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Abstract

Our aim in this paper is to characterize local Muckenhoupt weighted Lebesgue spaces with variable exponent by compactly supported smooth wavelets. We also investigate necessary and sufficient conditions for the corresponding modular inequalities to hold. One big achievement is that the weights with exponetial growth can be handled in the framework of variable exponents.

Key words: variable exponent, wavelet, local Muckenhoupt weight, modular inequality
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1 Introduction

Wavelets with proper decay and smoothness give us characterizations of various function spaces. In fact we can obtain norms equivalent to those spaces by using some square functions involving wavelet coefficients. The first author [13] and Kopaliani [17] have initially and independently obtained the wavelet characterizations of Lebesgue spaces with variable exponent. Later the characterizations have been generalized by [15] to the Muckenhoupt weighted setting.

The theory of Lebesgue spaces with variable exponent goes back to [27]. After that, Nakano investigated Lebesgue spaces with variable exponent in his Japanese books [23, 24]. The theory of Lebesgue spaces with variable exponent developed after Kováčik and Rákosník investigated Sobolev spaces with variable exponent in 90's [18]. Among others, Diening investigated the boundedness of the Hardy–Littlewood maximal operator in [10], which paved the way to exhaustive investigation of variable exponent Lebesgue spaces. For example, Cruz-Uribe, SFO, Fiorenza and Neugebauer further studied the boundedness of the Hardy–Littlewood maximal operator in [5, 6]. We refer to [3, 14] as well as [29, p. 447] for more details. Moreover the study on generalization of the classical Muckenhoupt weights in terms of variable exponent has been developed [2, 7]. The second and fourth authors [25] have defined the class of local Muckenhoupt weights and obtained boundedness of the local Hardy–Littlewood maximal operator motivated by Rychkov [28].

The goal of this paper is to establish the theory of wavelets on local weighted Lebesgue spaces with variable exponent, which is a follow-up of the paper [25]. We seek to characterize the spaces in terms of the inhomogeneous wavelet expansion.

In this paper we use the following notation of variable exponents. Let $p(\cdot)$: $\mathbb{R}^n \to [1, \infty)$ be a measurable function, and let w be a weight, that is, a measurable function which is positive almost everywhere. Then, we define the weighted variable Lebesgue space $L^{p(\cdot)}(w)$ to be the set of all measurable functions f such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} w(x) \, \mathrm{d}x < \infty.$$

When we investigate the boundedness of the Hardy–Littlewood maximal operator M defined by (1.10), the following two conditions seem standard:

(1) An exponent $r(\cdot)$ satisfies the local log-Hölder continuity condition if there exists $C > 0$ such that

$$\mathrm{LH}_0 \, : \, |r(x) - r(y)| \leq \frac{C}{-\log |x - y|}, \quad x, y \in \mathbb{R}^n, \quad |x - y| \leq \frac{1}{2}. \tag{1.1}$$

The set LH_0 collects all exponents $r(\cdot)$ which satisfies (1.1).

(2) An exponent $r(\cdot)$ satisfies the log-Hölder continuity condition at ∞ if there exist $C > 0$ and $r_\infty \in [0, \infty)$ such that

$$\mathrm{LH}_\infty \, : \, |r(x) - r_\infty| \leq \frac{C}{\log(e + |x|)}, \quad x \in \mathbb{R}^n. \tag{1.2}$$

The set LH_∞ collects all exponents $r(\cdot)$ which satisfies (1.2).

We also recall the theory of wavelets. Based on the fundamental wavelet theory (see [9, 20, 22, 29, 30]), we can construct compactly supported C^1-functions φ and ψ^l $(l = 1, 2, \ldots, 2^n - 1)$ so that the following conditions are satisfied:

1. For any $J \in \mathbb{Z}$, the system

$$\left\{ \varphi_{J,k}, \, \psi^l_{j,k} \, : \, k \in \mathbb{Z}^n, \, j \geq J, \, l = 1, 2, \ldots, 2^n - 1 \right\}$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. Here given a function F defined on $\mathbb{R}^n$, we write

$$F_{j,k} \equiv 2^{\frac{jn}{2}} F(2^j \cdot - k)$$

for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$.

2. The functions φ and ψ^l $(l = 1, 2, \ldots, 2^n - 1)$ belong to $C^1(\mathbb{R}^n)$. In addition, they are real-valued and compactly supported with $\mathrm{supp}\varphi = \mathrm{supp}\psi^l = [0, 2N - 1]^n$ for some $N \in \mathbb{N}$.

We also define $\chi_{j,k} \equiv 2^{\frac{jn}{2}} \chi_{Q_{j,k}}$ for $j \in \mathbb{Z}$ and $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, where $Q_{j,k}$ is the dyadic cube given by (1.9). Then using the L^2-inner product $\langle \cdot, \cdot \rangle$, we define the three square functions by

$$V f \equiv \left(\sum_{k \in \mathbb{Z}^n} |\langle f, \varphi_{J,k} \rangle \varphi_{J,k}|^2 \right)^{\frac{1}{2}},$$

$$W_1 f \equiv \left(\sum_{l=1}^{2^n-1} \sum_{j=J}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| \langle f, \psi^l_{j,k} \rangle \psi^l_{j,k} \right|^2 \right)^{\frac{1}{2}},$$

$$W_2 f \equiv \left(\sum_{l=1}^{2^n-1} \sum_{j=J}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| \langle f, \psi^l_{j,k} \rangle \chi_{j,k} \right|^2 \right)^{\frac{1}{2}}.$$

Here J is a fixed integer.

