbb{R}^N \setminus \{\mathbf{0}\}$ , we think of them as functions of  $\boldsymbol{\sigma} \in \mathbb{S}^{N-1}$  for  $r = |\mathbf{x}|$  fixed, when we apply  $\nabla_\sigma$  or  $\Delta_\sigma$  to them. As a simple example, the operation of (3) and (4) on the scalar field  $r = |\mathbf{x}|$  or its powers gives

$$(5) \quad \nabla r = \boldsymbol{\sigma} \quad \text{and} \quad \Delta r^s = \alpha_s r^{s-2}, \quad \text{where} \quad \alpha_s = s(s + N - 2)$$

for all  $s \in \mathbb{R}$ .

For later use, we prove the following lemma:

**Lemma 3.** *For any  $f \in C^\infty(\mathbb{S}^{N-1})$ ,*

$$(6) \quad \begin{cases} \Delta_\sigma(\boldsymbol{\sigma} f) - \boldsymbol{\sigma} \Delta_\sigma f = (2\nabla_\sigma - (N-1)\boldsymbol{\sigma})f, \\ \Delta_\sigma \nabla_\sigma f - \nabla_\sigma \Delta_\sigma f = ((N-3)\nabla_\sigma - 2\boldsymbol{\sigma} \Delta_\sigma)f \end{cases}$$

*holds for all  $\boldsymbol{\sigma} \in \mathbb{S}^{N-1}$ .*

*Proof.* Take any  $f \in C^\infty(\mathbb{S}^{N-1})$ . We identify  $f$  with  $\tilde{f} \in C^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})$  by the formula  $\tilde{f}(\mathbf{x}) = f(\boldsymbol{\sigma})$  where  $\boldsymbol{\sigma} = \frac{\mathbf{x}}{|\mathbf{x}|} \in \mathbb{S}^{N-1}$ . Note that  $\Delta_\sigma \boldsymbol{\sigma} = -(N-1)\boldsymbol{\sigma}$  since

$$0 = \Delta \mathbf{x} = \left( \partial_r^2 + \frac{N-1}{r} \partial_r + \frac{1}{r^2} \Delta_\sigma \right) (r\boldsymbol{\sigma}) = \frac{N-1}{r} \boldsymbol{\sigma} + \frac{1}{r} \Delta_\sigma \boldsymbol{\sigma}.$$

Thus we compute

$$\begin{aligned} \Delta_\sigma(\boldsymbol{\sigma} f) &= (\Delta_\sigma \boldsymbol{\sigma})f + \boldsymbol{\sigma}(\Delta_\sigma f) + 2(\nabla_\sigma f \cdot \nabla_\sigma) \boldsymbol{\sigma} \\ &= -(N-1)\boldsymbol{\sigma} f + (\Delta_\sigma f) \boldsymbol{\sigma} + 2\nabla_\sigma f, \end{aligned}$$

where we have used  $(\nabla_\sigma f \cdot \nabla_\sigma) \boldsymbol{\sigma} = (\nabla_\sigma f \cdot \nabla) \mathbf{x} = \nabla_\sigma f$ . This proves the first identity of (6).

To prove the second identity, we note from (3) resp. (4) that  $\nabla_\sigma f = r \nabla f$  resp.  $\Delta_\sigma f = r^2 \Delta f$ , since  $f = \tilde{f}$  is independent of the radial variable  $r$ . Also recalling from (5)<sub>s=1</sub> the formulae  $\nabla r = \boldsymbol{\sigma}$  and  $\Delta r = (N-1)r^{-1}$ , we have

$$\begin{aligned} (\Delta_\sigma \nabla_\sigma - \nabla_\sigma \Delta_\sigma) f &= r^2 \Delta(r \nabla f) - r \nabla(r^2 \Delta f) \\ &= r^2((\Delta r) \nabla f + 2(\nabla r \cdot \nabla) \nabla f) - r(\nabla r^2) \Delta f \\ &= (N-1)r \nabla f + 2r^2 \partial_r r^{-1} \nabla_\sigma f - 2r^2 \boldsymbol{\sigma} \Delta f \\ &= (N-3) \nabla_\sigma f - 2\boldsymbol{\sigma} \Delta_\sigma f, \end{aligned}$$

as desired.  $\square$

The *curl* of a vector field  $\mathbf{u} = (u_1, \dots, u_N) \in C^\infty(\mathbb{R}^N)^N$  is defined as the differential 2-form

$$\operatorname{curl} \mathbf{u} = d(\mathbf{u} \cdot d\mathbf{x}) = d\left(\sum_{k=1}^N u_k dx_k\right),$$

where  $d$  denotes the exterior differential. This can be expressed in terms of the standard Euclidean coordinates as

$$d(\mathbf{u} \cdot d\mathbf{x}) = \sum_{k=1}^N du_k \wedge dx_k = \sum_{j < k} \sum \left( \frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} \right) dx_j \wedge dx_k.$$

Thus the curl-free condition  $d(\mathbf{u} \cdot d\mathbf{x}) = 0$  holds if and only if

$$(7) \quad \frac{\partial u_k}{\partial x_j} = \frac{\partial u_j}{\partial x_k} \quad \text{for all } j, k \in \{1, \dots, N\}.$$

Here we claim that any curl-free vector fields  $\mathbf{u}$  can be represented by

$$(8) \quad \mathbf{u}(\mathbf{x}) = \nabla \phi(\mathbf{x}), \quad \phi(\mathbf{x}) = \int_0^{|\mathbf{x}|} \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \mathbf{u}\left(\rho \frac{\mathbf{x}}{|\mathbf{x}|}\right) d\rho \quad \text{for all } \mathbf{x} \in \mathbb{R}^N,$$

which we say that  $\mathbf{u}$  has the scalar potential  $\phi \in C^\infty(\mathbb{R}^N)$ . Conversely, the existence of such a potential implies  $d(\mathbf{u} \cdot d\mathbf{x}) = d(\nabla \phi \cdot d\mathbf{x}) = dd\phi = 0$ , that is,  $\mathbf{u}$  is curl-free.

The proof of the claim (8) is standard: For every  $i \in \{1, \dots, N\}$ , we have

$$\begin{aligned} u_i(\mathbf{x}) &= \int_0^1 \frac{d}{dt} \{tu_i(t\mathbf{x})\} dt = \int_0^1 \left\{ u_i(t\mathbf{x}) + t \sum_{j=1}^N \frac{\partial u_i(t\mathbf{x})}{\partial x_j} x_j \right\} dt \\ &= \int_0^1 \left\{ u_i(t\mathbf{x}) + t \sum_{j=1}^N \frac{\partial u_j(t\mathbf{x})}{\partial x_i} x_j \right\} dt \\ &= \int_0^1 \frac{\partial}{\partial x_i} \left( \sum_{j=1}^N u_j(t\mathbf{x}) x_j \right) dt = \frac{\partial}{\partial x_i} \int_0^1 \mathbf{u}(t\mathbf{x}) \cdot \mathbf{x} dt \quad \forall \mathbf{x} \in \mathbb{R}^N, \end{aligned}$$

here we have used (7) in the third equality. Thus we see that  $\phi(\mathbf{x}) = \int_0^1 \mathbf{u}(t\mathbf{x}) \cdot \mathbf{x} dt$  is a scalar potential of  $\mathbf{u}$ . An easy change of variables leads to (8).  $\square$

**2.2. Radial-spherical decomposition of curl-free fields.** In the following,  $\lambda \in \mathbb{R}$  denotes a fixed real number. Let  $\mathbf{u}$  be a curl-free field on  $\mathbb{R}^N$ , and let  $\phi$  be its scalar potential (8). We define a new vector field  $\mathbf{v}$  and two scalar fields  $f, \varphi$  on  $\mathbb{R}^N \setminus \{0\}$  by the formulae

$$(9) \quad \begin{cases} \mathbf{v}(\mathbf{x}) = |\mathbf{x}|^{1-\lambda} \mathbf{u}(\mathbf{x}), \\ f(\mathbf{x}) = |\mathbf{x}|^{1-\lambda} \omega_{N-1}^{-1} \int_{\mathbb{S}^{N-1}} \boldsymbol{\sigma} \cdot \mathbf{u}(|\mathbf{x}|\boldsymbol{\sigma}) d\boldsymbol{\sigma}, \\ \varphi(\mathbf{x}) = |\mathbf{x}|^{-\lambda} \left( \phi(\mathbf{x}) - \omega_{N-1}^{-1} \int_{\mathbb{S}^{N-1}} \phi(|\mathbf{x}|\boldsymbol{\sigma}) d\boldsymbol{\sigma} \right). \end{cases}$$

