FOUR-VARIABLE p-ADIC TRIPLE PRODUCT L-FUNCTIONS AND THE TRIVIAL ZERO CONJECTURE

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MING-LUN HSIEH AND SHUNSUKE YAMANA

Abstract. We construct the four-variable primitive p-adic L-functions associated with the triple product of Hida families and prove the explicit interpolation formulae at all critical values in the balanced range. Our construction is to carry out the p-adic interpolation of Garrett's integral representation of triple product L-functions via the p-adic Rankin-Selberg convolution method. As an application, we obtain the cyclotomic p-adic L-function for the motive associated with the triple product of elliptic curves and prove the trivial zero conjecture for this motive.

Contents

1. Introduction

The aim of this paper is to construct the four-variable p-adic triple product L-functions for the triple product of Hida families of elliptic newforms with explicit interpolation formulae at all critical specializations in the balanced region. Let p be an *odd* prime, $\mathcal{O}$ a valuation ring finite flat over $\mathbf{Z}_p$ and $\mathbf{I}$ a normal domain finite flat over the Iwasawa algebra $\Lambda = \mathcal{O}[\![\Gamma]\!]$ of the topological group $\Gamma = 1 + p\mathbf{Z}_p$. Let

$$\boldsymbol{F} = (\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$$

be a triplet of primitive Hida families of tame conductor (N_1, N_2, N_3) and nebentypus (χ_1, χ_2, χ_3) with coefficients in $\mathbf{I}$. Roughly speaking, we construct a four-variable Iwasawa function that interpolates the algebraic part of critical values of the triple product L-function attached to $\boldsymbol{F}$ at all balanced critical specializations twisted by Dirichlet characters. Our formulae completely comply with the conjectural form described in [CPR89], [Coa89a] and [Coa89b]. In order to state our result precisely, we need to introduce some notation from Hida theory for elliptic modular forms and technical items such as the modified Euler factors at p and the canonical periods of Hida families in the theory of p-adic L-functions.

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1.1. **Galois representations attached to Hida families.** Given a field F, we denote its separable closure by $\overline{F}$ and put $G_F = \mathrm{Gal}(\overline{F}/F)$. If $\mathcal{F} = \sum_{n=1}^{\infty} \mathbf{a}(n,\mathcal{F})q^n \in \mathbf{I}[\![q]\!]$ is a primitive cuspidal Hida family of tame conductor $N_{\mathcal{F}}$ and nebentypus $\chi_{\mathcal{F}}$, let $\rho_{\mathcal{F}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathrm{Frac}\,\mathbf{I})$ be the associated big Galois representation such that $\mathrm{Tr}\,\rho_{\mathcal{F}}(\mathrm{Frob}_\ell) = \mathbf{a}(\ell,\mathcal{F})$ for primes $\ell \nmid N_{\mathcal{F}}$, where Frob_ℓ is the geometric Frobenius at ℓ and $V_{\mathcal{F}}$ is the natural realization of $\rho_{\mathcal{F}}$ inside the étale cohomology groups of modular curves. Thus $V_{\mathcal{F}}$ is a lattice in $(\mathrm{Frac}\,\mathbf{I})^2$ with the continuous Galois action via $\rho_{\mathcal{F}}$, and the $G_{\mathbf{Q}_p}$-invariant subspace $\mathrm{Fil}^0 V_{\mathcal{F}} := V_{\mathcal{F}}^{I_p}$ fixed by the inertia group I_p at p is free of rank one over $\mathbf{I}$ ([Oht00, Corollary, page 558]). A point $Q \in \mathrm{Spec}\,\mathbf{I}(\overline{\mathbf{Q}}_p)$ is called an arithmetic point if $Q|_\Gamma : \Gamma \hookrightarrow \Lambda^\times \xrightarrow{Q} \overline{\mathbf{Q}}_p^\times$ is given by $Q(x) = x^{k_Q}\epsilon_Q(x)$ for some integer $k_Q \geq 2$ and a finite order character $\epsilon_Q : \Gamma \to \overline{\mathbf{Q}}_p^\times$. Let $\mathfrak{X}_{\mathbf{I}}^+$ be the set of arithmetic points of $\mathbf{I}$. For each arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}}^+$, the specialization $V_{\mathcal{F}_Q} := V_{\mathcal{F}} \otimes_{\mathbf{I},Q} \overline{\mathbf{Q}}_p$ is the geometric p-adic Galois representation associated with the p-stabilized newform $\mathcal{F}_Q = \sum_{n=1}^{\infty} Q(\mathbf{a}(n,\mathcal{F}))q^n$.

1.2. **Triple product L-functions.** We denote by $\mathbf{Q}_\infty$ the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$, by $\boldsymbol{\omega} : G_{\mathbf{Q}} \to \mu_{p-1} \hookrightarrow \mathbf{Z}_p^\times$ the Teichmüller character, by $\boldsymbol{\varepsilon}_{\mathrm{cyc}} : \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \xrightarrow{\sim} 1 + p\mathbf{Z}_p = \Gamma$ the p-adic cyclotomic character and by $\langle \boldsymbol{\varepsilon}_{\mathrm{cyc}} \rangle_T : G_{\mathbf{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \hookrightarrow \mathbf{Z}_p[\![\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]\!]^\times$ the universal cyclotomic character. Let

$$\mathbf{I}_3 = \mathbf{I}\widehat{\otimes}_{\mathcal{O}}\mathbf{I}\widehat{\otimes}_{\mathcal{O}}\mathbf{I}, \qquad \mathbf{I}_4 = \mathbf{I}_3[\![\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]\!]$$

be finite extensions of the three and four-variable Iwasawa algebras.

Fix $a \in \mathbf{Z}/(p-1)\mathbf{Z}$. The main object of this paper is a construction of the p-adic L-function for the triple tensor product Galois representation

$$\mathcal{V} = V_{\boldsymbol{f}}\widehat{\otimes}_{\mathcal{O}}V_{\boldsymbol{g}}\widehat{\otimes}_{\mathcal{O}}V_{\boldsymbol{h}}, \qquad \mathbf{V} = \mathcal{V}\widehat{\otimes}_{\mathcal{O}}\boldsymbol{\omega}^a \langle \boldsymbol{\varepsilon}_{\mathrm{cyc}} \rangle_T$$

of rank eight over $\mathbf{I}_4$. If (k_1,k_2,k_3) is a triplet of positive integers, we say (k_1,k_2,k_3) is balanced if $k_1+k_2+k_3 > 2k^*$ with $k^* := \max\{k_1,k_2,k_3\}$. Let $\mathfrak{X}_{\mathbf{I}_3}^{\mathrm{bal}}$ denote the set of balanced arithmetic points of $(\mathfrak{X}_{\mathbf{I}}^+)^3$. An integer k is said to be critical for (k_1,k_2,k_3) if

$$k^* \leq k \leq k_1 + k_2 + k_3 - k^* - 2.$$

We define the weight space $\mathfrak{X}_{\mathbf{I}_4}^{\mathrm{bal}} \subset \mathrm{Spec}\,\mathbf{I}_4(\overline{\mathbf{Q}}_p)$ to be the set of balanced critical points of $\mathbf{I}_4$ given by

$$\mathfrak{X}_{\mathbf{I}_4}^{\mathrm{bal}} = \{(Q_1,Q_2,Q_3,P) \in \mathfrak{X}_{\mathbf{I}_3}^{\mathrm{bal}} \times \mathfrak{X}_{\Lambda}^+ \mid k_P \text{ is critical for } (k_{Q_1},k_{Q_2},k_{Q_3})\}.$$

For each point $(\underline{Q},P) = (Q_1,Q_2,Q_3,P) \in \mathfrak{X}_{\mathbf{I}_4}^{\mathrm{bal}}$, the specialization $\mathbf{V}_{(\underline{Q},P)} = \mathcal{V}_{\underline{Q}} \otimes \boldsymbol{\varepsilon}_{\mathrm{cyc}}^{k_P}\epsilon_P\boldsymbol{\omega}^{a-k_P}$ is a p-adic geometric Galois representation, where $\mathcal{V}_{\underline{Q}} = V_{\boldsymbol{f}_{Q_1}} \otimes V_{\boldsymbol{g}_{Q_2}} \otimes V_{\boldsymbol{h}_{Q_3}}$ and ϵ_P is regarded as a Galois character via $\epsilon_P \circ \boldsymbol{\varepsilon}_{\mathrm{cyc}}$.

Next we briefly recall the motivic L-function associated with the specialization $\mathbf{V}_{(\underline{Q},P)}$. To the geometric p-adic Galois representation $\mathbf{V}_{(\underline{Q},P)}$, we can associate the Weil-Deligne representation $\mathrm{WD}_\ell(\mathbf{V}_{(\underline{Q},P)})$ of the Weil-Deligne group of $\mathbf{Q}_\ell$ over $\overline{\mathbf{Q}}_p$ (See [Tat79, (4.2.1)] for $\ell \neq p$ and [Fon94, (4.2.3)] for $\ell = p$). Fixing an isomorphism $\iota_p : \overline{\mathbf{Q}}_p \simeq \mathbf{C}$ once and for all, we define the motive L-function of $\mathbf{V}_{(\underline{Q},P)}$ by the Euler product

$$L(\mathbf{V}_{(\underline{Q},P)},s) = \prod_{\ell<\infty} L_\ell(\mathbf{V}_{(\underline{Q},P)},s)$$

of the local L-factors $L_\ell(\mathbf{V}_{(\underline{Q},P)},s)$ attached to $\mathrm{WD}_\ell(\mathbf{V}_{(\underline{Q},P)}) \otimes_{\overline{\mathbf{Q}}_p,\iota_p} \mathbf{C}$ (cf. [Del79, (1.2.2)], [Tay04, page 85]). On the other hand, we denote by $\pi_{\boldsymbol{f}_{Q_1}}$ (resp. $\pi_{\boldsymbol{g}_{Q_2}}, \pi_{\boldsymbol{h}_{Q_3}}$) the irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ associated with $\boldsymbol{f}_{Q_1}$ (resp. $\boldsymbol{g}_{Q_2}, \boldsymbol{h}_{Q_3}$) Let $L(s,\pi_{\boldsymbol{f}_{Q_1}} \times \pi_{\boldsymbol{g}_{Q_2}} \times \pi_{\boldsymbol{h}_{Q_3}} \otimes \epsilon_P\boldsymbol{\omega}^{a-k_P})$ be the automorphic L-function attached to the triple product of $\pi_{\boldsymbol{f}_{Q_1}}$, $\pi_{\boldsymbol{g}_{Q_2}}$, and $\pi_{\boldsymbol{h}_{Q_3}} \otimes \epsilon_P\boldsymbol{\omega}^{a-k_P}$, as constructed by Garrett [Gar87] in the classical setting and by Piatetski-Shapiro and Rallis [PSR87] in the adèlic setting. The analytic theory of $L(s,\pi_{\boldsymbol{f}_{Q_1}} \times \pi_{\boldsymbol{g}_{Q_2}} \times \pi_{\boldsymbol{h}_{Q_3}} \otimes \epsilon_P\boldsymbol{\omega}^{a-k_P})$ such as meromorphic continuation and a functional equation has been explored extensively in the literatures (*cf.* [PSR87, Ike89, Ike92]), and thanks to [Ram00, Theorem 4.4.1], we have

$$L(s + k_P - w_{\underline{Q}}/2, \pi_{\boldsymbol{f}_{Q_1}} \times \pi_{\boldsymbol{g}_{Q_2}} \times \pi_{\boldsymbol{h}_{Q_3}} \otimes \epsilon_P\boldsymbol{\omega}^{a-k_P}) = \Gamma_{\mathbf{V}_{(\underline{Q},P)}}(s) \cdot L(\mathbf{V}_{(\underline{Q},P)},s),$$

where $w_{\underline{Q}} := k_{Q_1} + k_{Q_2} + k_{Q_3} - 3$ and $\Gamma_{\mathbf{V}_{(\underline{Q},P)}}(s)$ is the Gamma factor of $\mathbf{V}_{(\underline{Q},P)}$ as given by

$$\Gamma_{\mathbf{V}_{(\underline{Q},P)}}(s) := \Gamma_{\mathbf{C}}(s + k_P)\Gamma_{\mathbf{C}}(s + 1 + k_P - k_{Q_1})\Gamma_{\mathbf{C}}(s + 1 + k_P - k_{Q_2})\Gamma_{\mathbf{C}}(s + 1 + k_P - k_{Q_3}).$$

Here $\Gamma_{\mathbf{C}}(s)=2(2\pi)^{-s}\Gamma(s)$. Hence we have a good understanding of the analytic properties of the motivic L-function $L(\mathbf{V}_{(\underline{Q},P)},s)$. The rationality of its critical L-values in the balanced region was proved in [Orl87] and [GH93], where the authors verify that the Deligne's period for $\mathbf{V}_{(\underline{Q},P)}$ is the product of Petersson norms of $\boldsymbol{f}_{Q_1}$, $\boldsymbol{g}_{Q_2}$, $\boldsymbol{h}_{Q_3}$. In this article we shall investigate the arithmetic of critical values $L(\mathbf{V}_{(\underline{Q},P)},0)$ for $(\underline{Q},P)\in\mathfrak{X}^{\rm bal}_{\mathbf{I}_4}$ and study the p-adic analytic behavior of its algebraic part viewed as a function on the weight space $\mathfrak{X}^{\rm bal}_{\mathbf{I}_4}$.

1.3. **The modified Euler factors at p and ∞.** Let $G_{\mathbf{Q}_p}$ denote the decomposition group at p. Define the rank four $G_{\mathbf{Q}_p}$-invariant subspace of $\mathbf{V}$ by

$$\mathrm{Fil}^+\mathbf{V}:=\mathrm{Fil}^+\mathcal{V}\otimes\boldsymbol{\omega}^a\left\langle\varepsilon_{\rm cyc}\right\rangle_T,$$

where

$$\mathrm{Fil}^+\mathcal{V}:=\mathrm{Fil}^0 V_{\boldsymbol{f}}\otimes\mathrm{Fil}^0 V_{\boldsymbol{g}}\otimes V_{\boldsymbol{h}}+V_{\boldsymbol{f}}\otimes\mathrm{Fil}^0 V_{\boldsymbol{g}}\otimes\mathrm{Fil}^0 V_{\boldsymbol{h}}+\mathrm{Fil}^0 V_{\boldsymbol{f}}\otimes V_{\boldsymbol{g}}\otimes\mathrm{Fil}^0 V_{\boldsymbol{h}}.$$

The pair $(\mathrm{Fil}^+\mathbf{V},\mathfrak{X}^{\rm bal}_{\mathbf{I}_4})$ satisfies the *Panchishkin condition* in [Gre94a, page 217] in the sense that for each arithmetic point $(\underline{Q},P)\in\mathfrak{X}^{\rm bal}_{\mathbf{I}_4}$, the Hodge-Tate numbers of $\mathrm{Fil}^+\mathbf{V}_{(\underline{Q},P)}$ are all positive, while none of the Hodge-Tate numbers of $\mathbf{V}_{(\underline{Q},P)}/\mathrm{Fil}^+\mathbf{V}_{(\underline{Q},P)}$ is positive. Here the Hodge-Tate number of $\mathbf{Q}_p(1)$ is one in our convention. Now we can define the modified p-Euler factor by

$$\mathcal{E}_p(\mathrm{Fil}^+\mathbf{V}_{(\underline{Q},P)}):=\frac{L_p(\mathrm{Fil}^+\mathbf{V}_{(\underline{Q},P)},0)}{\varepsilon(\mathrm{WD}_p(\mathrm{Fil}^+\mathbf{V}_{(\underline{Q},P)}))\cdot L_p((\mathrm{Fil}^+\mathbf{V}_{(\underline{Q},P)})^\vee,1)}\cdot\frac{1}{L_p(\mathbf{V}_{(\underline{Q},P)},0)}.$$

We note that this modified p-Euler factor is precisely the ratio between the factor $\mathcal{L}_p^{(\rho)}(\mathbf{V}_{(\underline{Q},P)})$ in [Coa89b, page 109, (18)] and the local L-factor $L_p(\mathbf{V}_{(\underline{Q},P)},0)$.

In the theory of p-adic L-functions, we also need the modified Euler factor $\mathcal{E}_\infty(\mathbf{V}_{(\underline{Q},P)})$ at the archimedean place observed by Deligne. It is defined to be the ratio between the factor $\mathcal{L}_\infty^{(\sqrt{-1})}(\mathbf{V}_{(\underline{Q},P)})$ in [Coa89b, page 103 (4)] and the Gamma factor $\Gamma_{\mathbf{V}_{(\underline{Q},P)}}(0)$. In our current case it is explicitly given by

$$\mathcal{E}_\infty(\mathbf{V}_{(\underline{Q},P)})=(\sqrt{-1})^{k_{Q_1}+k_{Q_2}+k_{Q_3}-3}.$$

1.4. **Hida's canonical periods.** To give the precise definition of periods for the motive $\mathbf{V}_{(\underline{Q},P)}$, we recall Hida's canonical period of an $\mathbf{I}$-adic primitive cuspidal Hida family $\mathcal{F}$ of tame conductor $N_{\mathcal{F}}$. Let $\mathfrak{m}_{\mathbf{I}}$ be the maximal ideal of $\mathbf{I}$. We say $\mathcal{F}$ is *controllable* if the following hypothesis holds:

Hypothesis (CR)**.** The residual Galois representation $\bar\rho_{\mathcal{F}}:=\rho_{\mathcal{F}}\,(\mathrm{mod}\ \mathfrak{m}_{\mathbf{I}}):G_{\mathbf{Q}}\to\mathrm{GL}_2(\bar{\mathbb{F}}_p)$ is absolutely irreducible and is p-distinguished.