Denote by $\mathcal{Q}$ the set of all compact cubes whose edges are parallel to coordinate axes. We will mix the notions considered in [7, 28] to define the local Muckenhoupt class as follows:

Definition 1.1. Given an exponent $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ and a weight w, we say that $w \in A^{\rm loc}_{p(\cdot)}$ if $[w]_{A^{\rm loc}_{p(\cdot)}} \equiv \sup_{Q \in \mathcal{Q}, |Q| \le 1} |Q|^{-1} \|\chi_Q\|_{L^{p(\cdot)}(w)} \|\chi_Q\|_{L^{p'(\cdot)}(\sigma)} < \infty$, where $\sigma \equiv w^{-\frac{1}{p(\cdot)-1}}$ and the supremum is taken over all cubes $Q \in \mathcal{Q}$ with volume less than 1.

Unlike the class of A_p and $A_{p(\cdot)}$, we can consider $w(x) = \exp(\alpha |x|)$ for any $\alpha \in \mathbb{R}$. Another typical example is $w(x) = (1 + |x|)^A$ for any $A \in \mathbb{R}$.

Theorem 1.2. *Let* $p(\cdot) \in {\rm LH}_0 \cap {\rm LH}_\infty$ *satisfy* $1 < p_- \equiv {\rm essinf}_{x \in \mathbb{R}^n} p(x) \le p_+ \equiv {\rm esssup}_{x \in \mathbb{R}^n} p(x) < \infty$, *and let* $w \in A^{\rm loc}_{p(\cdot)}$. *Fix* $J \in \mathbb{Z}$ *arbitrarily. Then there exists a constant* $C > 0$ *such that*

$$C^{-1} \|f\|_{L^{p(\cdot)}(w)} \le \|V f\|_{L^{p(\cdot)}(w)} + \|W_1 f\|_{L^{p(\cdot)}(w)} \le C \, \|f\|_{L^{p(\cdot)}(w)}, \tag{1.3}$$

$$C^{-1} \|f\|_{L^{p(\cdot)}(w)} \le \|V f\|_{L^{p(\cdot)}(w)} + \|W_2 f\|_{L^{p(\cdot)}(w)} \le C \, \|f\|_{L^{p(\cdot)}(w)} \tag{1.4}$$

for all $f \in L^1_{\rm loc}$.

Our result extends the one in [19] to the variable exponent setting and the one in [15] to the local weight setting. Also, the corresponding modular inequality fails, which can be proved in a similar way to [13]; see Section 5. Remark that $L^{p(\cdot)}(w)$ is a subset of $L^1_{\rm loc}$, since $w \in A^{\rm loc}_{p(\cdot)}$; see Section 2 for more.

The proof of Theorem 1.2 uses the boundedness of generalized local Calderón–Zygmund operators. Here and below, given a function space X, $X_{\rm c}$ denotes the set of all functions $f \in X$ with compact support. An L^2-bounded linear operator T is a (generalized) local Calderón–Zygmund operator (with the kernel K), if it satisfies the following conditions:

(1) There exists $K \in L^1_{\rm loc}(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\})$ such that, for all $f \in L^2_{\rm c}(\mathbb{R}^n)$, we have

$$T f(x) = \int_{\mathbb{R}^n} K(x, y) f(y) {\rm d}y \text{ for almost all } x \notin {\rm supp}(f). \tag{1.5}$$

(2) There exist constants γ, D_1 and D_2 such that the two conditions below hold for all $x, y, z \in \mathbb{R}^n$;

  (i) local size condition:

$$|K(x,y)| \le D_1 |x-y|^{-n} \chi_{[-\gamma,\gamma]^n}(x-y) \tag{1.6}$$

  if $x \ne y$,

  (ii) Hörmander's condition:

$$|K(x,z) - K(y,z)| + |K(z,x) - K(z,y)| \le D_2 \frac{|x-y|}{|x-z|^{n+1}} \tag{1.7}$$

  if $0 < 2|x-y| < |z-x|$.

This is an analogue of generalized singular integral operators, which requires

$$|K(x,y)| \le D_1 |x-y|^{-n} \tag{1.8}$$

instead of (1.6) if $x \ne y$. It is known that all generalized singular integral operators, initially defined on L^2, can be extended to a bounded linear operator on L^p for any $1 < p < \infty$.

The structure of the remaining part of this paper is as follows: First of all, in Section 2, we collect some preliminary facts. Section 3 handles generalized local singular integral operators acting on Lebesgue spaces with variable exponents. Section 4 proves Theorem 1.2. In Section 5 we show that the modular inequality fails unless $p(\cdot)$ is constant.