The transformation of the field  $\mathbf{u} \mapsto \mathbf{v}$  by the multiplication of  $|\mathbf{x}|^{1-\lambda}$  stems from an idea of Brezis-Vázquez [1] and Maz'ya [11]. Now let us denote by

$$\bar{\phi}(r) = \omega_{N-1}^{-1} \int_{\mathbb{S}^{N-1}} \phi(r\boldsymbol{\sigma}) d\boldsymbol{\sigma}, \quad r = |\mathbf{x}|$$

the spherical mean of the scalar potential  $\phi$  in (8), together with its radial derivative

$$\frac{\partial \bar{\phi}}{\partial r} = \omega_{N-1}^{-1} \int_{\mathbb{S}^{N-1}} \frac{\partial \phi}{\partial r}(r\boldsymbol{\sigma}) d\boldsymbol{\sigma} = \omega_{N-1}^{-1} \int_{\mathbb{S}^{N-1}} (\boldsymbol{\sigma} \cdot \nabla \phi)(r\boldsymbol{\sigma}) d\boldsymbol{\sigma}.$$

Then we see that (9) can be rewritten simply in terms of  $\phi$  as

$$(10) \quad \begin{cases} \mathbf{v}(\mathbf{x}) = r^{1-\lambda} \nabla \phi(\mathbf{x}), \\ f(r) = r^{1-\lambda} \frac{\partial \bar{\phi}}{\partial r}, \\ \varphi(\mathbf{x}) = r^{-\lambda} (\phi(\mathbf{x}) - \bar{\phi}(r)), \end{cases}$$

and that  $f$  is a spherical mean part of  $r^{1-\lambda} u_R$ , while  $\varphi$  has zero-spherical mean. Furthermore, the scalar representation of  $\mathbf{u}(\mathbf{x})$  in (8) is transformed into that of  $\mathbf{v}(\mathbf{x})$  by the following computation using (10):

$$\begin{aligned} \mathbf{v} &= r^{1-\lambda} \nabla \phi \\ &= r^{1-\lambda} (\nabla(\phi - \bar{\phi}) + \nabla \bar{\phi}) \\ &= r^{1-\lambda} \left( \nabla(r^\lambda \varphi) + \frac{\partial \bar{\phi}}{\partial r} \boldsymbol{\sigma} \right) \\ &= r^{1-\lambda} \nabla(r^\lambda \varphi) + f(r) \boldsymbol{\sigma} \\ &= r^{1-\lambda} (\lambda r^{\lambda-1} \varphi \boldsymbol{\sigma} + r^\lambda \nabla \varphi) + f(r) \boldsymbol{\sigma} \\ &= (\lambda \varphi + f) \boldsymbol{\sigma} + r \nabla \varphi \\ &= (\lambda \varphi + f + \partial_t \varphi) \boldsymbol{\sigma} + \nabla_\sigma \varphi. \end{aligned}$$

Here and hereafter we employ the notation  $t = \log r$  which obeys the differential identities

$$(11) \quad \begin{cases} \partial_t = r \partial_r, & dt = \frac{dr}{r}, \\ r \nabla = \boldsymbol{\sigma} \partial_t + \nabla_\sigma, \\ r^2 \Delta = \partial_t^2 + (N-2) \partial_t + \Delta_\sigma. \end{cases}$$

In view of the above computation result, we can say that  $f$  and  $\varphi$  are radial and spherical scalar potentials of  $\mathbf{v}$ , respectively.

In summary, we obtain the following proposition:

**Proposition 4.** *Let  $\lambda \in \mathbb{R}$ . Then a vector field  $\mathbf{u} \in C^\infty(\mathbb{R}^N)^N$  is curl-free if and only if there exist two scalar fields  $f, \varphi \in C^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})$  satisfying*

$$(12) \quad \begin{cases} f \text{ is radially symmetric and } \int_{\mathbb{S}^{N-1}} \varphi(r\boldsymbol{\sigma}) d\sigma = 0 & \forall r > 0, \\ \mathbf{v} = \boldsymbol{\sigma} (f + (\lambda + \partial_t) \varphi) + \nabla_\sigma \varphi & \text{on } \mathbb{R}^N \setminus \{\mathbf{0}\}, \end{cases}$$

where  $\mathbf{v} \in C_c^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})^N$  is the vector field given by the same equation  $\mathbf{v} = r^{1-\lambda} \mathbf{u}$  as in (9). Moreover, such  $f$  and  $\varphi$  are uniquely determined and explicitly given by the equations in (9); in particular, if  $\mathbf{u}$  has a compact support on  $\mathbb{R}^N \setminus \{\mathbf{0}\}$ , then so do  $f$  and  $\varphi$ .

For later use, we give an expression of the vector field  $\Delta_\sigma \mathbf{v}$  in terms of the scalar potentials:

**Lemma 5.** *Let  $\mathbf{v}$  be as in (12). Then*

$$(13) \quad \begin{aligned} \Delta_\sigma \mathbf{v} &= \boldsymbol{\sigma} (\partial_t + \lambda - 2) \Delta_\sigma \varphi \\ &\quad + \nabla_\sigma (2\partial_t + \Delta_\sigma + 2\lambda + 2N - 4) \varphi - (N-1) \mathbf{v}. \end{aligned}$$

*Proof.* By using Lemma 3 and Proposition 4, we compute

$$\begin{aligned}
\Delta_\sigma \mathbf{v} &= \Delta_\sigma \left( \boldsymbol{\sigma} (f + (\partial_t + \lambda)\varphi) \right) + \Delta_\sigma (\nabla_\sigma \varphi) \\
&= \left( \boldsymbol{\sigma} \Delta_\sigma + 2\nabla_\sigma - (N-1)\boldsymbol{\sigma} \right) (f + (\partial_t + \lambda)\varphi) \\
&\quad + \left( \nabla_\sigma \Delta_\sigma + (N-3)\nabla_\sigma - 2\boldsymbol{\sigma} \Delta_\sigma \right) \varphi \\
&= \boldsymbol{\sigma} \left( (\partial_t + \lambda - 2)\Delta_\sigma \varphi \right) - (N-1) \underbrace{\boldsymbol{\sigma} (f + (\partial_t + \lambda)\varphi)}_{\mathbf{v} - \nabla_\sigma \varphi} \\
&\quad + \nabla_\sigma (2\partial_t + \Delta_\sigma + 2\lambda + N - 3)\varphi \\
&= \boldsymbol{\sigma} (\partial_t + \lambda - 2)\Delta_\sigma \varphi \\
&\quad + \nabla_\sigma (2\partial_t + \Delta_\sigma + 2\lambda + 2N - 4)\varphi - (N-1)\mathbf{v}.
\end{aligned}$$

□

### 3. PROOF OF THEOREM 1

We assume that the left-hand side of (2) is finite, since otherwise there is nothing to prove. Then the integrability of  $|\nabla \mathbf{u}|^2 |\mathbf{x}|^{2\gamma}$  together with the smoothness of  $\mathbf{u}$  implies the existence of an integer  $m > -\frac{N}{2} - \gamma$  such that

$$\nabla \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^m) \quad \text{as } |\mathbf{x}| \rightarrow 0.$$

Moreover, in view of the assumption that  $\mathbf{u}(\mathbf{0}) = \mathbf{0}$  if  $\gamma \leq 1 - \frac{N}{2}$ , we see that  $\mathbf{u}$  satisfies

$$|\mathbf{x}|^{\gamma + \frac{N}{2} - 1} \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^\varepsilon) \quad \text{as } |\mathbf{x}| \rightarrow 0$$

for  $\varepsilon > 0$  given by

$$\varepsilon = \begin{cases} m + \frac{N}{2} + \gamma & \text{for } \gamma \leq 1 - \frac{N}{2}, \\ \gamma + \frac{N}{2} - 1 & \text{for } \gamma > 1 - \frac{N}{2}. \end{cases}$$

Hence also the scalar potential  $\phi(\mathbf{x}) = \int_0^{|\mathbf{x}|} u_R(r\mathbf{x}/|\mathbf{x}|) dr$  in (8) satisfies

$$|\mathbf{x}|^{\gamma + \frac{N}{2} - 1} \phi(\mathbf{x}) = O(|\mathbf{x}|^{1+\varepsilon}) \quad \text{as } |\mathbf{x}| \rightarrow 0.$$

Consequently, we have further obtained the integrability conditions

$$(14) \quad \int_{\mathbb{R}^N} |\mathbf{u}|^2 |\mathbf{x}|^{2\gamma-2} dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^N} \phi^2 |\mathbf{x}|^{2\gamma-4} dx < \infty$$

