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AUSTERE AND ARID PROPERTIES FOR PF SUBMANIFOLDS IN HILBERT SPACES

MASAHIRO MORIMOTO

ABSTRACT. Austere submanifolds and arid submanifolds constitute respectively two different classes of minimal submanifolds in finite dimensional Riemannian manifolds. In this paper we introduce these two notions into a class of proper Fredholm (PF) submanifolds in Hilbert spaces, discuss their relation and show examples of infinite dimensional austere PF submanifolds and arid PF submanifolds in Hilbert spaces. We also mention a classification problem of minimal orbits in hyperpolar PF actions on Hilbert spaces.

INTRODUCTION

An *austere submanifold* is a minimal submanifold of a Riemannian manifold which has a local symmetry. More precisely a submanifold M immersed in a finite dimensional Riemannian manifold $\bar{M}$ is called *austere* if for each normal vector ξ the set of eigenvalues with multiplicities of the shape operator A_ξ is invariant under the multiplication by (-1). This notion was originally introduced by Harvey and Lawson [5] in the study of calibrated geometry. Except for the case of surfaces the austere condition is much stronger than the minimal one. It is an interesting problem to classify austere submanifolds under suitable conditions (e.g. [1], [2], [13], [12], [14]).

In [12] Ikawa Sakai and Tasaki introduced a certain kind of austere submanifold which has a global symmetry, which they call a *weakly reflective submanifold*. A submanifold M immersed in a finite dimensional Riemannian manifold $\bar{M}$ is called *weakly reflective* if for each normal vector ξ at each $p \in M$ there exists an isometry ν_ξ of $\bar{M}$ which satisfies

$$\nu_\xi(p) = p, \quad d\nu_\xi(\xi) = -\xi, \quad \nu_\xi(M) = M.$$

Here we call ν_ξ a reflection with respect to ξ. A reflective submanifold ([19]), defined as a connected component of the fixed point set of an involutive isometry on $\bar{M}$, is an example of a weakly reflective submanifold. Another example is a singular orbit of a cohomogeneity one action, which was essentially shown to be weakly reflective by Podestà [27]. It is an interesting problem to study

submanifold geometry of orbits under isometric actions of Lie groups and to determine their weakly reflective orbits (e.g. [12], [23], [3]).

Recently Taketomi [29] introduced a generalized concept of weakly reflective submanifolds, namely *arid submanifolds*. A submanifold M immersed in a finite dimensional Riemannian manifold $\bar{M}$ is called *arid* if for each nonzero normal vector ξ at each $p \in M$ there exists an isometry φ_ξ of $\bar{M}$ which satisfies

$$\varphi_\xi(p) = p, \quad d\varphi_\xi(\xi) \neq \xi, \quad \varphi_\xi(M) = M.$$

Here we call φ_ξ an isometry with respect to ξ. From this definition we have:

$$\text{reflective} \;\Rightarrow\; \text{weakly reflective} \;\begin{array}{c} \nearrow \\ \searrow \end{array}\; \begin{array}{c} \text{austere} \\ \text{arid} \end{array}\; \begin{array}{c} \searrow \\ \nearrow \end{array}\; \text{minimal}.$$

In [29] he gave an example of an arid submanifold which is *not* an austere submanifold (therefore *not* a weakly reflective submanifold). Also he showed that any isolated orbit of a proper isometric action is an arid submanifold. It is a problem for a given proper isometric action to determine their arid orbits which are not isolated.

It is also interesting to study these submanifolds and problems in the infinite dimensional case.

A fundamental class of infinite dimensional submanifolds is given by *proper Fredholm* (PF) submanifolds in Hilbert spaces (Terng [30]), where the shape operators are compact operators and the distance functions satisfy the Palais-Smale condition ([24], [28]). It is known that many examples of PF submanifolds are obtained through a certain Riemannian submersion $\Phi_K : V_{\mathfrak{g}} \to G/K$ which is called the *parallel transport map* ([15], [32]). Here G/K is a compact normal homogeneous space and $V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g})$ the Hilbert space of all L^2-paths with values in the Lie algebra $\mathfrak{g}$ of G (cf. Section 2). It was proved ([32]) that if N is a closed submanifold of G/K then its inverse image $\Phi_K^{-1}(N)$ is a PF submanfold of $V_{\mathfrak{g}}$. A fundamental problem is to study the geometrical relation between N and $\Phi_K^{-1}(N)$. For example, it was shown ([15], [16], [7]) that if N is minimal then $\Phi_K^{-1}(N)$ is also minimal (in some sense).

Recently the author [22] introduced the concept of weakly reflective submanifolds into a class of PF submanifolds in Hilbert spaces and studied the geometrical relation between submanifolds N and $\Phi_K^{-1}(N)$. To simplify explanation here we suppose that G/K is an irreducible Riemannian symmetric space of compact type. The author showed ([22, Theorem 8]):

If N is weakly reflective, then $\Phi_K^{-1}(N)$ is also weakly reflective.

It is noted that even if N is reflective, $\Phi_K^{-1}(N)$ can *not* be reflective ([22, Remark 5]). Then there are two questions:

Question A: *If N is austere, is $\Phi_K^{-1}(N)$ austere ?*
Question B: *If N is arid, is $\Phi_K^{-1}(N)$ arid ?*

The purpose of this paper is to give answers to Questions A and B as far as possible. Main results are Theorem 4.1 and Theorem 5.1. Here Theorem 4.1 gives an affirmative answer to Question A under the assumption that G/K is a sphere, while in other cases it remains unsolved because there is no simple

relation between the principal curvatures of N and $\Phi_K^{-1}(N)$. On the other hand Theorem 5.1 is just an affirmative answer to Question B. Note that Theorem 5.1 is proved similarly to the weakly reflective case [22]. As an application we show examples of arid PF submanifolds which are *not* austere (therefore *not* weakly reflective) PF submanifolds (Examples 5.4, 5.7, 5.9). Also we mention a classification problem of minimal orbits in the $P(G, H)$-*action*, which is an isometric PF action of a Hilbert Lie group $P(G, H)$ on the Hilbert space $V_{\mathfrak{g}}$.

This paper is organized as follows. In Section 1 we define the arid property for PF submanifolds in Hilbert spaces. In Section 2 we prepare the setting of $P(G, H)$-actions and the parallel transport map. In Section 3 we calculate the principal curvatures of PF submanifolds obtained through the parallel transport map. In Section 4, to answer Question A, we study the austere property of so obtained PF submanifolds. In section 5, to answer Question B, we study the arid property of so obtained PF submanifolds. In Section 6 we mention the classification problem of minimal orbits in hyperpolar $P(G, H)$-actions.

1. PF submanifolds with symmetries

Let V be a separable Hilbert space over $\mathbb{R}$ and M a Hilbert manifold immersed in V. The end point map $Y : T^{\perp}M \to V$ is defined by $Y(\xi) := p + \xi$ for $\xi \in T_p^{\perp}M$. M is called *proper Fredholm* (PF) ([30]) if it has finite codimension and the restriction of Y to a normal disc bundle of any finite radius is proper and Fredholm. It is known ([30, Proposition 2.7]) that for each $p \in M$ and each $\xi \in T_p^{\perp}M$ the shape operator A_{ξ} is a self-adjoint compact operator on T_pM which is not of trace class in general.

At present there are three kinds of definitions for 'minimal' PF submanifolds, namely ζ-minimal ([15]), r-minimal ([7]) and f-minimal ([16]). Here we recall the first two which are used in this paper. Let M be a PF submanifold of V and $\xi \in T^{\perp}M$. We denote by

$$\mu_1 \le \mu_2 \le \cdots < 0 < \cdots \le \lambda_2 \le \lambda_1.$$

the eigenvalues repeated with multiplicities of the shape operator A_{ξ}.

We say that A_{ξ} is ζ-*regularizable* ([15]) if $\sum_k \lambda_k^s + \sum_k |\mu_k|^s < \infty$ for all $s > 1$ and

$$\operatorname{tr}_{\zeta} A_{\xi} := \lim_{s \searrow 1} \left(\sum_k \lambda_k^s - \sum_k |\mu_k|^s \right)$$

exists. Then we call $\operatorname{tr}_{\zeta} A_{\xi}$ the ζ-*regularized mean curvature* in the direction of ξ. M is called ζ-*regularizable* if A_{ξ} is ζ-regularizable for all $\xi \in T^{\perp}M$. If M is ζ-regularizable and $\operatorname{tr}_{\zeta} A_{\xi}$ vanishes for all $\xi \in T^{\perp}M$, we say that M is ζ-*minimal*.

We say that A_{ξ} is *regularizable* ([7]) if $\operatorname{tr} A_{\xi}^2 < \infty$ and

$$\operatorname{tr}_r A_{\xi} := \sum_{k=1}^{\infty} (\lambda_k + \mu_k)$$

converges, where we regard λ_k or μ_k as zero if there are less than k positive or negative eigenvalues, respectively. Then we call $\mathrm{tr}_r A_\xi$ the *regularized mean curvature* in the direction of ξ. M is called *regularizable* if A_ξ is regularizable for all $\xi \in T^\perp M$. If M is regularizable and $\mathrm{tr}_r A_\xi$ vanishes for all $\xi \in T^\perp M$, we say that M is *r-minimal*.

In [22] the concepts of reflective submanifolds, weakly reflective submanifolds and austere submanifolds were introduced into a class of PF submanifolds in Hilbert spaces. Similarly we can define arid PF submanifolds:

Definition. Let M be a PF submanifold of V. M is called *arid* if for each $p \in M$ and each $\xi \in T_p^\perp M \setminus \{0\}$ there exists an isometry φ_ξ of V which satisfies

$$\varphi_\xi(p) = p, \quad d\varphi_\xi(\xi) \neq \xi, \quad \varphi_\xi(M) = M.$$

We have the following relation:

$$\text{reflective PF} \quad \Rightarrow \quad \text{weakly reflective PF} \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \quad \begin{array}{c} \text{austere PF} \\ \text{arid PF.} \end{array}$$

The ζ-minimality and r-minimality of austere PF submanifolds were discussed in [22, Section 1]. We do not know in general whether arid PF submanifolds are ζ-minimal or r-minimal because it is not clear that the mean curvature vector of a PF submanifold is well-defined or not. However it will not interfere our purpose because we will give attention to PF submanifolds obtained through the parallel transport map ([15], [32]), where the mean curvature vector is well-defined ([15, Theorem 4.12], [7, Lemma 5.2]) and thus the arid property implies both ζ-minimality and r-minimality.

2. $P(G, H)$-actions and the parallel transport map

In this section we prepare the setting of $P(G, H)$-actions and the parallel transport map. The related and detailed facts can be found in [22, Section 2] and references therein.

Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$. Choose an $\mathrm{Ad}(G)$-invariant inner product of $\mathfrak{g}$ and equip the corresponding bi-invariant Riemannian metric with G. Denote by $\mathcal{G} := H^1([0,1], G)$ the Hilbert Lie group of all Sobolev H^1-paths in G parametrized on $[0,1]$ and by $V_\mathfrak{g} := H^0([0,1], \mathfrak{g})$ the Hilbert space of all Sobolev H^0-paths in $\mathfrak{g}$ parametrized on $[0,1]$. For each $a \in G$ (resp. $x \in \mathfrak{g}$) we denote by $\hat{a} \in \mathcal{G}$ (resp. $\hat{x} \in V_\mathfrak{g}$) the constant path which values at a (resp. x). Then $\mathcal{G}$ acts on $V_\mathfrak{g}$ via the gauge transformations:

$$(2.1) \qquad g * u := gug^{-1} - g'g^{-1}, \quad g \in \mathcal{G}, \ u \in V_\mathfrak{g},$$

where g' denotes the weak derivative of g with respect to the parameter on $[0,1]$. We know that this action is isometric, transitive, proper and Fredholm ([25, Theorem 5.8.1]).