In this paper we use the following notation:

1. The set $\mathbb{N}_0 \equiv \{0, 1, \ldots\}$ consists of all non-negative integers.

2. Write
$$\mathcal{D}_j \equiv \{Q_{j,k} \,:\, k \in \mathbb{Z}^n\}.$$

3. Let $Q_0 \equiv \prod_{m=1}^{n} [a_m, b_m]$ be a cube. A dyadic cube with respect to Q_0 is the set of the form
$$\prod_{m=1}^{n} \left[a_m + \frac{k_m - 1}{2^j}(b_m - a_m), a_m + \frac{k_m}{2^j}(b_m - a_m) \right]$$
for some $j \in \mathbb{N}_0$ and $k = (k_1, k_2, \ldots, k_n) \in \{1, 2, \ldots, 2^j\}^n$. The set $\mathcal{D}(Q_0)$ collects all dyadic cubes with respect to a cube Q_0.

4. The letter C denotes positive constants that may change from one occurrence to another. Let $A, B \ge 0$. Then $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \le CB$, where C depends only on the parameters of importance. The symbol $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$ happen simultaneously.

5. Let $\mathcal{M}$ be the set of all complex-valued measurable functions defined on $\mathbb{R}^n$. Likewise, for a measurable set E, let $\mathcal{M}(E)$ be the set of all complex-valued measurable functions defined on E.

6. Let E be a set. Then we denote its indicator function by χ_E.

7. The symbol $\langle f, g \rangle$ stands for the L^2-inner product. That is, we write

$$\langle f, g \rangle \equiv \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,\mathrm{d}x$$

for all complex-valued measurable L^2-functions f, g defined on $\mathbb{R}^n$.

8. A set S is said to be a dyadic cube if

$$S = Q_{j,k} \equiv \prod_{m=1}^{n} \left[2^{-j}k_m, 2^{-j}(k_m+1)\right] \tag{1.9}$$

for some $j \in \mathbb{Z}$ and $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$.

9. The set $\mathcal{P}$ consists of all $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that $1 < p_- \leq p_+ < \infty$.

10. Given a cube Q, we denote by $c(Q)$ the center of Q and by $\ell(Q)$ the sidelength of Q: $\ell(Q) = |Q|^{\frac{1}{n}}$, where $|Q|$ denotes the volume of the cube Q. In addition, $|E|$ is the Lebesgue measure for general measurable set $E \subset \mathbb{R}^n$.

11. Let f be a measurable function. We consider the local maximal operator given by

$$M^{\mathrm{loc}} f(x) \equiv \sup_{Q \in \mathcal{Q}, |Q| \leq 1} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)|\mathrm{d}y \quad (x \in \mathbb{R}^n).$$

Needless to say, this is an analogue of the Hardy–Littlewood maximal operator given by

$$Mf(x) \equiv \sup_{Q \in \mathcal{Q}} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)|\mathrm{d}y \quad (x \in \mathbb{R}^n). \tag{1.10}$$

12. Let $K \in \mathbb{N}$. The operator $(M^{\mathrm{loc}})^K$ is the K-fold composition of M^{loc}.

13. Let E be a measurable set in $\mathbb{R}^n$. For a function $f : E \to \mathbb{C}$, f^* denotes its decreasing rearrangement.

2 Preliminaries

Here we collect some preliminary facts used in this paper. First, we recall the structure of weighed Banach function spaces and then we concentrate on weighted Lebesgue spaces with variable exponent.

2.1 Definition of function spaces

Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function, and let w be a weight. We have defined the weighted Lebesgue space $L^{p(\cdot)}(w)$ with variable exponent $p(\cdot)$ in Introduction. In addition, the space $L^{p(\cdot)}(w)$ is a Banach space equipped with the norm given by

$$\|f\|_{L^{p(\cdot)}(w)} \equiv \inf \left\{ \lambda > 0 \,:\, \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} w(x)\mathrm{d}x \leq 1 \right\}.$$

If $w = 1$ almost everywhere, then $L^{p(\cdot)}(w)$ is a non-weighted variable Lebesgue and we write $L^{p(\cdot)} \equiv L^{p(\cdot)}(w)$. Moreover if $p(\cdot)$ equals to a constant p, then $L^{p(\cdot)} = L^p$, that is the usual L^p space. When we consider non-weighted function spaces defined on $\mathbb{R}^n$, we may simply write $L^2 \equiv L^2(1)$, $L^1_{\mathrm{loc}} \equiv L^1_{\mathrm{loc}}(1)$ and so on.

2.2 Weighted Banach function spaces

We define Banach function spaces and then state their fundamental properties. For further information on the theory of Banach function space including the proof of Lemma 2.2 below we refer to [1, 26]. We additionally show some properties of Banach function spaces in terms of boundedness of the Hardy–Littlewood maximal operator. We will also consider the weighted case based on Karlovich and Spitkovsky [16].

Definition 2.1. Let X be a linear subspace of $\mathcal{M}$.