Suppose that $\mathcal{F}$ is controllable. Then the local component of the universal cuspidal ordinary Hecke algebra corresponding to $\mathcal{F}$ is known to be Gorenstein by [MW86, Prop. 2, §9] and [Wil95, Corollary 2, page 482], and with this Gorenstein property, Hida proved in [Hid88a, Theorem 0.1] that the congruence module for $\mathcal{F}$ is isomorphic to $\mathbf{I}/(\eta_{\mathcal{F}})$ for some non-zero element $\eta_{\mathcal{F}}\in\mathbf{I}$ if $p>3$. Moreover, for any arithmetic point $Q\in\mathfrak{X}^+_{\mathbf{I}}$, the specialization $\eta_{\mathcal{F}_Q}=Q(\eta_{\mathcal{F}})$ generates the congruence ideal of $\mathcal{F}_Q$. We denote by $\mathcal{F}_Q^\circ$ the normalized newform of weight k_Q, conductor $N_Q=N_{\mathcal{F}}p^{n_Q}$ with nebentypus χ_Q corresponding to $\mathcal{F}_Q$. There is a unique decomposition $\chi_Q=\chi'_Q\chi_{Q,(p)}$, where χ'_Q and $\chi_{Q,(p)}$ are Dirichlet characters modulo $N_{\mathcal{F}}$ and p^{n_Q} respectively. Let $\alpha_Q=\mathbf{a}(p,\mathcal{F}_Q)$. Define the modified Euler factor $\mathcal{E}_p(\mathcal{F}_Q,\mathrm{Ad})$ for the adjoint motive of $\mathcal{F}_Q$ by

$$\mathcal{E}_p(\mathcal{F}_Q,\mathrm{Ad})=\alpha_Q^{-2n_Q}\times\begin{cases}(1-\alpha_Q^{-2}\chi_Q(p)p^{k_Q-1})(1-\alpha_Q^{-2}\chi_Q(p)p^{k_Q-2}) & \text{if } n_Q=0,\\ -1 & \text{if } n_Q=1,\chi_{Q,(p)}=1\ (\text{so } k_Q=2),\\ \mathfrak{g}(\chi_{Q,(p)})\chi_{Q,(p)}(-1) & \text{if } n_Q>0,\ \chi_{Q,(p)}\neq 1.\end{cases}$$

Here $\mathfrak{g}(\chi_{Q,(p)})$ is the usual Gauss sum. Fixing the choice of a generator $\eta_{\mathcal{F}}$ and letting $\|\mathcal{F}_Q^\circ\|^2_{\Gamma_0(N_Q)}$ be the usual Petersson norm of $\mathcal{F}_Q^\circ$, we define the *canonical period* $\Omega_{\mathcal{F}_Q}$ of $\mathcal{F}$ at Q by

$$\Omega_{\mathcal{F}_Q}:=(-2\sqrt{-1})^{k_Q+1}\cdot\|\mathcal{F}_Q^\circ\|^2_{\Gamma_0(N_Q)}\cdot\frac{\mathcal{E}_p(\mathcal{F}_Q,\mathrm{Ad})}{\iota_p(\eta_{\mathcal{F}_Q})}\in\mathbf{C}^\times.$$

By [Hid16, Corollary 6.24, Theorem 6.28], one can show that for each arithmetic point Q, up to a p-adic unit, the period $\Omega_{\mathcal{F}_Q}$ is equal to the product of the plus/minus canonical periods $\Omega(+\,;\mathcal{F}_Q^\circ)\Omega(-\,;\mathcal{F}_Q^\circ)$ introduced in [Hid94, page 488].

1.5. **Statement of the main result.** We impose the following technical assumption:

(sf) N_i is square-free and $\chi_i = \boldsymbol{\omega}^{a_i}$ is a power of the Teichmüller character for $i = 1, 2, 3$.

Our main result is a construction of the balanced p-adic triple product L-functions:

Theorem A. *In addition to* (sf)*, we further suppose that $p > 3$ and that $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{h}$ satisfy Hypothesis (CR). Fix generators $(\eta_{\boldsymbol{f}}, \eta_{\boldsymbol{g}}, \eta_{\boldsymbol{h}})$ of the congruence ideals of $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. Then for each $a \in \mathbf{Z}/(p-1)\mathbf{Z}$, there exists a unique element $L^*_{\boldsymbol{F},(a)} \in \mathbf{I}_4$ such that for each arithmetic point $(\underline{Q}, P) = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}^{\rm bal}_{\mathbf{I}_4}$, we have*

$$L^*_{\boldsymbol{F},(a)}(\underline{Q}, P) = \Gamma_{\mathbf{V}_{(\underline{Q},P)}}(0) \cdot \frac{L(\mathbf{V}_{(\underline{Q},P)}, 0)}{\Omega_{\boldsymbol{f}_{Q_1}}\Omega_{\boldsymbol{g}_{Q_2}}\Omega_{\boldsymbol{h}_{Q_3}}} \cdot (\sqrt{-1})^{k_{Q_1}+k_{Q_2}+k_{Q_3}-3} \cdot \mathcal{E}_p(\mathrm{Fil}^+\,\mathbf{V}_{(\underline{Q},P)}).$$

In the literature, the three weight variable p-adic L-function for the triple product of Hida families in the balanced case has been extensively studied by Greenberg-Seveso [GS16], the first author [Hsi19] and so on. These works, based on Ichino's formula [Ich08], focuses on the p-adic interpolation of central values and hence the cyclotomic variable is excluded. Our four-variable p-adic L-function $L^*_{\boldsymbol{F},(a)}$ specializes to this three variable p-adic L-function along the central critical line (see Remark 7.8). The first attempt to construct the cyclotomic p-adic triple product L-functions was made by Böcherer and Panchishkin [BP06, BP09], where they constructed one-variable p-adic L-functions associated with three primitive elliptic newforms. Their construction is not restricted to the ordinary case but the interpolation formula is less complete and the p-integrality of the p-adic L-function is not discussed. Without the Hypothesis (CR), we construct a canonical four-variable p-adic triple product L-functions but with denominators (see Corollary 7.9).

1.6. **Application to the trivial zero conjecture.** Let E_i be a p-ordinary elliptic curve over the rationals $\mathbf{Q}$ of square-free conductor M_i. We write $L(\boldsymbol{E}, s)$ for the degree eight motivic L-function for the triple product

$$\mathbf{V}_{\boldsymbol{E}} = \mathrm{H}^1_{\acute{e}t}(E_{1/\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes \mathrm{H}^1_{\acute{e}t}(E_{2/\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes \mathrm{H}^1_{\acute{e}t}(E_{3/\overline{\mathbf{Q}}}, \mathbf{Q}_p) \tag{1.1}$$

realized in the middle cohomology of the abelian variety $\boldsymbol{E} = E_1 \times E_2 \times E_3$ by the Künneth formula. Hence

$$L(\mathrm{H}^3_{\acute{e}t}(\boldsymbol{E}_{/\overline{\mathbf{Q}}}, \mathbf{Q}_p), s) = L(\boldsymbol{E}, s)\prod_{i=1}^{3} L(E_i, s-1)^2.$$

Our four-variable p-adic L-function yields a cyclotomic p-adic L-function

$$L_p(\boldsymbol{E}) \in \mathbf{Z}_p[\![\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]\!] \otimes \mathbf{Q}_p,$$

which roughly interpolates the algebraic part of central values $\frac{L(\boldsymbol{E}\otimes\chi,2)}{\Omega}$ with a fixed period Ω for all finite order characters χ of $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. Define an analytic function $L_p(\boldsymbol{E}, s) := \boldsymbol{\varepsilon}^{s-2}_{\rm cyc}(L_p(\boldsymbol{E}))$ for $s \in \mathbf{Z}_p$ (See Proposition 8.2 for the precise statement). The Euler-like factor $\mathcal{E}_p(\mathrm{Fil}^+\,\mathbf{V}_{\boldsymbol{E}}(2))$ can possibly vanish. In this case the interpolation formula forces $L_p(\boldsymbol{E}, 2)$ to be zero. Such a zero is called a trivial zero. For example, it appears if all E_i have split multiplicative reduction at p (see Remark 8.3). In this particular case, the trivial zero conjecture predicts that the leading coefficient of $L_p(\boldsymbol{E}, s)$ is the product of the $\mathscr{L}$-invariants for E_i and the algebraic part of the complex central value $L(\boldsymbol{E}, 2)$ (*cf.* [Gre94b, (25), p. 166] and [Ben11, p. 1579]). Using the method of Greenberg-Stevens [GS93] and [BDJ17], we establish the trivial zero conjecture for the triple product of elliptic curves. The following result is a special case of our more general result (see Theorem 8.4).

Theorem B. *If E_1, E_2, E_3 are split multiplicative at p, then $L_p(\boldsymbol{E}, s)$ has at least a triple zero at $s = 2$ and*

$$\lim_{s\to 2}\frac{L_p(\boldsymbol{E}, s)}{(s-2)^3} = \prod_{i=1}^{3}\mathscr{L}_p(E_i) \cdot \frac{L(\boldsymbol{E}, 2)}{\Omega},$$

where $\mathscr{L}_p(E_i) = \frac{\log_p q_{E_i}}{\mathrm{ord}_p q_{E_i}}$ is the $\mathscr{L}$-invariant of E_i with Tate's p-adic period q_{E_i} attached to E_i.

In the case of a p-adic L-function $L_p(E,s)$ of an elliptic curve E over $\mathbf{Q}$ the trivial zero arises if and only if E is split multiplicative at p. An analogus formula for $L_p'(E,1)$ was experimentally discovered in [MTT86] and proved in [GS93], and for Hilbert modular forms in [Mok09], [Spi14] and [BDJ17]. Our result proves the first cases of the trivial zero conjecture where multiple trivial zeros are present and the Galois representation is not of GL(2)-type.

1.7. **The construction of $L^*_{\boldsymbol{F},(a)}$.** We give a sketch of the construction of $L^*_{\boldsymbol{F},(a)}$. Our method is the combination of Garrett's integral representation of the triple product L-function, an integrality result of critical L-values for triple products in [Miz90] and Hida's p-adic Rankin-Selberg method. We begin with a constriction of the four-variable p-adic family of the pull-back of Siegel-Eisenstein series. For each point $x=(Q_1,Q_2,Q_3,P)\in\mathfrak{X}^{\mathrm{bal}}_{\mathbf{I}_4}$, we reorder the weights $\{k_{Q_1},k_{Q_2},k_{Q_3}\}=\{k_x,l_x,m_x\}$ so that $k_x\geq l_x\geq m_x$. For each $\nu_1,\nu_2\in\{0,1\}$, we put

$$\mathfrak{X}^{\mathrm{bal}}_{(\nu_1,\nu_2)}=\left\{x\in\mathfrak{X}^{\mathrm{bal}}_{\mathbf{I}_4}\mid k_x\equiv l_x+\nu_1\equiv m_x+\nu_2 \pmod 2\right\}.$$

Hence we have the partition of the weight space

$$\mathfrak{X}^{\mathrm{bal}}_{\mathbf{I}_4}=\coprod_{\nu_1,\nu_2\in\{0,1\}}\mathfrak{X}^{\mathrm{bal}}_{(\nu_1,\nu_2)}.$$

Let $N=\mathrm{lcm}(N_1,N_2,N_3)$. For each $x\in\mathfrak{X}^{\mathrm{bal}}_{(\nu_1,\nu_2)}$, we shall construct a nearly holomorphic Siegel-Eisenstein series $\mathbf{E}^{(\nu_1,\nu_2)}_x(Z,s)$ of degree three, weight $(k_x,k_x-\nu_1,k_x-\nu_2)$ and level $\Gamma^{(3)}_1(Np^\infty)$ and consider the pull-back given by

$$G^{(\nu_1,\nu_2)}_x(z_1,z_2,z_3):=e_{\mathrm{ord}}\,\mathrm{Hol}\left(\lambda_{z_2}^{\frac{k_x-l_x-\nu_1}{2}}\lambda_{z_3}^{\frac{k_x-m_x-\nu_2}{2}}\mathbf{E}^{(\nu_1,\nu_2)}_x\left(\mathrm{diag}(z_1,z_2,z_3),k_P-\frac{w_{\underline{Q}}+1}{2}\right)\right),$$

where $\lambda_z:=-\frac{1}{2\pi\sqrt{-1}}(\mathrm{Im}\,z)^2\frac{\partial}{\partial\bar z}$ is the weight-lowering differential operator, Hol is the holomorphic projection and e_{ord} is Hida's ordinary projector. Then we show that $G^{(\nu_1,\nu_2)}_x$ is an ordinary cusp form of weight (k_x,l_x,m_x) on $\mathfrak{H}^3_1$ the product of three copies of the upper half plane.

The most crucial (and perhaps surprising) point is that the four classes of Siegel-Eisenstein series $\mathbf{E}^{(\nu_1,\nu_2)}_x$ can be constructed so that $G^{(\nu_1,\nu_2)}_x$ can be put into a single four-variable Hida family of triple product modular forms. More precisely, let $S^{\mathrm{ord}}(N,\chi)$ denote the space of ordinary Λ-adic modular forms of tame level N and character χ. In the following we will associate to $a\in\mathbf{Z}/(p-1)\mathbf{Z}$ and $\underline{\chi}=(\chi_1,\chi_2,\chi_3)$ an explicit triple product ordinary Λ-adic form

$$\mathcal{G}^{(a)}_{\underline{\chi}}\in S^{\mathrm{ord}}(N,\chi_1,\mathbf{Z}_p[\![X_1]\!])\widehat{\otimes}_{\mathbf{Z}_p}S^{\mathrm{ord}}(N,\chi_2,\mathbf{Z}_p[\![X_2]\!])\widehat{\otimes}_{\mathbf{Z}_p}S^{\mathrm{ord}}(N,\chi_3,\mathbf{Z}_p[\![X_3]\!])\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{Z}_p[\![\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]\!].$$

By an explicit calculation of Fourier coefficients of $G^{(\nu_1,\nu_2)}_x$, we prove in Proposition 6.8 that the specialization $\mathcal{G}^{(a)}_{\underline{\chi}}(x)$ at every $x\in\mathfrak{X}^{\mathrm{bal}}_{\mathbf{I}_4}$ is the q-expansion of $G^{(\nu_1,\nu_2)}_x$.

Let T^+_3 be the set of positive definite half-integral matrices of size 3. The Siegel series attached to $B\in T^+_3$ and a rational prime ℓ is defined by

$$b_\ell(B,s)=\sum_{z\in\mathrm{Sym}_3(\mathbf{Q}_\ell)/\mathrm{Sym}_3(\mathbf{Z}_\ell)}\boldsymbol{\psi}(-\mathrm{tr}(Bz))\nu[z]^{-s},$$

where $\boldsymbol{\psi}$ is an arbitrarily fixed additive character on $\mathbf{Q}_\ell$ of order 0 and $\nu[z]$ is the product of denominators of elementary divisors of z. There exists a polynomial $F_{B,\ell}(X)\in\mathbf{Z}[X]$ such that

$$b_\ell(T,s)=(1-\ell^{-s})(1-\ell^{2-2s})F_{B,\ell}(\ell^{-s}).$$

Let $z\mapsto[z]$ denote the inclusion of group-like elements $1+p\mathbf{Z}_p\hookrightarrow\mathbf{Z}_p[\![1+p\mathbf{Z}_p]\!]^\times$. Fix a topological generator $\mathbf{u}\in 1+p\mathbf{Z}_p$ and identify $\mathbf{Z}_p[\![1+p\mathbf{Z}_p]\!]$ with $\mathbf{Z}_p[\![X]\!]$, where $X=[\mathbf{u}]-1$. Define a character $\langle\cdot\rangle:\mathbf{Z}_p^\times\to 1+p\mathbf{Z}_p$ by $\langle x\rangle=x\boldsymbol{\omega}(x)^{-1}$ and write $\langle x\rangle_X=[\langle x\rangle]=(1+X)^{\log_p z/\log_p\mathbf{u}}\in\mathbf{Z}_p[\![X]\!]$. Let Ξ_p be a set of symmetric matrices of size 3 over $\mathbf{Z}_p$ whose off-diagonal entries are p-units but whose diagonal entries are not. Now the seven-variable formal power series is presented by

$$\mathcal{G}^{(a)}_{\underline{\chi}}=\sum_{B=(b_{ij})\in T^+_3\cap\Xi_p}\mathcal{Q}^{(a)}_B(X_1,X_2,X_3,T)\cdot\mathcal{F}^{(a)}_B(X_1,X_2,X_3,T)\cdot q_1^{b_{11}}q_2^{b_{22}}q_3^{b_{33}},$$

where $\mathcal{Q}_B^{(a)}, \mathcal{F}_B^{(a)} \in \mathbf{Z}_p[\![X_1, X_2, X_3, T]\!]$ are given by

$$\mathcal{Q}_B^{(a)}(X_1, X_2, X_3, T) = \frac{\boldsymbol{\omega}^a(8b_{23}b_{31}b_{12})\,\langle 8b_{23}b_{31}b_{12}\rangle_T}{\chi_1(2b_{23})\chi_2(2b_{31})\chi_3(2b_{12})\,\langle 2b_{23}\rangle_{X_1}\langle 2b_{31}\rangle_{X_2}\langle 2b_{12}\rangle_{X_3}},$$
$$\mathcal{F}_B^{(a)}(X_1, X_2, X_3, T) = \prod_{\ell\nmid pN} F_{B,\ell}(\langle \ell\rangle_T^{-2}(\boldsymbol{\omega}^{-2a}\chi_1\chi_2\chi_3)(\ell)\,\langle \ell\rangle_{X_1}\langle \ell\rangle_{X_2}\langle \ell\rangle_{X_3}\,\ell^{-4}).$$