Let H be a closed subgroup of $G \times G$ with Lie algebra $\mathfrak{h}$. Define a Lie subgroup $P(G, H)$ of $\mathcal{G}$ by

$$P(G, H) := \{g \in \mathcal{G} \mid (g(0), g(1)) \in H\}$$

with Lie algebra

$$\operatorname{Lie} P(G, H) = \{Z \in H^1([0, 1], \mathfrak{g}) \mid (Z(0), Z(1)) \in \mathfrak{h}\}.$$

The induced action of $P(G, H)$ on $V_{\mathfrak{g}}$ is called the $P(G, H)$-*action*. We know that $P(G, H)$-action is isometric, proper and Fredholm ([31, p. 132]). Thus each orbit of the $P(G, H)$-action is a PF submanifold of $V_{\mathfrak{g}}$ ([25, Theorem 7.1.6]). Also we know ([32, Corollary 4.2]) that both $P(G, \{e\} \times G)$-action on $V_{\mathfrak{g}}$ and $P(G, G \times \{e\})$-action on $V_{\mathfrak{g}}$ are simply transitive.

The *parallel transport map* ([15], [31]) $\Phi : V_{\mathfrak{g}} \to G$ is a Riemannian submersion defined by

$$\Phi(u) := E_u(1), \quad u \in V_{\mathfrak{g}},$$

where $E_u \in \mathcal{G}$ is the unique solution to the linear ordinary differential equation

$$\begin{cases} E_u^{-1} \dot{E}_u = u, \\ E_u(0) = e. \end{cases}$$

Note that $G \times G$ acts on G isometrically by

$$(2.2) \qquad\qquad (b_1, b_2) \cdot a := b_1 a b_2^{-1}, \quad a, b_1, b_2 \in G.$$

We know ([31, Proposition 1.1]) that for $g \in \mathcal{G}$ and $u \in V_{\mathfrak{g}}$

$$(2.3) \qquad\qquad \Phi(g * u) = (g(0), g(1)) \cdot \Phi(u)$$

and that for any closed subgroup H of $G \times G$,

$$(2.4) \qquad\qquad P(G, H) * u = \Phi^{-1}(H \cdot \Phi(u)).$$

More generally, if N is a closed submanifold of G then the inverse image $\Phi^{-1}(N)$ is a PF submanifold of $V_{\mathfrak{g}}$ ([32, Lemma 5.1]).

The differential $(d\Phi)_{\hat{0}} : T_{\hat{0}} V_{\mathfrak{g}} \to T_e G \cong \mathfrak{g}$ of Φ at $\hat{0} \in V_{\mathfrak{g}}$ (cf. [32, p. 685]) is given by

$$(d\Phi)_{\hat{0}}(X) = \int_0^1 X(t) dt, \quad X \in T_{\hat{0}} V_{\mathfrak{g}} \cong V_{\mathfrak{g}}.$$

Using this we obtain the orthogonal direct sum decomposition

$$T_{\hat{0}} V_{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \operatorname{Ker}(d\Phi)_{\hat{0}}, \quad X = \left(\int_0^1 X(t) dt \right) \oplus \left(X - \int_0^1 X(t) dt \right).$$

Let K be a closed subgroup of G with Lie algebra $\mathfrak{k}$. Denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the orthogonal direct sum decomposition. Restricting the $\operatorname{Ad}(G)$-invariant inner product of $\mathfrak{g}$ to $\mathfrak{m}$ we define the induced G-invariant Riemannian metric on the homogeneous space G/K. Such a metric is called a normal homogeneous metric and G/K a compact normal homogeneous space. We write $\pi : G \to G/K$ for the natural projection, which is a Riemannian submersion with totally geodesic fiber. The *parallel transport map* Φ_K *over* G/K is a Riemannian submersion defined by

$$(2.5) \qquad\qquad \Phi_K := \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K.$$

Note that for each $g \in P(G, G \times \{e\})$ and $a \in G$ the diagrams

$$
(2.6) \quad \text{(i)} \quad
\begin{array}{ccc}
V_{\mathfrak{g}} & \xrightarrow{\ g* \ } & V_{\mathfrak{g}} \\
\Phi \downarrow & & \Phi \downarrow \\
G & \xrightarrow{(g(0),\, e)} & G
\end{array}
\qquad \text{(ii)} \quad
\begin{array}{ccc}
G & \xrightarrow{\ l_a \ } & G \\
\pi \downarrow & & \pi \downarrow \\
G/K & \xrightarrow{\ L_a \ } & G/K
\end{array}
$$

commute, where l_a denotes the left translation by a and L_a the isometry on G/K defined by $L_a(bK) := (ab)K$. Thus setting $a := g(0)$ we obtain a commutative diagram

$$
(2.7) \quad
\begin{array}{ccc}
V_{\mathfrak{g}} & \xrightarrow{\ g* \ } & V_{\mathfrak{g}} \\
\Phi_K \downarrow & & \Phi_K \downarrow \\
G/K & \xrightarrow{\ L_a \ } & G/K.
\end{array}
$$

Let G, K be as above. Denote by $F_K := \Phi_K^{-1}(eK) = \Phi^{-1}(K)$ the fiber of Φ_K at $eK \in G/K$ and by $F := \Phi^{-1}(e)$ the fiber of Φ at $e \in G$. Since

$$
F_K = \Phi^{-1}((\{e\} \times K) \cdot e) = P(G, \{e\} \times K) * \hat{0}
$$

we can write (cf. [22, equation (9)])

$$
(2.8) \qquad T_{\hat{0}} F_K = \{ -Z' \mid Z \in H^1([0,1], \mathfrak{g}), \ Z(0) = 0, \ Z(1) \in \mathfrak{k} \}.
$$

Similarly we have

$$
(2.9) \qquad T_{\hat{0}} F = \{ -Q' \mid Q \in H^1([0,1], \mathfrak{g}), \ Q(0) = Q(1) = 0 \}.
$$

Suppose that a closed submanifold N of G/K through $eK \in G/K$ is given. Since Φ and Φ_K are Riemannian submersions we have the orthogonal direct sum decompositions

$$
T_{\hat{0}} \Phi_K^{-1}(N) \cong T_{\hat{0}} F_K \oplus T_{eK} N \cong T_{\hat{0}} F \oplus \mathfrak{k} \oplus T_{eK} N.
$$

From now on $A^{\Phi_K^{-1}(N)}$, $A^{\pi^{-1}(N)}$ and A^N denote the shape operators of $\Phi_K^{-1}(N)$, $\pi^{-1}(N)$ and N, respectively. We fix a normal vector ξ of N at eK. Then its horizontal lift at $\hat{0} \in \Phi_K^{-1}(N)$ is given by the constant path $\hat{\xi}$.

Proposition 2.1. *Suppose $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. For $-Q' \in T_{\hat{0}} F$, $x \in \mathfrak{k}$, $y \in T_{eK} N$*

$$
(2.10) \qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(-Q') = [\hat{\xi}, Q] - \left[\xi, \int_0^1 Q(t)\, dt \right]^{\perp},
$$

$$
(2.11) \qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(x) = \frac{1}{2} [\xi, x]^{\perp} - t[\xi, x],
$$

$$
(2.12) \qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y) = A_{\xi}^N(y) + (1 - t)[\xi, y],
$$

where $\perp$ denote the projection from $\mathfrak{g}$ onto $T_{eK}^{\perp} N (\subset \mathfrak{m})$.

Proof. The formula (2.10) follows from [22, Corollary 1 (ii)]. Also from [22, Corollary 1 (i)] we have the following formula: for $v \in T_e \pi^{-1}(N)$

$$
A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(v) = A_{\xi}^{\pi^{-1}(N)}(v) - \left(t - \frac{1}{2} \right) [\xi, v].
$$

Further by [22, Remark 1 (ii)], for $v = x \oplus y \in T_e \pi^{-1}(N) = \mathfrak{k} \oplus T_{eK} N$

$$(2.13) \qquad A_\xi^{\pi^{-1}(N)}(v) = A_\xi^N(y) + \frac{1}{2}[\xi, y]_{\mathfrak{k}} - \frac{1}{2}[\xi, x]^\top,$$

where the subscript $\mathfrak{k}$ and $\top$ denote the projections onto $\mathfrak{k}$ and $T_e N$ respectively. Since we are supposing $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ these formulas imply (2.11) and (2.12). $\square$

Corollary 2.2. *For* $-Z' \in T_{\hat{0}} F_K$

$$(2.14) \qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(-Z') = [\hat{\xi}, Z] - \left[\xi, \int_0^1 Z(t) dt\right]^\perp.$$

Proof. Set $Q := Z - tZ(1)$ and $x := -Z(1)$ so that $-Z' = -Q' + x$. By (2.10) and (2.11) the desired formula follows. $\square$

3. PRINCIPAL CURVATURES

In this section we calculate the principal curvatures of PF submanifolds obtained through the parallel transport map. For technical reasons, here we will restrict our attention to PF submanifolds obtained from *curvature adapted* submanifolds in compact *symmetric* spaces. Although such a subject has been studied by Koike [16] there are some inaccuracies in his eigenspace decomposition and so here we give the corrected formula with another elementary proof by using the formulas for shape operators prepared in the last section.

Recall that a submanifold M immersed in a Riemannian manifold $\bar{M}$ is called *curvature adapted* if for each $p \in M$ and each $\xi \in T_p^\perp M$ the Jacobi operator $R_\xi := \bar{R}(\cdot, \xi)\xi : T_p \bar{M} \to T_p \bar{M}$, where $\bar{R}$ denotes the curvature tensor of $\bar{M}$, satisfies

$$R_\xi(T_p M) \subset T_p M \quad \text{and} \quad A_\xi^M \circ R_\xi|_{T_p M} = R_\xi|_{T_p M} \circ A_\xi^M,$$

where A_ξ^M denotes the shape operator of M in the direction of ξ.

Let G be a connected compact Lie group and K a closed subgroup of G. Suppose that K is symmetric, that is, there exists an involutive automorphism θ of G such that $G_0^\theta \subset K \subset G^\theta$, where G^θ is the fixed point subgroup of θ and G_0^θ the identity component. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively and by $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the direct sum decomposition into the ± 1-eigenspaces of $d\theta$. We fix an inner product $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ which is invariant under both $Ad(G)$ and θ. Then the above direct sum decomposition is orthogonal with respect to this inner product $\langle \cdot, \cdot \rangle$. We equip the corresponding bi-invariant Riemannian metric with G and a normal homogeneous metric with G/K. Then G/K is a compact symmetric space and the natural projection $\pi : G \to G/K$ is a Riemannian submersion with totally geodesic fiber. We denote by $\Phi_K : V_{\mathfrak{g}} \to G/K$ the parallel transport map.

Let N be a curvature adapted closed submanifold of G/K. Note that in order to calculate the principal curvatures of a PF submanifold $\Phi_K^{-1}(N)$ we can assume without loss of generality that N contains eK and moreover it suffices to consider normal vectors only at $\hat{0} \in \Phi_K^{-1}(N)$ because of the commutativity (2.7). Thus in the rest of this section we fix $\xi \in T_{eK}^\perp N$ and calculate the

principal curvatures of $\Phi_K^{-1}(N)$ in the direction of $\hat{\xi} \in T_{\hat{0}}^\perp \Phi_K^{-1}(N)$. Note that in this case the Jacobi operator is given by $R_\xi = -\operatorname{ad}(\xi)^2 : \mathfrak{m} \to \mathfrak{m}$.