1. The space X is said to be a Banach function space if there exists a functional $\|\cdot\|_X : \mathcal{M} \to [0, \infty]$ satisfying the following properties: Let f, g, h, $f_j \in \mathcal{M}$ ($j = 1, 2, \ldots$) and $\lambda \in \mathbb{C}$ be arbitrary.
   (a) $f \in X$ holds if and only if $\|f\|_X < \infty$.
   (b) Norm property:
      i. Positivity: $\|f\|_X \geq 0$.
      ii. Strict positivity: $\|f\|_X = 0$ holds if and only if $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.
      iii. Homogeneity: $\|\lambda f\|_X = |\lambda| \cdot \|f\|_X$ holds.
      iv. Triangle inequality: $\|f + g\|_X \leq \|f\|_X + \|g\|_X$.
   (c) Symmetry: $\|f\|_X = \||f|\|_X$.
   (d) Lattice property: If $0 \leq g(x) \leq f(x)$ for almost every $x \in \mathbb{R}^n$, then $\|g\|_X \leq \|f\|_X$.
   (e) Fatou property: If $0 \leq f_j(x) \leq f_{j+1}(x)$ for all j and $f_j(x) \to f(x)$ as $j \to \infty$ for almost every $x \in \mathbb{R}^n$, then $\lim_{j\to\infty} \|f_j\|_X = \|f\|_X$.
   (f) For every measurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$, $\|\chi_E\|_X$ is finite. Additionally there exists a constant $C_E > 0$ depending only on E such that
   $$\int_E |h(x)|\,\mathrm{d}x \leq C_E \|h\|_X.$$

2. Suppose that X is a Banach function space equipped with a norm $\|\cdot\|_X$. The associate space X' is defined by $X' \equiv \{f \in \mathcal{M} : \|f\|_{X'} < \infty\}$, where $\|f\|_{X'} \equiv \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x\right| : \|g\|_X \le 1\right\}$.

Lemma 2.2. *Let X be a Banach function space.*

1. *(The Lorentz–Luxemberg theorem.) The associate space X' is again a Banach function space, whose associate space X'' coincides with X, in particular, the norms $\|\cdot\|_{X''}$ and $\|\cdot\|_X$ are equivalent.*

2. *(The generalized Hölder inequality.)*

$$\|f \cdot g\|_1 \le \|f\|_X \|g\|_{X'},$$

whenever $f \in X$ and $g \in X'$.

Kováčik and Rákosník [18] have proved that the Lebesgue space $L^{p(\cdot)}$ with variable exponent $p(\cdot)$ is a Banach function space and the associate space equals to $L^{p'(\cdot)}$ with norm equivalence.

Below we define weighted Banach function spaces and give some of their properties. Let X be a Banach function space. The set $X_{\mathrm{loc}}(\mathbb{R}^n)$ consists of all $f \in \mathcal{M}$ such that $f\chi_{[-j,j]^n} \in X$ for $j \in \mathbb{N}$. Given a function W satisfying $0 < W(x) < \infty$ for almost every $x \in \mathbb{R}^n$, $W \in X_{\mathrm{loc}}(\mathbb{R}^n)$ and $W^{-1} \in (X')_{\mathrm{loc}}(\mathbb{R}^n)$, we define the weighted Banach function space $X(\mathbb{R}^n, W)$ by

$$X(\mathbb{R}^n, W) \equiv \{f \in \mathcal{M} : fW \in X\}.$$

We summarize the properties of the weighted Banach function space $X(\mathbb{R}^n, W)$.

Lemma 2.3. *Let W be a weight as above, and let X be a Banach function space.*

1. *The weighted Banach function space $X(\mathbb{R}^n, W)$, which is equipped with the norm $\|f\|_{X(\mathbb{R}^n,W)} \equiv \|fW\|_X$ for $f \in \mathcal{M}$, is a Banach function space.*

2. *The associate space of $X(\mathbb{R}^n, W)$ coincides with $X'(\mathbb{R}^n, W^{-1})$.*

The properties above naturally arise from those of usual Banach function spaces and their proofs are found in [16].

2.3 Some observations on weighted Lebesgue spaces with variable exponents

First, we apply Lemma 2.3 to local weighted Lebesgue spaces with variable exponent. We use the following duality principle.

Proposition 2.4. *Let $p(\cdot) \in \mathcal{P}$ and w be a weight whose dual weight $\sigma = w^{-\frac{1}{p(\cdot)-1}}$ is locally integrable. Then $L^{p(\cdot)}(w)$ is a Banach function space and the associate space $\left(L^{p(\cdot)}(w)\right)'$ is $L^{p'(\cdot)}(\sigma)$ with norm equivalence.*

Proof. As we mentioned, the associate space of $L^{p(\cdot)}$ is $L^{p'(\cdot)}$ with norm equivalence [18]. Since $f \in L^{p(\cdot)}(w) \mapsto f \cdot w^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}$ and $f \in L^{p'(\cdot)}(\sigma) \mapsto f \cdot \sigma^{\frac{1}{p'(\cdot)}} \in L^{p'(\cdot)}$ are isomorphisms, the conclusions are clear. $\square$

In [25], the maximal inequality (Proposition 2.5) and its vector-valued extension (Proposition 2.6) were obtained.

Proposition 2.5. [25] *Suppose that* $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. *Then given any* $w \in A_{p(\cdot)}^{\mathrm{loc}}$, *there exists a constant* $C > 0$ *such that*

$$\|M^{\mathrm{loc}} f\|_{L^{p(\cdot)}(w)} \le C\|f\|_{L^{p(\cdot)}(w)}$$

for all $f \in L^{p(\cdot)}(w)$.

This result corresponds to the ones obtained in [2, 4].

By adapting an extrapolation result in [8] to our local weight setting, the second and fourth authors obtained the following vector-valued inequality.