Now we apply the p-adic Rankin-Selberg method to define the p-adic L-function. Denote the universal ordinary cuspidal Hecke algebra by $T(N,\chi,\mathbf{I})$. For each $? \in \{\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}\}$ we write $1_? \in T(N_1,\chi_1,\mathbf{I})\otimes_{\mathbf{I}}\mathrm{Frac}\mathbf{I}$ for the idempotent corresponding to ?. We define

$$L_{\boldsymbol{F},(a)} := \text{the first Fourier coefficient of } \eta_{\boldsymbol{f}}\eta_{\boldsymbol{g}}\eta_{\boldsymbol{h}}(\mathbf{1}_{\boldsymbol{f}}\otimes\mathbf{1}_{\boldsymbol{g}}\otimes\mathbf{1}_{\boldsymbol{h}}(\mathrm{Tr}_{N/N_1}\otimes\mathrm{Tr}_{N/N_2}\otimes\mathrm{Tr}_{N/N_3}(\mathcal{G}_{\underline{\chi}}^{(a)}))\in\mathbf{I}_3[\![T]\!],$$

where $\mathrm{Tr}_{N/N_i} : S^{\mathrm{ord}}(N,\chi_i,\mathbf{I}) \to S^{\mathrm{ord}}(N_i,\chi_i,\mathbf{I})$ is the usual trace map, and then the p-adic triple product L-function is defined to be $L^*_{\boldsymbol{F},(a)} = L_{\boldsymbol{F},(a)}\cdot\mathfrak{f}^{-1}_{\underline{\chi},a,N_1,N_2,N_3}$, where $\mathfrak{f}_{\underline{\chi},a,N_1,N_2,N_3}\in\mathbf{I}_4^\times$ is a fudge factor which is essentially a product of epsilon factors at prime-to-p finite places. The p-adic Rankin-Selberg method tells us that the interpolation formula for the value $L_{\boldsymbol{F},(a)}(x)$ at $x\in\mathfrak{X}^{\mathrm{bal}}_{\mathbf{I}_4}$ is roughly given by

$$\lim_{s\to k_P-\frac{w_Q+1}{2}}\eta_{\boldsymbol{f}_{Q_1}}\eta_{\boldsymbol{g}_{Q_2}}\eta_{\boldsymbol{h}_{Q_3}}\frac{\langle \boldsymbol{f}_{Q_1}\otimes\boldsymbol{g}_{Q_2}\otimes\boldsymbol{h}_{Q_3},\mathbf{E}_x^{(\nu_1,\nu_2)}(s)\rangle}{\|\boldsymbol{f}_{Q_1}\|^2\|\boldsymbol{g}_{Q_2}\|^2\|\boldsymbol{h}_{Q_3}\|^2}$$

(cf. Lemma 7.3), where $\langle\,,\,\rangle$ is the Petersson pairing on $\mathfrak{H}_1^3$ and $\|\cdot\|$ is the Petersson norm on $\mathfrak{H}_1$. The series $\mathbf{E}_x^{(\nu_1,\nu_2)}(Z,s)$ is constructed from a factorizable section of a certain family of induced representations. By means of the generalization of Garrett's work, carried out in [PSR87, Ike89] (see Lemma 7.1) the pairing can be unfolded and written as a product of $L\bigl(s+\frac{1}{2},\pi_{\boldsymbol{f}_{Q_1}}\times\pi_{\boldsymbol{g}_{Q_2}}\times\pi_{\boldsymbol{h}_{Q_3}}\otimes\epsilon_P\boldsymbol{\omega}^{a-k_P}\bigr)$ and the normalized local zeta integrals at primes dividing pN. It turns out that these local zeta integrals are essentially given by the modified Euler factor $\mathcal{E}_p(\mathrm{Fil}^+\mathbf{V}_{(\underline{Q},P)})$ at p and the local epsilon factors $\mathfrak{f}_{\underline{\chi},a,N_1,N_2,N_3}$ at primes $\ell|N$. In both calculations the key ingredients are Lemma 2.1 and the local functional equations for GL_1 and GL_2, by which we can generalize Proposition 4.2 of [GK92] without brute force calculations (see Remark 3.3).

This paper is organized as follows. In §2, §3 and §4, we make the choices of local datum for Siegel Eisenstein series $\mathbf{E}_x^{(\nu_1,\nu_2)}(Z,s)$ and carry out the explicit computation of local zeta integrals that appear in Garrett's integral representation of triple product L-functions. After preparing some notation in Hida theory in §5, we show that the Fourier expansion of $G_x^{(\nu_1,\nu_2)}$ can be p-adically interpolated by the power series $\mathcal{G}_{\underline{\chi}}^{(a)}$ in §6. The key ingredient is Proposition 6.3 about the computation of Fourier coefficients of $G_x^{(\nu_1,\nu_2)}$. In §7, we put all the local computations in §2, 3, and 4 and prove the main interpolation formulae in Theorem 7.6. Finally, in §8 we construct some improved p-adic L-functions in Lemmas 8.5 and 8.6 and use them to prove the trivial zero conjecture for the triple product of elliptic curves in Theorem 8.4.

Notation. The following notations will be used frequently throughout the paper. For an associative ring R with identity element, we denote by $R^\times$ the group of all its invertible elements, and by $\mathrm{M}_{m,n}(R)$ the module of all $m\times n$ matrices with entries in R. Put $\mathrm{M}_n(R)=\mathrm{M}_{n,n}(R)$ and $\mathrm{GL}_n(R)=\mathrm{M}_n(R)^\times$ particularly when we view the set as a ring. The identity and zero elements of the ring $\mathrm{M}_n(R)$ are denoted by $\mathbf{1}_n$ and $\mathbf{0}_n$ (when n needs to be stressed) respectively. The transpose of a matrix x is denoted by x^{t}. Let $\mathrm{Sym}_n(R)=\{z\in\mathrm{M}_n(R)\mid z^{\mathrm{t}}=z\}$ be the space of symmetric matrices of size n over R. For any set X we denote by $\mathbb{I}_X$ the characteristic function of X. When X is a finite set, we denote by $\sharp X$ the number of elements in X. When X is a totally disconnected locally compact topological space or a smooth real manifold, we write $\mathcal{S}(X)$ for the space of Schwartz-Bruhat functions on X. If x is a real number, then we put $\lceil x\rceil=\max\{i\in\mathbf{Z}\mid i\le x\}$.

If R is a commutative ring and $G=\mathrm{GL}_2(R)$, we denote by ρ the right translation of G on the space of $\mathbf{C}$-valued functions on G. Thus $(\rho(g)f)(g')=f(g'g)$. We write $\mathbf{1}:G\to\mathbf{C}$ for the constant function $\mathbf{1}(g)=1$. For a function $f:G\to\mathbf{C}$ and a character $\chi:R^\times\to\mathbf{C}^\times$, let $f\otimes\chi:G\to\mathbf{C}$ denote the function $f\otimes\chi(g)=f(g)\chi(\det g)$.

2. Computation of the local zeta integral: the p-adic case

2.1. **The local zeta integral.** Let T_n be the subgroup of diagonal matrices in GL_n, U_n the subgroup of upper triangular unipotent matrices in GL_n, Z_n the subgroup of scalar matrices in GL_n and $B_n=T_nU_n$ the standard Borel subgroup of GL_n. The symplectic similitude group of degree n is defined by

$$\mathrm{GSp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid gJ_n g^{\mathrm{t}} = \nu_n(g)J_n,\ \nu_n(g) \in \mathrm{GL}_1\}, \qquad J_n = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}.$$

We define the homomorphisms

$$\mathbf{m} : \mathrm{GL}_n \times \mathrm{GL}_1 \to \mathrm{GSp}_{2n}, \qquad \mathbf{n},\ \mathbf{n}^- : \mathrm{Sym}_n \to \mathrm{GSp}_{2n}$$

by

$$\mathbf{m}(A,\nu) = \begin{pmatrix} A & 0 \\ 0 & \nu(A^{\mathrm{t}})^{-1} \end{pmatrix}, \qquad \mathbf{n}(z) = \begin{pmatrix} \mathbf{1}_n & z \\ 0 & \mathbf{1}_n \end{pmatrix}, \qquad \mathbf{n}^-(z) = \begin{pmatrix} \mathbf{1}_n & 0 \\ z & \mathbf{1}_n \end{pmatrix}.$$

We write

$$\mathbf{m}(A) = \mathbf{m}(A,1), \qquad \mathbf{d}(\nu) = \mathbf{m}(\mathbf{1}_n,\nu).$$

A maximal parabolic subgroup $\mathcal{P}_n = \mathcal{M}_n N_n$ of GSp_{2n} is defined by

$$\mathcal{M}_n = \mathbf{m}(\mathrm{GL}_n \times \mathrm{GL}_1), \qquad N_n = \mathbf{n}(\mathrm{Sym}_n).$$

Define algebraic groups of $U^0 \subset U \subset H$ by

$$\begin{aligned} H &= \{(g_1,g_2,g_3) \in (\mathrm{GL}_2)^3 \mid \det g_1 = \det g_2 = \det g_3\}, \\ U &= \{(\mathbf{n}(x_1),\mathbf{n}(x_2),\mathbf{n}(x_3)) \mid x_1,x_2,x_3 \in \mathrm{M}_1\}, \\ U^0 &= \{(\mathbf{n}(x_1),\mathbf{n}(x_2),\mathbf{n}(x_3)) \mid x_1+x_2+x_3 = 0\}. \end{aligned}$$

We define the embedding $\iota : H \hookrightarrow \mathrm{GSp}_6$ by

$$\iota\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}\right) = \left(\begin{array}{ccc|ccc} a_1 & & & b_1 & & \\ & a_2 & & & b_2 & \\ & & a_3 & & & b_3 \\ \hline c_1 & & & d_1 & & \\ & c_2 & & & d_2 & \\ & & c_3 & & & d_3 \end{array}\right).$$

We identity $Z = Z_6$ with the center of GSp_6. Put

$$\boldsymbol{\eta} = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right).$$

Let F be a local field of characteristic zero. In the nonarchimedean case F contains a ring $\mathfrak{o}$ of integers having a single prime ideal $\mathfrak{p}$ and the absolute value $\boldsymbol{\alpha}_F = |\cdot|$ on F is normalized via $|\varpi| = q^{-1}$ for any generator ϖ of $\mathfrak{p}$, where q denotes the order of the residue field $\mathfrak{o}/\mathfrak{p}$. Fix an additive character $\boldsymbol{\psi}$ on F which is trivial on $\mathfrak{o}$ but non-trivial on $\mathfrak{p}^{-1}$. When $F = \mathbf{R}$, we define $\boldsymbol{\psi}(x) = e^{2\pi\sqrt{-1}x}$ for $x \in \mathbf{R}$.

Let K be a standard maximal compact subgroup of $\mathrm{GSp}_6(F)$. For quasi-characters $\hat{\omega}, \chi : F^\times \to \mathbf{C}^\times$ we let $I_3(\hat{\omega},\chi) := \mathrm{Ind}_{\mathcal{P}_3}^{\mathrm{GSp}_6(F)} \chi^2\hat{\omega} \boxtimes \chi^{-3}\hat{\omega}^{-1}$ be the space of all right K-finite functions f on $\mathrm{GSp}_6(F)$ which satisfy

$$f(\mathbf{m}(A,\lambda)\mathbf{n}(z)g) = \hat{\omega}(\lambda^{-2}\det A)\chi(\lambda^{-3}(\det A)^2)|\lambda^{-3}(\det A)^2|f(g)$$

for $A \in \mathrm{GL}_3(F)$, $\lambda \in F^\times$, $z \in \mathrm{Sym}_3(F)$ and $g \in \mathrm{GSp}_6(F)$. The group $\mathrm{GSp}_6(F)$ acts on $I_3(\hat{\omega},\chi)$ by right translation ρ_3. It is important to note that for $t = \mathrm{diag}(a,d) \in T_2$

$$f(\boldsymbol{\eta}\iota(tg_1,tg_2,tg_3)) = \hat{\omega}(d)^{-1}\chi(ad^{-1})|ad^{-1}|f(\boldsymbol{\eta}\iota(g_1,g_2,g_3)). \tag{2.1}$$

It is well worthy of notice that

$$I_3(\hat{\omega},\chi) \otimes \mu \circ \nu_3 = I_3(\hat{\omega}\mu^{-2},\chi\mu). \tag{2.2}$$

We call a K-finite function $(s,g)\mapsto f_s(g)$ on $\mathbf{C}\times\mathrm{GSp}_6(F)$ a holomorphic section of $I_3(\hat{\omega},\chi\boldsymbol{\alpha}_F^s)$ if $f_s(g)$ is holomorphic in s for each $g\in\mathrm{GSp}_6(F)$ and $f_s\in I_3(\hat{\omega},\chi\boldsymbol{\alpha}_F^s)$ for each $s\in\mathbf{C}$. We associate to a non-degenerate symmetric matrix B of size 3 the degenerate Whittaker functional

$$\mathcal{W}_B: I_3(\hat{\omega},\chi\boldsymbol{\alpha}_F^s)\to\mathbf{C},\qquad \mathcal{W}_B(f_s)=\int_{\mathrm{Sym}_3(F)} f_s(J_3\mathbf{n}(z))\boldsymbol{\psi}(-\operatorname{tr}(Bz))\,\mathrm{d}z.$$

The integral converges if $\operatorname{Re}s$ is sufficiently large and can be continued to an entire function.

Given an irreducible admissible infinite dimensional representation π of $\mathrm{GL}_2(F)$, we denote by $\mathscr{W}(\pi)$ the Whittaker model of π with respect to $\boldsymbol{\psi}$. Let π_1,π_2,π_3 be a triplet of irreducible admissible infinite dimensional representations of $\mathrm{GL}_2(F)$. We denote the central character of π_i by ω_i. Set $\hat{\omega}=\omega_1\omega_2\omega_3$. We associate to a holomorphic section f_s of $I(\hat{\omega},\chi\boldsymbol{\alpha}_F^s)$ and Whittaker functions $W_i\in\mathscr{W}(\pi_i)$ the local zeta integral

$$Z(W_1,W_2,W_3,f_s)=\int_{U^0Z\backslash H} W_1(g_1)W_2(g_2)W_3(g_3)f_s(\boldsymbol{\eta}\iota(g_1,g_2,g_3))\,\mathrm{d}g_1\mathrm{d}g_2\mathrm{d}g_3,$$

which converges absolutely if $\operatorname{Re}s$ is sufficiently large.

We define a map $\iota_0: H\hookrightarrow\mathrm{GSp}_6$ by

$$\iota_0(g_1,g_2,g_3)=\boldsymbol{\eta}\iota(g_1,g_2J_1,g_3J_1).$$

As a preliminary step, we choose a coordinate system on an open dense subset of $U^0Z\backslash H$.