Denote by $\{\sqrt{-1}\,\nu\}$ the set of all distinct eigenvalues of the skew adjoint operator $\operatorname{ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$. Consider the complexification $\operatorname{ad}(\xi) : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ and the eigenspace decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0^{\mathbb{C}} + \sum_{\nu \neq 0} \mathfrak{g}_\nu,$$

$$\mathfrak{g}_0 := \{x \in \mathfrak{g} \mid \operatorname{ad}(\xi)(x) = 0\},$$
$$\mathfrak{g}_\nu := \{z \in \mathfrak{g}^{\mathbb{C}} \mid \operatorname{ad}(\xi)(z) = \sqrt{-1}\,\nu z\}.$$

Since $\bar{\mathfrak{g}}_\nu = \mathfrak{g}_{-\nu}$ we can write

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0^{\mathbb{C}} + \sum_{\nu > 0} (\mathfrak{g}_\nu + \mathfrak{g}_{-\nu})$$

and thus we obtain

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\nu > 0} (\mathfrak{g}_\nu + \mathfrak{g}_{-\nu})_{\mathbb{R}},$$

$$(\mathfrak{g}_\nu + \mathfrak{g}_{-\nu})_{\mathbb{R}} = \{x \in \mathfrak{g} \mid \operatorname{ad}(\xi)^2(x) = -\nu^2 x\},$$

which is nothing but the eigenspace decomposition with respect to $\operatorname{ad}(\xi)^2 : \mathfrak{g} \to \mathfrak{g}$. Since $\operatorname{ad}(\xi)^2$ commutes with involution θ we have the simultaneous eigenspace decomposition

$$(3.1) \qquad \mathfrak{k} = \mathfrak{k}_0 + \sum_{\nu > 0} \mathfrak{k}_\nu, \qquad \mathfrak{m} = \mathfrak{m}_0 + \sum_{\nu > 0} \mathfrak{m}_\nu,$$

$$\mathfrak{k}_0 := \{x \in \mathfrak{k} \mid \operatorname{ad}(\xi)(x) = 0\},$$
$$\mathfrak{m}_0 := \{y \in \mathfrak{m} \mid \operatorname{ad}(\xi)(y) = 0\},$$
$$\mathfrak{k}_\nu := \{x \in \mathfrak{k} \mid \operatorname{ad}(\xi)^2(x) = -\nu^2 x\},$$
$$\mathfrak{m}_\nu := \{y \in \mathfrak{m} \mid \operatorname{ad}(\xi)^2(y) = -\nu^2 y\}.$$

By similar arguments as in [18, p. 60], for each $\nu > 0$ we can take bases $\{x_1^\nu, \cdots, x_{m(\nu)}^\nu\}$ of $\mathfrak{k}_\nu$ and $\{y_1^\nu, \cdots, y_{m(\nu)}^\nu\}$ of $\mathfrak{m}_\nu$ where $m(\nu) := \dim \mathfrak{k}_\nu = \dim \mathfrak{m}_\nu$ such that

$$(3.2) \qquad [\xi, x_k^\nu] = -\nu y_k^\nu, \qquad [\xi, y_k^\nu] = \nu x_k^\nu.$$

Thus a linear isomorphism $\varphi_\nu : \mathfrak{k}_\nu \to \mathfrak{m}_\nu$ is defined by

$$(3.3) \qquad \varphi_\nu(x) := -\frac{1}{\nu}[\xi, x].$$

Let $\{\lambda\}$ denote the set of all distinct eigenvalues of the shape operator A_ξ^N. Set

$$S_\lambda := \operatorname{Ker}(A_\xi^N - \lambda \operatorname{id}).$$

Since N is curvature adapted, for each $\nu \geq 0$ we have the decomposition

$$
\begin{array}{ccccc}
\mathfrak{m} & = & T_{eK}N & \oplus & T_{eK}^{\perp}N \\
\cup & & \cup & & \cup \\
\mathfrak{m}_{\nu} & = & \mathfrak{m}_{\nu} \cap T_{eK}N & \oplus & \mathfrak{m}_{\nu} \cap T_{eK}^{\perp}N \\
& & \| & & \\
& & \sum_{\lambda}(\mathfrak{m}_{\nu} \cap S_{\lambda}). & &
\end{array}
$$

For each $\nu \geq 0$ we set

$$
m(\nu, \lambda) := \dim(\mathfrak{m}_{\nu} \cap S_{\lambda}), \qquad m(\nu, \perp) := \dim(\mathfrak{m}_{\nu} \cap T_{eK}^{\perp}N).
$$

For each $\nu \geq 0$ and λ, choose

$$
\{y_1^{(\nu,\lambda)}, \cdots, y_{m(\nu,\lambda)}^{(\nu,\lambda)}\} : \text{ a basis of } \mathfrak{m}_{\nu} \cap S_{\lambda},
$$

$$
\{y_1^{(\nu,\perp)}, \cdots, y_{m(\nu,\perp)}^{(\nu,\perp)}\} : \text{ a basis of } \mathfrak{m}_{\nu} \cap T_{eK}^{\perp}N.
$$

Then for each $\nu \geq 0$ we obtain

$$
\{y_1^{(\nu,\lambda)}, \cdots, y_{m(\nu,\lambda)}^{(\nu,\lambda)}\}_{\lambda} \cup \{y_1^{(\nu,\perp)}, \cdots, y_{m(\nu,\perp)}^{(\nu,\perp)}\} : \text{ a basis of } \mathfrak{m}_{\nu}.
$$

Thus for each $\nu > 0$, via an isomorphism (3.3) we obtain

$$
\{x_1^{(\nu,\lambda)}, \cdots, x_{m(\nu,\lambda)}^{(\nu,\lambda)}\}_{\lambda} \cup \{x_1^{(\nu,\perp)}, \cdots, x_{m(\nu,\perp)}^{(\nu,\perp)}\} : \text{ a basis of } \mathfrak{k}_{\nu}.
$$

For $\nu = 0$ we choose and denote by

$$
\{x_1^0, \cdots, x_{\dim \mathfrak{k}_0}^0\} : \text{ a basis of } \mathfrak{k}_0.
$$

Note that these satisfy

$$
\begin{array}{ll}
[\xi, x_i^0] = 0, & [\xi, y_j^{(0,\lambda)}] = [\xi, y_l^{(0,\perp)}] = 0, \\
[\xi, x_k^{(\nu,\lambda)}] = -\nu\, y_k^{(\nu,\lambda)}, & [\xi, y_k^{(\nu,\lambda)}] = \nu\, x_k^{(\nu,\lambda)}, \\
[\xi, x_r^{(\nu,\perp)}] = -\nu\, y_r^{(\nu,\perp)}, & [\xi, y_r^{(\nu,\perp)}] = \nu\, x_r^{(\nu,\perp)}.
\end{array}
$$

Set $V(\mathfrak{g}) := V_{\mathfrak{g}} = H^0([0,1], \mathfrak{g})$. We decompose

$$
V(\mathfrak{g}) = \sum_{\nu \geq 0} V(\mathfrak{k}_{\nu}) + \sum_{\nu \geq 0} \left(V(\mathfrak{m}_{\nu} \cap T_{eK}N) + V(\mathfrak{m}_{\nu} \cap T_{eK}^{\perp}N) \right)
$$

and equip a basis with each term above. Recall that there are well-known three kinds of orthonormal bases in $H^0([0,1], \mathbb{R})$:

(3.4) $\qquad\qquad \{1, \ \sqrt{2}\cos 2n\pi t, \ \sqrt{2}\sin 2n\pi t\}_{n=1}^{\infty}$,

(3.5) $\qquad\qquad \{1, \ \sqrt{2}\cos n\pi t\}_{n=1}^{\infty}$,

(3.6) $\qquad\qquad \{\sqrt{2}\sin n\pi t\}_{n=1}^{\infty}$.

For $\nu = 0$ we consider the following bases:

$$
\begin{array}{lll}
\text{a basis of } V(\mathfrak{k}_0) & : & \{x_i^0 \sin n\pi t\}_{i,n}, \\
\text{a basis of } V(\mathfrak{m}_0 \cap T_{eK}N) & : & \{y_j^{(0,\lambda)}\}_{\lambda,j} \ \cup \ \{y_j^{(0,\lambda)}\cos n\pi t\}_{\lambda,j,n}, \\
\text{a basis of } V(\mathfrak{m}_0 \cap T_{eK}^{\perp}N) & : & \{y_l^{(0,\perp)}\}_l \ \cup \ \{y_l^{(0,\perp)}\cos n\pi t\}_{l,n}.
\end{array}
$$

For each $\nu > 0$ we consider the following bases:

a basis of $V(\mathfrak{k}_\nu)$: $\{x_k^{(\nu,\lambda)} \sin n\pi t\}_{\lambda,k,n}$ $\cup$ $\{x_r^{(\nu,\perp)} \sin n\pi t\}_{r,n}$,

a basis of $V(\mathfrak{m}_\nu \cap T_{eK} N)$: $\{y_k^{(\nu,\lambda)}\}_{\lambda,k}$ $\cup$ $\{y_k^{(\nu,\lambda)} \cos n\pi t\}_{\lambda,n,k}$,

a basis of $V(\mathfrak{m}_\nu \cap T_{eK}^\perp N)$: $\{y_r^{(\nu,\perp)}\}_r$ $\cup$ $\{y_r^{(\nu,\perp)} \cos n\pi t\}_{n,r}$.

Clearly all these bases form a basis of $V(\mathfrak{g}) = V_\mathfrak{g}$. Identifying $T_{\hat{0}} V_\mathfrak{g} \cong V_\mathfrak{g}$ and considering the orthogonal direct sum decomposition

$$T_{\hat{0}} V_\mathfrak{g} = T_{\hat{0}} \Phi_K^{-1}(N) \oplus T_{eK}^\perp N, \qquad X = \left(X - \int_0^1 X(t)^\perp dt \right) \oplus \int_0^1 X(t)^\perp dt$$

we consequently obtain the following basis of $T_{\hat{0}} \Phi_K^{-1}(N)$:

$$\{x_i^0 \sin n\pi t\}_{i,n} \cup \{y_j^{(0,\lambda)}\}_{\lambda,j} \cup \{y_j^{(0,\lambda)} \cos n\pi t\}_{\lambda,j,n} \cup \{y_r^{(0,\perp)} \cos n\pi t\}_{r,n}$$

$$\cup \bigcup_{\nu > 0} \left(\{x_k^{(\nu,\lambda)} \sin n\pi t\}_{\lambda,k,n} \cup \{y_k^{(\nu,\lambda)}\}_{\lambda,k} \cup \{y_k^{(\nu,\lambda)} \cos n\pi t\}_{\lambda,k,n} \right)$$

$$\cup \bigcup_{\nu > 0} \left(\{x_r^{(\nu,\perp)} \sin n\pi t\}_{r,n} \cup \{y_r^{(\nu,\perp)} \cos n\pi t\}_{r,n} \right).$$

Lemma 3.1.