The proof of the theorem is carried out in the following steps:

#### 3.1. Reduction to the case of compact support distinct from the origin.

We can further assume that the curl-free field  $\mathbf{u} = \nabla \phi$  is compactly supported on  $\mathbb{R}^N \setminus \{\mathbf{0}\}$ : Indeed, let  $\{\mathbf{u}_n\} \subset C_c^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})^N$  denote the sequence of curl-free fields defined by

$$\mathbf{u}_n(\mathbf{x}) = \nabla \left( \zeta(|\mathbf{x}|^{\frac{1}{n}}) \phi(\mathbf{x}) \right) \quad \text{for every } n \in \mathbb{N},$$

where we fix  $\zeta \in C_c^\infty(\mathbb{R}_+)$  such that  $\zeta(r) = \begin{cases} 0 & \text{for } 0 < r < 1/2 \\ 1 & \text{for } 1 \leq r \end{cases}$ . Then we see that  $\bigcup_{n=1}^\infty \text{supp } \mathbf{u}_n$  is bounded, and that the asymptotic formulae

$$\begin{aligned}
\mathbf{u}_n(\mathbf{x}) &= \zeta(|\mathbf{x}|^{\frac{1}{n}}) \mathbf{u}(\mathbf{x}) + \boldsymbol{\sigma} n^{-1} |\mathbf{x}|^{\frac{1}{n}-1} \zeta'(|\mathbf{x}|^{\frac{1}{n}}) \phi(\mathbf{x}) \\
&= \mathbf{u}(\mathbf{x}) + o(1) \mathbf{u}(\mathbf{x}) + O(1/n) |\mathbf{x}|^{-1} \phi(\mathbf{x}), \\
\nabla \mathbf{u}_n(\mathbf{x}) &= \nabla \mathbf{u}(\mathbf{x}) + o(1) \nabla \mathbf{u}(\mathbf{x}) + O(1/n) \boldsymbol{\sigma} |\mathbf{x}|^{-1} \mathbf{u}(\mathbf{x}) + O(1/n) \boldsymbol{\sigma} \boldsymbol{\sigma} |\mathbf{x}|^{-2} \phi(\mathbf{x})
\end{aligned}$$

hold as  $n \rightarrow \infty$ . Therefore, taking the  $L^2$  integration on both sides gives

$$\begin{aligned} \int_{\mathbb{R}^N} |\mathbf{u}_n|^2 |\mathbf{x}|^{2\gamma-2} dx &= \int_{\mathbb{R}^N} |\mathbf{u}|^2 |\mathbf{x}|^{2\gamma-2} dx + o(1), \\ \int_{\mathbb{R}^N} |\nabla \mathbf{u}_n|^2 |\mathbf{x}|^{2\gamma} dx &= \int_{\mathbb{R}^N} |\nabla \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx + o(1), \end{aligned}$$

thanks to the integrability conditions (14). This result shows that the integrals in the inequality (2) can be approximated by curl-free fields with compact support on  $\mathbb{R}^N \setminus \{\mathbf{0}\}$ .

**3.2. Calculation of the integrals in the Hardy-Leray inequality.** In the rest of the present section, we choose

$$(15) \quad \lambda = 2 - \frac{N}{2} - \gamma$$

in view of § 2.2. Then, with respect to the measure

$$(16) \quad |\mathbf{x}|^{2\gamma} dx = r^{2\gamma+N-1} dr d\sigma = r^{4-2\lambda} \frac{dr}{r} d\sigma = r^{4-2\lambda} dt d\sigma,$$

the  $L^2$  integration of  $\mathbf{u}(\mathbf{x})/|\mathbf{x}| = r^{\lambda-2}\mathbf{v}$  can be expressed in terms of  $f$  and  $\varphi$  (in Proposition 4) as

$$\begin{aligned} (17) \quad \int_{\mathbb{R}^N} \frac{|\mathbf{u}|^2}{|\mathbf{x}|^2} |\mathbf{x}|^{2\gamma} dx &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\mathbf{v}|^2 dt d\sigma \\ &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( (f + \partial_t \varphi + \lambda \varphi)^2 + |\nabla_\sigma \varphi|^2 \right) dt d\sigma \\ &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( f^2 + (\partial_t \varphi)^2 + \lambda^2 \varphi^2 + |\nabla_\sigma \varphi|^2 \right) dt d\sigma, \end{aligned}$$

where the last equality follows from the integration by parts together with the support compactness and  $\int_{\mathbb{S}^{N-1}} \varphi d\sigma = 0$ . On the other hand, the integration of  $|\nabla \mathbf{u}|^2 = |\partial_r \mathbf{u}|^2 + r^{-2} |\nabla_\sigma \mathbf{u}|^2$  with respect to the measure (16) yields

$$\begin{aligned} (18) \quad \int_{\mathbb{R}^N} |\nabla \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx &= \int_{\mathbb{R}^N} \left( |\partial_r \mathbf{u}|^2 + r^{-2} |\nabla_\sigma \mathbf{u}|^2 \right) |\mathbf{x}|^{2\gamma} dx \\ &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( |\partial_r (r^{\lambda-1} \mathbf{v})|^2 + r^{-2} |\nabla_\sigma (r^{\lambda-1} \mathbf{v})|^2 \right) r^{4-2\lambda} dt d\sigma \\ &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( |(\lambda-1)\mathbf{v} + \partial_t \mathbf{v}|^2 + |\nabla_\sigma \mathbf{v}|^2 \right) dt d\sigma \\ &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( (\lambda-1)^2 |\mathbf{v}|^2 + |\partial_t \mathbf{v}|^2 \right) dt d\sigma + \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\nabla_\sigma \mathbf{v}|^2 dt d\sigma. \end{aligned}$$

To evaluate the last integral, let us take the  $L^2$ -inner product of  $\Delta_\sigma \mathbf{v}$  in (13) and  $\mathbf{v} = \boldsymbol{\sigma}(f + (\partial_t + \lambda)\varphi) + \nabla_\sigma \varphi$ ; then integration by parts gives

$$\begin{aligned}
 (19) \quad & \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\nabla_\sigma \mathbf{v}|^2 dt d\sigma = - \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \mathbf{v} \cdot (\Delta_\sigma \mathbf{v}) dt d\sigma \\
 & = - \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} (f + (\partial_t + \lambda)\varphi) (\partial_t + \lambda - 2) \Delta_\sigma \varphi dt d\sigma \\
 & \quad + \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( -\nabla_\sigma \varphi \cdot \nabla_\sigma (2\partial_t + \Delta_\sigma + 2\lambda + 2N - 4)\varphi + (N-1)|\mathbf{v}|^2 \right) dt d\sigma \\
 & = \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( (\Delta_\sigma \varphi)^2 + (\lambda^2 - 4\lambda - 2N + 4)|\nabla_\sigma \varphi|^2 \right) dt d\sigma \\
 & \quad + \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} (|\partial_t \nabla_\sigma \varphi|^2 + (N-1)|\mathbf{v}|^2) dt d\sigma.
 \end{aligned}$$

Here we note that the spectrum of  $-\Delta_\sigma$  is given by the set

$$\{\alpha_\nu = \nu(N + \nu - 2) ; \nu \in \mathbb{N} \cup \{0\}\},$$

and hence the estimate

$$\begin{aligned}
 & \frac{1}{\int_{\mathbb{S}^{N-1}} \varphi^2 d\sigma} \int_{\mathbb{S}^{N-1}} \left( (\Delta_\sigma \varphi)^2 + (\lambda^2 - 4\lambda - 2N + 4)|\nabla_\sigma \varphi|^2 \right) d\sigma \\
 & \geq \min_{\nu \in \mathbb{N}} \{ \alpha_\nu^2 + (\lambda^2 - 4\lambda - 2N + 4)\alpha_\nu ; \nu \in \mathbb{N} \} \\
 & = \alpha_1^2 + (\lambda^2 - 4\lambda - 2N + 4)\alpha_1 \\
 & = (N-1)((\lambda-2)^2 - N-1)
 \end{aligned}$$

holds for all  $\varphi \in C^\infty(\mathbb{S}^{N-1}) \setminus \{0\}$  such that  $\int_{\mathbb{S}^{N-1}} \varphi d\sigma = 0$ . Also by using  $\int_{\mathbb{S}^{N-1}} \partial_t \varphi d\sigma = 0$ , we have the estimate

$$\int_{\mathbb{S}^{N-1}} |\nabla_\sigma (\partial_t \varphi)|^2 d\sigma \geq (N-1) \int_{\mathbb{S}^{N-1}} |\partial_t \varphi|^2 d\sigma$$

as an  $L^2(\mathbb{S}^{N-1})$  version of the Poincaré inequality. Combine the above two estimates with the right-hand side of (19), and we obtain

$$\iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\nabla_\sigma \mathbf{v}|^2 dt d\sigma \geq (N-1) \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( |\mathbf{v}|^2 + ((\lambda-2)^2 - N-1)\varphi^2 + (\partial_t \varphi)^2 \right) dt d\sigma$$

to evaluate the last integral in (18); hence we get

$$\begin{aligned}
 (20) \quad & \int_{\mathbb{R}^N} |\nabla \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx \\
 & \geq ((\lambda-1)^2 + N-1) \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\mathbf{v}|^2 dt d\sigma + \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\partial_t \mathbf{v}|^2 dt d\sigma \\
 & \quad + (N-1) \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( ((\lambda-2)^2 - N-1)\varphi^2 + (\partial_t \varphi)^2 \right) dt d\sigma.
 \end{aligned}$$