Lemma 2.1. *If $(x_1,u_1,u_2,u_3,a_2,a_3)\in F^4\oplus F^{\times 2}$, then*

$$\iota_0(\mathbf{n}^-(u_1)\mathbf{n}(x_1),\mathbf{m}(a_2)\mathbf{n}^-(u_2),\mathbf{m}(a_3)\mathbf{n}^-(u_3))=\begin{pmatrix}A & B\\ \mathbf{0}_3 & (A^{\mathrm{t}})^{-1}\end{pmatrix}J_3\mathbf{n}(-z),$$

where

$$A=\begin{pmatrix}1 & a_2u_1 & a_3u_1\\ 0 & a_2 & 0\\ 0 & 0 & a_3\end{pmatrix},\qquad B=\begin{pmatrix}-u_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\qquad z=\begin{pmatrix}-x_1 & a_2 & a_3\\ a_2 & u_2+a_2^2u_1 & a_2a_3u_1\\ a_3 & a_2a_3u_1 & u_3+a_3^2u_1\end{pmatrix}.$$

Proof. We can prove Lemma 2.1 by the matrix expression of ι_0. □

2.2. **The unramified case.** When π_i is unramified, we write $W_i^0\in\mathscr{W}(\pi_i)$ for the unique Whittaker function which takes the value 1 on $\mathrm{GL}_2(\mathfrak{o})$. Assume that $\hat{\omega}$ and χ are unramified. Then we define the holomorphic section $f_s^0(\chi)$ of $I_3(\hat{\omega},\chi\boldsymbol{\alpha}_F^s)$ by the condition that $f_s^0(k,\chi)=1$ for $k\in\mathrm{GSp}_6(\mathfrak{o})$. Garrett has proved that

$$Z(W_1^0,W_2^0,W_3^0,f_s^0(\chi))=\frac{L\left(s+\frac{1}{2},\pi_1\times\pi_2\times\pi_3\otimes\chi\right)}{L(2s+2,\chi^2\hat{\omega})L(4s+2,\chi^4\hat{\omega}^2)}.$$

We associate to a half-integral symmetric matrix B the series defined by

$$b(B,s)=\sum_{z\in\mathrm{Sym}_3(F)/\mathrm{Sym}_3(\mathfrak{o})}\boldsymbol{\psi}(-\operatorname{tr}(Bz))\nu[z]^{-s},$$

where $\boldsymbol{\psi}$ is an arbitrarily fixed additive character on F of order 0 and $\nu[z]=[z\mathfrak{o}^3+\mathfrak{o}^3:\mathfrak{o}^3]$. If $\det B\neq 0$, then there exists a polynomial $F_B(X)\in\mathbf{Z}[X]$ such that

$$b(B,s)=(1-q^{-s})(1-q^{2-2s})F_B(q^{-s}).$$

The following relation is well-known (cf. [Shi97, Proposition 19.2, page 158]):

$$\mathcal{W}_B(f_s^0(\chi))=\frac{F_B(\chi^2\hat{\omega}(\varpi)q^{-2s-2})}{L(2s+2,\chi^2\hat{\omega})L(4s+2,\chi^4\hat{\omega}^2)}.\tag{2.3}$$

2.3. **The $\mathfrak{p}$-adic case.** Let St stand for the Steinberg representation of $\mathrm{GL}_2(F)$. For quasi-characters μ,ν of $F^\times$ the representation $I(\mu,\nu)$ is realized on the space of functions $f:\mathrm{GL}_2(F)\to\mathbf{C}$ which satisfy

$$f\left(\begin{pmatrix} a & b\\ 0 & d\end{pmatrix} g\right)=\mu(a)\nu(d)\left|\frac{a}{d}\right|^{1/2} f(g)$$

for $a,d\in F^\times$, $b\in F$ and $g\in\mathrm{GL}_2(F)$, where $\mathrm{GL}_2(F)$ acts by right translation ρ. Hereafter we assume that π_i are not supercuspidal and are infinite dimensional. Then π_i is a quotient of a principal series representation $I(\mu_i,\nu_i)$ with quasi-characters μ_i,ν_i. If $\mu_i\nu_i^{-1}\neq\boldsymbol{\alpha}_F^{-1}$, then $\pi_i\simeq I(\mu_i,\nu_i)$. If $\mu_i\nu_i^{-1}=\boldsymbol{\alpha}_F^{-1}$, then $\pi_i\simeq \mathrm{St}\otimes\mu_i\boldsymbol{\alpha}_F^{1/2}$. Let $W_i^{\mathrm{ord}}\in\mathscr{W}(\pi_i)$ be the unique Whittaker function characterized by

$$W_i^{\mathrm{ord}}(\mathbf{t}(a))=\nu_i(a)|a|^{1/2}\mathbb{I}_{\mathfrak{o}}(a)$$

for $a\in F^\times$, where $\mathbf{t}(a)=\mathrm{diag}(a,1)$. Fix a prime element ϖ of $\mathfrak{o}$. For each non-negative integer n we put

$$m_n=\mathbf{m}(\varpi^n), \qquad t_n=J_1^{-1}m_n, \qquad W_i^{(n)}=\pi_i(t_n)W_i^{\mathrm{ord}}.$$

Given a character μ of $\mathfrak{o}^\times$, we define $\varphi_\mu\in\mathcal{S}(F)$ by

$$\varphi_\mu(x)=\mu(x)\mathbb{I}_{\mathfrak{o}^\times}(x).$$

We write $c(\mu)$ for the smallest integer n such that μ is trivial on $\mathfrak{o}^\times\cap(1+\mathfrak{p}^n)$. Define the open compact subgroup $\mathcal{K}_0^{(g)}(\mathfrak{p}^n)$ of $\mathrm{GSp}_{2g}(F)$ by

$$K_0^{(g)}(\mathfrak{p}^n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{GSp}_{2g}(\mathfrak{o})\;\middle|\; c\in\mathrm{M}_g(\mathfrak{p}^n)\right\}.$$

We can define characters $\mu^\uparrow$ and $\mu^\downarrow$ of $K_0^{(1)}(\mathfrak{p}^n)$ by

$$\mu^\uparrow\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)=\mu(a), \qquad \mu^\downarrow\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)=\mu(d), \tag{2.4}$$

provided that $n\geq c(\mu)$. We define the Fourier transform of $\varPhi\in\mathcal{S}(\mathrm{Sym}_g(F))$ with respect to $\boldsymbol{\psi}$ by

$$\widehat{\varPhi}(w)=\int_{\mathrm{Sym}_g(F)}\varPhi(z)\boldsymbol{\psi}(\mathrm{tr}(zw))\,\mathrm{d}z.$$

Given a Schwartz function $\varPhi\in\mathcal{S}(\mathrm{Sym}_3(F))$, we can define a section $f_\varPhi(\chi)$ of $I_3(\hat\omega,\chi)$ by requiring that

$$f_\varPhi(J_3\mathbf{n}(z),\chi)=\varPhi(z) \tag{2.5}$$

for $z\in\mathrm{Sym}_3(F)$. Lemma 2.1 gives

$$\begin{aligned}(2.6)\quad & f_\varPhi(\iota_0(\mathbf{n}^-(u_1)\mathbf{n}(x_1),\mathbf{m}(a_2)\mathbf{n}^-(u_2),\mathbf{m}(a_3)\mathbf{n}^-(u_3)),\chi)\\ &=\hat\omega(a_2a_3)\chi(a_2a_3)^2|a_2a_3|^2\varPhi\left(\begin{pmatrix} x_1 & -a_2 & -a_3\\ -a_2 & -u_2-a_2^2u_1 & -a_2a_3u_1\\ -a_3 & -a_2a_3u_1 & -u_3-a_3^2u_1\end{pmatrix}\right).\end{aligned}$$

Now we define $\Phi\in\mathcal{S}(\mathrm{Sym}_3(F))$ by

$$\Phi\left(\begin{pmatrix} u_1 & x_3 & x_2\\ x_3 & u_2 & x_1\\ x_2 & x_1 & u_3\end{pmatrix}\right)=\prod_{i=1}^3\phi_i(u_i)\varphi_i(x_i), \tag{2.7}$$

where we define $\phi_1,\phi_2,\phi_3,\varphi_1,\varphi_2,\varphi_3\in\mathcal{S}(F)$ by

$$\phi_1=\phi_2=\phi_3=\widehat{\mathbb{I}_{\mathfrak{p}}}, \qquad \varphi_1=\widehat{\varphi_{\chi\mu_1\nu_2\nu_3}}, \qquad \varphi_2=\widehat{\varphi_{\chi\nu_1\mu_2\nu_3}}, \qquad \varphi_3=\widehat{\varphi_{\chi\nu_1\nu_2\mu_3}}.$$

Lemma 2.2. *If $n\geq\max\{1,c(\chi),c(\mu_i),c(\nu_i)\mid i=1,2,3\}$, then*

$$\rho_3(\iota(g_1,g_2,g_3))f_\Phi(\chi)=f_\Phi(\chi)\prod_{i=1}^3\mu_i^\uparrow(g_i)^{-1}\nu_i^\downarrow(g_i)^{-1}$$

for $g_1,g_2,g_3\in K_0^{(1)}(\mathfrak{p}^{2n})$ with $\det g_1=\det g_2=\det g_3$.

Proof. One can easily check that

$$\widehat{\varphi_\mu} \in \mathcal{S}(\mathfrak{p}^{-c(\mu)}), \qquad \widehat{\varphi_\mu}(ax) = \mu(a)^{-1}\widehat{\varphi_\mu}(x), \qquad \widehat{\varphi_\mu}(x+b) = \widehat{\varphi_\mu}(x) \tag{2.8}$$

for $a \in \mathfrak{o}^\times$, $b \in \mathfrak{o}$ and $x \in F$. Simply because $\phi_i = \mathbb{I}_{\mathfrak{p}^{-1}}$, we see that $\Phi(z+c) = \Phi(z)$ for $c \in \mathrm{Sym}_3(\mathfrak{o})$, which means that $f_\Phi(\chi)$ is fixed by the action of $\mathbf{n}(\mathrm{Sym}_3(\mathfrak{o}))$. Put

$$\chi_1 = \chi\mu_1\nu_2\nu_3, \qquad \chi_2 = \chi\nu_1\mu_2\nu_3, \qquad \chi_3 = \chi\nu_1\nu_2\mu_3.$$

Since

$$\begin{pmatrix} \frac{1}{a_1} & & \\ & \frac{1}{a_2} & \\ & & \frac{1}{a_3} \end{pmatrix} \begin{pmatrix} u_1 & x_2 & x_3 \\ x_2 & u_2 & x_1 \\ x_3 & x_1 & u_3 \end{pmatrix} \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix} = \begin{pmatrix} \frac{d_1u_1}{a_1} & \frac{d_2x_2}{a_1} & \frac{d_3x_3}{a_1} \\ \frac{d_1x_2}{a_2} & \frac{d_2u_2}{a_2} & \frac{d_3x_1}{a_2} \\ \frac{d_1x_3}{a_3} & \frac{d_2x_1}{a_3} & \frac{d_3u_3}{a_3} \end{pmatrix},$$

if $a_i, d_i \in \mathfrak{o}^\times$ and $\lambda = a_1d_1 = a_2d_2 = a_3d_3$, then by (2.8)

$$\begin{aligned} &\rho_3(\iota(\mathrm{diag}(a_1,d_1), \mathrm{diag}(a_2,d_2), \mathrm{diag}(a_3,d_3)))f_\Phi(\chi) \\ =&\hat{\omega}\left(\frac{d_1d_2d_3}{\lambda^2}\right)\chi\left(\frac{(d_1d_2d_3)^2}{\lambda^3}\right)\chi_1\left(\frac{a_2}{d_3}\right)\chi_2\left(\frac{a_1}{d_3}\right)\chi_3\left(\frac{a_1}{d_2}\right)f_\Phi(\chi) \\ =&\hat{\omega}\left(\frac{d_1d_2d_3}{\lambda^2}\right)(\mu_1\nu_2\nu_3)\left(\frac{a_2}{d_3}\right)(\nu_1\mu_2\nu_3)\left(\frac{a_1}{d_3}\right)(\nu_1\nu_2\mu_3)\left(\frac{a_1}{d_2}\right)f_\Phi(\chi) \\ =&\hat{\omega}\left(\frac{d_1d_2d_3}{\lambda^2}\right)\omega_1\left(\frac{a_2}{d_3}\right)\omega_2\left(\frac{a_1}{d_3}\right)\omega_3\left(\frac{a_1}{d_2}\right)f_\Phi(\chi)\prod_{i=1}^3\nu_i\left(\frac{a_i}{d_i}\right) \\ =&f_\Phi(\chi)\prod_{i=1}^3\omega_i(a_i)^{-1}\nu_i\left(\frac{a_i}{d_i}\right) = f_\Phi(\chi)\prod_{i=1}^3\mu_i(a_i)^{-1}\nu_i(d_i)^{-1}. \end{aligned}$$

Let $w \in \mathrm{Sym}_3(\mathfrak{p}^{2n})$. If $f_\Phi(g\mathbf{n}^-(w),\chi) \neq 0$, then since $g\mathbf{n}^-(w) \in \mathcal{P}_3J_3\mathbf{n}(z)$ with $z \in \mathrm{Sym}_3(\mathfrak{p}^{-n})$ and since

$$\mathbf{n}(z)\mathbf{n}^-(-w) = \begin{pmatrix} \mathbf{1}_3 - zw & \mathbf{0}_3 \\ -w & (\mathbf{1}_3 - wz)^{-1} \end{pmatrix}\mathbf{n}((\mathbf{1}_3 - zw)^{-1}z),$$

we have $g \in \mathcal{P}_3J_3\mathbf{n}(\mathrm{Sym}_3(\mathfrak{p}^{-n}))$. We see by the identity above that

$$f_\Phi(J_3\mathbf{n}(z)\mathbf{n}^-(w),\chi) = f_\Phi(J_3\mathbf{n}((\mathbf{1}_3 + zw)^{-1}z),\chi) = f_\Phi(J_3\mathbf{n}(z),\chi)$$

for $z \in \mathrm{Sym}_3(\mathfrak{p}^{-n})$ and $w \in \mathrm{Sym}_3(\mathfrak{p}^{2n})$. We conclude that $f_\Phi(\chi)$ is fixed by right translation by $\mathbf{n}^-(\mathrm{Sym}_3(\mathfrak{p}^{2n}))$. The proof is complete by $K_0^{(1)}(\mathfrak{p}^m) = \mathbf{n}(\mathfrak{o})\mathbf{d}(\mathfrak{o}^\times)\mathbf{m}(\mathfrak{o}^\times)\mathbf{n}^-(\mathfrak{p}^m)$. □

2.4. The $\mathfrak{p}$-adic zeta integral.

Proposition 2.3. *If $n \geq \max\{1, c(\chi), c(\mu_i), c(\nu_i) \mid i = 1,2,3\}$, then*

$$\begin{aligned} Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f_\Phi(\chi)) = (1+q^{-1})^{-3}\prod_{j=1}^3\left(\frac{\beta_j}{q\alpha_j}\right)^n \\ \times(\chi\nu_1\nu_2\nu_3)(-1)\gamma\left(\frac{1}{2}, \pi_1\otimes\chi\nu_2\nu_3, \boldsymbol{\psi}\right)^{-1}\prod_{i=2,3}\gamma\left(\frac{1}{2}, \chi\nu_1\nu_i\mu_{5-i}, \boldsymbol{\psi}\right)^{-1}, \end{aligned}$$

where $\alpha_i = \mu_i(\varpi)$ and $\beta_i = \nu_i(\varpi)$.

Proof. We associate to $f_i \in I(\mu_i,\nu_i)$ a function $W(f_i) \in \mathscr{W}(\pi_i)$ by

$$W(g, f_i) = \int_F f_i(J_1\mathbf{n}(u)g)\boldsymbol{\psi}(-u)\,\mathrm{d}u = \lim_{k\to\infty}\int_{\mathfrak{p}^{-k}} f_i(J_1\mathbf{n}(u)g)\boldsymbol{\psi}(-u)\,\mathrm{d}u.$$

Here the limit stabilizes and the integral makes sense for any $f_i \in \pi_i$. The integral W factors through the quotient $I(\mu_i,\nu_i) \twoheadrightarrow \mathrm{St}\otimes\mu_i\boldsymbol{\alpha}_F^{1/2}$ when $\mu_i\nu_i^{-1} = \boldsymbol{\alpha}_F^{-1}$. Let $f_i^{\mathrm{ord}} \in I(\mu_i,\nu_i)$ be such that $f_i^{\mathrm{ord}}(g) = 0$ unless $g \in B_2J_1U_2$ and such that $f_i^{\mathrm{ord}}(J_1\mathbf{n}(x)) = \mathbb{I}_\mathfrak{o}(x)$ for $x \in F$. One can easily check

$$W_i^{\mathrm{ord}} = W(f_i^{\mathrm{ord}}).$$

For $i=2,3$ we put

$$f_i'=\rho(m_n)f_i^{\rm ord}\in I(\mu_i,\nu_i), \qquad\qquad W_i'=W(f_i').$$

Recall that $m_n=\operatorname{diag}(\varpi^n,\varpi^{-n})$. Then

$$Z(W_1^{(n)},W_2^{(n)},W_3^{(n)},f)=\int_{U^0Z\backslash H}W_1^{(n)}(g_1)W_2'(g_2)W_3'(g_3)f(\iota_0(g_1,g_2,g_3))\,\mathrm{d}g_1\mathrm{d}g_2\mathrm{d}g_3.$$

Observe that

$$W_1^{(n)}(g_1)W_2'(g_2)W_3'(g_3)=\int_{U^0}\mathbf{W}_1(u_0(g_1,g_2,g_3);f_2',f_3')\,\mathrm{d}u_0,$$

where

$$\mathbf{W}_1(g_1,g_2,g_3;f_2',f_3')=W_1^{(n)}(g_1)f_2'(J_1g_2)f_3'(J_1g_3).$$

Substituting this expression, we are led to

$$Z(W_1^{(n)},W_2^{(n)},W_3^{(n)},f_\Phi(\chi))=\int_{Z\backslash H}\mathbf{W}_1(g;f_2',f_3')\,f_\Phi(\iota_0(g),\chi)\,\mathrm{d}g.$$

Define a function $\mathcal{F}$ on $\mathrm{SL}_2(F)$ by

$$\mathcal{F}(g)=\int_{\mathrm{SL}_2(F)^2}f_2'(J_1g_2)f_3'(J_1g_3)f_\Phi(\iota_0(g,g_2,g_3),\chi)\,\mathrm{d}g_2\mathrm{d}g_3.$$