(i) $\qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(x_i^0 \sin n\pi t) = 0, \qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_j^{(0,\lambda)}) = \lambda y_j^{(0,\lambda)},$

(ii) $\qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_j^{(0,\lambda)} \cos n\pi t) = A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_l^{(0,\perp)} \cos n\pi t) = 0,$

(iii) $\qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(x_r^{(\nu,\perp)} \sin n\pi t) = -\dfrac{\nu}{n\pi} y_r^{(\nu,\perp)} \cos n\pi t,$

$\qquad\qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_r^{(\nu,\perp)} \cos n\pi t) = -\dfrac{\nu}{n\pi} x_r^{(\nu,\perp)} \sin n\pi t,$

(iv) $\qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_k^{(\nu,\lambda)}) = \lambda y_k^{(\nu,\lambda)} + \dfrac{2\nu}{\pi} \sum_{n=1}^\infty \dfrac{1}{n}(x_k^{(\nu,\lambda)} \sin n\pi t),$

(v) $\qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(x_k^{(\nu,\lambda)} \sin n\pi t) = -\dfrac{\nu}{n\pi} y_k^{(\nu,\lambda)}(-1 + \cos n\pi t),$

(vi) $\qquad A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_k^{(\nu,\lambda)} \cos n\pi t) = -\dfrac{\nu}{n\pi} x_k^{(\nu,\lambda)} \sin n\pi t.$

Proof. (i) and (ii): The second equality of (i) follows from (2.12). Let Q be

$$\frac{1}{n\pi} x_i^0 \cos n\pi t, \quad -\frac{1}{n\pi} y_j^{(0,\lambda)} \sin n\pi t \quad \text{or} \quad -\frac{1}{n\pi} y_l^{(0,\perp)} \sin n\pi t.$$

By (2.10) we obtain the first formula of (i) and formulas in (ii). (iii): Set

$$Z_1 := \frac{1}{n\pi} x_r^{(\nu,\perp)}(-1 + \cos n\pi t), \quad Z_2 := -\frac{1}{n\pi} y_r^{(\nu,\perp)} \sin n\pi t.$$

Then we have

$$[\xi, Z_1] = -\frac{\nu}{n\pi} y_r^{(\nu,\perp)}(-1 + \cos n\pi t), \quad [\xi, Z_2] = -\frac{\nu}{n\pi} x_r^{(\nu,\perp)} \sin n\pi t.$$

Thus we have

$$\left[\xi, \int_0^1 Z_1(t) dt \right]^\perp = \frac{\nu}{n\pi} y_r^{(\nu,\perp)}, \quad \left[\xi, \int_0^1 Z_2(t) dt \right]^\perp = 0.$$

Hence by (2.14) the desired equalities follow. (iv): By (2.12) we have

$$A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(y_k^{(\nu,\lambda)}) = \lambda y_k^{(\nu,\lambda)} + (1-t)\,\nu x_k^{(\nu,\lambda)}.$$

Since $\int_0^1 (1-t)\sin n\pi t\,dt = (n\pi)^{-1}$ the Fourier expansion with respect to a basis (3.6) of a function $f : [0,1] \to \mathbb{R}$, $t \mapsto 1 - t$ is given by

$$f = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\sin n\pi t).$$

This shows the desired equality. (v), (vi): Set

$$Z_1 := \frac{1}{n\pi}x_k^{(\nu,\lambda)}(-1 + \cos n\pi t), \quad Z_2 := -\frac{1}{n\pi}y_k^{(\nu,\lambda)}\sin n\pi t.$$

Then we have

$$[\hat{\xi}, Z_1] = -\frac{\nu}{n\pi}y_k^{(\nu,\lambda)}(-1 + \cos n\pi t), \quad [\hat{\xi}, Z_2] = -\frac{\nu}{n\pi}x_k^{(\nu,\lambda)}\sin n\pi t,$$

Thus we have

$$\left[\xi, \int_0^1 Z_1(t)dt\right]^{\perp} = \left[\xi, \int_0^1 Z_2(t)dt\right]^{\perp} = 0.$$

Hence by (2.14) we obtain the desired formulas. $\qquad\square$

We come now to the principal curvatures of a PF submanifold $\Phi_K^{-1}(N)$:

Theorem 3.2. *Let G/K be a compact symmetric space, $\Phi_K : V_{\mathfrak{g}} \to G/K$ the parallel transport map, N a curvature adapted closed submanifold of G/K through $eK \in G/K$, and $\xi \in T_{eK}^{\perp}N \subset \mathfrak{m}$. Denote by $\{\sqrt{-1}\,\nu\}$ the set of all distinct eigenvalues of $\mathrm{ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$ and by $\{\lambda\}$ the set of all distinct eigenvalues of the shape operator A_{ξ}^N. For each $\nu > 0$, each λ and each $m \in \mathbb{Z}$ we set*

$$\mu = \mu(\nu, \lambda, m) := \frac{\nu}{\arctan\frac{\nu}{\lambda} + m\pi},$$

where we set $\arctan(\nu/\lambda) := \pi/2$ if $\lambda = 0$. Then the principal curvatures of a PF submanifold $\Phi_K^{-1}(N)$ in the direction of $\hat{\xi} \in T_{\hat{0}}^{\perp}\Phi^{-1}(N)$ are given by

$$\left\{0,\ \lambda,\ \frac{\nu}{n\pi},\ \mu(\nu, \lambda, m)\right\}_{\lambda,\,\nu>0,\,n\in\mathbb{Z}\setminus\{0\},\,m\in\mathbb{Z}}.$$

The eigenfunctions and the multiplicities are given in the following table.

eigenvalue	basis of eigenfunctions	multiplicity
0	$\{x_i^0\sin n\pi t, y_j^{(0,\lambda)}\cos n\pi t, y_l^{(0,\perp)}\cos n\pi t\}_{n\in\mathbb{Z}_{\geq 1},\lambda,i,j,l}$	∞
λ	$\{y_j^{(0,\lambda)}\}_j$	$m(0,\lambda)$
$\dfrac{\nu}{n\pi}$	$\{x_r^{(\nu,\perp)}\sin n\pi t - y_r^{(\nu,\perp)}\cos n\pi t\}_r$	$m(\nu,\perp)$
$\mu(\nu,\lambda,m)$	$\left\{\sum_{n\in\mathbb{Z}}\frac{\nu}{n\pi\mu+\nu}(x_k^{(\nu,\lambda)}\sin n\pi t + y_k^{(\nu,\lambda)}\cos n\pi t)\right\}_k$	$m(\nu,\lambda)$

Remark 3.3. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{m}$ and Δ^+ the set of positive roots satisfying $\alpha(\xi) \geq 0$ for each $\alpha \in \Delta^+$. Then for each $\nu > 0$ there exists $\alpha \in \Delta^+$ such that $\nu = \alpha(\xi)$ and thus the above eigenvalues coincide with those given by Koike [16, Theorem 3.3]. However note that the eigenspace decomposition [16, p. 73, line 3] does not hold in general.

Proof of Theorem 3.2. By Lemma 3.1 (i) - (iii) it follows that 0, λ and $\frac{\nu}{n\pi}$ are eigenvalues of $A_{\hat{\xi}}^{\Phi_K^{-1}(N)}$ with eigenfunctions described above. Let W denote a subspace of $T_{\hat{0}}\Phi_K^{-1}(N)$ spanned by all such eigenfunctions and consider its orthogonal complements $W^\perp$ in $T_{\hat{0}}\Phi_K^{-1}(N)$. We know that one basis of $W^\perp$ is given by

$$\bigcup_{\nu > 0}\left(\{y_k^{(\nu,\lambda)}\}_{\lambda, k} \cup \{x_k^{(\nu,\lambda)}\sin n\pi t,\ y_k^{(\nu,\lambda)}\cos n\pi t\}_{\lambda,\ k,\ n\in\mathbb{Z}_{\geq 1}}\right).$$

In particular Lemma 3.1 (iv) - (vi) show that for each $\nu > 0$, λ and k, a subspace of $T_{\hat{0}}\Phi_K^{-1}(N)$ spanned by

$$\{y_k^{(\nu,\lambda)}\} \cup \{x_k^{(\nu,\lambda)}\sin n\pi t,\ y_k^{(\nu,\lambda)}\cos n\pi t\}_{n\in\mathbb{Z}_{\geq 1}}$$

is invariant under $A_{\hat{\xi}}^{\Phi_K^{-1}(N)}$. We denote such a subspace by $W^\perp_{(\lambda,\nu,k)}$. Suppose that for constants $a_n, b_n, c \in \mathbb{R}$

$$\varphi := c y_k^{(\nu,\lambda)} + \sum_{n=1}^{\infty}\{a_n(x_k^{(\nu,\lambda)}\sin n\pi t) + b_n(y_k^{(\nu,\lambda)}\cos n\pi t)\} \quad \in W^\perp_{(\lambda,\,\nu,\,k)}$$

is a (nonzero) eigenfunction of $A_{\hat{\xi}}^{\Phi_K(N)}$ for some eigenvalue μ. By Lemma 3.1 (iv) - (vi) we have

$$A_{\hat{\xi}}^{\Phi_K^{-1}(N)}(\varphi) = \left(c\lambda + \frac{\nu}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}\right)y_k^{(\nu,\lambda)}$$

$$+ \frac{\nu}{\pi}\sum_{n=1}^{\infty}\frac{2c - b_n}{n}(x_k^{(\nu,\lambda)}\sin n\pi t) - \frac{\nu}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}(y_k^{(\nu,\lambda)}\cos n\pi t).$$

Comparing with

$$\mu\varphi = \mu c y_k^{(\nu,\lambda)} + \sum_{n=1}^{\infty}\{\mu a_n(x_k^{(\nu,\lambda)}\sin n\pi t) + \mu b_n(y_k^{(\nu,\lambda)}\cos n\pi t)\}$$

we obtain a system of equations

$$(3.7) \qquad\qquad\qquad c\lambda + \frac{\nu}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n} = c\mu,$$

$$(3.8) \qquad\qquad\qquad \frac{\nu}{\pi}\frac{2c - b_n}{n} = \mu a_n,$$

$$(3.9) \qquad\qquad\qquad -\frac{\nu}{\pi}\frac{a_n}{n} = \mu b_n.$$

Summing (3.9) with respect to $n \in \mathbb{Z}_{\geq 1}$ we have

$$-\frac{\nu}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n} = \mu \sum_{n=1}^{\infty} b_n.$$

Applying this to (3.7) we obtain

$$(3.10) \qquad c\lambda - \mu \sum_{n=1}^{\infty} b_n = c\mu.$$

On the other hand, multiplying (3.9) by μ we have

$$-\frac{\nu}{n\pi} \mu a_n = \mu^2 b_n.$$

Applying (3.8) to this we obtain

$$-2c \left(\frac{\nu}{n\pi}\right)^2 = b_n \left(\mu^2 - \left(\frac{\nu}{n\pi}\right)^2\right)$$

for *all* $n \in \mathbb{Z}_{\geq 1}$. Note that this implies $c \neq 0$, $\mu \neq 0$ and $\mu^2 - (\frac{\nu}{n\pi})^2 \neq 0$ for all $n \in \mathbb{Z}$. Therefore without loss of generality we can (and will) assume $c = 1$ from now on, and

$$(3.11) \qquad b_n = \frac{-2r^2}{n^2 - r^2}, \quad \text{where} \quad r := \frac{\nu}{\pi\mu}.$$

From (3.10), (3.11) and the standard formula (see [21, 1.449 - 4]):

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} = \frac{1}{2a^2} - \frac{\pi}{2a} \cot(\pi a), \quad a \in \mathbb{R} \backslash \mathbb{Z}$$

we have

$$\lambda - \mu = \mu \sum_{n=1}^{\infty} b_n = -\mu + \nu \cot(\nu/\mu).$$

Thus

$$\mu = \frac{\nu}{\arctan(\nu/\lambda) + m\pi}, \quad m \in \mathbb{Z}.$$

By (3.11) we have

$$b_n = \frac{-2\nu^2}{(n\pi\mu)^2 - \nu^2} = \frac{-2(\arctan(\nu/\lambda) + m\pi)^2}{(n\pi)^2 - (\arctan(\nu/\lambda) + m\pi)^2}$$

$$= (\arctan(\nu/\lambda) + m\pi)$$

$$\times \left(\frac{1}{n\pi + (\arctan(\nu/\lambda) + m\pi)} - \frac{1}{n\pi - (\arctan(\nu/\lambda) + m\pi)}\right).$$

By (3.9) we have

$$a_n = -n\pi b_n \mu \nu^{-1} = \frac{2n\pi(\arctan(\nu/\lambda) + m\pi)}{(n\pi)^2 - (\arctan(\nu/\lambda) + m\pi)^2}$$

$$= (\arctan(\nu/\lambda) + m\pi)$$

$$\times \left(\frac{1}{n\pi + (\arctan(\nu/\lambda) + m\pi)} + \frac{1}{n\pi - (\arctan(\nu/\lambda) + m\pi)} \right).$$

Therefore we obtain

$$\varphi = y_k^{(\nu,\lambda)} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\arctan(\nu/\lambda) + m\pi}{n\pi + (\arctan(\nu/\lambda) + m\pi)} \{ (x_k^{(\nu,\lambda)} \sin n\pi t) + (y_k^{(\nu,\lambda)} \cos n\pi t) \}$$

$$= \sum_{n \in \mathbb{Z}} \frac{\nu}{n\pi\mu + \nu} \{ (x_k^{(\nu,\lambda)} \sin n\pi t) + (y_k^{(\nu,\lambda)} \cos n\pi t) \}.$$