Here the equality holds if and only if  $-\Delta_\sigma \varphi = \alpha_1 \varphi$ ; note that this equation also produces for  $\partial_t \varphi$  the same equation  $-\Delta_\sigma (\partial_t \varphi) = \alpha_1 (\partial_t \varphi)$  since  $\partial_t$  and  $\Delta_\sigma$  commutes.

To further proceed, we have the following two cases according to the sign of  $(\lambda-2)^2 - N-1$ :

**3.3. The case  $|\lambda - 2| \geq \sqrt{N+1}$ .** Discarding the last two integrals in (20) and recalling the first equation of (17), we get the Hardy–Leray inequality

$$\int_{\mathbb{R}^N} |\nabla \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx \geq H_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\mathbf{u}|^2}{|\mathbf{x}|^2} |\mathbf{x}|^{2\gamma} dx$$

for curl-free fields  $\mathbf{u}$ , with the constant

$$H_{N,\gamma} = (\lambda - 1)^2 + N - 1 = \left( \gamma + \frac{N}{2} - 1 \right)^2 + N - 1.$$

To show that this number is the best possible, let us choose the sequence of curl-free fields  $\{\mathbf{u}_n = r^{\lambda-1} \mathbf{v}_n\}_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^N \setminus \{0\})^N$  by the formula

$$\mathbf{v}_n = \sigma h\left(\frac{t}{n}\right) \quad \left( \text{or equivalently } \mathbf{u}_n(\mathbf{x}) = \mathbf{x} |\mathbf{x}|^{\lambda-2} h\left(\log |\mathbf{x}|^{\frac{1}{n}}\right) \right)$$

for all  $(t, \sigma) \in \mathbb{R} \times \mathbb{S}^{N-1}$ , where  $h \in C_c^\infty(\mathbb{R})$  such that  $h \not\equiv 0$ . Then, noticing that the triplet  $(\mathbf{u}, \mathbf{v}, \varphi) = (\mathbf{u}_n, \mathbf{v}_n, 0)$  attains the equality of the inequality (20), we get

$$\begin{aligned} \frac{\int_{\mathbb{R}^N} |\nabla \mathbf{u}_n|^2 |\mathbf{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\mathbf{u}_n|^2 |\mathbf{x}|^{2\gamma-2} dx} &= H_{N,\gamma} + \frac{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} |\partial_t \mathbf{v}_n|^2 dt d\sigma}{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} |\mathbf{v}_n|^2 dt d\sigma} = H_{N,\gamma} + \frac{1}{n^2} \frac{\int_{\mathbb{R}} (h'(t))^2 dt}{\int_{\mathbb{R}} (h(t))^2 dt} \\ &\longrightarrow H_{N,\gamma} \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which proves the best possibility of  $H_{N,\gamma}$ .

**3.4. The case  $|\lambda - 2| < \sqrt{N+1}$ .** By using the  $L^2(\mathbb{S}^{N-1})$ -Poincaré inequality and equation (17), we have

$$\begin{aligned} \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \varphi^2 dt d\sigma &\leq \frac{1}{\lambda^2 + \alpha_1} \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} (\lambda^2 \varphi^2 + |\nabla_\sigma \varphi|^2) dt d\sigma \\ &= \frac{1}{\lambda^2 + N - 1} \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} (|\mathbf{v}|^2 - (f^2 + (\partial_t \varphi)^2)) dt d\sigma, \end{aligned}$$

where the first equality holds if and only if  $-\Delta_\sigma \varphi = \alpha_1 \varphi$ . Combining this estimate with (20) and noting that  $(\lambda - 2)^2 - N - 1 < 0$ , we get

(21)

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx \\ &\geq \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( ((\lambda - 1)^2 + N - 1) |\mathbf{v}|^2 + |\partial_t \mathbf{v}|^2 + (N - 1)(\partial_t \varphi)^2 \right) dt d\sigma \\ &\quad - \frac{(N - 1)(N + 1 - (\lambda - 2)^2)}{\lambda^2 + N - 1} \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} (|\mathbf{v}|^2 - (f^2 + (\partial_t \varphi)^2)) dt d\sigma \\ &= \frac{(\lambda - 1)^2 (\lambda^2 + 3(N - 1))}{\lambda^2 + N - 1} \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\mathbf{v}|^2 dt d\sigma + \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\partial_t \mathbf{v}|^2 dt d\sigma \\ &\quad + (N - 1) \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( \frac{N + 1 - (\lambda - 2)^2}{\lambda^2 + N - 1} f^2 + \frac{4(\lambda + \frac{N}{2} - 1)}{\lambda^2 + N - 1} (\partial_t \varphi)^2 \right) dt d\sigma, \end{aligned}$$

where the first equality holds if and only if  $-\Delta_\sigma \varphi = \alpha_1 \varphi$ . In the same way as before, discard the last two integrals in (21) and recall the first equation of (17); then the Hardy–Leray inequality for curl-free fields

$$\int_{\mathbb{R}^N} |\nabla \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx \geq H_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\mathbf{u}|^2}{|\mathbf{x}|^2} |\mathbf{x}|^{2\gamma} dx$$

holds with the constant  $H_{N,\gamma}$  given by

$$H_{N,\gamma} = \frac{(\lambda - 1)^2(\lambda^2 + 3(N - 1))}{\lambda^2 + N - 1} = \left(\gamma + \frac{N}{2} - 1\right)^2 \frac{(\gamma + \frac{N}{2} - 2)^2 + 3(N - 1)}{(\gamma + \frac{N}{2} - 2)^2 + N - 1}.$$

To show that this  $H_{N,\gamma}$  is sharp, let us choose the sequence of curl-free fields  $\{\mathbf{u}_n = r^{\lambda-1} \mathbf{v}_n\}_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})^N$  by the formulae

$$\begin{cases} \mathbf{v}_n = \boldsymbol{\sigma}(\partial_t + \lambda)\varphi_n + \nabla_\sigma \varphi_n & \left(\text{or equivalently } \mathbf{u}_n(\mathbf{x}) = \nabla(|\mathbf{x}|^\lambda \varphi_n(\mathbf{x}))\right) \\ \varphi_n = h\left(\frac{t}{n}\right) Y_1(\boldsymbol{\sigma}), \end{cases}$$

for all  $(t, \boldsymbol{\sigma}) \in \mathbb{R} \times \mathbb{S}^{N-1}$ , where  $h \in C_c^\infty(\mathbb{R}) \setminus \{0\}$  and where  $Y_1 \in C^\infty(\mathbb{S}^{N-1})$  denotes the eigenfunction of  $-\Delta_\sigma$  associated with the eigenvalue  $\alpha_1 = N - 1$ . Then a straightforward calculation yields

$$\begin{aligned} \frac{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} (\partial_t \varphi_n)^2 dt d\sigma}{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} |\mathbf{v}_n|^2 dt d\sigma} &= \frac{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} (\partial_t \varphi_n)^2 dt d\sigma}{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} ((\lambda^2 + \alpha_1)\varphi_n^2 + (\partial_t \varphi_n)^2) dt d\sigma} \\ &= \frac{\int_{\mathbb{R}} n^{-2}(h'(t))^2 dt}{\int_{\mathbb{R}} ((\lambda^2 + \alpha_1)(h(t))^2 + n^{-2}(h'(t))^2) dt} \xrightarrow{(n \rightarrow \infty)} 0, \\ \frac{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} |\partial_t \mathbf{v}_n|^2 dt d\sigma}{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} |\mathbf{v}_n|^2 dt d\sigma} &= \frac{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} ((\lambda^2 + \alpha_1)(\partial_t \varphi_n)^2 + (\partial_t^2 \varphi_n)^2) dt d\sigma}{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} ((\lambda^2 + \alpha_1)\varphi_n^2 + (\partial_t \varphi_n)^2) dt d\sigma} \\ &= \frac{\int_{\mathbb{R}} (n^{-2}(\lambda^2 + \alpha_1)(h'(t))^2 + n^{-4}(h''(t))^2) dt}{\int_{\mathbb{R}} ((\lambda^2 + \alpha_1)(h(t))^2 + n^{-2}(h'(t))^2) dt} \xrightarrow{(n \rightarrow \infty)} 0. \end{aligned}$$