Proposition 2.6. [25] *Suppose that* $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. *Let also* $w \in A_{p(\cdot)}^{\mathrm{loc}}$ *and* $1 < q \le \infty$. *Then for any sequence* $\{f_j\}_{j=1}^\infty \subset \mathcal{M}$, *we have*

$$\left\| \left(\sum_{j=1}^\infty \left[M^{\mathrm{loc}} f_j \right]^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}(w)} \le C \left\| \left(\sum_{j=1}^\infty |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}(w)}. \tag{2.1}$$

A natural modification is made when $q = \infty$.

Let $E \subset \mathbb{R}^n$ be a measurable set. The set $C_{\mathrm{c}}^\infty(E)$ consists of all infinitely differentiable functions defined on $\mathbb{R}^n$ whose support is compact and contained in E. Once we obtain the boundedness of M^{loc}, with ease we can obtain the density of $C_{\mathrm{c}}^\infty(\mathbb{R}^n)$.

Corollary 2.7. *Suppose that* $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, *and let* $w \in A_{p(\cdot)}^{\mathrm{loc}}$. *Then* $C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ *is dense in* $L^{p(\cdot)}(w)$.

Remark that this is an analogue of [15, Theorem 2.8].

Proof. By the Lebesgue convergence theorem obtained in [14], we have only to approximate functions in L_{c}^∞, the set of all essentially bounded functions with compact support. Let $f \in L_{\mathrm{c}}^\infty$. Choose a non-negative function $\tau \in C_{\mathrm{c}}^\infty([-1,1]^n)$ with L^1-norm 1 and consider $j^n\tau(j\cdot) * f$ for each $j \in \mathbb{N}$. Then we know that $|j^n\tau(j\cdot) * f| \lesssim M^{\mathrm{loc}} f$ for all $j \in \mathbb{N}$. We know that $j^n\tau(j\cdot) * f(x) \to f(x)$ for almost all $x \in \mathbb{R}^n$ as $j \to \infty$ by the Lebesgue differentiation theorem. Thus, once again we can use the Lebesgue convergence theorem mentioned above to have $j^n\tau(j\cdot)*f \to f$ as $j \to \infty$. $\square$

3 Generalized local singular integral operators

We prove that the generalized local singular integral operators are bounded on $L^{p(\cdot)}(w)$ if the postulates in Proposition 2.5 are satisfied. There are several ways to prove the boundedness of the generalized (local) singular integral operators. One technique is to transform the sharp maximal inequality obtained by Fefferman and Stein [11] to the form adapted to our function spaces. Here, we use the modified version considered by Hytönen [12] and Lerner [21]. For $Q \in \mathcal{Q}$, the median of $f \in \mathcal{M}(Q)$, which is close to $(\chi_Q f)^*(2^{-1}|Q|)$, will be an important role in the sequel. We use the following notation: The mean oscillation of f over a cube Q of level $\lambda \in (0,1)$ is given by $\omega_\lambda(f;Q) \equiv \inf\limits_{c\in\mathbb{C}}((f-c)\chi_Q)^*(\lambda|Q|)$.

Definition 3.1 (Median)**.** Let $Q \in \mathcal{Q}$ and $f \in \mathcal{M}(Q)$ be a real-valued function. Define $\mathrm{MED}(f;Q) \equiv \{a \in \mathbb{R} : a \text{ satisfies } (3.1)\}$, where condition (3.1) is given by

$$|\{x \in Q : f(x) > a\}|, \quad |\{x \in Q : f(x) < a\}| \le \frac{1}{2}|Q|. \tag{3.1}$$

Denote by $\mathrm{Med}(f;Q)$ any element in $\mathrm{MED}(f;Q)$. The quantity $\mathrm{Med}(f;Q)$ is called the median of f over Q. One can regard $\mathrm{Med}(f;Q)$ as if it were a mapping $Q \mapsto \mathrm{Med}(f;Q)$.

We now recall the notion of sparseness. A set of cubes $\mathfrak{A}$ is sparse, if there exists a disjoint collection $\{K(Q)\}_{Q\in\mathfrak{A}}$ such that $K(Q)$ contained in Q and $2|K(Q)| \ge |Q|$ for each $Q \in \mathfrak{A}$. Each $K(Q)$ is called the nutshell of Q. We invoke the following result obtained by Hytönen [12] and Lerner [21].

Lemma 3.2. *Let $Q \in \mathcal{D}$, and let $g \in \mathcal{M}(Q)$ be a real-valued function. Then there exists a sparse family $\mathcal{S}(Q) \subset \mathcal{D}(Q)$, which depends on g, such that $Q \in \mathcal{S}(Q)$ and*

$$\chi_Q|g - \mathrm{Med}(g;Q)| \le \sum_{S\in\mathcal{S}(Q)} \omega_{2^{-n-2}}(g;S)\chi_S$$

almost everywhere.

We estimate the mean oscillation of Tf in the next lemma.