Let $T'=\mathbf{m}(F^\times)$ be the diagonal torus of $\mathrm{SL}_2(F)$. Then

$$\begin{aligned}Z(W_1^{(n)},W_2^{(n)},W_3^{(n)},f_\Phi(\chi))=&\int_{F^\times}\mathrm{d}^\times a\int_{T'\backslash\mathrm{SL}_2(F)}W_1^{(n)}(\mathbf{t}(a)g)\int_{\mathrm{SL}_2(F)^2}\\ &f_2'(J_1\mathbf{t}(a)g_2)f_3'(J_1\mathbf{t}(a)g_3)f_\Phi(\iota_0(\mathbf{t}(a)g,\mathbf{t}(a)g_2,\mathbf{t}(a)g_3),\chi)\,\mathrm{d}g_2\mathrm{d}g_3\mathrm{d}g\\ =&\int_{F^\times}\int_{T'\backslash\mathrm{SL}_2(F)}W_1^{(n)}(\mathbf{t}(a)g)\chi(a)\nu_2(a)\nu_3(a)\mathcal{F}(g)\,\mathrm{d}g\mathrm{d}^\times a\end{aligned}\tag{2.9}$$

by (2.1). To justify the manipulations we show that the integral

$$\int_{F^\times}\int_{\mathrm{SL}_2(\mathfrak{o})}\int_F|W_1^{(n)}(\mathbf{t}(a)k)\chi(a)\nu_2(a)\nu_3(a)\mathcal{F}(\mathbf{n}(x)k)|\,\mathrm{d}x\mathrm{d}k\mathrm{d}^\times a$$

is convergent for $\operatorname{Re}\chi\gg0$. The integral

$$\int_{F^\times}|W_1^{(n)}(\mathbf{t}(a)k)\chi(a)\nu_2(a)\nu_3(a)|\,\mathrm{d}^\times a$$

is absolutely convergent. We frequently use the integration formula

$$\int_{\mathrm{SL}_2(F)}h(g)\,\mathrm{d}g=\frac{\zeta(2)}{\zeta(1)}\int_F\int_F\int_{F^\times}h(\mathbf{m}(a)\mathbf{n}^-(u)\mathbf{n}(x))\,\mathrm{d}^\times a\mathrm{d}u\mathrm{d}x$$

for an integrable function h on $\mathrm{SL}_2(F)$. Observe that

$$\begin{aligned}\mathcal{F}(g)=&\frac{\zeta(2)^2}{\zeta(1)^2}\int_{F^2}\mathrm{d}x_2\mathrm{d}x_3\,f_2'(J_1\mathbf{n}(x_2))f_3'(J_1\mathbf{n}(x_3))\int_{F^{\times2}}\prod_{i=2,3}(\nu_i\mu_i^{-1})(a_i)\frac{\mathrm{d}^\times a_i}{|a_i|}\\ &\times\int_{F^2}f_\Phi(\iota_0(g,\mathbf{m}(a_2)\mathbf{n}^-(u_2)\mathbf{n}(x_2),\mathbf{m}(a_3)\mathbf{n}^-(u_3)\mathbf{n}(x_3)),\chi)\,\mathrm{d}u_2\mathrm{d}u_3.\end{aligned}$$

Recall that

$$\mathbf{t}(a)=\begin{pmatrix}a&0\\0&1\end{pmatrix},\qquad \mathbf{m}(a)=\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix},\qquad \mathbf{n}(x)=\begin{pmatrix}1&x\\0&1\end{pmatrix},\qquad \mathbf{n}^-(u)=\begin{pmatrix}1&0\\u&1\end{pmatrix}.$$

Observe that

$$f_i'(J_1\mathbf{n}(x))=f_i^{\rm ord}(J_1m_n\mathbf{n}(x\varpi^{-2n}))=\beta_i^n\alpha_i^{-n}q^n\mathbb{I}_{\mathfrak{p}^{2n}}(x).$$

Lemma 2.2 shows that

$$\rho_3(\iota(\mathbf{1}_2,\mathbf{n}(x_2),\mathbf{n}(x_3))J_3)f_\Phi(\chi)=\rho_3(J_3)f_\Phi(\chi)\qquad\qquad (x_2,x_3\in\mathfrak{p}^{2n}).$$

It follows that $\mathcal{F}(g)$ equals the product of $\frac{f_2'(J_1)f_3'(J_1)}{q^{4n}(1+q^{-1})^2}$ and

$$\int_{F^{\times 2}\oplus F^2} f_\Phi(\iota_0(g,\mathbf{m}(a_2)\mathbf{n}^-(u_2),\mathbf{m}(a_3)\mathbf{n}^-(u_3)),\chi)\prod_{i=2,3}\frac{\nu_i(a_i)\mathrm{d}^\times a_i}{\mu_i(a_i)|a_i|}\mathrm{d}u_i.$$

In particular, $\mathcal{F}(\mathbf{n}^-(u)\mathbf{n}(x))$ equals the product of $\frac{f_2'(J_1)f_3'(J_1)}{q^{4n}(1+q^{-1})^2}$ and

$$\begin{aligned}&\int_{F^{\times 2}\oplus F^2}\Phi\left(\begin{pmatrix} x & -a_2 & -a_3\\ -a_2 & -u_2 & -a_2a_3u\\ -a_3 & -a_2a_3u & -u_3\end{pmatrix}\right)\prod_{i=2,3}(\hat{\omega}\chi^2\nu_i\mu_i^{-1})(a_i)|a_i|\mathrm{d}^\times a_i\mathrm{d}u_i\\ =&\int_{F^{\times 2}\oplus F^2}\varphi_1(-a_2a_3u)\varphi_2(-a_2)\varphi_3(-a_3)\phi_1(x)\phi_3(-u_2-a_2^2u)\phi_2(-u_3-a_3^2u)\prod_{i=2,3}(\hat{\omega}\chi^2\nu_i\mu_i^{-1})(a_i)|a_i|\mathrm{d}^\times a_i\mathrm{d}u_i\end{aligned}$$

by (2.6). Its integral over $x,u\in F$ converges absolutely if $\operatorname{Re}\chi$ is large.

Recall the functional equations

$$\begin{aligned}\gamma\left(\frac{1}{2},\pi_1\otimes\chi,\boldsymbol{\psi}\right)\int_{F^\times}W_1(\mathbf{t}(a)g)\chi(a)\mathrm{d}^\times a&=\int_{F^\times}W_1(\mathbf{t}(a)J_1^{-1}g)(\chi\omega_1)^{-1}(a)\mathrm{d}^\times a,\\ \gamma(s,\chi,\boldsymbol{\psi})\int_{F^\times}\varphi(a)\chi(a)|a|^s\,\mathrm{d}^\times a&=\int_{F^\times}\widehat{\varphi}(a)\chi(a)^{-1}|a|^{1-s}\,\mathrm{d}^\times a\end{aligned}\tag{2.10}$$

for every $W_1\in\mathscr{W}(\pi_1)$ and $\varphi\in\mathcal{S}(F)$. It follows from (2.9) that

$$\begin{aligned}&\gamma\left(\frac{1}{2},\pi_1\otimes\chi\nu_2\nu_3,\boldsymbol{\psi}\right)Z(W_1^{(n)},W_2^{(n)},W_3^{(n)},f_\Phi(\chi))\\ =&\int_{F^\times}\int_{T'\backslash\mathrm{SL}_2(F)}W_1^{(n)}(\mathbf{t}(a)J_1^{-1}g)(\chi\nu_2\nu_3\omega_1)(a)^{-1}\mathcal{F}(g)\,\mathrm{d}g\mathrm{d}^\times a\\ =&\int_{F^\times}\int_F W_1^{(n)}(\mathbf{t}(a)J_1^{-1}\mathbf{n}(x))(\chi\nu_2\nu_3\omega_1)(a)^{-1}\mathcal{F}_{\boldsymbol{\psi}}(a,x)\,\mathrm{d}x\mathrm{d}^\times a,\end{aligned}\tag{2.11}$$

where

$$\mathcal{F}_{\boldsymbol{\psi}}(a,x)=(1+q^{-1})^{-1}\int_F\mathcal{F}(\mathbf{n}^-(u)\mathbf{n}(x))\boldsymbol{\psi}(-au)\,\mathrm{d}u.$$

We have seen that

$$\begin{aligned}\frac{q^{4n}(1+q^{-1})^3}{f_2'(J_1)f_3'(J_1)}\mathcal{F}_{\boldsymbol{\psi}}(a,x)=&\int_{F^{\times 2}}\hat{\omega}(a_2a_3)\chi(a_2a_3)^2|a_2a_3|^2\varphi_3(-a_2)\varphi_2(-a_3)\\ &\times\int_F\varphi_1(-a_2a_3u)\overline{\boldsymbol{\psi}(au)}\mathrm{d}u\,\phi_1(x)\prod_{i=2,3}\frac{\nu_i(a_i)\mathrm{d}^\times a_i}{\mu_i(a_i)|a_i|}\int_F\phi_i(u_i)\,\mathrm{d}u_i\\ =&\phi_1(x)\int_{F^{\times 2}}\widehat{\varphi_1}\left(\frac{a}{a_2a_3}\right)\prod_{i=2,3}\widehat{\phi}_i(0)(\hat{\omega}\chi^2\nu_i\mu_i^{-1})(a_i)\varphi_{5-i}(-a_i)\,\mathrm{d}^\times a_i.\end{aligned}$$

If $xu\neq-1$, then

$$J_1\mathbf{n}(x)\mathbf{n}^-(u)=\mathbf{m}((1+ux)^{-1})\mathbf{n}(-(1+xu)u)J_1\mathbf{n}((1+xu)^{-1}x),$$

which implies that

$$\rho(\mathbf{n}^-(u))f_1^{\mathrm{ord}}=f_1^{\mathrm{ord}},\qquad\qquad \pi_1(\mathbf{n}^-(u))W_1^{\mathrm{ord}}=W_1^{\mathrm{ord}}$$

for $u\in\mathfrak{p}^n$. If $\phi_1(x)\neq 0$, then since $x\in\mathfrak{p}^{-1}$,

$$W_1^{(n)}(\mathbf{t}(a)J_1\mathbf{n}(x))=W_1^{\mathrm{ord}}(\mathbf{t}(a)m_n\mathbf{n}^-(-\varpi^{2n}x))=q^{-n}\beta_1^n\alpha_1^{-n}\nu_1(a)|a|^{1/2}\mathbb{I}_{\mathfrak{o}}(a\varpi^{2n}).$$

We conclude by (2.11) that

$$\begin{aligned}
&\gamma\left(\frac{1}{2}, \pi_1 \otimes \chi\nu_2\nu_3, \boldsymbol{\psi}\right) Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f_\Phi(\chi)) \\
=&\int_{F^\times}\int_F W_1^{(n)}(\mathbf{t}(a)J_1^{-1}\mathbf{n}(x))(\chi\nu_2\nu_3\omega_1)(a)^{-1}\frac{f_2'(J_1)f_3'(J_1)}{q^{4n}(1+q^{-1})^3}\mathcal{F}_{\boldsymbol{\psi}}(a,x)\,\mathrm{d}x\mathrm{d}^\times a \\
=&\int_{F^\times} W_1^{(n)}(\mathbf{t}(a)J_1^{-1})(\chi\nu_2\nu_3\omega_1)(a)^{-1}\frac{f_2'(J_1)f_3'(J_1)}{q^{4n}(1+q^{-1})^3}\int_F \mathcal{F}_{\boldsymbol{\psi}}(a,x)\,\mathrm{d}x\mathrm{d}^\times a \\
=&\frac{f_2'(J_1)f_3'(J_1)}{q^{4n}(1+q^{-1})^3}\widehat{\phi_1}(0)\widehat{\phi_2}(0)\widehat{\phi_3}(0)\int_{F^{\times 3}}\frac{W_1^{(n)}(\mathbf{t}(a)J_1^{-1})}{(\chi\nu_2\nu_3\omega_1)(a)}\widehat{\varphi_1}\left(\frac{a}{a_2a_3}\right)\mathrm{d}^\times a\prod_{i=2,3}(\hat{\omega}\chi^2\nu_i\mu_i^{-1})(a_i)\varphi_{5-i}(-a_i)\,\mathrm{d}^\times a_i.
\end{aligned}$$

The last integral is equal to

$$\begin{aligned}
&\hat{\omega}(-1)\int_{F^{\times 3}}\frac{W_1^{(n)}(\mathbf{t}(aa_2a_3)J_1)}{(\chi\nu_2\nu_3\omega_1)(a)}\widehat{\varphi_1}(a)\,\mathrm{d}^\times a\prod_{i=2,3}(\chi\nu_i\mu_{5-i})(a_i)\varphi_{5-i}(a_i)\,\mathrm{d}^\times a_i \\
=&\hat{\omega}(-1)W_1^{(n)}(J_1)\int_{F^{\times 3}}\frac{\nu_1(aa_2a_3)}{(\chi\nu_2\nu_3\omega_1)(a)}|aa_2a_3|^{1/2}\mathbb{I}_{\mathfrak{o}}(aa_1a_2\varpi^{2n})\varphi_{\chi\mu_1\nu_2\nu_3}(-a)\,\mathrm{d}^\times a\prod_{i=2,3}(\chi\nu_i\mu_{5-i})(a_i)\varphi_{5-i}(a_i)\,\mathrm{d}^\times a_i \\
=&\hat{\omega}(-1)W_1^{(n)}(J_1)(\chi\mu_1\nu_2\nu_3)(-1)\prod_{i=2,3}\int_{F^\times}\varphi_{5-i}(a_i)(\chi\nu_1\nu_i\mu_{5-i})(a_i)|a_i|^{1/2}\,\mathrm{d}^\times a_i.
\end{aligned}$$

In the last line we employ the fact that if $\varphi_{5-i}(a_i) \neq 0$, then $a_i \in \mathfrak{p}^{-n}$. The proof is now complete by $f_i'(J_1) = \beta_i^n\alpha_i^{-n}q^n$, $W_1^{(n)}(J_1) = \beta_1^n\alpha_1^{-n}q^{-n}$ and the functional equation (2.10). □

2.5. **Degenerate Whittaker functions at $\mathfrak{p}$.** Let $\Xi_{\mathfrak{p}}$ be a subset of $\mathrm{Sym}_3(F)$ which consists of symmetric matrices whose the diagonal entries belong to $\mathfrak{p}$ and whose off-diagonal entries belong to $\frac{1}{2}\mathfrak{o}^\times$.

Proposition 2.4. *Let $B = (b_{ij}) \in \mathrm{Sym}_3(F)$. Put $y_i = b_{jk}$ whenever $\{i,j,k\} = \{1,2,3\}$. Then*

$$\mathcal{W}_B(f_\Phi(\chi)) = \chi(8y_1y_2y_3)\prod_{i=1}^{3}\mu_i(2y_i)\mathbb{I}_{\mathfrak{o}^\times}(2y_i)\mathbb{I}_{\mathfrak{p}}(b_{ii})\prod_{j\in\{1,2,3\}\smallsetminus\{i\}}\nu_j(2y_i).$$

In particular, $\mathcal{W}_B(f_\Phi(\chi)) \neq 0$ if and only if $B \in \Xi_{\mathfrak{p}}$.

Proof. Observe that

$$\mathcal{W}_B(f_\Phi(\chi)) = \int_{\mathrm{Sym}_3(F)} f_\Phi(J_3\mathbf{n}(z),\chi)\boldsymbol{\psi}(-\operatorname{tr}(Bz))\,\mathrm{d}z = \widehat{\Phi}(-B) \tag{2.12}$$

for any $\Phi \in \mathcal{S}(\mathrm{Sym}_3(F))$. We have

$$\widehat{\Phi}\left(-\begin{pmatrix} b_{11} & y_3 & y_2 \\ y_3 & b_{22} & y_1 \\ y_2 & y_1 & b_{33}\end{pmatrix}\right) = \prod_{i=1}^{3}\widehat{\varphi}_i(-2y_i)\widehat{\phi}_i(-b_{ii}) = \varphi_{\chi\mu_1\nu_2\nu_3}(2y_1)\varphi_{\chi\nu_1\mu_2\nu_3}(2y_2)\varphi_{\chi\nu_1\nu_2\mu_3}(2y_3)\prod_{i=1}^{3}\mathbb{I}_{\mathfrak{p}}(b_{ii})$$

by definition. □

2.6. **Restatements.** We rewrite Propositions 2.3 and 2.4 in a form which is suitable for our later discussion. Suppose that π_i is a subrepresentation of $I(\mu_i,\nu_i)$ with μ_i unramified. Thus $\omega_i = \mu_i\nu_i$ coincides with ν_i on $\mathfrak{o}^\times$. Let

$$\breve{W}_i(\mathrm{diag}(a,1)) = \nu_i(a)^{-1}|a|^{1/2}\mathbb{I}_{\mathfrak{o}}(a).$$

Definition 2.5. We associate to the quadruplet of characters of $\mathfrak{o}^\times$

$$\mathcal{D} = (\chi,\omega_1,\omega_2,\omega_3)$$

a holomorphic section $f_{\mathcal{D},s} = f_{\Phi_{\mathcal{D}}}(\chi\hat{\omega}\boldsymbol{\alpha}_F^s)$ of $I_3(\hat{\omega}^{-1},\chi\hat{\omega}\boldsymbol{\alpha}_F^s)$ by

$$\Phi_{\mathcal{D}}\left(\begin{pmatrix} u_1 & x_3 & x_2 \\ x_3 & u_2 & x_1 \\ x_2 & x_1 & u_3\end{pmatrix}\right) = \prod_{i=1}^{3}\widehat{\mathbb{I}_{\mathfrak{p}}}(u_i)\widehat{\varphi_{\chi\omega_i}}(x_i).$$

For each quadruplet $(\chi_0,\chi_1,\chi_2,\chi_3)$ of characters of $\mathfrak{o}^\times$, valued in a commutative ring R we set

$$\mathcal{Q}_B(\chi_0,\chi_1,\chi_2,\chi_3):=\chi_0(8b_{12}b_{23}b_{13})\cdot\chi_1(2b_{23})\chi_2(2b_{13})\chi_3(2b_{12})\mathbb{I}_{\Xi_\mathfrak{p}}(B).$$

Given a section f_s of $I_3(\hat{\omega}^{-1},\chi\hat{\omega}\boldsymbol{\alpha}_F^s)$, we are interested in the quantity

$$Z^*_\mathfrak{p}(f_s)=\frac{Z(\rho(t_n)\breve{W}_1,\rho(t_n)\breve{W}_2,\rho(t_n)\breve{W}_3,f_s)}{L\big(s+\frac{1}{2},\pi_1\times\pi_2\times\pi_3\otimes\chi\big)}\prod_{i=1}^3\frac{\zeta(1)}{\zeta(2)}\left(\frac{\omega_i(\varpi)q}{\mu_i(\varpi)^2}\right)^n. \tag{2.13}$$