Since the quadruple  $(\mathbf{u}, \mathbf{v}, \varphi, f) = (\mathbf{u}_n, \mathbf{v}_n, \varphi_n, 0)$  attains the equality in (21), the above calculation directly gives

$$\begin{aligned} &\frac{\int_{\mathbb{R}^N} |\nabla \mathbf{u}_n|^2 |\mathbf{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\mathbf{u}_n|^2 |\mathbf{x}|^{2\gamma-2} dx} \\ &= H_{N,\gamma} + \frac{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} (|\partial_t \mathbf{v}_n|^2 dt d\sigma + \frac{4(N-1)(\lambda + \frac{N}{2} - 1)}{\lambda^2 + N - 1} (\partial_t \varphi_n)^2) dt d\sigma}{\int_{\mathbb{R} \times \mathbb{S}^{N-1}} |\mathbf{v}_n|^2 dt d\sigma} \\ &\longrightarrow H_{N,\gamma} \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which proves the sharpness of  $H_{N,\gamma}$ .

**3.5. Conclusion of the proof of Theorem 1.** In view of the inequalities (20) and (21), we have already proved in §3.3 and §3.4 that every curl-free field  $\mathbf{u} \in C_c^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})^N$  satisfies the inequality

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx &\geq H_{N,\gamma} \int_{\mathbb{R}^N} |\mathbf{u}|^2 |\mathbf{x}|^{2\gamma-2} dx \\ &\quad + \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\partial_t \mathbf{v}|^2 dt d\sigma + (N - 1) \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \mathcal{E}_{N,\gamma}[\mathbf{u}] dt d\sigma \end{aligned}$$

with the constant  $H_{N,\gamma}$  in Theorem B and the remainder function  $\mathcal{E}_{N,\gamma}[\mathbf{u}]$  given by

$$\mathcal{E}_{N,\gamma}[\mathbf{u}](\mathbf{x}) = \begin{cases} ((\lambda - 2)^2 - N - 1)\varphi^2 + (\partial_t \varphi)^2 & \text{for } |\lambda - 2| \geq \sqrt{N + 1}, \\ \frac{N + 1 - (\lambda - 2)^2}{\lambda^2 + N - 1} \varphi^2 + \frac{4(\lambda + \frac{N}{2} - 1)}{\lambda^2 + N - 1} (\partial_t \varphi)^2 & \text{for } |\lambda - 2| < \sqrt{N + 1}. \end{cases}$$

Moreover, the equality in the above integral inequality holds if and only if  $-\Delta_\sigma \varphi = \alpha_1 \varphi$ . Finally, restoring the notations

$$\lambda = 2 - \frac{N}{2} - \gamma, \quad \partial_t = \mathbf{x} \cdot \nabla, \quad dt d\sigma = |\mathbf{x}|^{-N} dx, \quad \mathbf{v} = |\mathbf{x}|^{\gamma + \frac{N}{2} - 1} \mathbf{u},$$

we complete the proof.  $\square$

#### 4. A PROOF OF THE SHARP RELICH-LERAY INEQUALITY FOR CURL-FREE FIELDS

The same approach to prove Theorem 1 can also be applied to treat other inequalities involving higher-order derivatives. The following sharp Rellich-Leray inequality for curl-free fields was first proven in [7].

**Theorem C.** ([7]) *Let  $\mathbf{u} \in C_c^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})^N$  be a curl-free vector field. Then the inequality*

$$(22) \quad R_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\mathbf{u}|^2}{|\mathbf{x}|^4} |\mathbf{x}|^{2\gamma} dx \leq \int_{\mathbb{R}^N} |\Delta \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx$$

holds with the best constant  $R_{N,\gamma}$  given by

$$(23) \quad R_{N,\gamma} = \min \left\{ \left( \alpha_{\gamma - \frac{N}{2}} - N + 1 \right)^2, \min_{\nu \in \mathbb{N}} \frac{\left( \gamma + \frac{N}{2} - 1 \right)^2 + \alpha_\nu}{\left( \gamma + \frac{N}{2} - 3 \right)^2 + \alpha_\nu} \left( \alpha_{\gamma - \frac{N}{2} - 1} - \alpha_\nu \right)^2 \right\}$$

in terms of the same notation  $\alpha_s = s(s + N - 2)$  as in (5).

In this section, we prove the following improvement of Theorem C.

**Theorem 6.** *Let  $R_{N,\gamma}$  be the same as in (23). Then the inequality (22) can be further improved to be*

$$\int_{\mathbb{R}^N} |\Delta \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx \geq R_{N,\gamma} \int_{\mathbb{R}^N} \frac{|\mathbf{u}|^2}{|\mathbf{x}|^4} |\mathbf{x}|^{2\gamma} dx + c_{N,\gamma} \int_{\mathbb{R}^N} |\mathbf{x} \cdot \nabla (|\mathbf{x}|^{\gamma + \frac{N}{2} - 2} \mathbf{u})|^2 |\mathbf{x}|^{-N} dx$$

for some positive constant  $c_{N,\gamma} > 0$ .

As a direct consequence of this fact, the equality sign of inequality (22) is never attained by any non-zero curl-free field  $\mathbf{u}$ .

*Proof.* Let  $\mathbf{u} \in C_c^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})^N$  be a curl-free field. Applying the replacement

$$(24) \quad \gamma \longmapsto \gamma - 1$$

to equation (15), we choose

$$\lambda = 3 - N/2 - \gamma.$$

By this choice, let us calculate the integrals in inequality (22): Apply the replacement (24) to the equations in (17), and we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|\mathbf{u}|^2}{|\mathbf{x}|^4} |\mathbf{x}|^{2\gamma} dx &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\mathbf{v}|^2 dt d\sigma \\ &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( f^2 + (\partial_t \varphi)^2 + \lambda^2 \varphi^2 + |\nabla_\sigma \varphi|^2 \right) dt d\sigma \\ &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( f^2 + \varphi (\lambda^2 - \partial_t^2 - \Delta_\sigma) \varphi \right) dt d\sigma, \end{aligned}$$

where the last equality follows from integration by parts together with the support compactness. Also we notice that the condition (12) in Proposition 4 is invariant under the following replacement of the triplet:

$$(25) \quad (\mathbf{v}, f, \varphi) \longmapsto (\partial_t^k \mathbf{v}, \partial_t^k f, \partial_t^k \varphi)$$

for  $k = 1, 2$ . Hence we have

$$(26) \quad \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\partial_t^k \mathbf{v}|^2 dt d\sigma = \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( (\partial_t^k f)^2 + \varphi (\lambda^2 - \partial_t^2 - \Delta_\sigma) (-\partial_t^2)^k \varphi \right) dt d\sigma.$$