This proves the theorem. $\square$

Considering the case that $N = \{eK\}$ we obtain:

Corollary 3.4. *Let G/K be a compact symmetric space. Take $\xi \in T_{eK}(G/K) = \mathfrak{m}$. Denote by $\{\sqrt{-1}\,\nu\}$ the set of all distinct eigenvalues of $\mathrm{ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$. Then the principal curvatures of the fiber $\Phi_K^{-1}(eK)$ in the direction of $\hat{\xi} \in T_{\hat{0}}^{\perp} \Phi_K^{-1}(eK)$ are given by*

$$\left\{ 0, \frac{\nu}{n\pi} \right\}_{\nu > 0, \, n \in \mathbb{Z} \setminus \{0\}}.$$

Denoting by $\{x_i^0\}_i$ a basis of $\mathfrak{k}_0$, by $\{y_j^0\}_j$ a basis of $\mathfrak{m}_0$ and by $\{x_k^\nu\}_k$, $\{y_k^\nu\}_k$ the bases defined by (3.2), the eigenfunctions and the multiplicities are given in the following table.

eigenvalue	basis of eigenfunctions	multiplicity
0	$\{x_i^0 \sin n\pi t, y_j^0 \cos n\pi t\}_{n \in \mathbb{Z}_{\geq 1}, \lambda, i, j}$	∞
$\dfrac{\nu}{n\pi}$	$\{x_k^\nu \sin n\pi t - y_k^\nu \cos n\pi t\}_k$	$m(\nu)$

The principal curvatures of $\pi^{-1}(N)$ are given by the following proposition, which is proved by the straightforward computations using the formula (2.13).

Proposition 3.5. *With notation as in Theorem 3.2, for each $\nu > 0$ and each λ we set*

$$\kappa_+ = \kappa_+(\nu, \lambda) := \frac{1}{2}(\lambda + \sqrt{\lambda^2 + \nu^2}),$$

$$\kappa_- = \kappa_-(\nu, \lambda) := \frac{1}{2}(\lambda - \sqrt{\lambda^2 + \nu^2}).$$

Then the principal curvatures of $\pi^{-1}(N)$ in the direction of $\xi \in T_e^{\perp} \pi^{-1}(N) \cong T_{eK}^{\perp} N$ are given by

$$\{0, \, \lambda, \, \kappa_+(\nu, \lambda), \, \kappa_-(\nu, \lambda)\}_{\lambda, \, \nu > 0}.$$

The eigenfunctions and the multiplicities are given in the following table.

eigenvalue	basis of eigenfunctions	multiplicity
0	$\{x_i^0\}_i,\ \{x_r^{(\nu,\perp)}\}_{r,\nu}$	$\dim \mathfrak{k}_0 + \dim T_{eK}^{\perp} N$
λ	$\{y_j^{(0,\lambda)}\}_j$	$m(0,\lambda)$
$\kappa_+(\nu,\lambda)$	$\{\nu x_k^{(\nu,\lambda)} + 2\kappa_+ y_k^{(\nu,\lambda)}\}_k$	$m(\nu,\lambda)$
$\kappa_-(\nu,\lambda)$	$\{\nu x_k^{(\nu,\lambda)} + 2\kappa_- y_k^{(\nu,\lambda)}\}_k$	$m(\nu,\lambda)$

4. The austere property

In this section we study the austere property of PF submanifolds obtained through the parallel transport map Φ_K. Notice that even if N is an austere curvature adapted submanifold of a compact symmetric space G/K, it is not clear in general whether the inverse image $\Phi_K^{-1}(N)$ is austere or not; according to Theorem 3.2 it is not clear whether the set of eigenvalues with *multiplicities* of the shape operator is invariant under the multiplication by (-1) or not. From this reason here we will restrict our attention further to the case that G/K is a sphere and show that in this case N is austere if and only if $\Phi_K^{-1}(N)$ is austere (Theorem 4.1).

Let $S^l(r) = G/K$ denote the l-dimensional sphere of radius $r > 0$, where $l \in \mathbb{Z}_{\geq 1}$ and $(G,K) = (SO(l+1), SO(l))$. Note that in this case any submanifold of G/K is automatically curvature adapted. Let N be a closed submanifold of G/K. Suppose $eK \in N$ and fix $\xi \in T_{eK}^{\perp} N$. Then for $v \in T_{eK}N$ and $\eta \in T_{eK}^{\perp}N$ the Jacobi operator R_ξ satisfies

$$R_\xi(v) := \frac{\|\xi\|^2}{r^2} v, \qquad R_\xi(\eta) := \frac{1}{r^2}\{\|\xi\|^2 \eta - \langle \eta, \xi \rangle \xi\}.$$

Thus in this case the eigenspace decomposition (3.1) of $\mathfrak{m}$ is given by

$$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_\nu, \qquad \text{where} \quad \nu := \|\xi\|/r,$$

$$\mathfrak{m}_0 = \mathbb{R}\xi \subset T_{eK}^{\perp}N,$$
$$\mathfrak{m}_\nu = T_{eK}N \oplus \{\eta \in T_{eK}^{\perp}N \mid \eta \perp \xi\}.$$

In particular we have

$$\mathfrak{m}_0 \cap T_{eK}N = \{0\}, \qquad \mathfrak{m}_0 \cap T_{eK}^{\perp}N = \mathbb{R}\xi,$$
$$\mathfrak{m}_\nu \cap T_{eK}N = T_{eK}N, \quad \mathfrak{m}_\nu \cap T_{eK}^{\perp}N = \{\eta \in T_{eK}^{\perp}N \mid \eta \perp \xi\}.$$

Hence by Proposition 3.5 the principal curvatures of $\pi^{-1}(N)$ in the direction of $\xi \in T_e^{\perp}\pi^{-1}(N) \cong T_{eK}^{\perp}N$ are given by

$$(4.1) \qquad \{0,\ \kappa_+(\|\xi\|/r, \lambda),\ \kappa_-(\|\xi\|/r, \lambda)\}_\lambda.$$

Further by Theorem 3.2 the principal curvatures of a PF submanifold $\Phi_K^{-1}(N)$ in the direction of $\hat{\xi} \in T_{\hat{0}}^{\perp} \Phi^{-1}(N)$ are given by

$$(4.2) \qquad \left\{ 0, \ \frac{\|\xi\|}{rn\pi}, \ \mu(\|\xi\|/r, \lambda, m) \right\}_{\lambda, \ n \in \mathbb{Z} \setminus \{0\}, \ m \in \mathbb{Z}}.$$

Notice that in this case the multiplicities of

$$\lambda, \quad \kappa_+(\|\xi\|/r, \lambda), \quad \kappa_-(\|\xi\|/r, \lambda), \quad \mu(\|\xi\|/r, \lambda, m)$$

are the same for each λ.

Theorem 4.1. *Let N be a closed submanifold of the l-dimensional sphere $S^l(r) = G/K$ of radius $r > 0$, where $l \in \mathbb{Z}_{\geq 1}$ and $(G, K) = (SO(l+1), SO(l))$. Then the the following are equivalent:*

- (i) *N is an austere submanifold of G/K,*
- (ii) *$\pi^{-1}(N)$ is an austere submanifold of G,*
- (iii) *$\Phi_K^{-1}(N)$ is an austere PF submanifold of $V_{\mathfrak{g}}$.*

Proof. "(i) $\Rightarrow$ (ii)": Take $a \in \pi^{-1}(N)$ and $w \in T_a^{\perp}\pi^{-1}(N)$. Set $\eta := d\pi(w) \in T_{aK}^{\perp}N$, $N' := L_a^{-1}(N)$ and $\xi := dL_a^{-1}(\eta) \in T_{eK}^{\perp}N'$. Denote by $v \in T_e^{\perp}\pi^{-1}(N')$ the horizontal lift of ξ. By commutativity of (2.6) (ii), we have $l_a(\pi^{-1}(N')) = \pi^{-1}(N)$ and $dl_a(v) = w$. Thus in order to show the austerity of $A_w^{\pi^{-1}(N)}$ it suffices to show that of $A_v^{\pi^{-1}(N')}$. For each eigenvalue λ of $A_\xi^{N'}$ it follows from the austerity of $A_\xi^{N'}$ that $-\lambda$ is also an eigenvalue of $A_\xi^{N'}$ and

$$(-1) \times \kappa_+(\|\xi\|/r, \lambda) = \kappa_-(\|\xi\|/r, -\lambda),$$
$$(-1) \times \kappa_-(\|\xi\|/r, \lambda) = \kappa_+(\|\xi\|/r, -\lambda).$$

Note that these identities still hold even if the multiplicities are taking account of. This shows that the set (4.1) with multiplicities is invariant under the multiplication by (-1) and (ii) follows.

"(ii) $\Rightarrow$ (i)": Take $aK \in N$ and $\eta \in T_{aK}^{\perp}N$. Denote by $w \in T_a^{\perp}\pi^{-1}(N)$ the horizontal lift of η. Defining N', ξ, v by the above way it suffices to show the austerity of $A_\xi^{N'}$. Let λ be an eigenvalue of $A_\xi^{N'}$. Since the set (4.1) is invariant under the multiplication by (-1) there exist eigenvalues λ' and λ'' of $A_\xi^{N'}$ such that

$$(-1) \times \kappa_+(\|\xi\|/r, \lambda) = \kappa_-(\|\xi\|/r, \lambda'),$$
$$(-1) \times \kappa_-(\|\xi\|/r, \lambda) = \kappa_+(\|\xi\|/r, \lambda'').$$

Note that the function $\mathbb{R} \to \mathbb{R}_{>0}$, $x \mapsto \kappa_+(\|\xi\|/r, x)$ is monotonically increasing. Thus the relation

$$\kappa_-(\|\xi\|/r, x) = -\kappa_+(\|\xi\|/r, -x)$$

shows that also the function $\mathbb{R} \to \mathbb{R}_{<0}$, $x \mapsto \kappa_-(\|\xi\|/r, x)$ is monotonically increasing. From these we obtain $\lambda' = \lambda'' = -\lambda$. Note that this identity still holds even if the multiplicities are taken account of. This shows (i).

"(i) $\Rightarrow$ (iii)": Take $u \in \Phi_K^{-1}(N)$ and $X \in T_u^{\perp}\Phi_K^{-1}(N)$. Also take $g \in P(G, G \times \{e\})$ so that $u = g * \hat{0}$. Set $a := g(0) = \Phi(u)$, $\eta := d\Phi_K(X) \in T_{aK}^{\perp}N$,

$N' := L_a^{-1}(N)$ and $\xi := dL_a^{-1}(\eta) \in T_{eK}^\perp N'$. Denote by $\hat{\xi} \in T_{\hat{0}}^\perp \Phi_K^{-1}(N')$ the horizontal lift of ξ. By commutativity of (2.7) we have $g * (\Phi_K^{-1}(N')) = \Phi_K^{-1}(N)$ and $dg * (\hat{\xi}) = X$. Thus in order to show the austerity of $A_X^{\Phi_K^{-1}(N)}$ it suffices to show that of $A_{\hat{\xi}}^{\Phi_K^{-1}(N')}$. For each λ it follows from the austerity of $A_\xi^{N'}$ that $-\lambda$ is also an eigenvalue of $A_\xi^{N'}$ and

$$(-1) \times \mu(\|\xi\|/r, \lambda, m) = \mu(\|\xi\|/r, -\lambda, -m).$$

Note that this identity still hold even if the multiplicities are taken account of. This shows that the set

$$\{\mu(\|\xi\|/r, \lambda, m)\}_{\lambda, \ m \in \mathbb{Z}}$$

with multiplicities is invariant under the multiplication by (-1). This together with (4.2) shows the austerity of $A_{\hat{\xi}}^{\Phi_K^{-1}(N')}$ and (iii) follows.