Lemma 3.3. *Let T be a generalized local singular integral operator, whose kernel K is supported on $\{(x,y) : x-y \in [-\gamma,\gamma]^n\}$ with some $\gamma \in \mathbb{N}$. Then for any cube Q with $|Q| \le 1$ and $f \in L^2$, we have*

$$\omega_{2^{-n-2}}(Tf;Q) \lesssim \inf_{y\in Q}(M^{\mathrm{loc}})^{2\gamma+3}f(y).$$

Proof. Let $Q(x,R)$ denote the cube of length $2R$ centered at x in $\mathbb{R}^n$ in the proof. First of all, we note that

$$Tf(x) = T[\chi_{Q(c(Q),\gamma+1)}f](x)$$

for all $x \in Q$. With this in mind, we set

$$\alpha \equiv \int_{Q(c(Q),\gamma+1)\setminus 3Q} K(c(Q),y)f(y)\mathrm{d}y.$$

Then

$$|Tf(x)-\alpha| \lesssim \int_{Q(c(Q),\gamma+1)\setminus 3Q} |K(x,y)-K(c(Q),y))||f(y)|dy + |T[\chi_{3Q}f](x)|$$
$$\lesssim \int_{Q(c(Q),\gamma+1)\setminus 3Q} \frac{\ell(Q)}{|z-c(Q)|^{n+1}}|f(z)|\mathrm{d}z + |T[\chi_{3Q}f](x)|.$$

Let $J_0 \in \mathbb{N}$ be the smallest integer such that $2^{J_0}Q \supset Q(c(Q),\gamma+1)$. We estimate

$$\int_{Q(c(Q),\gamma+1)\setminus 3Q} \frac{\ell(Q)}{|z-c(Q)|^{n+1}}|f(z)|\mathrm{d}z \le \sum_{j=1}^{J_0} \int_{2^jQ\setminus 2^{j-1}Q} \frac{\ell(Q)}{|z-c(Q)|^{n+1}}|f(z)|\mathrm{d}z$$
$$\lesssim \sum_{j=1}^{J_0} \frac{1}{2^{j(n+1)}|Q|}\int_{2^jQ}|f(z)|\mathrm{d}z.$$

Note that

$$\frac{1}{2^{jn}|Q|}\int_{2^jQ}|f(z)|\mathrm{d}z \lesssim \inf_{y\in Q}(M^{\mathrm{loc}})^{2\gamma+3}f(y)$$

for all $j=1,2,\ldots,J_0$. Meanwhile,

$$\omega(T[\chi_{3Q}f];Q) \lesssim \frac{1}{|Q|}\int_Q |f(x)|\mathrm{d}x \lesssim \inf_{y\in Q}(M^{\mathrm{loc}})^{2\gamma+3}f(y)$$

by virtue of the weak-L^1 boundedness of T. Inserting these estimates into the above chain of inequalities, we obtain the desired result. □

We will prove the boundedness of generalized local Calderón–Zygmund operators, based on Lemma 3.2.

Theorem 3.4. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Let T be a generalized local Calderón–Zygmund operator. Let also $w \in A^{\mathrm{loc}}_{p(\cdot)}$. Then T, initially defined for L^∞_{c}, can be extended to a bounded linear operator on $L^{p(\cdot)}(w)$.*

Proof. Let $f \in L^\infty_{\mathrm{c}}$. By decomposing the integral kernel and the function f we may assume that these are real-valued. Let $\mathcal{D}_0$ be the set of all dyadic cubes of volume 1. We decompose

$$|Tf(x)| = \sum_{Q\in\mathcal{D}_0} |Tf(x)|\chi_Q(x)$$
$$\le \sum_{Q\in\mathcal{D}_0} (|Tf(x)-\mathrm{Med}(Tf;Q)| + |\mathrm{Med}(Tf;Q)|)\chi_Q(x).$$

Fix $Q \in \mathcal{D}_0$ for the time being. For the first term, we use the Lerner–Hytönen decomposition to have a sparse family $\mathcal{S}(Q)$ such that

$$|Tf(x)-\mathrm{Med}(Tf;Q)| \le \sum_{S\in\mathcal{S}(Q)} \omega_{2^{-n-2}}(Tf;S)\chi_S(x)$$

for almost all $x \in \mathbb{R}^n$. Since T is a generalized local singular integral operator, there exists $\gamma \in \mathbb{N}$ such that the kernel is supported on $\{(x,y) : x-y \in [-\gamma,\gamma]^n\}$. For this $\gamma \in \mathbb{N}$,

$$\omega_{2^{-n-2}}(Tf;S) \lesssim \inf_{y\in S}(M^{\rm loc})^{2\gamma+3}f(y).$$

thanks to Lemma 3.3. Meanwhile, by the definition of median and the weak L^1-boundedness of T, we have

$$\begin{aligned}|{\rm Med}(Tf;Q)| &= |{\rm Med}(T[\chi_{(\gamma+1)Q}f];Q)| \\ &\lesssim \frac{1}{|(\gamma+1)Q|}\int_{(\gamma+1)Q}|f(x)|{\rm d}x \\ &\lesssim \inf_{y\in Q}(M^{\rm loc})^{2\gamma+3}f(y).\end{aligned}$$