Proposition 2.6. *Notations and assumptions being as above, we have*

$$\rho_3(\iota(g_1,g_2,g_3))f_{\mathcal{D},s}=f_{\mathcal{D},s}\prod_{i=1}^3\omega_i^\downarrow(g_i),\qquad g_1,g_2,g_3\in K_0^{(1)}(\mathfrak{p}^{2n})$$

if $\det g_1=\det g_2=\det g_3$ *and* $n\geq\max\{1,c(\chi),c(\omega_i)\}$. *Moreover,*

$$\mathcal{W}_B(f_{\mathcal{D},s})=\mathcal{Q}_B(\mathcal{D}),\qquad Z^*_\mathfrak{p}(f_{\mathcal{D},s})=\chi(-1)E_\mathfrak{p}\left(s+\frac{1}{2},\pi_1\times\pi_2\times\pi_3\otimes\chi\right),$$

where

$$E_\mathfrak{p}(s,\pi_1\times\pi_2\times\pi_3\otimes\chi)^{-1}=L(s,\pi_1\times\pi_2\times\pi_3\otimes\chi)\gamma(s,\pi_1\otimes\chi\mu_2\mu_3,\boldsymbol{\psi})\prod_{i=2,3}\gamma(s,\chi\mu_1\mu_i\nu_{5-i},\boldsymbol{\psi}).$$

Proof. Since ω_i coincides with ν_i on $\mathfrak{o}^\times$, we apply Proposition 2.4 and get the formula for $\mathcal{W}_B(f_{\mathcal{D},s})$ by replacing π_i, μ_i, ω_i, ν_i, χ by $\pi_i^\vee\simeq\pi_i\otimes\omega_i^{-1}$, ω_i^{-1}, μ_i^{-1}, ν_i^{-1}, $\chi\hat{\omega}$, respectively. Proposition 2.3 applied to $\breve{W}_i^{\mathrm{ord}}$ and $I_3(\hat{\omega}^{-1},\chi\hat{\omega})$ gives

$$\begin{aligned}Z(\rho(t_n)\breve{W}_1,\rho(t_n)\breve{W}_2,\rho(t_n)\breve{W}_3,f_{\mathcal{D},s})&=(1+q^{-1})^{-3}\prod_{i=1}^3\left(\frac{\nu_i(\varpi)^{-1}}{q\mu_i(\varpi)^{-1}}\right)^n\\&\times\chi(-1)\gamma\left(s+\frac{1}{2},\pi_1^\vee\otimes(\chi\hat{\omega})\nu_2^{-1}\nu_3^{-1},\boldsymbol{\psi}\right)^{-1}\prod_{i=2,3}\gamma\left(s+\frac{1}{2},(\chi\hat{\omega})\nu_1^{-1}\nu_i^{-1}\mu_{5-i}^{-1},\boldsymbol{\psi}\right)^{-1},\end{aligned}$$

from which the formula for $Z^*_\mathfrak{p}(s)$ readily follows. □

We will use the following lemma to achieve the functional equation of the p-adic L-function in §7.7.

Lemma 2.7. *Put* $\check{\chi}=\chi^{-1}\hat{\omega}^{-1}$. *Then*

$$E_\mathfrak{p}(1-s,\pi_1\times\pi_2\times\pi_3\otimes\check{\chi})=\hat{\omega}(-1)E_\mathfrak{p}(s,\pi_1\times\pi_2\times\pi_3\otimes\chi)\varepsilon(s,\pi_1\times\pi_2\times\pi_3\otimes\chi,\boldsymbol{\psi}).$$

Proof. Since $\pi_i\otimes\omega_i^{-1}\simeq\pi_i^\vee$, we get

$$E_\mathfrak{p}(s,\pi_1\times\pi_2\times\pi_3\otimes\check{\chi})^{-1}=L(s,\pi_1^\vee\times\pi_2^\vee\times\pi_3^\vee\otimes\overline{\chi})\gamma(s,\pi_1^\vee\otimes\overline{\chi\nu_2\nu_3},\boldsymbol{\psi})\prod_{i=2,3}\gamma(s,\overline{\chi\nu_1\nu_i\mu_{5-i}},\boldsymbol{\psi}),$$

where $\pi_i\simeq I(\mu_i,\nu_i)$. By definition we arrive at

$$\begin{aligned}&\varepsilon(s,\pi_1\times\pi_2\times\pi_3\otimes\chi,\boldsymbol{\psi})L(s,\pi_1\times\pi_2\times\pi_3\otimes\chi)^{-1}E_\mathfrak{p}\left(1-s,\pi_1\times\pi_2\times\pi_3\otimes\check{\chi}\right)^{-1}\\=&\gamma(s,\pi_1\times\pi_2\times\pi_3\otimes\chi,\boldsymbol{\psi})\gamma(1-s,\pi_1^\vee\otimes\overline{\chi\mu_2\mu_3},\boldsymbol{\psi})\prod_{i=2,3}\gamma(1-s,\overline{\chi\mu_1\mu_i\nu_{5-i}},\boldsymbol{\psi}).\end{aligned}$$

The statement can now be deduced from multiplicativity and the functional equation of gamma factors. □

3. Computation of the local zeta integral: the ramified case

Recall that St denotes the Steinberg representation of $\mathrm{GL}_2(F)$. Let π_i be either an irreducible unramified principal series representation or the Steinberg representation. Since

$$Z(W_1\otimes\chi_1,W_2\otimes\chi_2,W_3\otimes\chi_3,f)=Z(W_1,W_2,W_3,f\otimes(\chi_1\chi_2\chi_3)\circ\nu_3) \tag{3.1}$$

for characters χ_1,χ_2,χ_3 of $F^\times$, where

$$(W_i\otimes\chi_i)(g_i)=W_i(g_i)\chi_i(\det g_i),\qquad (f\otimes\chi\circ\nu_3)(g)=f(g)\chi(\nu_3(g)),$$

there is no harm in assuming that $\pi_i \simeq I(\boldsymbol{\alpha}_F^{-t_i}, \boldsymbol{\alpha}_F^{t_i})$ with $t_i \in \mathbf{C}$ or $\pi_i \simeq \mathrm{St}$. When $\pi_i \simeq I(\boldsymbol{\alpha}_F^{-t_i}, \boldsymbol{\alpha}_F^{t_i})$, we denote the unique Whittaker function which takes the value 1 on $\mathrm{GL}_2(\mathfrak{o})$ by $W_i^0 \in \mathscr{W}(\pi_i)$ and let $W_i^\pm \in \mathscr{W}(\pi_i)$ be the unique Whittaker function characterized by

$$W_i^\pm(\mathbf{t}(a)) = |a|^{(\pm 2t_i+1)/2}\mathbb{I}_{\mathfrak{o}}(a)$$

for $a \in F^\times$. When $\pi_i \simeq \mathrm{St}\otimes\boldsymbol{\alpha}_F^{s_i}$, we define $W_i^+ \in \mathscr{W}(\mathrm{St}\otimes\boldsymbol{\alpha}_F^{s_i})$ by

$$W_i^+(\mathbf{t}(a)) = |a|^{s_i+1}\mathbb{I}_{\mathfrak{o}}(a)$$

and set $t_i = s_i + \frac{1}{2}$ to be uniform. We define $f_i^{\mathrm{ord}} \in I(\boldsymbol{\alpha}_F^{-t_i}, \boldsymbol{\alpha}_F^{t_i})$ as before. Recall that $W_i^+ = W(f_i^{\mathrm{ord}})$. Put

$$\eta_1 = \begin{pmatrix} 0 & -1 \\ \varpi & 0 \end{pmatrix}, \qquad \mathcal{W}_i^\pm = \pi_i(\eta_1)W_i^\pm, \qquad \mathcal{W}_i^0 = \pi_i(\eta_1)W_i^0.$$

Lemma 3.1. *If π_i is an irreducible unramified principal series, then*

$$W_i^0 = q^{1/2}\frac{\mathcal{W}_i^+ - \mathcal{W}_i^-}{q^{-t_i} - q^{t_i}}.$$

PROOF. The relation $W_i^\pm = W_i^0 - q^{(\pm 2t_i - 1)/2}\mathcal{W}_i^0$ implies the stated identity in view of $\pi_i(\eta_1)\mathcal{W}_i^0 = W_i^0$. $\square$

Fix an unramified character $\chi = \boldsymbol{\alpha}_F^s$ of $F^\times$. We will abbreviate $I_3(\chi) = I_3(1,\chi)$. Take $\Phi = \mathbb{I}_{\mathrm{Sym}_3(\mathfrak{o})}$ and put $h^0(\chi) = f_\Phi(\chi)$. Since

$$\mathcal{P}_3 J_3 \mathbf{n}(\mathrm{Sym}_3(\mathfrak{o})) = \mathcal{P}_3 J_3 K_0^{(3)}(\mathfrak{p}) = \mathcal{P}_3 K_0^{(3)}(\mathfrak{p}) J_3 K_0^{(3)}(\mathfrak{p}),$$

the restriction of the section $h^0(\chi)$ to $\mathrm{GSp}_6(\mathfrak{o})$ is the characteristic function of $K_0^{(3)}(\mathfrak{p})J_3K_0^{(3)}(\mathfrak{p})$. In particular,

$$\rho_3(k)h^0(\chi) = h^0(\chi)$$

for $k \in K_0^{(3)}(\mathfrak{p})$ (cf. Lemma 2.2).

Lemma 3.2. *Assume that $\pi_1 \simeq \mathrm{St}$. Then*

$$Z(\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_3^+, h^0(\chi)) = -\frac{q^{s-2}}{(1+q^{-1})^3}\zeta(s+1+t_2+t_3)\prod_{i=2,3}\zeta(s+1+t_i-t_{5-i}).$$

Remark 3.3. Lemma 3.2 is compatible with the computation [GK92]. Let $\Phi^0(\chi) \in I_3(\chi)$ be the function whose restriction to $\mathrm{GSp}_6(\mathfrak{o})$ is the characteristic function of $K_0^{(3)}(\mathfrak{p})$. Put $\eta_3 = \iota(\eta_1,\eta_1,\eta_1)$. Then

$$\eta_3 K_0^{(3)}(\mathfrak{p})\eta_3^{-1} = K_0^{(3)}(\mathfrak{p}), \qquad h^0(\chi) = q^{3+3s}\rho_3(\eta_3)\Phi^0(\chi)$$

by Lemma 3.1 of [GK92]. We obtain

$$Z(\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_3^+, h^0(\chi)) = q^{3+3s}Z(W_1^+, W_2^+, W_3^+, \Phi^0).$$

When $\pi_1 \simeq \pi_2 \simeq \pi_3 \simeq \mathrm{St}$, Proposition 4.2 of [GK92] gives

$$Z(W_1^+, W_2^+, W_3^+, \Phi^0) = -(q+1)^{-3}q^{-2s-2}L\left(s+\frac{1}{2}, \mathrm{St}\times\mathrm{St}\times\mathrm{St}\right).$$

PROOF. On account of (2.9) we have

$$Z(\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_3^+, h^0(\chi)) = \int_{F^\times}\int_{T'\backslash \mathrm{SL}_2(F)} \mathcal{W}_1^+(\mathbf{t}(a)g)|a|^{s+t_2+t_3}\mathcal{F}'(g)\,\mathrm{d}g\mathrm{d}^\times a.$$

Put $f_i'' = \pi_i(\mathbf{t}(\varpi))f_i^{\mathrm{ord}}$. Since

$$f_i''(J_1\mathbf{n}(x)) = f_i^{\mathrm{ord}}(J_1\mathbf{t}(\varpi)\mathbf{n}(x/\varpi)) = q^{(1-2t_i)/2}\mathbb{I}_{\mathfrak{p}}(x),$$

we get

$$\mathcal{F}'(g) = (1+q^{-1})^{-2}\int_{F^2} \mathrm{d}x_2\mathrm{d}x_3\, f_2''(J_1\mathbf{n}(x_2))f_3''(J_1\mathbf{n}(x_3))\int_{F^{\times 2}}\prod_{i=2,3}|a_i|^{2t_i}\frac{\mathrm{d}^\times a_i}{|a_i|}$$
$$\times\int_{F^2} h^0(\iota_0(g,\mathbf{m}(a_2)\mathbf{n}^-(u_2)\mathbf{n}(x_2),\mathbf{m}(a_3)\mathbf{n}^-(u_3)\mathbf{n}(x_3)),\chi)\,\mathrm{d}u_2\mathrm{d}u_3$$
$$=\int_{F^{\times 2}\oplus F^2} h^0(\iota_0(g,\mathbf{m}(a_2)\mathbf{n}^-(u_2),\mathbf{m}(a_3)\mathbf{n}^-(u_3)),\chi)\frac{\prod_{i=2,3}|a_i|^{2t_i-1}\mathrm{d}^\times a_i\mathrm{d}u_i}{q^{1+t_2+t_3}(1+q^{-1})^2}.$$

In view of (2.6)

$$\mathcal{F}'(\mathbf{n}^-(u)\mathbf{n}(x)) = \int_{F^{\times 2}\oplus F^2}\varPhi\left(\begin{pmatrix} x & -a_2 & -a_3\\ -a_2 & -u_2 & -a_2a_3u\\ -a_3 & -a_2a_3u & -u_3\end{pmatrix}\right)\frac{\prod_{i=2,3}|a_i|^{1+2s+2t_i}\mathrm{d}^\times a_i\mathrm{d}u_i}{q^{1+t_2+t_3}(1+q^{-1})^2}$$
$$= q^{-1-t_2-t_3}(1+q^{-1})^{-2}\mathbb{I}_{\mathfrak{o}}(x)\int_{\mathfrak{o}^2}\mathbb{I}_{\mathfrak{o}}(a_2a_3u)\prod_{i=2,3}|a_i|^{1+2s+2t_i}\mathrm{d}^\times a_i.$$

Owing to (2.11) we arrive at

$$\gamma\left(s+\frac{1}{2},\pi_1\otimes\boldsymbol{\alpha}_F^{t_2+t_3},\boldsymbol{\psi}\right)Z(\mathcal{W}_1^+,\mathcal{W}_2^+,\mathcal{W}_3^+,h^0(\chi))$$
$$=\int_{F^\times}\int_F \mathcal{W}_1^+(\mathbf{t}(a)J_1^{-1}\mathbf{n}(x))|a|^{-s-t_2-t_3}\mathcal{F}'_{\boldsymbol{\psi}}(a,x)\,\mathrm{d}x\mathrm{d}^\times a,$$

where

$$\mathcal{F}'_{\boldsymbol{\psi}}(a,x) = (1+q^{-1})^{-1}\int_F \mathcal{F}'(\mathbf{n}^-(u)\mathbf{n}(x))\boldsymbol{\psi}(-au)\,\mathrm{d}u$$
$$= q^{-1-t_2-t_3}(1+q^{-1})^{-3}\mathbb{I}_{\mathfrak{o}}(x)\int_{\mathfrak{o}^2}\mathbb{I}_{\mathfrak{o}}\left(\frac{a}{a_2a_3}\right)\prod_{i=2,3}|a_i|^{2s+2t_i}\mathrm{d}^\times a_i.$$

We conclude that

$$q^{1+t_2+t_3}(1+q^{-1})^3\gamma\left(s+\frac{1}{2},\pi_1\otimes\boldsymbol{\alpha}_F^{t_2+t_3},\boldsymbol{\psi}\right)Z(\mathcal{W}_1^+,\mathcal{W}_2^+,\mathcal{W}_3^+,h^0(\chi))$$
$$=\int_{F^\times}\int_F \mathrm{d}x\mathrm{d}^\times a\,\frac{\mathcal{W}_1^+(\mathbf{t}(a)J_1^{-1}\mathbf{n}(x))}{|a|^{s+t_2+t_3}}\mathbb{I}_{\mathfrak{o}}(x)\int_{\mathfrak{o}^2}\mathbb{I}_{\mathfrak{o}}\left(\frac{a}{a_2a_3}\right)\prod_{i=2,3}|a_i|^{2s+2t_i}\mathrm{d}^\times a_i$$
$$=\int_{F^\times}\mathrm{d}^\times a\,\frac{\mathcal{W}_1^+(\mathbf{t}(a_2a_3a)J_1^{-1})}{|a_2a_3a|^{s+t_2+t_3}}\int_{\mathfrak{o}^2}\mathbb{I}_{\mathfrak{o}}(a)\prod_{i=2,3}|a_i|^{2s+2t_i}\mathrm{d}^\times a_i$$
$$=\int_{\mathfrak{o}}\mathrm{d}^\times a\,\frac{W_1^+(\mathbf{t}(a_2a_3a\varpi))}{|a|^{s+t_2+t_3}}\prod_{i=2,3}\int_{\mathfrak{o}}|a_i|^{s+t_i-t_{5-i}}\mathrm{d}^\times a_i$$
$$=\frac{\zeta\left(\frac{1}{2}-s+t_1-t_2-t_3\right)}{q^{(2t_1+1)/2}}\prod_{i=2,3}\zeta\left(s+\frac{1}{2}+t_1+t_i-t_{5-i}\right).$$

Assume that $\pi_1\simeq\mathrm{St}$. Then $t_1=\frac{1}{2}$ and

$$\gamma\left(s+\frac{1}{2},\pi_1\otimes\boldsymbol{\alpha}_F^{t_2+t_3},\boldsymbol{\psi}\right) = -q^{-s-t_2-t_3}\frac{\zeta(1-s-t_2-t_3)}{\zeta(s+1+t_2+t_3)},$$

from which we complete our proof. □

Proposition 3.4. *Let π_i be either an unramified principal series representation or the Steinberg representation twisted by an unramified character. Set $W_i=W_i^0$ in the former case and $W_i=W_i^+$ in the latter case. Put*