"(iii) $\Rightarrow$ (i)": Take $aK \in N$ and $\eta \in T_{aK}^\perp N$. Choose $u \in \Phi_K^{-1}(a)$. Denote by $X \in T_u^\perp \Phi_K^{-1}(N)$ the horizontal lift of η. Defining N', ξ, $\hat{\xi}$ by the above way it suffices to show the austerity of $A_\xi^{N'}$. From (4.2) and the assumption the set

$$\{\mu(\|\xi\|/r, \lambda, m)\}_{\lambda, \ m \in \mathbb{Z}}$$

with multiplicities is invariant under the multiplication by (-1). Thus for each eigenvalue λ of $A_\xi^{N'}$ and each $m \in \mathbb{Z}$ there exists an eigenvalue λ' of $A_\xi^{N'}$ and $m' \in \mathbb{Z}$ such that

$$(-1) \times \mu(\|\xi\|/r, \lambda, m) = \mu(\|\xi\|/r, \lambda', m').$$

That is,

$$-\arctan \frac{\|\xi\|}{r\lambda} - m\pi = \arctan \frac{\|\xi\|}{r\lambda'} + m'\pi.$$

Since $-\pi/2 < \arctan x < \pi/2$, the above equality shows $m' = -m$ and

$$\lambda' = -\lambda.$$

Note that this identity still holds even if the multiplicities are taking account of. This shows (i). □

Example 4.2. Ikawa, Sakai and Tasaki ([12, Theorem 5.1]) classified austere submanifolds of the standard sphere given as orbits of s-representations of irreducible Riemannian symmetric pairs. Applying Theorem 4.1 to their result we obtain austere PF submanifolds as follows. Let (U, L) be a compact Riemannian symmetric pair, where L is connected. Denote by $\mathfrak{u} = \mathfrak{l} \oplus \mathfrak{p}$ the canonical decomposition and by $\mathrm{Ad} : L \to SO(\mathfrak{p})$ the isotropy representation. If an orbit $\mathrm{Ad}(L) \cdot x$ through $x \in \mathfrak{p}$ is an austere submanifold of the hypersphere $S(\|x\|)$ in $\mathfrak{p}$ then the orbit $P(SO(\mathfrak{p}), \mathrm{Ad}(L) \times SO(\mathfrak{p})_x) * \hat{0}$ is an austere PF submanifold of the Hilbert space $V_{\mathfrak{o}(\mathfrak{p})}$.

5. The arid property

In this section we study the arid property of PF submanifolds obtained through the parallel transport map Φ_K. The main theorem is the following:

Theorem 5.1. *Let G be a connected compact semisimple Lie group equipped with a bi-invariant Riemannian metric induced from a negative multiple of the Killing form and K a symmetric subgroup of G such that the pair (G, K) effective. If N is an arid submanifold of the symmetric space G/K then*

(i) *$\pi^{-1}(N)$ is an arid submanifold of G, and*
(ii) *$\Phi_K^{-1}(N)$ is an arid PF submanifold of $V_{\mathfrak{g}}$.*

From this theorem we obtain the following corollary:

Corollary 5.2. *Let M be an irreducible Riemannian symmetric space of compact type. Denote by G the identity component of the group of isometries of M and by K the isotropy subgroup of G at a fixed $p \in M$. If N is an arid submanifold of $M = G/K$ then $\Phi_K^{-1}(N)$ is an arid PF submanifold of $V_{\mathfrak{g}}$.*

To prove Theorem 5.1 we need the following lemma:

Lemma 5.3. *Let $\mathcal{M}$ and $\mathcal{B}$ be Riemannian Hilbert manifolds, $\phi : \mathcal{M} \to \mathcal{B}$ a Riemannian submersion and N a closed submanifold of $\mathcal{B}$. Fix $p \in \phi^{-1}(N)$ and $X \in (T_p^\perp \phi^{-1}(N)) \backslash \{0\}$. Suppose that $\varphi_{\mathcal{M}}$ is an isometry of $\mathcal{M}$ fixing p, that $\varphi_{\mathcal{B}}$ is an isometry of $\mathcal{B}$ fixing $\phi(p)$ and that the diagram*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\varphi_{\mathcal{M}}} & \mathcal{M} \\ \phi \downarrow & & \phi \downarrow \\ \mathcal{B} & \xrightarrow{\varphi_{\mathcal{B}}} & \mathcal{B} \end{array}$$

commutes. Then the following are equivalent:

(i) *$\varphi_{\mathcal{M}}$ satisfies $\varphi_{\mathcal{M}}(\phi^{-1}(N)) = \phi^{-1}(N)$ and $d\varphi_{\mathcal{M}}(X) \neq X$.*
(ii) *$\varphi_{\mathcal{B}}$ satisfies $\varphi_{\mathcal{B}}(N) = N$ and $d\varphi_{\mathcal{B}}(d\phi(X)) \neq d\phi(X)$.*

Proof. It is easily seen that the condition $\varphi_{\mathcal{M}}(\phi^{-1}(N)) = \phi^{-1}(N)$ is equivalent to the condition $\varphi_{\mathcal{B}}(N) = N$. Then by commutativity of the diagram

$$\begin{array}{ccc} T_p^\perp \phi^{-1}(N) & \xrightarrow{d\varphi_{\mathcal{M}}} & T_p^\perp \phi^{-1}(N) \\ d\phi \downarrow & & d\phi \downarrow \\ T_{\phi(p)}^\perp N & \xrightarrow{d\varphi_{\mathcal{B}}} & T_{\phi(p)}^\perp N, \end{array}$$

the condition $d\varphi_{\mathcal{M}}(X) \neq X$ is equivalent to the condition $d\varphi_{\mathcal{B}}(d\phi(X)) \neq d\phi(X)$. This proves the lemma. $\square$

Proof of Theorem 5.1. (i) Take $a \in \pi^{-1}(N)$ and $w \in (T_a^\perp \pi^{-1}(N)) \backslash \{0\}$. Set $\eta := d\pi(w) \in T_{aK}^\perp N$, $N' := L_a^{-1}(N)$ and $\xi := dL_a^{-1}(\eta) \in (T_{eK}^\perp N') \backslash \{0\}$. Denote by $v \in (T_e^\perp \pi^{-1}(N')) \backslash \{0\}$ the horizontal lift of ξ. By commutativity of (2.6) (ii) we have $l_a(\pi^{-1}(N')) = \pi^{-1}(N)$ and $dl_a(v) = w$. Thus in order to show the existence of an isometry φ_w with respect to w it suffices to construct an isometry φ_v with respect to v. Let φ_η be an isometry with respect to η. Then

an isometry φ_ξ with respect to ξ is defined by $\varphi_\xi := L_a^{-1} \circ \varphi_\eta \circ L_a$. Now we define φ_v as follows. Denote by $I(G/K)$ the group of isometries of G/K. From the assumption the map $L : G \to I(G/K)$, $a \mapsto L_a$ is a Lie group isomorphism onto the identity component $I_0(G/K)$ ([9, Theorem 4.1 in Chapter V]). Since $I_0(G/K)$ is a normal subgroup of $I(G/K)$ an automorphism $\varphi_v : G \to G$, $b \mapsto \varphi_v(b)$ is defined by

$$(5.1) \qquad\qquad L_{\varphi_v(b)} := \varphi_\xi \circ L_b \circ \varphi_\xi^{-1}.$$

Since the bi-invariant Riemannian metric on G is induced from the Killing form of $\mathfrak{g}$ the automorphism φ_v is an isometry of G. Moreover since

$$\varphi_\xi \circ \pi(b) = \varphi_\xi(bK) = \varphi_\xi \circ L_b(eK) = \varphi_\xi \circ L_b \circ \varphi_\xi^{-1}(eK) \quad \text{and}$$

$$\pi \circ \varphi_v(b) = \varphi_v(b)K = L_{\varphi_v(b)}(eK) = \varphi_\xi \circ L_b \circ \varphi_\xi^{-1}(eK)$$

hold for all $b \in G$ it follows from Lemma 5.3 that φ_v is an isometry with respect to v. This proves (i).

(ii) Take $u \in \Phi_K^{-1}(N)$ and $X \in (T_u^\perp \Phi_K^{-1}(N)) \backslash \{0\}$. Also take $g \in P(G, G \times \{e\})$ so that $u = g * \hat{0}$. Set $a := \Phi(u) = g(0)$ and $\eta := d\Phi_K(X) \in (T_{aK}^\perp N) \backslash \{0\}$. Define N', ξ, v as in the above (i). Denote by $\hat{\xi} \in (T_{\hat{0}}^\perp \Phi_K^{-1}(N')) \backslash \{0\}$ the horizontal lift of ξ with respect to the Riemannian submersion $\Phi_K : V_\mathfrak{g} \to G/K$. By commutativity of (2.7) we have $g * \Phi_K^{-1}(N') = \Phi_K^{-1}(N)$ and $d(g*)\hat{\xi} = X$. Thus in order to show the existence of an isometry φ_X with respect to X it suffices to construct an isometry $\varphi_{\hat{\xi}}$ with respect to $\hat{\xi}$. By the same way as in (i) we can define an isometry φ_v with respect to $v \in (T_e^\perp \pi^{-1}(N')) \backslash \{0\}$. Moreover we define a linear orthogonal transformation $\varphi_{\hat{\xi}}$ of $V_\mathfrak{g}$ by

$$(5.2) \qquad\qquad \varphi_{\hat{\xi}}(u) := d\varphi_v \circ u, \quad u \in V_\mathfrak{g}.$$

Since φ_v is an automorphism of G we have $\varphi_{\hat{\xi}}(g * \hat{0}) = (\varphi_v \circ g) * \hat{0}$ for all $g \in \mathcal{G}$. This together with (2.3) implies that the following diagram commutes:

$$
\begin{array}{ccc}
V_\mathfrak{g} & \xrightarrow{\varphi_{\hat{\xi}}} & V_\mathfrak{g} \\
\Phi \downarrow & & \Phi \downarrow \\
G & \xrightarrow{\varphi_v} & G.
\end{array}
$$

Thus by Lemma 5.3 $\varphi_{\hat{\xi}}$ is an isometry with respect to $\hat{\xi}$ and (ii) follows. $\qquad\square$

Example 5.4. Let $m, n \in \mathbb{Z}_{\geq 2}$. Set $\bar{M} := S^{mn-1}(\sqrt{m}) \subset \mathbb{R}^{mn}$ and

$$M := \underbrace{S^{n-1}(1) \times \cdots \times S^{n-1}(1)}_{m \text{ times}} \subset \bar{M}.$$

Ikawa, Sakai and Tasaki [12, Example 2.3] showed that if $m = 2$ then M is a weakly reflective submanifold of $\bar{M}$. Taketomi [29, Proposition 3.1] showed that if $m \geq 3$ then M is an arid submanifold of $\bar{M}$ and is not an austere submanifold (therefore not a weakly reflective submanifold) of $\bar{M}$.

Set $G := SO(mn)$ and $K := SO(mn - 1)$ so that $\bar{M} = G/K$. If $m = 2$ then $\Phi_K^{-1}(M)$ is a weakly reflective PF submanifold of $V_\mathfrak{g}$ by [22, Theorem 8] and is not a totally geodesic PF submanifold of $V_\mathfrak{g}$ by [22, Theorem 3]. If $m \geq 3$

then $\Phi_K^{-1}(M)$ is an arid PF submanifold of $V_{\mathfrak{g}}$ by Theorem 5.1 and is not an austere PF submanifold (therefore not a weakly reflective PF submanifold) of $V_{\mathfrak{g}}$ by Theorem 4.1.