We define

$$K(S) \equiv S \setminus \bigcup_{R\in\mathcal{S}(Q)} R.$$

Then we have $2|K(S)| \ge |S|$. Putting these observations all together, we obtain

$$\begin{aligned}|Tf(x)| &\lesssim \sum_{Q\in\mathcal{D}_0}\sum_{S\in\mathcal{S}(Q)}\inf_{y\in S}(M^{\rm loc})^{2\gamma+3}f(y)\chi_S(x) + \sum_{Q\in\mathcal{D}_0}\inf_{y\in Q}(M^{\rm loc})^{2\gamma+3}f(y)\chi_Q(x) \\ &\lesssim \sum_{Q\in\mathcal{D}_0}\sum_{S\in\mathcal{S}(Q)}\inf_{y\in S}(M^{\rm loc})^{2\gamma+3}f(y)M^{\rm loc}\chi_{K(S)}(x)^\theta + (M^{\rm loc})^{2\gamma+3}f(x)\end{aligned}$$

for any $\theta > 0$. Thus, choosing $\theta > 0$ suitably, by virtue of Proposition 2.6 we obtain

$$\begin{aligned}&\|Tf\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\|\sum_{Q\in\mathcal{D}_0}\sum_{S\in\mathcal{S}(Q)}\inf_{y\in S}(M^{\rm loc})^{2\gamma+3}f(y)(M^{\rm loc}\chi_{K(S)})^\theta\right\|_{L^{p(\cdot)}(w)} + \|(M^{\rm loc})^{2\gamma+3}f\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\|\sum_{Q\in\mathcal{D}_0}\sum_{S\in\mathcal{S}(Q)}\inf_{y\in S}(M^{\rm loc})^{2\gamma+3}f(y)\chi_{K(S)}\right\|_{L^{p(\cdot)}(w)} + \|(M^{\rm loc})^{2\gamma+3}f\|_{L^{p(\cdot)}(w)} \\ &\lesssim \|(M^{\rm loc})^{2\gamma+3}f\|_{L^{p(\cdot)}(w)} \\ &\lesssim \|f\|_{L^{p(\cdot)}(w)},\end{aligned}$$

as required. □

4 Proof of Theorem 1.2

We will suppose that f is a real-valued function keeping in mind that φ and ψ are real-valued. Once we prove the right-hand estimate, then the left-hand estimate is automatically obtained by a duality argument. In fact, we have

$$\int_{\mathbb{R}^n} f(x)g(x){\rm d}x = \sum_{k\in\mathbb{Z}^n}\langle f,\varphi_{J,k}\rangle\langle g,\varphi_{J,k}\rangle + \sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n}\langle f,\psi^l_{j,k}\rangle\langle g,\psi^l_{j,k}\rangle$$

for all real-valued functions $f, g \in L^\infty_{\rm c}$ and hence by using Hölder's inequality twice, we have

$$\left|\int_{\mathbb{R}^n} f(x)g(x){\rm d}x\right| \lesssim \|Vf\|_{L^{p(\cdot)}(w)} \|Vg\|_{L^{p'(\cdot)}(\sigma)} + \|W_1 f\|_{L^{p(\cdot)}(w)} \|W_1 g\|_{L^{p'(\cdot)}(\sigma)}.$$

Since we know that

$$\|Vg\|_{L^{p'(\cdot)}(\sigma)} + \|W_1 g\|_{L^{p'(\cdot)}(\sigma)} \lesssim \|g\|_{L^{p'(\cdot)}(\sigma)},$$

we obtain

$$\|f\|_{L^{p(\cdot)}(w)} \lesssim \|Vf\|_{L^{p(\cdot)}(w)} + \|W_1 f\|_{L^{p(\cdot)}(w)}.$$

A similar argument also yields

$$\|f\|_{L^{p(\cdot)}(w)} \lesssim \|Vf\|_{L^{p(\cdot)}(w)} + \|W_2 f\|_{L^{p(\cdot)}(w)}.$$

By a simple limiting argument, we may assume $f \in L^\infty_{\rm c}$ to prove

$$\|Vf\|_{L^{p(\cdot)}(w)} + \|W_1 f\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{L^{p(\cdot)}(w)}.$$

Note that

$$|{\rm supp}\varphi_{J,k}| = \left|\prod_{m=1}^n \left[2^{-J}k_m, 2^{-J}(2N-1+k_m)\right]\right| = \left(2^{-J}(2N-1)\right)^n.$$

Then we see that

$$Vf(x) \lesssim (M^{\rm loc})^{4N+10} f(x).$$

Thus, it follows from Proposition 2.5 that

$$\|Vf\|_{L^{p(\cdot)}(w)} \lesssim \|f\|_{L^{p(\cdot)}(w)}.$$

We will show

$$\|W_1 f\|_{L^{p(\cdot)}(w)} \le C\|f\|_{L^{p(\cdot)}(w)}. \tag{4.1}$$

Let $j \in \mathbb{Z} \cap [J, \infty)$ and $k \in \mathbb{Z}^n$. Since

$$2^{jn} \int_{Q_{j,k'}} |\varphi_{j,k}(x)|\,{\rm d}x \ge C 2^{\frac{jn}{2}}$$

as long as $k' \in \mathbb{Z}^n$ satisfies $Q_{j,k'} \cap Q_{j,k} \ne \emptyset$, we have $\chi_{j,k} \le C(M^{\rm loc})^{4N+10}\varphi_{j,k}$. Thus, the proof of

$$\|W_2 f\|_{L^{p(\cdot)}(w)} \le C\|f\|_{L^{p(\cdot)}(w)}$$

follows immediately from

$$\|W_1 f\|_{L^{p(\cdot)}(w)} \le C\|f\|_{L^{p(\cdot)}(w)}.$$

We remark that

$$\left\|\left(\sum_{l=1}^{2^n-1} \sum_{j=J}^{\infty} \sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi^l_{j,k}\rangle \psi^l_{j,k}\right|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}(w)} \le C\|f\|_{L^{p(\cdot)}(w)}.$$