On the other hand, with the aid of (11), we have

$$\begin{aligned}\Delta \mathbf{u} &= \Delta(r^{\lambda-1} \mathbf{v}) = (\Delta r^{\lambda-1}) \mathbf{v} + 2((\nabla r^{\lambda-1}) \cdot \nabla) \mathbf{v} + r^{\lambda-1} \Delta \mathbf{v} \\ &= \alpha_{\lambda-1} r^{\lambda-3} \mathbf{v} + 2(\lambda-1) r^{\lambda-3} \partial_t \mathbf{v} + r^{\lambda-3} (\partial_t^2 + (N-2) \partial_t + \Delta_\sigma) \mathbf{v} \\ &= r^{\lambda-3} (\alpha_{\lambda-1} \mathbf{v} + (2\lambda + N - 4) \partial_t \mathbf{v} + \partial_t^2 \mathbf{v} + \Delta_\sigma \mathbf{v}),\end{aligned}$$

where in the second line we have used the same formula  $\Delta r^{\lambda-1} = \alpha_{\lambda-1} r^{\lambda-3}$  as in (5)<sub>s=λ-1</sub>. Then the  $L^2$  integration (by parts) of this result yields

$$\begin{aligned}(27) \quad \int_{\mathbb{R}^N} |\Delta \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\alpha_{\lambda-1} \mathbf{v} + (2\lambda + N - 4) \partial_t \mathbf{v} + \partial_t^2 \mathbf{v} + \Delta_\sigma \mathbf{v}|^2 dt d\sigma \\ &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\partial_t^2 \mathbf{v}|^2 dt d\sigma + ((N-2)^2 + 2\alpha_{\lambda-1}) \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\partial_t \mathbf{v}|^2 dt d\sigma \\ &\quad + 2 \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\nabla_\sigma \partial_t \mathbf{v}|^2 dt d\sigma + \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\Delta_\sigma \mathbf{v} + \alpha_{\lambda-1} \mathbf{v}|^2 dt d\sigma,\end{aligned}$$

where the second equality follows from the identity  $(2\lambda + N - 4)^2 - 2\alpha_{\lambda-1} = (N-2)^2 + 2\alpha_{\lambda-1}$ . To calculate the second last integral in (27), we apply the replacement (25) to the equation in (19):

$$\begin{aligned}(28) \quad \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\partial_t \nabla_\sigma \mathbf{v}|^2 dt d\sigma &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( (\Delta_\sigma \partial_t \varphi)^2 + |\nabla_\sigma \partial_t^2 \varphi|^2 + ((\lambda-2)^2 - 2N) |\nabla_\sigma \partial_t \varphi|^2 \right. \\ &\quad \left. + (N-1) |\partial_t \mathbf{v}|^2 \right) dt d\sigma \\ &= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( \varphi \left( -\Delta_\sigma^2 \partial_t^2 - \Delta_\sigma \partial_t^4 + ((\lambda-2)^2 - 2N) \Delta_\sigma \partial_t^2 \right) \varphi \right. \\ &\quad \left. + (N-1) \left( (\partial_t f)^2 + \varphi (\lambda^2 - \partial_t^2 - \Delta_\sigma) (-\partial_t^2 \varphi) \right) \right) dt d\sigma.\end{aligned}$$

Also to calculate the last integral in (27), let us compute from (13) and (12) that

$$\begin{aligned}\Delta_\sigma \mathbf{v} + \alpha_{\lambda-1} \mathbf{v} &= \boldsymbol{\sigma} (\partial_t + \lambda - 2) \Delta_\sigma \varphi + \nabla_\sigma (2\partial_t + \Delta_\sigma + 2(\lambda + N - 2)) \varphi \\ &\quad + (\alpha_{\lambda-1} - (N-1)) (\boldsymbol{\sigma} (f + (\partial_t + \lambda) \varphi) + \nabla_\sigma \varphi) \\ &= \boldsymbol{\sigma} \left( (\Delta_\sigma + \alpha_{\lambda-1} - N + 1) \partial_t \varphi + (\lambda - 2) \Delta_\sigma \varphi + (\alpha_{\lambda-1} - N + 1) (f + \lambda \varphi) \right) \\ &\quad + \nabla_\sigma (2\partial_t + \Delta_\sigma + \alpha_\lambda) \varphi,\end{aligned}$$

here we have used  $\alpha_{\lambda-1} - (N-1) + 2(\lambda + N - 2) = \alpha_\lambda$  in the second equality. Hence the  $L^2$  integration by parts of this result yields

$$\begin{aligned}
(29) \quad & \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\triangle_\sigma \mathbf{v} + \alpha_{\lambda-1} \mathbf{v}|^2 dt d\sigma \\
&= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left| \begin{aligned} & (\triangle_\sigma + \alpha_{\lambda-1} - N + 1) \partial_t \varphi \\ & + (\lambda - 2) \triangle_\sigma \varphi + (\alpha_{\lambda-1} - N + 1)(f + \lambda \varphi) \end{aligned} \right|^2 dt d\sigma \\
&+ \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} |\nabla_\sigma (2\partial_t + \triangle_\sigma + \alpha_\lambda) \varphi|^2 dt d\sigma \\
&= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( \begin{aligned} & ((\triangle_\sigma + \alpha_{\lambda-1} - N + 1) \partial_t \varphi)^2 + 4 |\partial_t \nabla_\sigma \varphi|^2 \\ & + |(\lambda - 2) \triangle_\sigma \varphi + (\alpha_{\lambda-1} - N + 1) \lambda \varphi|^2 \\ & + |\nabla_\sigma \triangle_\sigma \varphi + \alpha_\lambda \nabla_\sigma \varphi|^2 + (\alpha_{\lambda-1} - N + 1)^2 f^2 \end{aligned} \right) dt d\sigma \\
&= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( \begin{aligned} & \varphi \left( (\triangle_\sigma + \alpha_{\lambda-1} - N + 1)^2 - 4 \triangle_\sigma \right) (-\partial_t^2 \varphi) \\ & + \varphi \left( (\lambda - 2)^2 - \triangle_\sigma \right) (\triangle_\sigma + \alpha_\lambda)^2 \varphi \\ & + (\alpha_{\lambda-1} - N + 1)^2 f^2 \end{aligned} \right) dt d\sigma.
\end{aligned}$$

Substitute (26), (28) and (29) into (27), and after some lengthy algebraic calculations, we obtain

$$\begin{aligned}
(30) \quad & \int_{\mathbb{R}^N} |\triangle \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx = (26)_{k=2} + ((N-2)^2 + 2\alpha_{\lambda-1}) (26)_{k=1} + 2 \times (28) + (29) \\
&= \iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( \varphi Q_1(-\partial_t^2, -\triangle_\sigma) \varphi + f Q_0(-\partial_t^2) f \right) dt d\sigma,
\end{aligned}$$

where  $Q_1(\cdot, \cdot)$  and  $Q_0(\cdot)$  are the polynomials given by

$$\begin{aligned}
(31) \quad & Q_1(\tau, \alpha) = (\lambda^2 + \tau + \alpha) \tau^2 + ((N-2)^2 + 2\alpha_{\lambda-1}) (\lambda^2 + \tau + \alpha) \tau \\
&+ 2 \left( \alpha^2 \tau + \alpha \tau^2 + ((\lambda-2)^2 - 2N) \alpha \tau + (N-1)(\lambda^2 + \tau + \alpha) \tau \right) \\
&+ ((-\alpha + \alpha_{\lambda-1} - N + 1)^2 + 4\alpha) \tau + ((\lambda-2)^2 + \alpha) (-\alpha + \alpha_\lambda)^2 \\
&= (\tau + \alpha + (\lambda-2)^2) \left( \begin{aligned} & (\tau + \alpha + \lambda^2) (\tau + \alpha + (\lambda + N - 2)^2) \\ & - (2\lambda + N - 2)^2 \alpha \end{aligned} \right),
\end{aligned}$$

$$\begin{aligned}
(32) \quad & Q_0(\tau) = \tau^2 + ((N-2)^2 + 2\alpha_{\lambda-1}) \tau + 2(N-1)\tau + (\alpha_{\lambda-1} - N + 1)^2 \\
&= (\tau + (\lambda-2)^2) (\tau + (\lambda + N - 2)^2).
\end{aligned}$$

Therefore, we get

$$(33) \quad \frac{\int_{\mathbb{R}^N} |\triangle \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\mathbf{u}|^2 |\mathbf{x}|^{2\gamma-4} dx} = \frac{\iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( \varphi Q_1(-\partial_t^2, -\triangle_\sigma) \varphi + f Q_0(-\partial_t^2) f \right) dt d\sigma}{\iint_{\mathbb{R} \times \mathbb{S}^{N-1}} \left( \varphi (\lambda^2 - \partial_t^2 - \triangle_\sigma) \varphi + f^2 \right) dt d\sigma}$$

as far as  $\mathbf{u} \neq \mathbf{0}$ .