$\breve{W}_i = W_i \otimes \omega_i^{-1}$. *If not all π_i are principal series, then for an unramified character χ of $F^\times$*

$$\begin{aligned} Z(W_1, W_2, W_3, h^0(\chi)) &= Z(\breve{W}_1, \breve{W}_2, \breve{W}_3, h^0(\chi\hat{\omega})) \\ &= -(\hat{\omega}^2\chi^4)(\varpi) q(1+q)^{-3} \frac{L\left(\frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi\right)}{\varepsilon\left(\frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi, \boldsymbol{\psi}\right)}. \end{aligned}$$

Remark 3.5. If π_1 and π_2 are irreducible unramified principal series representations, then

$$\begin{aligned} L(s, \pi_1 \times \pi_2 \times \mathrm{St}) &= L\left(s + \frac{1}{2}, \pi_1 \times \pi_2\right), & \varepsilon(s, \pi_1 \times \pi_2 \times \mathrm{St}, \boldsymbol{\psi}) &= q^{-4s+2}\omega_1(\varpi)^2\omega_2(\varpi)^2, \\ L(s, \pi_1 \times \mathrm{St} \times \mathrm{St}) &= L(s, \pi_1)L(s+1, \pi_1), & \varepsilon(s, \pi_1 \times \mathrm{St} \times \mathrm{St}, \boldsymbol{\psi}) &= q^{-4s+2}\omega_1(\varpi)^2, \\ L(s, \mathrm{St} \times \mathrm{St} \times \mathrm{St}) &= \zeta\left(s + \frac{3}{2}\right)\zeta\left(s + \frac{1}{2}\right)^2, & \varepsilon(s, \mathrm{St} \times \mathrm{St} \times \mathrm{St}, \boldsymbol{\psi}) &= -q^{-(10s-5)/2}. \end{aligned}$$

Proof. In view of [Ike89, Lemma 3.1] and (3.1) we may assume that $\pi_1 \simeq \mathrm{St}$ and π_i is a quotient of $I(\boldsymbol{\alpha}_F^{-t_i}, \boldsymbol{\alpha}_F^{t_i})$ for $i = 2, 3$. If all π_i are discrete series representations, then since $\mathcal{W}_1 = -W_1$, the result follows from Lemma 3.2. Let $\chi = \boldsymbol{\alpha}_F^s$ and $\pi_3 \simeq I(\boldsymbol{\alpha}_F^{-t_3}, \boldsymbol{\alpha}_F^{t_3})$. Lemma 3.2 gives

$$Z(W_1^+, \mathcal{W}_2^+, \mathcal{W}_3^\pm, h^0(\chi)) = \frac{q^{s-2}}{(1+q^{-1})^3} L(s+1+t_2, \pi_3)\zeta(s+1-t_2 \pm t_3).$$

Thanks to Lemma 3.1 we obtain

$$\begin{aligned} \frac{Z(W_1^+, \mathcal{W}_2^+, W_3^0, h^0(\chi))}{q^{s-2}L(s+1+t_2, \pi_3)} &= q^{1/2} \frac{\zeta(s+1-t_2+t_3) - \zeta(s+1-t_2-t_3)}{(1+q^{-1})^3(q^{-t_3} - q^{t_3})} \\ &= (1+q^{-1})^{-3} q^{1/2} q^{-s-1+t_2} L(s+1-t_2, \pi_3). \end{aligned}$$

If $\pi_2 \simeq I(\boldsymbol{\alpha}_F^{-t_2}, \boldsymbol{\alpha}_F^{t_2})$, then

$$Z(W_1^+, \mathcal{W}_2^+, W_3^0, h^0(\chi)) = (1+q^{-1})^{-3} q^{(2t_2-5)/2} L(s+1, \pi_2 \times \pi_3),$$

and so again by Lemma 3.1,

$$Z(W_1^+, W_2^0, W_3^0, h^0(\chi)) = (1+q^{-1})^{-3} L(s+1, \pi_2 \times \pi_3) \frac{q^{t_2-2} - q^{-t_2-2}}{q^{-t_2} - q^{t_2}} = -(1+q^{-1})^{-3} q^{-2} L(s+1, \pi_2 \times \pi_3).$$

If $\pi_2 \simeq \mathrm{St}$, we obtain the claimed result by letting $t_2 = \frac{1}{2}$. □

4. Computation of the local zeta integral: the archimedean case

4.1. **Archimedean sections.** We define the sign character $\mathrm{sgn} : \mathbf{R}^\times \to \{\pm 1\}$ by $\mathrm{sgn}(x) = \frac{x}{|x|}$. Let $\mathrm{Sym}_n^+(\mathbf{R})$ denote the set of positive definite symmetric matrices of rank n. The Siegel upper half-space $\mathfrak{H}_n$ of degree n consists of complex symmetric matrices of size n with positive definite imaginary part. The Lie group $\mathrm{GSp}_n^+(\mathbf{R}) = \{g \in \mathrm{GSp}_n(\mathbf{R}) \mid \nu_n(g) > 0\}$ acts on the space $\mathfrak{H}_n$ by $gZ = (AZ+B)(CZ+D)^{-1}$, where $Z \in \mathfrak{H}_n$ and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with matrices A, B, C, D of size n. Let $C^\infty(\mathfrak{H}_n)$ be the space of $\mathbf{C}$-valued smooth functions on the upper half complex plane $\mathfrak{H}_n$. For an integer k and $f \in C^\infty(\mathfrak{H}_n)$ we define

$$(4.1) \qquad f|_k g(Z) = f(gZ)J(g, Z)^{-k}, \qquad J(g, Z) = \nu_n(g)^{-n/2}\det(CZ+D).$$

Put $\mathbf{i} = \sqrt{-1}\mathbf{1}_n$. We will identity the compact unitary group $\mathrm{U}(n) = \{u \in \mathrm{GL}_n(\mathbf{C}) \mid \bar{u}^{\mathrm{t}} u = \mathbf{1}_n\}$ with the fixator $\{g \in Sp_n(\mathbf{R}) \mid g(\mathbf{i}) = \mathbf{i}\}$ via the map $g \mapsto \overline{J(g, \mathbf{i})}$.

For $1 \le i, j \le 3$ and $u \in \mathrm{U}(3)$ we define $H_{ij}(u)$ to be the (i,j)-entry of the matrix $u^{\mathrm{t}} u$. By definition, H_{ij} is a function on $\mathrm{O}(3)\backslash\mathrm{U}(3)$, and hence we can extend it to a unique function on $\mathrm{GSp}_6(\mathbf{R})$ such that

$$H_{ij}(\mathbf{n}(z)\mathbf{m}(A, \nu)u) = H_{ij}(u) \qquad (z \in \mathrm{Sym}_3(\mathbf{R}),\ A \in \mathrm{GL}_3(\mathbf{R}),\ \nu \in \mathbf{R}^\times,\ u \in \mathrm{U}(3)).$$

A parity type is a triplet $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ of integers which belongs to one of the following triplets

$$\lambda \in \{(0,0,0),\ (0,1,1),\ (1,0,1),\ (1,1,2)\}.$$

Fix a parity type λ and a character χ_∞ of $\mathbf{R}^\times$. Put

$$H_\lambda := \begin{cases} 1 & \text{if } \lambda = (0,0,0), \\ \overline{H_{23}} & \text{if } \lambda = (0,1,1), \\ H_{12} & \text{if } \lambda = (1,0,1), \\ H_{12}\overline{H_{23}} & \text{if } \lambda = (1,1,2). \end{cases}$$

For each integer k we define $f^{[k,\lambda]}_{\infty,s} \in I_3(\mathrm{sgn}^{k-\lambda_1}, \chi_\infty\,\mathrm{sgn}^{k-\lambda_1}\,\boldsymbol{\alpha}^s_{\mathbf{R}})$ by

$$f^{[k,\lambda]}_{s,\infty}(g) := H_\lambda(g)\chi_\infty(\nu_3(g)) \cdot J(g,\mathbf{i})^{-k+\lambda_1}\,|J(g,\mathbf{i})|^{k-\lambda_1-2s-2}.$$

Since

$$H_{ij}(g\iota(\kappa_{\theta_1},\kappa_{\theta_2},\kappa_{\theta_3})) = e^{\sqrt{-1}(\theta_i+\theta_j)}H_{ij}(g), \qquad \kappa_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

we have

$$f^{[k,\lambda]}_{s,\infty}(g\iota(\kappa_{\theta_1},\kappa_{\theta_2},\kappa_{\theta_3})) = f^{[k,\lambda]}_{s,\infty}(g)e^{\sqrt{-1}\{k\theta_1+(k-\lambda_2)\theta_2+(k-\lambda_3)\theta_3\}}. \tag{4.2}$$

4.2. **Archimedean degenerate Whittaker functions.** For a positive integer m we put

$$\Gamma_m(s) = \pi^{m(m-1)/4}\prod_{j=0}^{m-1}\Gamma\left(s-\frac{j}{2}\right).$$

If h is positive definite and $\alpha,\beta \in \mathbf{C}$, then the integral

$$\omega(h;\alpha,\beta) = \frac{\det(h)^\beta}{\Gamma_3(\beta)}\int_{\mathrm{Sym}^+_3(\mathbf{R})} e^{-\mathrm{tr}(uh)}\det(u+\mathbf{1}_3)^{\alpha-2}(\det u)^{\beta-2}\,\mathrm{d}u$$

is absolutely convergent for $\mathrm{Re}\,\beta > 2$ and can be continued to a holomorphic function on $\mathbf{C}\times\mathbf{C}$ by Theorem 3.1 of [Shi82]. It is convenient to introduce the function $\omega^\star(h;\alpha,\beta)$ given by

$$\begin{aligned} \omega^\star(h;\alpha,\beta) &:= \det(4\pi h)^{\alpha-2}\cdot\omega(4\pi h;\alpha,\beta) \\ &= \frac{1}{\Gamma_3(\beta)}\int_{\mathrm{Sym}^+_3(\mathbf{R})} e^{-\mathrm{tr}(u)}\det(u+4\pi h)^{\alpha-2}(\det u)^{\beta-2}\,\mathrm{d}u. \end{aligned} \tag{4.3}$$

It follows from this expression that if $\alpha \in \mathbf{Z}$ and $\alpha \geq 2$, then $\omega^\star(h;\alpha,\beta)$ is a polynomial function in h of degree at most $\alpha - 2$ and makes sense for an arbitrary symmetric matrix h.

Lemma 4.1. *For $x \in \mathrm{Sym}_3(\mathbf{R})$ we have*

$$H_{23}(J_3\mathbf{n}(x)) = 2\sqrt{-1}(x_{11}x_{23} - x_{12}x_{13} + \sqrt{-1}x_{23})/\det(x+\mathbf{i}),$$
$$H_{12}(J_3\mathbf{n}(x)) = 2\sqrt{-1}(x_{12}x_{33} - x_{23}x_{13} + \sqrt{-1}x_{12})/\det(x+\mathbf{i}).$$

PROOF. The Iwasawa decomposition of $J_3\mathbf{n}(x)$ can be written as

$$J_3\mathbf{n}(x) = \begin{pmatrix} z^{\mathrm{t}} & * \\ 0 & z^{-1} \end{pmatrix}\begin{pmatrix} zx & -z \\ z & zx \end{pmatrix}, \qquad z \in \mathrm{GL}_3(\mathbf{R})$$

with $z^{\mathrm{t}}z = (\mathbf{1}_3 + x^2)^{-1}$. Let $u = z(x-\mathbf{i}) \in \mathrm{U}(3)$. Then $u^{\mathrm{t}}u = (x-\mathbf{i})(x+\mathbf{i})^{-1}$. We denote the adjugate of a matrix $A \in \mathrm{M}_3(\mathbf{R})$ by $\mathrm{adj}(A)$. Since $A\cdot\mathrm{adj}(A) = (\det A)\mathbf{1}_3$, we have

$$u^{\mathrm{t}}u = \det(x+\mathbf{i})^{-1}(x-\mathbf{i})\mathrm{adj}(x+\mathbf{i}) = -2\sqrt{-1}\det(x+\mathbf{i})^{-1}\mathrm{adj}(x+\mathbf{i}) + \mathbf{1}_3.$$

By definition we find that

$$\begin{aligned} H_{23}(J_3\mathbf{n}(x)) = H_{23}(u) &= \det(x+\mathbf{i})^{-1}\cdot 2\sqrt{-1}\det\begin{pmatrix} x_{11}+\sqrt{-1} & x_{12} \\ x_{13} & x_{23} \end{pmatrix} \\ &= \det(x+\mathbf{i})^{-1}\cdot 2\sqrt{-1}(x_{11}x_{23} - x_{12}x_{13} + \sqrt{-1}x_{23}). \end{aligned}$$

One can compute $H_{12}(J_3\mathbf{n}(x))$ in the same way. □

Definition 4.2. We associate to a parity type λ the differential operator $\mathscr{D}_\lambda$ on $T=(T_{ij})\in \mathrm{Sym}_3(\mathbf{R})$ by

$$\mathscr{D}_{(0,1,1)} := \frac{1}{2\pi^2\sqrt{-1}}\{\partial_{13}\partial_{12}-\partial_{23}(\partial_{11}-4\pi)\}, \qquad \mathscr{D}_{(0,0,0)}=\mathrm{id},$$
$$\mathscr{D}_{(1,0,1)} := \frac{1}{2\pi^2\sqrt{-1}}\{\partial_{12}\partial_{33}-\partial_{23}\partial_{13}\}, \qquad \mathscr{D}_{(1,1,2)} := \mathscr{D}_{(0,1,1)}\mathscr{D}_{(1,0,1)}.$$

Here

$$\partial_{ij} := \frac{\partial}{\partial T_{ij}}\cdot \begin{cases} 1 & \text{if } i=j,\\ \frac{1}{2} & \text{if } i\neq j.\end{cases}$$

Definition 4.3. For each parity type λ and an integer $\lambda_2\le r\le k-2$ we put $M=k-r-2$ and define

$$\mathbf{K}^M_{\mathscr{D}_\lambda}(T;u) := \mathscr{D}_\lambda\{\det(4\pi T+u)^M\},$$
$$\omega^M_{\mathscr{D}_\lambda}(T,s) := \frac{1}{\Gamma_3(s)}\int_{\mathrm{Sym}_3^+(\mathbf{R})} e^{-\operatorname{tr}(u)}\mathbf{K}^M_{\mathscr{D}_\lambda}(T;u)(\det u)^{s-2}\,\mathrm{d}u = \mathscr{D}_\lambda\omega^\star(T;M+2,s).$$

Lemma 4.4. *Let $A\in \mathrm{GL}_3(\mathbf{R})^+$ and $B\in\mathrm{Sym}_3(\mathbf{Q})$ with $\det B\neq 0$. If B is positive definite, then*

$$\lim_{s\mapsto \frac{k-\lambda_1}{2}-r-1}\mathcal{W}_B(\mathbf{m}(A),f^{[k,\lambda]}_{s,\infty}) = C_1^{[k,r,\lambda]}e^{-2\pi\operatorname{tr}(A^{\mathrm{t}}BA)}\frac{\omega^M_{\mathscr{D}_\lambda}(A^{\mathrm{t}}BA;\lambda_2-r)}{(\det A)^{k-\lambda_1-2r-4}},$$

where

$$C_1^{[k,r,\lambda]} = (\sqrt{-1})^{k-\lambda_2}\frac{2^{3(3+2r-k-\lambda_2)}\pi^6}{\Gamma_3(k-r)}.$$

If B is not positive definite, then for any integer $0\le r<k-1$,

$$\lim_{s\to \frac{k-\lambda_1}{2}-r-1}\mathcal{W}_B(\mathbf{m}(A),f^{[k,\lambda]}_{s,\infty})=0.$$

Proof. For each parity type λ we define another differential operator $\mathcal{D}_\lambda$ on $\mathrm{Sym}_3(\mathbf{R})$ by

$$\mathcal{D}_{(0,1,1)} := \frac{1}{2\pi^2\sqrt{-1}}\{\partial_{13}\partial_{12}-\partial_{23}(\partial_{11}-2\pi)\}, \qquad \mathcal{D}_{(0,0,0)}:=\mathrm{id},$$
$$\mathcal{D}_{(1,0,1)} := \frac{1}{2\pi^2\sqrt{-1}}\{\partial_{12}(\partial_{33}+2\pi)-\partial_{23}\partial_{13}\}, \qquad \mathcal{D}_{(1,1,2)} := \mathcal{D}_{(0,1,1)}\mathcal{D}_{(1,0,1)}.$$

It should be remarked that by Lemma 4.1

$$\mathcal{D}_\lambda(e^{-2\pi\sqrt{-1}\operatorname{tr}(Tx)}) = \det(x+\mathbf{i})^{\lambda_1}\det(x-\mathbf{i})^{\lambda_2}H_\lambda(J_3\mathbf{n}(x))e^{-2\pi\sqrt{-1}\operatorname{tr}(Tx)}.$$