In general it is not clear that conversely the arid property of $\Phi_K^{-1}(N)$ implies the arid property of N or not. However the next theorem shows that under suitable assumptions the arid property of $\Phi_K^{-1}(N)$ is equivalent to that of N. To explain this we now introduce some terminologies which are used in a context slightly wider than [29]. Let M be a submanifold immersed in a finite dimensional Riemannian manifold $\bar{M}$. Denote by $I(\bar{M})$ the group of isometries of $\bar{M}$. For a closed subgroup $\mathfrak{G}$ of $I(\bar{M})$ we say that M is $\mathfrak{G}$-*arid* (resp. $\mathfrak{G}$-*weakly reflective*) if for each $p \in M$ and each $\xi \in T_p^\perp M \backslash \{0\}$ there exists an isometry φ_ξ (resp. reflection ν_ξ) with respect to ξ satisfying $\varphi_\xi \in \mathfrak{G}_p$, where $\mathfrak{G}_p$ denotes the isotropy subgroup of $\mathfrak{G}$ at p. Clearly $\mathfrak{G}$-weakly reflective submanifols are $\mathfrak{G}$-arid submanifolds. If $\mathfrak{G} = I(\bar{M})$ then "$\mathfrak{G}$-arid" (resp. "$\mathfrak{G}$-weakly reflective") is nothing but "arid" (resp. "weakly reflective"). The same concepts and relation are also valid for PF submanifolds in Hilbert spaces.

From now on, as in Section 2, we denote by G a connected compact Lie group equipped with a bi-invariant Riemannian metric, K a closed subgroup of G and G/K the compact normal homogeneous space. Recall that the Hilbert Lie group $\mathcal{G} := H^1([0,1], G)$ acts on $V_{\mathfrak{g}}$ via the gauge transformations (2.1). We denote by $\mathcal{G}_u$ the isotropy subgroup of $\mathcal{G}$ at $u \in V_{\mathfrak{g}}$. If $u = \hat{0}$ then $\mathcal{G}_{\hat{0}}$ is the set of constant paths $\hat{G} := \{\hat{b} \in G \mid b \in G\}$. Thus if $u = g * \hat{0}$ for some $g \in \mathcal{G}$ then $\mathcal{G}_u = g\hat{G}g^{-1}$. Recall also that $G \times G$ acts on G by the formula (2.2). We denote by $(G \times G)_a = (a,e)\Delta G(a,e)^{-1}$ the isotropy subgroup of $G \times G$ at $a \in G$, where $\Delta G := \{(b,b) \mid b \in G\}$. Finally we recall the G-action on G/K defined by $b \cdot (aK) := (ba)K$ for $a, b \in G$. We denote by $G_{aK} = aKa^{-1}$ the isotropy subgroup of G at $aK \in G/K$.

Theorem 5.5.
(i) *Let N be a closed submanifold of a compact Lie group G equipped with a bi-invariant Riemannian metric. Then the following are equivalent:*
   (a) *N is a $(G \times G)$-arid submanifold of G.*
   (b) *$\Phi^{-1}(N)$ is a $\mathcal{G}$-arid PF submanifold of $V_{\mathfrak{g}}$.*
(ii) *Let N be a closed submanifold of a compact normal homogeneous space G/K. Then the following are equivalent:*
   (a) *N is a G-arid submanifold of G/K.*
   (b) *$\pi^{-1}(N)$ is a $(G \times K)$-arid submanifold of G.*
   (c) *$\Phi_K^{-1}(N)$ is a $P(G, G \times K)$-arid PF submanifold of $V_{\mathfrak{g}}$.*

Proof. (i) "(a) $\Rightarrow$ (b)": Take $u \in \Phi^{-1}(N)$ and $X \in (T_u^\perp \Phi^{-1}(N)) \backslash \{0\}$. Also take $g \in P(G, G \times \{e\})$ so that $u = g * \hat{0}$. Set $a := \Phi(u) = g(0)$, $\eta := d\Phi(X) \in (T_a^\perp N) \backslash \{0\}$, $N' := a^{-1}N$ and $\xi := dL_a^{-1}(\eta) \in (T_e^\perp N') \backslash \{0\}$. Denote by $\hat{\xi} \in (T_{\hat{0}}^\perp \Phi^{-1}(N')) \backslash \{0\}$ the horizontal lift of ξ. By commutativity of (2.6) (i) we have $g * (\Phi^{-1}(N')) = \Phi^{-1}(N)$ and $(dg*)\hat{\xi} = X$. Thus in order to show the existence of an isometry φ_X with respect to X satisfying $\varphi_X \in \mathcal{G}_u$ it suffices to construct an isometry $\varphi_{\hat{\xi}}$ with respect to $\hat{\xi}$ satisfying $\varphi_{\hat{\xi}} \in \mathcal{G}_{\hat{0}}$.

Let φ_η be an isometry with respect to η which is given by $\varphi_\eta(c) = b'cb^{-1}$ for some $(b', b) \in (G \times G)_a$. Then an isometry φ_ξ with respect to ξ is defined by $\varphi_\xi := (a, e)^{-1} \circ \varphi_\eta \circ (a, e)$, that is, $\varphi_\xi(c) := bcb^{-1}$ for $c \in G$. Now we define a linear orthogonal transformation $\varphi_{\hat{\xi}}$ of $V_{\mathfrak{g}}$ by

$$\varphi_{\hat{\xi}}(u) := d\varphi_\xi \circ u = bub^{-1} = \hat{b} * u, \quad u \in V_{\mathfrak{g}}.$$

Clearly $\varphi_{\hat{\xi}} \in \mathcal{G}_{\hat{0}}$. Moreover by (2.3) the following diagram commutes:

$$
\begin{array}{ccc}
V_{\mathfrak{g}} & \xrightarrow{\varphi_{\hat{\xi}}} & V_{\mathfrak{g}} \\
\Phi \downarrow & & \Phi \downarrow \\
G & \xrightarrow{\varphi_\xi} & G.
\end{array}
$$

From Lemma 5.3 $\varphi_{\hat{\xi}}$ is an isometry with respect to $\hat{\xi}$ and (b) follows.

(i) "(b) $\Rightarrow$ (a)": Take $a \in N$ and $\eta \in (T_a^\perp N)\backslash\{0\}$. Fix $u \in \Phi^{-1}(a)$. Denote by $X \in (T_u^\perp \Phi^{-1}(N))\backslash\{0\}$ the horizontal lift of η. Take $g \in P(G, G \times \{e\})$ so that $g * \hat{0} = u$ and define N', ξ, $\hat{\xi}$ as in the above (i). Let φ_X be an isometry with respect to X satisfying $\varphi_X \in \mathcal{G}_u$. Then an isometry $\varphi_{\hat{\xi}}$ with respect to $\hat{\xi} \in (T_{\hat{0}}^\perp \Phi^{-1}(N'))\backslash\{0\}$ is defined by $\varphi_{\hat{\xi}} := (g*)^{-1} \circ \varphi_X \circ (g*)$. By definition $\varphi_{\hat{\xi}} \in \mathcal{G}_{\hat{0}}$ and thus there exists $b \in G$ such that $\varphi_{\hat{\xi}}(u) = bub^{-1}$. Hence defining an isometry φ_ξ of G by $\varphi_\xi(c) := bcb^{-1}$ for $c \in G$ it follows from Lemma 5.3 that φ_ξ is an isometry with respect to ξ satisfying $\varphi_\xi \in (G \times G)_e$. Therefore an isometry φ_η with respect to η is defined by $\varphi_\eta := l_a \circ \varphi_\xi \circ l_a^{-1}$ so that $\varphi_\eta \in (G \times G)_a$. This shows (a).

(ii) "(a) $\Rightarrow$ (b)": Take $a \in \pi^{-1}(N)$ and $w \in (T_a^\perp \pi^{-1}(N))\backslash\{0\}$. Set $\eta := d\pi(w) \in (T_{aK}^\perp N)\backslash\{0\}$, $N' := L_a^{-1}(N)$ and $\xi := dL_a^{-1}(\eta) \in (T_{eK}^\perp N')\backslash\{0\}$. Denote by $v \in (T_e^\perp \pi^{-1}(N'))\backslash\{0\}$ the horizontal lift of ξ. By commutativity of (2.6) (ii), we have $l_a(\pi^{-1}(N')) = \pi^{-1}(N)$ and $dl_a(v) = w$. Thus in order to show the existence of an isometry φ_w with respect to w satisfying $\varphi_w \in (G \times K)_a = (a, e)\Delta K(a, e)^{-1}$ it suffices to construct an isometry φ_v with respect to v satisfying $\varphi_v \in (G \times K)_e = \Delta K$. Let φ_η be an isometry with respect to η which is given by $\varphi_\eta(cK) = (bc)K$ for some $b \in G_{aK}$. Then there exists $k \in K$ such that $b = aka^{-1}$. Thus an isometry φ_ξ with respect to ξ is defined by $\varphi_\xi := L_a^{-1} \circ \varphi_\xi \circ L_a$, that is, $\varphi_\xi = L_k$. Define an isometry φ_v of G by

$$\varphi_v(c) := kck^{-1}, \quad c \in G.$$

Clearly $\varphi_v \in (G \times K)_e$. Moreover the following diagram commutes:

$$
\begin{array}{ccc}
G & \xrightarrow{\varphi_v} & G \\
\pi \downarrow & & \pi \downarrow \\
G/K & \xrightarrow{\varphi_\xi} & G/K.
\end{array}
$$

Thus by Lemma 5.3 φ_v is an isometry with respect to v and (b) follows.

(ii) "(b) $\Rightarrow$ (a)": Take $aK \in N$ and $\eta \in T_{aK}^\perp N$. Denote by $w \in T_a^\perp \pi^{-1}(N)$ the horizontal lift of η. Define N', ξ, v as above. Let φ_w be an isometry with

respect to w satisfying $\varphi_w \in (G \times K)_a$. Then an isometry with respect to v is defined by $\varphi_v := l_a^{-1} \circ \varphi_w \circ l_a$ so that $\varphi_v \in (G \times K)_e$. Then there exists $k \in K$ such that $\varphi_v(c) = kck^{-1}$. Thus defining an isometry φ_ξ of G/K by $\varphi_\xi := L_k$ it follows from Lemma 5.3 that φ_ξ is an isometry with respect to ξ satisfying $\varphi_\xi \in G_{eK}$. Hence an isometry φ_η with respect to η is defined by $\varphi_\eta := l_a \circ \varphi_\xi \circ l_a^{-1}$ so that $\varphi_\eta \in G_{aK}$. This shows (b).

Using the fact $gP(G, G \times K)_{\hat{0}}g^{-1} = P(G, G \times K)_{g*\hat{0}}$ for $g \in P(G, G \times \{e\})$ the equivalence of (b) and (c) of (ii) follows by the similar arguments to (i). □

Corollary 5.6. *Let G, G/K be as in Theorem 5.5.*

(i) *Let H be a closed subgroup of $G \times G$. Then the following are equivalent:*
   (a) *an orbit $H \cdot a$ through $a \in G$ is an H-arid submanifold of G,*
   (b) *an orbit $P(G, H) * u$ through $u \in \Phi(a)$ is a $P(G, H)$-arid PF submanifold of $V_{\mathfrak{g}}$.*
(ii) *Let K' be a closed subgroup of G. Then the following are equivalent:*
   (a) *an orbit $K' \cdot aK$ through $aK \in G/K$ is a K'-arid submanifold of G/K,*
   (b) *an orbit $(K' \times K) \cdot a$ through $a \in G$ is a $(K' \times K)$-arid submanifold of G,*
   (c) *an orbit $P(G, K' \times K) * u$ through $u \in \Phi^{-1}(a)$ is a $P(G, K' \times K)$-arid PF submanifold of $V_{\mathfrak{g}}$.*

Proof. (i) Set $H' := (a, e)^{-1}H(a, e)$. Then we have $H \cdot a = l_a(H' \cdot e)$. Take $g \in P(G, G \times \{e\})$ so that $g * \hat{0} = u$. From (2.4) and (2.6) (i) it follows that $P(G, H) * u = g * (P(G, H') * \hat{0})$. From this fact we can assume without loss of generality that $a = e$. Moreover by homogeneity it suffices to consider normal vectors only at $e \in G$ or $\hat{0} \in V_{\mathfrak{g}}$. By similar arguments as in Theorem 5.5 (i) our claim follows. (ii) Similarly we can reduce the case $a = e$ and the assertion follows by similar arguments as in Theorem 5.5 (ii). □

Example 5.7. Let (U, L) be a compact Riemannian symmetric pair where L connected. Denote by $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ the canonical decomposition, by $\mathrm{Ad} : L \to SO(\mathfrak{p})$ the isotropy representation and by S the standard sphere in $\mathfrak{p}$.