From now on, we evaluate the right-hand side of (33). We apply to  $\varphi$  and  $f$  the 1-D Fourier transformation with respect to  $t$ : we set

$$\widehat{\varphi}(\tau, \boldsymbol{\sigma}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau t} \varphi(e^t \boldsymbol{\sigma}) dt, \quad \widehat{f}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau t} f(e^t \boldsymbol{\sigma}) dt$$

for  $(\tau, \boldsymbol{\sigma}) \in \mathbb{R} \times \mathbb{S}^{N-1}$ . Furthermore, we apply to  $\widehat{\varphi}$  the spherical harmonics expansion:

$$\widehat{\varphi}(\tau, \boldsymbol{\sigma}) = \sum_{\nu \in \mathbb{N}} \widehat{\varphi}_\nu(\tau) Y_\nu(\boldsymbol{\sigma}), \quad \begin{cases} -\Delta_\sigma Y_\nu = \alpha_\nu Y_\nu, \\ \alpha_\nu = \nu(\nu + N - 2) \quad \forall \nu \in \mathbb{N}, \end{cases}$$

with the normalization  $\int_{\mathbb{S}^{N-1}} |Y_\nu(\boldsymbol{\sigma})|^2 d\boldsymbol{\sigma} = 1$ . Substituting these formulae into (33) and noticing the  $L^2(\mathbb{R})$  isometry of the Fourier transformation, we have

$$(34) \quad \frac{\int_{\mathbb{R}^N} |\Delta \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\mathbf{u}|^2 |\mathbf{x}|^{2\gamma-4} dx} = \frac{\sum_{\nu \in \mathbb{N}} \int_{\mathbb{R}} Q_1(\tau^2, \alpha_\nu) |\widehat{\varphi}_\nu(\tau)|^2 d\tau + \omega_{N-1} \int_{\mathbb{R}} Q_0(\tau^2) |\widehat{f}(\tau)|^2 d\tau}{\sum_{\nu \in \mathbb{N}} \int_{\mathbb{R}} (\lambda^2 + \tau^2 + \alpha_\nu) |\widehat{\varphi}_\nu(\tau)|^2 d\tau + \omega_{N-1} \int_{\mathbb{R}} |\widehat{f}(\tau)|^2 d\tau} \\ \geq \min \left\{ \inf_{\nu \in \mathbb{N}} \inf_{\tau \in \mathbb{R}} \frac{Q_1(\tau^2, \alpha_\nu)}{\lambda^2 + \tau^2 + \alpha_\nu}, \inf_{\tau \in \mathbb{R}} Q_0(\tau^2) \right\} \\ = \min \left\{ \min_{\nu \in \mathbb{N}} \frac{Q_1(0, \alpha_\nu)}{\lambda^2 + \alpha_\nu}, Q_0(0) \right\}.$$

Here the last equality follows from that in view of (31) and (32) the functions

$$\frac{Q_1(\tau, \alpha_\nu)}{\lambda^2 + \tau + \alpha_\nu} = (\tau + \alpha_\nu + (\lambda - 2)^2) \left( \tau + \alpha_\nu \left( 1 - \frac{(2\lambda + N - 2)^2}{\lambda^2 + \tau + \alpha_\nu} \right) + (\lambda + N - 2)^2 \right)$$

and  $Q_0(\tau)$  are monotonically increasing in  $\tau \in [0, \infty)$  for each  $\nu \in \mathbb{N}$ . Therefore, we have proved the Rellich-Leray inequality for curl-free fields (22):

$$\int_{\mathbb{R}^N} |\Delta \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx \geq R_{N,\gamma} \int_{\mathbb{R}^N} |\mathbf{u}|^2 |\mathbf{x}|^{2\gamma-4} dx$$

holds with the constant  $R_{N,\gamma}$  given by

$$R_{N,\gamma} = \min \left\{ \min_{\nu \in \mathbb{N}} \frac{Q_1(0, \alpha_\nu)}{\lambda^2 + \alpha_\nu}, Q_0(0) \right\} \\ = \min \left\{ \min_{\nu \in \mathbb{N}} \frac{(\lambda - 2)^2 + \alpha_\nu}{\lambda^2 + \alpha_\nu} (\alpha_\nu - \alpha_{\lambda-1})^2, (\alpha_{\lambda-1} - N + 1)^2 \right\} \\ = \min \left\{ \min_{\nu \in \mathbb{N}} \frac{(\gamma + \frac{N}{2} - 1)^2 + \alpha_\nu}{(\gamma + \frac{N}{2} - 3)^2 + \alpha_\nu} (\alpha_{\gamma - \frac{N}{2} - 1} - \alpha_\nu)^2, (\alpha_{\gamma - \frac{N}{2}} - N + 1)^2 \right\}.$$

Now, we prove the sharpness of  $R_{N,\gamma}$ . For this purpose, we choose  $\nu_0 \in \mathbb{N} \cup \{0\}$  to be such that

$$\begin{cases} \nu_0 = 0, & \text{if } \min \left\{ \min_{\nu \in \mathbb{N}} \frac{Q_1(0, \alpha_\nu)}{\lambda^2 + \alpha_\nu}, Q_0(0) \right\} = Q_0(0), \\ \min_{\nu \in \mathbb{N}} \frac{Q_1(0, \alpha_\nu)}{\lambda^2 + \alpha_\nu} = \frac{Q_1(0, \alpha_{\nu_0})}{\lambda^2 + \alpha_{\nu_0}}, & \text{otherwise,} \end{cases}$$

and define the sequence of vector fields  $\{\mathbf{u}_n = r^{\lambda-1} \mathbf{v}_n\}_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})^N$  by the formulae

$$\mathbf{v}_n = \begin{cases} \boldsymbol{\sigma} f_n & \text{if } \nu_0 = 0, \\ \boldsymbol{\sigma}(\partial_t + \lambda)\varphi_n + \nabla_\sigma \varphi_n & \text{otherwise.} \end{cases}$$

Here

$$\begin{cases} f_n(\mathbf{x}) = h\left(\frac{\log |\mathbf{x}|}{n}\right) & \text{if } \nu_0 = 0, \\ \varphi_n(\mathbf{x}) = h\left(\frac{\log |\mathbf{x}|}{n}\right) Y_{\nu_0}(\mathbf{x}/|\mathbf{x}|) & \text{otherwise,} \end{cases}$$

with  $h \in C_c^\infty(\mathbb{R}) \setminus \{0\}$ , and  $Y_{\nu_0} \in C^\infty(\mathbb{S}^{N-1})$  denotes the eigenfunction of  $-\Delta_\sigma$  associated with the eigenvalue  $\alpha_{\nu_0} = \nu_0(\nu_0 + N - 2)$ . Notice from Proposition 4 that  $\mathbf{u}_n$  is curl-free. Then applying the formula (33) to  $(\mathbf{u}, f, \varphi) = (\mathbf{u}_n, f_n, 0)$  or  $(\mathbf{u}, f, \varphi) = (\mathbf{u}_n, 0, \varphi_n)$  gives

$$\frac{\int_{\mathbb{R}^N} |\Delta \mathbf{u}_n|^2 |\mathbf{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\mathbf{u}_n|^2 |\mathbf{x}|^{2\gamma-4} dx} = \begin{cases} \frac{\int_{\mathbb{R}} h(\frac{t}{n}) Q_0(-\partial_t^2) h(\frac{t}{n}) dt}{\int_{\mathbb{R}} (h(\frac{t}{n}))^2 dt} & \text{if } \nu_0 = 0, \\ \frac{\int_{\mathbb{R}} h(\frac{t}{n}) Q_1(-\partial_t^2, \alpha_{\nu_0}) h(\frac{t}{n}) dt}{\int_{\mathbb{R}} h(\frac{t}{n}) (\lambda^2 - \partial_t^2 + \alpha_{\nu_0}) h(\frac{t}{n}) dt} & \text{otherwise.} \end{cases}$$

Passing to  $n \rightarrow \infty$ , we get

$$\begin{aligned} \frac{\int_{\mathbb{R}^N} |\Delta \mathbf{u}_n|^2 |\mathbf{x}|^{2\gamma} dx}{\int_{\mathbb{R}^N} |\mathbf{u}_n|^2 |\mathbf{x}|^{2\gamma-4} dx} &= O(1/n^2) + \begin{cases} Q_0(0) & \text{if } \nu_0 = 0 \\ \frac{Q_1(0, \alpha_{\nu_0})}{\lambda^2 + \alpha_{\nu_0}} & \text{otherwise} \end{cases} \\ &\longrightarrow R_{N,\gamma}, \end{aligned}$$

which shows the desired sharpness of  $R_{N,\gamma}$ .