Let $\Theta = (\Theta_1, \Theta_2, \Theta_3) : \mathbb{N} \to (\mathbb{N} \cap [1, 2^n - 1]) \times (\mathbb{N} \cap [J, \infty)) \times \mathbb{Z}^n$ be a bijection. Then we have

$$\left(\sum_{l=1}^{2^n-1} \sum_{j=J}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| \langle f, \psi^l_{j,k} \rangle \psi^l_{j,k} \right|^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_{\mu=1}^{\infty} \left| \langle f, \psi^{\Theta_3(\mu)}_{\Theta_1(\mu),\Theta_2(\mu)} \rangle \psi^{\Theta_3(\mu)}_{\Theta_1(\mu),\Theta_2(\mu)} \right|^2 \right)^{\frac{1}{2}}$$

$$\lesssim \left(\frac{1}{\pi} \int_0^{2\pi} \left| \sum_{\mu=1}^{\infty} \operatorname{sgn}(\sin(2^\mu t)) \langle f, \psi^{\Theta_3(\mu)}_{\Theta_1(\mu),\Theta_2(\mu)} \rangle \psi^{\Theta_3(\mu)}_{\Theta_1(\mu),\Theta_2(\mu)} \right|^{p_-} \mathrm{d}t \right)^{\frac{1}{p_-}}$$

thanks to the property of the Rademacher sequence $\{\operatorname{sgn}(\sin(2^\mu \cdot))\}_{\mu=1}^{\infty}$ [29]. Thus, by Minkowski's inequality, we have only to show that

$$f \mapsto \sum_{\mu=1}^{\infty} \operatorname{sgn}(\sin(2^\mu t)) \langle f, \psi^{\Theta_3(\mu)}_{\Theta_1(\mu),\Theta_2(\mu)} \rangle \psi^{\Theta_3(\mu)}_{\Theta_1(\mu),\Theta_2(\mu)}$$

is a generalized local singular integral operator. However, it is a generalized singular integral operator with the constants independent of t and l from the result [30]. Since the functions φ and ψ^l $(l = 1, 2, \ldots, 2^n - 1)$ are compactly supported, it follows that the operator in question is a local generalized singular integral operator with the constants independent of t and l. Consequently, we obtain (4.1) by virtue of Theorem 3.4.

5 Modular inequalities for wavelet characterizations

Similar to [15, Theorem 4.3], we can prove that some modular inequalities for wavelet characterizations fail unless the exponent is constant.

Theorem 5.1. *Suppose that $p(\cdot) \in \mathcal{P} \cap \mathrm{LH}_0 \cap \mathrm{LH}_\infty$. Let $w \in A^{\mathrm{loc}}_{p(\cdot)}$ and $j^* \in \mathbb{Z}$. Then the following four conditions are equivalent:*

(X1) *$p(\cdot)$ is a constant.*

(X2) *For all $f \in L^{p(\cdot)}(w)$,*

$$\int_{\mathbb{R}^n} \left| \sum_{l=1}^{2^n-1} \sum_{k \in \mathbb{Z}^n} \langle f, \psi^l_{j^*,k} \rangle \psi^l_{j^*,k}(x) \right|^{p(x)} w(x) \mathrm{d}x \lesssim \int_{\mathbb{R}^n} |f(x)|^{p(x)} w(x) \mathrm{d}x.$$

(X3) *For all $f \in L^{p(\cdot)}(w)$,*

$$\int_{\mathbb{R}^n} Vf(x)^{p(x)} w(x) \mathrm{d}x + \int_{\mathbb{R}^n} W_1 f(x)^{p(x)} w(x) \mathrm{d}x \sim \int_{\mathbb{R}^n} |f(x)|^{p(x)} w(x) \mathrm{d}x.$$

(X4) *For all* $f \in L^{p(\cdot)}(w)$,

$$\int_{\mathbb{R}^n} Vf(x)^{p(x)} w(x) \mathrm{d}x + \int_{\mathbb{R}^n} W_2 f(x)^{p(x)} w(x) \mathrm{d}x \sim \int_{\mathbb{R}^n} |f(x)|^{p(x)} w(x) \mathrm{d}x.$$

Fix $j^* \in \mathbb{Z}$. Remark that

$$f \mapsto \sum_{l=1}^{2^n-1} \sum_{k \in \mathbb{Z}^n} \langle f, \psi^l_{j^*,k} \rangle \psi^l_{j^*,k}(x)$$

is the orthogonal projection from L^2 to a linear space spanned by

$$\{\psi^l_{j^*,k} : l = 1, 2, \ldots, 2^n - 1, k \in \mathbb{Z}^n\}.$$

Proof. From Theorem 1.2 or the result in [19], (X2)–(X4) hold once we assume (X1) holds. Conditions (X3) and (X4) are clearly stronger than (X2). We can prove that (X2) implies (X1) as we did in [15, Theorem 4.3]. □

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