Let us first show that there are examples of $\mathrm{Ad}(L)$-orbits which are $\mathrm{Ad}(L)$-arid submanifolds in S. By Taketomi's result ([29, Proposition 4.4]) an orbit $N := \mathrm{Ad}(L)w$ through $w \in S$ is an $\mathrm{Ad}(L)$-arid submanifold of S if and only if N is an isolated orbit of the $\mathrm{Ad}(L)$-action on S. One can find such isolated orbits by considering the fundamental Weyl Chamber. Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and denote by F the fundamental system of the restricted root system with respect to $\mathfrak{a}$. The fundamental Weyl chamber is defined by

$$C := \{w \in \mathfrak{a} \mid \forall \alpha \in F,\ \alpha(w) > 0\}$$

with closure

$$\bar{C} := \{w \in \mathfrak{a} \mid \forall \alpha \in F,\ \alpha(w) \geq 0\}.$$

It is known ([10, Lemma 1.2]) that $\bar{C}$ is decomposed by

$$\bar{C} = \coprod_{\Delta:\ \text{subset of } F} C^{\Delta} \qquad : \text{disjoint union,}$$

$$C^\Delta := \{w \in \mathfrak{a} \mid \forall \alpha \in \Delta, \ \alpha(w) > 0 \ \text{ and } \ \forall \beta \in F \backslash \Delta, \ \beta(w) = 0\}.$$

If a subset Δ consists of only one element then $\dim C^\Delta = 1$ and thus the intersection $S \cap C^\Delta$ consists of only one point, which implies that in this case the orbit $\mathrm{Ad}(L)w$ through $w \in C^\Delta$ is isolated. In this way we can obtain examples of $\mathrm{Ad}(L)$-arid submanifolds in the standard sphere S. Notice that from the classification result of austere $\mathrm{Ad}(L)$-orbits ([12, Theorem 5.1]), in particular we can choose $\mathrm{Ad}(L)$-arid orbits which are *not* austere.

Applying Corollary 5.6 (ii) to such examples we obtain the orbit

$$P(SO(\mathfrak{p}), \mathrm{Ad}(L) \times SO(\mathfrak{p})_w) * \hat{0}$$

which is an $P(SO(\mathfrak{p}), \mathrm{Ad}(L) \times SO(\mathfrak{p})_w)$-arid PF submanifolds in the Hilbert space $V_{\mathfrak{o}(\mathfrak{p})}$. Moreover by Theorem 4.1 such an arid PF submanifold is *not* an austere (therefore *not* a weakly reflective) PF submanifold in $V_{\mathfrak{o}(\mathfrak{p})}$.

Compared to Corollary 5.6 (i) the following proposition covers only H-orbits through $e \in G$. However the isometry φ_ξ with respect to each normal vector ξ at $e \in G$ need *not* belong to the isotropy subgroup H_e at $e \in G$.

Proposition 5.8. *Let G be a connected compact Lie group with a bi-invariant Riemannian metric and H be a closed subgroup of $G \times G$. Suppose that the orbit $H \cdot e$ through $e \in G$ is an arid submanifold of G such that for each $\xi \in (T_e^\perp(H \cdot e)) \backslash \{0\}$ an isometry φ_ξ with respect to ξ is an automorphism of G. Then the orbit $P(G, H) * \hat{0}$ through $\hat{0} \in V_{\mathfrak{g}}$ is an arid PF submanifold of $V_{\mathfrak{g}}$.*

Proof. Let φ_ξ be an isometry with respect to $\xi \in (T_e^\perp(H \cdot e)) \backslash \{0\}$ which is an automorphism of G. Then an isometry $\varphi_{\hat{\xi}}$ with respect to $\hat{\xi} \in (T_{\hat{0}}^\perp P(G, H) * \hat{0}) \backslash \{0\}$ is defined similarly to (5.2). By homogeneity of $P(G, H) * \hat{0}$ our claim follows. $\qquad\square$

Now we see an example of an arid submanifold $H \cdot e$ satisfying the condition in Proposition 5.8. Although the following $H \cdot e$ can be shown to be arid by applying Theorem 5.1 (i) to Taketomi's example [29, Proposition 3.1], here we give a direct proof in order to see an isometry with respect to each normal vector explicitly.

Example 5.9. Set $G := SO(9)$. Denote by E the 3×3 unit matrix. Define $Q \in G$ by

$$Q := \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2}E & \sqrt{3}E & E \\ \sqrt{2}E & -\sqrt{3}E & E \\ \sqrt{2}E & 0 & -2E \end{bmatrix}.$$

Set $K := Q(\{1\} \times SO(8))Q^{-1}$, $K' := SO(3) \times SO(3) \times SO(3)$ and $H := K' \times K$. Then the tangent space of the orbit $H \cdot e$ is given by

$$T_e(H \cdot e) = \mathfrak{k}' + \mathfrak{k} = \mathfrak{k}' + Q(0 \oplus \mathfrak{o}(8))Q^{-1} = Q(Q^{-1}\mathfrak{k}'Q + (0 \oplus \mathfrak{o}(8)))Q^{-1}.$$

Then it follows from the direct computations that each $X \in Q^{-1}T_e^{\perp}(H \cdot e)Q$ is written by

$$X = \begin{bmatrix} 0 & S & T \\ -S & 0 & 0 \\ -T & 0 & 0 \end{bmatrix},$$

$$S := \begin{bmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T := \begin{bmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad s,t \in \mathbb{R}.$$

The calculation of QXQ^{-1} shows that each $Y \in T_e^{\perp}(H \cdot e)$ is written by

$$Y = \begin{bmatrix} 0 & U & V \\ -U & 0 & W \\ -V & -W & 0 \end{bmatrix},$$

$$U := \begin{bmatrix} -2x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V := \begin{bmatrix} -x-y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad W := \begin{bmatrix} x-y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for $x, y \in \mathbb{R}$. For each $(i,j) \in \{(1,2),(1,3),(2,3)\}$ we define $P_{i,j} \in O(9)$ by

$$P_{1,2} := \begin{bmatrix} 0 & E & 0 \\ E & 0 & 0 \\ 0 & 0 & E \end{bmatrix}, \quad P_{1,3} := \begin{bmatrix} 0 & 0 & E \\ 0 & E & 0 \\ E & 0 & 0 \end{bmatrix}, \quad P_{2,3} := \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & E \\ 0 & E & 0 \end{bmatrix}$$

and define an automorphism φ_{ij} of G by

$$\varphi_{ij}(A) := P_{ij}AP_{ij}^{-1}, \quad A \in G.$$

Then for each $\xi \in T_e^{\perp}(H \cdot e)$ there exists (i,j) such that φ_{ij} is an isometry with respect to ξ. Thus $H \cdot e$ is an arid submanifold of G. Since φ_{ij} is an automorphism of G it follows from Proposition 5.8 that the orbit $P(G, H) * \hat{0}$ is an arid PF submanifold of $V_{\mathfrak{g}}$. Notice that we can not apply Corollary 5.6 (i) to this example since φ_{ij} is *not* an inner automorphism of G and thus not belong to the isotropy subgroup H_e at $e \in G$.

6. Open problems

Recall that an isometric action of a compact Lie group on a Riemannian manifold M is called *polar* if there exists a closed connected submanifold S of M which meets every orbit orthogonally. If S is flat in the induced metric then such an action is called *hyperpolar* ([8]). Similarly an isometric PF action of a Hilbert Lie group on a Hilbert space V is called *hyperpolar* ([8]) if there exists a closed affine subspace S of V which meets every orbit orthogonally. It was shown ([30]) that the $P(G, H)$-action is hyperpolar if the H-action on G is hyperpolar. Hyperpolar actions on irreducible Riemannian symmetric spaces of compact type were classified by Kollross [17].

In this paper and the previous paper [22] we have seen examples of minimal PF submanifolds with certain symmetries, such as weakly reflective PF submanifolds, austere PF submanifolds and arid PF submanifolds. At present all

such examples are obtained as orbits of hyperpolar $P(G, H)$-actions. Then it is interesting to study the following problem:

Problem. Determine minimal orbits in a hyperpolar $P(G, H)$-action and classify the symmetric properties they have.

It is noted ([8]) that a hyperpolar $P(G, H)$-action can be thought of the isotropy representation of an affine Kac-Moody symmetric space, which is an infinite dimensional version of a symmetric space proposed by Terng [31] and established by Heintze [6], Popescu [26] and Freyn [4]. In the finite dimensional case minimal orbits of the isotropy representation of symmetric spaces have been systematically studied (e.g. [11], [12]). It may be interesting to ask whether there are similar properties for minimal orbits in the isotropy representations of Kac-Moody symmetric spaces.

In order to study the above problem it is noted ([15], [7]) that a $P(G, H)$-orbit is minimal if and only if the H-orbit is minimal. This shows that the determination of minimal $P(G, H)$-orbits can be reduced to a finite dimensional problem. However the classification of symmetric properties seems not easy. As we have seen in Section 4 the austere property via the parallel transport map is not clear except for the spherical case. Moreover even if we classify all austere orbits in H- or $P(G, H)$-actions it seems very difficult in general to classify all weakly reflective orbits; it is very hard to assert one austere orbit is *not* weakly reflective.

In connection with such a classification problem the author is now giving attention to the structure of weakly reflective submanifolds. Let M be a weakly reflective submanifold of a Riemannian manifold $\bar{M}$. Denote by $I(\bar{M})$ the group of isometries of $\bar{M}$ and by $I_0(\bar{M})$ its identity component. There are at least two kinds of weakly reflective submanifolds:

  (a) for each $p \in M$ and each $\xi \in T_p^\perp M$ there exists $\nu_\xi \in I_0(\bar{M})$ (: *identity component*) such that $\nu_\xi(p) = p$, $d\nu_\xi(\xi) = -\xi$ and $\nu_\xi(M) = M$.
  (b) for each $p \in M$ there exists an *involutive* isometry $\nu_p \in I(\bar{M})$ which is *independent* of the choice of $\xi \in T_p^\perp M$ such that $\nu_p(p) = p$, $d\nu_p(\xi) = -\xi$ and $\nu_p(M) = M$.

For example, consider an isometric action of cohomogeneity one on $\bar{M}$. It is known ([27], [12]) that in this case any singular orbit M is a weakly reflective submanifold of $\bar{M}$. More precisely if M has one codimension in $\bar{M}$ then M satisfies the condition (b) and otherwise M satisfies the condition (a). Note that there exist examples of weakly reflective submanifolds which satisfy both conditions (a) and (b): a weakly reflective submanifold M reviewed in Example 5.4 (the case $m = 2$) satisfies (b) for all $n \in \mathbb{Z}_{\geq 2}$ and in particular if n is even then it also satisfies (a). Although examples by Ohno [23] and by Kimura-Mashimo [14] satisfy the condition (a), Enoyoshi's example ([3]) does not satisfy (a) but (b). It should be also noted ([22]) that under suitable assumptions if M satisfies the condition (b) then its inverse image under the parallel transport map is a weakly reflective PF submanifold satisfying the condition (b). It can be a problem to consider whether weakly reflective submanifolds which does not

satisfy both conditions (a) and (b) exist or not. Also for arid submanifolds similar types may work and might be useful to study their structure.

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(M. Morimoto) Department of Mathematics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumyoshi-ku, Osaka 558-8585 JAPAN.

Email address: mmasahiro0408@gmail.com