In order to obtain further improvement, we recall that the two integrals in (22) can be expressed in terms of  $\varphi$  and  $f$  (in Proposition 4) as

$$\begin{aligned} \int_{\mathbb{R}^N} |\Delta \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx &= \sum_{\nu \in \mathbb{N}} \int_{\mathbb{R}} Q_1(\tau^2, \alpha_\nu) |\widehat{\varphi}_\nu(\tau)|^2 d\tau + \omega_{N-1} \int_{\mathbb{R}} Q_0(\tau^2) |\widehat{f}(\tau)|^2 d\tau, \\ \int_{\mathbb{R}^N} |\mathbf{u}|^2 |\mathbf{x}|^{2\gamma-4} dx &= \sum_{\nu \in \mathbb{N}} \int_{\mathbb{R}} (\lambda^2 + \tau^2 + \alpha_\nu) |\widehat{\varphi}_\nu(\tau)|^2 d\tau + \omega_{N-1} \int_{\mathbb{R}} |\widehat{f}(\tau)|^2 d\tau \end{aligned}$$

for  $\lambda = 3 - N/2 - \gamma$ , together with the polynomials  $Q_1$  and  $Q_0$  given by (31) and (32). Also recall the expression  $R_{N,\gamma} = \min \left\{ \min_{\nu \in \mathbb{N}} \frac{Q_1(0, \alpha_\nu)}{\lambda^2 + \alpha_\nu}, Q_0(0) \right\}$  of the best constant of the inequality (22) and let  $\nu_1 \in \mathbb{N}$  be such that

$$\frac{Q_1(0, \alpha_{\nu_1})}{\lambda^2 + \alpha_{\nu_1}} = \min_{\nu \in \mathbb{N}} \frac{Q_1(0, \alpha_\nu)}{\lambda^2 + \alpha_\nu}.$$

Then the difference between the both sides of (22) has the following estimate:

$$\begin{aligned} &\int_{\mathbb{R}^N} |\Delta \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx - \min \left\{ \frac{Q_1(0, \alpha_{\nu_1})}{\lambda^2 + \alpha_{\nu_1}}, Q_0(0) \right\} \int_{\mathbb{R}^N} |\mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx \\ &\geq \int_{\mathbb{R}^N} |\Delta \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx \\ &\quad - \frac{Q_1(0, \alpha_{\nu_1})}{\lambda^2 + \alpha_{\nu_1}} \sum_{\nu \in \mathbb{N}} \int_{\mathbb{R}} (\tau^2 + \lambda^2 + \alpha_\nu) |\widehat{\varphi}_\nu(\tau)|^2 d\tau - Q_0(0) \omega_{N-1} \int_{\mathbb{R}} |\widehat{f}(\tau)|^2 d\tau \\ &= \sum_{\nu \in \mathbb{N}} \int_{\mathbb{R}} \left( Q_1(\tau^2, \alpha_\nu) - \frac{Q_1(0, \alpha_{\nu_1})}{\lambda^2 + \alpha_{\nu_1}} (\tau^2 + \lambda^2 + \alpha_\nu) \right) |\widehat{\varphi}_\nu(\tau)|^2 d\tau \\ &\quad + \omega_{N-1} \int_{\mathbb{R}} (Q_0(\tau^2) - Q_0(0)) |\widehat{f}(\tau)|^2 d\tau \\ &\geq c_{1,N,\gamma} \sum_{\nu \in \mathbb{N}} \int_{\mathbb{R}} (\lambda^2 + \tau^2 + \alpha_\nu) \tau^2 |\widehat{\varphi}_\nu(\tau)|^2 d\tau + c_{0,N,\gamma} \omega_{N-1} \int_{\mathbb{R}} \tau^2 |\widehat{f}(\tau)|^2 d\tau \\ &\geq \min \{c_{1,N,\gamma}, c_{0,N,\gamma}\} \int_{\mathbb{R}} \left( \sum_{\nu \in \mathbb{N}} (\lambda^2 + \tau^2 + \alpha_\nu) \tau^2 |\widehat{\varphi}_\nu(\tau)|^2 + \omega_{N-1} \tau^2 |\widehat{f}(\tau)|^2 \right) d\tau \\ &= \min \{c_{1,N,\gamma}, c_{0,N,\gamma}\} \int_{\mathbb{R}^N} |\partial_t \mathbf{v}|^2 r^{-N} dx, \end{aligned}$$

where we have defined the two constants  $c_{0,N,\gamma}$  and  $c_{1,N,\gamma}$  by

$$\begin{aligned} c_{0,N,\gamma} &= \inf_{\tau \in \mathbb{R} \setminus \{0\}} \frac{Q_0(\tau^2) - Q_0(0)}{\tau^2} = Q'_0(0) = (\lambda - 2)^2 + (\lambda + N - 2)^2 \\ &= N^2/2 + 2(1 - \gamma)^2 > 0, \\ c_{1,N,\gamma} &= \inf_{\nu \in \mathbb{N}} \inf_{\tau \in \mathbb{R} \setminus \{0\}} \frac{1}{\tau^2} \left( \frac{Q_1(\tau^2, \alpha_\nu)}{\tau^2 + \lambda^2 + \alpha_\nu} - \frac{Q_1(0, \alpha_{\nu_1})}{\lambda^2 + \alpha_{\nu_1}} \right). \end{aligned}$$

Hence it suffices to show  $c_{1,N,\gamma} > 0$ . To this end, notice that

$$c_{1,N,\gamma} \geq \inf_{\nu \in \mathbb{N}} \inf_{\tau \in \mathbb{R} \setminus \{0\}} \frac{1}{\tau^2} \left( \frac{Q_1(\tau^2, \alpha_\nu)}{\tau^2 + \lambda^2 + \alpha_\nu} - \frac{Q_1(0, \alpha_\nu)}{\lambda^2 + \alpha_\nu} \right) = \inf_{\nu \in \mathbb{N}} \inf_{\tau \geq 0} Q_2(\tau, \alpha_\nu)$$

for the rational polynomial  $Q_2(\cdot, \cdot)$  defined by the following algebraic calculation:

$$\begin{aligned} Q_2(\tau, a) &= \frac{1}{\tau} \left( \frac{Q_1(\tau, a)}{\tau + \lambda^2 + a} - \frac{Q_1(0, a)}{\lambda^2 + a} \right) \\ &= \frac{4(1 - \lambda)(2\lambda + N - 2)^2 a}{(\lambda^2 + a)(\tau + \lambda^2 + a)} + 2 \left( \lambda + \frac{N}{2} - 2 \right)^2 + \frac{N^2}{2} + 2a + \tau \\ &= \frac{16(1 - \lambda)(2 - \gamma)^2 a}{(\lambda^2 + a)(\tau + \lambda^2 + a)} + c_{0,N,\gamma} + 2a + \tau. \end{aligned}$$

In order to further estimate  $Q_2(\tau, \alpha_\nu)$  for  $\tau \geq 0$  and  $\nu \in \mathbb{N}$ , let us consider the following two cases: for  $\lambda \leq 1$ , it is clear that  $Q_2(\tau, \alpha_\nu) \geq c_{0,N,\gamma} + 2\alpha_\nu$ . For  $\lambda > 1$ , since it is clear that  $Q_2(\tau, \alpha_\nu)$  is monotone increasing in  $\tau$ , we have

$$\begin{aligned} Q_2(\tau, \alpha_\nu) &\geq Q_2(0, \alpha_\nu) = -\frac{16(\lambda - 1)(2 - \gamma)^2 \alpha_\nu}{(\lambda^2 + \alpha_\nu)^2} + c_{0,N,\gamma} + 2\alpha_\nu \\ &\geq -\frac{16(\lambda - 1)(2 - \gamma)^2 \alpha_\nu}{4\lambda^2 \alpha_\nu} + c_{0,N,\gamma} + 2\alpha_\nu \\ &\geq -(2 - \gamma)^2 + c_{0,N,\gamma} + 2\alpha_\nu \\ &= \gamma^2 + N^2/2 + 2(\alpha_\nu - 1) \geq N^2/2 + 2(\alpha_1 - 1), \end{aligned}$$

where the inequalities in the second and third lines follow from

$$(\lambda^2 + \alpha_\nu)^2 \geq 4\lambda^2 \alpha_\nu \quad \text{and} \quad -(\lambda - 1)/\lambda^2 \geq -1/4.$$

Hence it turns out that  $\inf_{\nu \in \mathbb{N}} \inf_{\tau \geq 0} Q_2(\tau, \alpha_\nu) > 0$ , and hence that  $c_{1,N,\gamma} > 0$ . Therefore, we have obtained the inequality

$$\int_{\mathbb{R}^N} |\Delta \mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx - R_{N,\gamma} \int_{\mathbb{R}^N} |\mathbf{u}|^2 |\mathbf{x}|^{2\gamma} dx \geq c_{N,\gamma} \int_{\mathbb{R}^N} |\mathbf{x} \cdot \nabla (|\mathbf{x}|^{\gamma + \frac{N}{2} - 1} \mathbf{u})|^2 |\mathbf{x}|^{-N} dx$$

for  $c_{N,\gamma} = \min\{c_{0,N,\gamma}, c_{1,N,\gamma}\} > 0$ . The proof of Theorem 6 is now complete, although the constant  $c_{N,\gamma}$  is not ensured to be optimal.  $\square$

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