Recall that

$$\mathcal{W}_B(\mathbf{m}(A),f^{[k,\lambda]}_{s,\infty}) = (\det A)^{-2s-2}\mathcal{W}_{A^{\mathrm{t}}BA}(\mathbf{1}_6,f^{[k,\lambda]}_{s,\infty}),$$

which reduces our computation to the case $A=\mathbf{1}_3$. We see that

$$\begin{aligned}\mathcal{W}_B(\mathbf{1}_6,f^{[k,\lambda]}_{s,\infty}) =& \int_{\mathrm{Sym}_3(\mathbf{R})}\det(x+\mathbf{i})^{-\alpha_0}\det(x-\mathbf{i})^{-\beta_0}H_\lambda(J_3\mathbf{n}(x))e^{-2\pi\sqrt{-1}\operatorname{tr}(Bx)}\,\mathrm{d}x\\ =&\mathcal{D}_\lambda(\xi(\mathbf{1}_3,T;\alpha_0+\lambda_1,\beta_0+\lambda_2))|_{T=B}\end{aligned}$$

with $\alpha_0=s+1+\frac{k-\lambda_1}{2}$ and $\beta_0=s+1-\frac{k-\lambda_1}{2}$. On the other hand, for any $h\in\mathrm{Sym}_3(\mathbf{R})$, we have

$$\xi(\mathbf{1}_3,h;\alpha,\beta) = (\sqrt{-1})^{3(\beta-\alpha)}\frac{(2\pi)^6e^{-2\pi\operatorname{tr}(h)}}{2^3\Gamma_3(\alpha)\Gamma_3(\beta)}\int_{u>0,\,u>-2\pi h}e^{-2\operatorname{tr}(u)}\det(u+2\pi h)^{\alpha-2}(\det u)^{\beta-2}\,\mathrm{d}u$$

by [Shi82, (1.29)]. If h is positive definite, then the last integral equals $2^{3(2-\alpha-\beta)}\omega^\star(h;\alpha,\beta)\cdot\Gamma_3(\beta)$. Observe that for every polynomial P on $\mathrm{Sym}_3(\mathbf{R})$

$$\mathcal{D}_\lambda(e^{-2\pi\operatorname{tr}(T)}P(T)) = e^{-2\pi\operatorname{tr}(T)}\mathscr{D}_\lambda P(T).$$

This proves the case where B is positive definite. If the signature of B is $(3-q,q)$, then Theorem 4.2 of [Shi82] gives that a holomorphic function $\widetilde{\omega}(\alpha,\beta)$ such that

$$\xi(\mathbf{1}_3,B;\alpha,\beta) = \frac{\Gamma_p\big(\beta-\frac{q}{2}\big)\Gamma_q\big(\alpha-\frac{p}{2}\big)}{\Gamma_3(\alpha)\Gamma_3(\beta)}\cdot\widetilde{\omega}(\alpha,\beta).$$

Thus $\xi(\mathbf{1}_3, B; k-r, -r) = 0$ for $0 \leq r \leq k-2$ unless B is positive definite. □

4.3. **The constant term of $\mathcal{W}_B(\mathbf{m}(\mathrm{diag}(\sqrt{y_1}, \sqrt{y_2}, \sqrt{y_3})), f_{s,\infty}^{[k,\lambda]})$ as a polynomial of y_1^{-1}.** Given $y = \mathrm{diag}(y_1, y_2, y_3) \in \mathbf{R}_+^3$, we put $A = \mathrm{diag}(\sqrt{y_1}, \sqrt{y_2}, \sqrt{y_3})$ and define

$$\mathbf{W}_B^{[k,r,\lambda]}(y) := (y_1y_2y_3)^{r-k+2}\sqrt{y_1}^{\lambda_1}\sqrt{y_2}^{\lambda_1+\lambda_2}\sqrt{y_3}^{2\lambda_1+\lambda_2}\cdot \omega_{\mathscr{D}_\lambda}^M(A^{\mathrm{t}}BA, \lambda_2 - r).$$

Now we write

$$\omega^\star(T; M+2, s) = \sum_{0\leq j_1,j_2,j_3\leq M} c_{j_1j_2j_3}T_{12}^{j_3}T_{23}^{j_1}T_{13}^{j_2}, \qquad c_{j_1j_2j_3} \in \mathbf{C}[T_{11}, T_{22}, T_{33}],$$

where $T = (T_{ij}) \in \mathrm{Sym}_3^+(\mathbf{R})$. Since

$$\omega^\star(\varepsilon^{\mathrm{t}}T\varepsilon; M+2, s) = \omega^\star(T; M+2, s), \qquad \varepsilon = \mathrm{diag}(-1, 1, 1)$$

in view of the expression (4.3), we get $c_{j_1j_2j_3} = (-1)^{j_2+j_3}c_{j_1j_2j_3}$. Thus $c_{j_1j_2j_3} = 0$ unless $j_2 \equiv j_3 \pmod 2$. By symmetry we conclude that $c_{j_1j_2j_3} = 0$ unless $j_1 \equiv j_2 \equiv j_3 \pmod 2$. Moreover, we can write

$$T_{23}^{\lambda_1}T_{12}^{\lambda_2}T_{13}^{\lambda_1+\lambda_2}\omega_{\mathscr{D}_\lambda}^\star(T, s) = \sum_{0\leq j_1,j_2,j_3\leq M,\ j_1\equiv j_2\equiv j_3 \pmod 2} a_{j_1j_2j_3}T_{12}^{j_3}T_{23}^{j_1}T_{13}^{j_2}, \qquad a_{j_1j_2j_3} \in \mathbf{C}[T_{11}, T_{22}, T_{33}].$$

Thus we can write

$$\mathbf{W}_B^{[k,r,\lambda]}(\mathrm{diag}(y_1, y_2, y_3)) = \sum_{0\leq a,b,c\leq M} Q_{a,b,c}^{[k,\lambda]}(B, r)y_1^{-a}y_2^{-b}y_3^{-c}. \tag{4.4}$$

We shall determine the coefficient $Q_{0,b,c}^{[k,\lambda]}(B, r)$ of $\mathbf{W}_B^{[k,r,\lambda]}(y)$ for matrices B with zero diagonal entries

$$B = \begin{pmatrix} 0 & b_3 & b_2 \\ b_3 & 0 & b_1 \\ b_2 & b_1 & 0 \end{pmatrix}.$$

Let $\mathcal{Y}$ be the matrix with variables Y_1, Y_2, Y_3 given by

$$\mathcal{Y} = \begin{pmatrix} 0 & \sqrt{Y_1Y_2} & \sqrt{Y_1Y_3} \\ \sqrt{Y_1Y_2} & 0 & \sqrt{Y_2Y_3} \\ \sqrt{Y_1Y_3} & \sqrt{Y_2Y_3} & 0 \end{pmatrix}.$$

For two functions $f, g : \mathbf{R}_+ \to \mathbf{C}$ and $c \in \mathbf{R}$ we say that $f(y) = g(y) + \mathrm{o}(y^c)$ if $\lim_{y\to\infty} \frac{f(y)-g(y)}{y^c} = 0$.

Lemma 4.5. *The polynomial $\mathbf{K}_{\mathscr{D}_\lambda}^M\left(\frac{\mathcal{Y}}{4\pi}; u\right) \in \mathbf{C}[\sqrt{Y_1}, \sqrt{Y_2}, \sqrt{Y_3}, u]$ in Definition 4.3 has the form*

$$\mathbf{K}_{\mathscr{D}_\lambda}^M((4\pi)^{-1}\mathcal{Y}; u) = C_2^{[k,r,\lambda]}\mathbf{c}_\lambda(Y_2, Y_3; u)\cdot Y_1^{M-\frac{\lambda_1}{2}} + \mathrm{o}(Y_1^{M-\frac{\lambda_1}{2}})$$

with $C_2^{[k,r,\lambda]} \in \mathbf{C}$ and $\mathbf{c}_\lambda(Y_2, Y_3; u) \in \mathbf{C}[\sqrt{Y_2}, \sqrt{Y_3}, u]$ give by

$$C_2^{[k,r,\lambda]} = \frac{(2M+\lambda_1)!}{(2M)!}\cdot\frac{2^{3(\lambda_1+\lambda_2)-\lambda_1}M!}{(\sqrt{-1})^{\lambda_2-\lambda_1}(M-\lambda_1-\lambda_2)!},$$

$$\mathbf{c}_\lambda(Y_2, Y_3; u) = \left(-u_{22}Y_3 - u_{33}Y_2 + 2Y_2Y_3 + 2u_{23}\sqrt{Y_2Y_3}\right)^{M-\lambda_1-\lambda_2}\cdot\sqrt{Y_2}^{\lambda_1+\lambda_2}\sqrt{Y_3}^{\lambda_2}.$$

Proof. This is proved by a direct computation. Note that

$$\begin{aligned}
\partial_{11}\det(T+u) &= (T_{22}+u_{22})(T_{33}+u_{33}) - (T_{23}+u_{23})^2,\\
\partial_{12}\det(T+u) &= -(T_{12}+u_{12})(T_{33}+u_{33}) + (T_{23}+u_{23})(T_{13}+u_{13}),\\
\partial_{13}\det(T+u) &= -(T_{13}+u_{13})(T_{22}+u_{22}) + (T_{12}+u_{12})(T_{23}+u_{23}),\\
\partial_{23}\det(T+u) &= -(T_{23}+u_{23})(T_{11}+u_{11}) + (T_{12}+u_{12})(T_{13}+u_{13}),\\
\partial_{33}\det(T+u) &= (T_{11}+u_{11})(T_{22}+u_{22}) - (T_{12}+u_{12})^2.
\end{aligned}$$

Put

$$\Delta = \det(T+u), \qquad R = (-u_{22}Y_3 - u_{33}Y_2 + 2Y_2Y_3 + 2u_{23}\sqrt{Y_2Y_3})Y_1.$$

Since $\Delta|_{T=\mathcal{Y}} = R + \mathrm{o}(Y_1)$, we have

$$\begin{aligned}
&\mathbf{K}^M_{\mathscr{D}_{(0,1,1)}}((4\pi)^{-1}\mathcal{Y}; u)\\
=&(2\pi^2\sqrt{-1})^{-1}\cdot(4\pi)^2\{\partial_{13}\partial_{12}-\partial_{23}(\partial_{11}-1)\}\Delta^M|_{T=\mathcal{Y}}\\
\equiv&-8\sqrt{-1}[M(M-1)R^{M-2}(\partial_{13}\Delta\partial_{12}\Delta-\partial_{23}\Delta\partial_{11}\Delta)+MR^{M-1}\{\partial_{13}\partial_{12}-\partial_{23}(\partial_{11}-1)\}\Delta]|_{T=\mathcal{Y}}\\
\equiv&-8\sqrt{-1}MR^{M-1}\partial_{23}\Delta|_{T=\mathcal{Y}}\\
\equiv&-8\sqrt{-1}MR^{M-1}\sqrt{Y_2Y_3}Y_1 \pmod{\mathrm{o}(Y_1^M)},
\end{aligned}$$

which verifies the case $\lambda = (0,1,1)$. When $\lambda = (1,0,1)$, we have

$$\begin{aligned}
&\mathbf{K}^M_{\mathscr{D}_{(1,0,1)}}((4\pi)^{-1}\mathcal{Y}; u)\\
\equiv&-8\sqrt{-1}\{M(M-1)R^{M-2}(\partial_{12}\Delta\partial_{33}\Delta-\partial_{13}\Delta\partial_{23}\Delta)+MR^{M-1}(\partial_{12}\partial_{33}\Delta-\partial_{13}\partial_{23}\Delta)\}|_{T=\mathcal{Y}}\\
\equiv&-8\sqrt{-1}\left\{M(M-1)R^{M-2}(-R\sqrt{Y_1Y_2})+MR^{M-1}\left(-\frac{3}{2}\sqrt{Y_1Y_2}\right)\right\}\\
\equiv&4\sqrt{-1}M(2M+1)R^{M-1}\sqrt{Y_1Y_2}\pmod{\mathrm{o}(Y_1^{M-\frac{1}{2}})}
\end{aligned}$$

as claimed. Since

$$\mathscr{D}_{(0,1,1)}\Delta^M|_{T=\mathcal{Y}} = -8\sqrt{-1}M\Delta^{M-1}T_{12}T_{13}|_{T=\mathcal{Y}} + \mathrm{o}(Y_1^M),$$

we have

$$\begin{aligned}
\mathbf{K}^M_{\mathscr{D}_{(1,1,2)}}((4\pi)^{-1}\mathcal{Y}; u) \equiv&32M(M-1)(2M-1)R^{M-2}\sqrt{Y_1Y_2}Y_1\sqrt{Y_2Y_3}\\
&-64M(M-1)\Delta^{M-2}(T_{13}\partial_{33}\Delta-T_{12}\partial_{23}\Delta)|_{T=\mathcal{Y}}\pmod{\mathrm{o}(Y_1^{M-\frac{1}{2}})},
\end{aligned}$$

which proves the case $\lambda = (1,1,2)$. □

Lemma 4.6. *Let $F(T)$ be a polynomial in $T = (T_{ij}) \in \mathrm{Sym}_3(\mathbf{R})$. Then we have*

$$\int_{\mathrm{Sym}_3^+(\mathbf{R})} e^{-\operatorname{tr}(u)}F(u)\frac{(\det u)^{s-2}}{\Gamma_3(s)}\mathrm{d}u = F(-\partial_{ij})(\det T)^{-s}|_{T=\mathbf{1}_3}.$$

Proof. If T is positive definite and $\operatorname{Re} s > 2$, then

$$\int_{\mathrm{Sym}_3^+(\mathbf{R})} e^{-\operatorname{tr}(Tu)}\frac{(\det u)^{s-2}}{\Gamma_3(s)}\,\mathrm{d}u = (\det T)^{-s}$$

by [Shi81, (1.14)]. The declared formula follows immediately from the fact that

$$F(-\partial_{ij})(e^{-\operatorname{tr}(Tu)}) = F(u)e^{-\operatorname{tr}(Tu)}.$$ □

Now let $k \geq l \geq m$ be a set of balanced integers. We say that (k,l,m) has the parity type λ if

$$\lambda_1,\lambda_2\in\{0,1\},\qquad \lambda_1\equiv l-m \pmod 2,\qquad \lambda_2\equiv k-l\pmod 2,\qquad \lambda_3=\lambda_1+\lambda_2.$$

Lemma 4.7. *Let λ be the parity type of $k \geq l \geq m$ and r an integer such that $k - \frac{l+m+\lambda_1}{2} \leq r \leq \frac{l+m}{2} - 2$. Put*

$$M = k-r-2,\qquad b=\frac{1}{2}(k-l-\lambda_2),\qquad c=\frac{1}{2}(k-m-\lambda_3),\qquad n=M+\frac{1}{2}(l+m-\lambda_1).$$

Then we have

$$Q^{[k,\lambda]}_{0,b,c}(B,r) = w_{0,b,c}\cdot(b_1b_2b_3)^n b_1^{-k}b_2^{-l}b_3^{-m},$$

where

$$w_{0,b,c} = (4\pi)^{3M-b-c-2\lambda_1-\lambda_2}2^{M+\lambda_1+2\lambda_2-b-c}\frac{(\sqrt{-1})^{\lambda_1-\lambda_2}(2M+\lambda_1)!M!}{(2M)!(M-\lambda_1-\lambda_2-b-c)!}\frac{(r-\lambda_2)!}{b!c!(r-\lambda_2-b-c)!}.$$

PROOF. Substitute $Y_i = 4\pi\frac{b_1b_2b_3}{b_i^2}y_i$ into the matrix $\mathcal{Y}$. Then $\mathcal{Y} = 4\pi A^{\mathrm{t}}BA$ and

$$\mathbf{W}_B^{[k,r,\lambda]}(y) = \sum_{a,b,c} Q_{a,b,c}^{[k,\lambda]}(B,r)(4\pi)^{a+b+c}Y_1^{-a}Y_2^{-b}Y_3^{-c}\cdot b_1^{b+c-a}b_2^{a+c-b}b_3^{a+b-c}.$$

On the other hand,

$$\mathbf{W}_B^{[k,r,\lambda]}(y) = \left(\frac{(4\pi)^3b_1b_2b_3}{Y_1Y_2Y_3}\right)^M \frac{\sqrt{Y_1}^{\lambda_1}\sqrt{Y_2}^{\lambda_1+\lambda_2}\sqrt{Y_3}^{2\lambda_1+\lambda_2}}{(4\pi)^{2\lambda_1+\lambda_2}b_1^{\lambda_1+\lambda_2}b_2^{\lambda_1}}\omega^\star_{\mathscr{D}_\lambda}((4\pi)^{-1}\mathcal{Y},\lambda_2-r)$$

by definition. The equations above give a complex number $w_{0,b,c}$ such that

$$Q_{0,b,c}^{[k,\lambda]}(B,r) = w_{0,b,c}\cdot(b_1b_2b_3)^nb_1^{-k}b_2^{-l}b_3^{-m}.$$

Our task is to determine $w_{0,b,c}$. It is the coefficient of $Y_2^{M-\lambda_1-\lambda_2-b}Y_3^{M-\lambda_1-\lambda_2-c}$ in the polynomial

$$\begin{aligned}
&\frac{(4\pi)^{3M-b-c-2\lambda_1-\lambda_2}}{\sqrt{Y_2}^{\lambda_1+\lambda_2}\sqrt{Y_3}^{\lambda_2}}\int_{\mathrm{Sym}_3^+(\mathbf{R})} e^{-\operatorname{tr}(u)}C_2^{[k,r,\lambda]}\mathbf{c}_\lambda(Y_2,Y_3;u)\frac{(\det u)^{s-2}}{\Gamma_3(s)}\,\mathrm{d}u\Big|_{s=\lambda_2-r}\\
=&(4\pi)^{3M-b-c-2\lambda_1-\lambda_2}C_2^{[k,r,\lambda]}\\
&\times\int_{\mathrm{Sym}_3^+(\mathbf{R})} e^{-\operatorname{tr}(u)}\left(-u_{22}Y_3-u_{33}Y_2+2Y_2Y_3+2u_{23}\sqrt{Y_2Y_3}\right)^{M-\lambda_1-\lambda_2}\frac{(\det u)^{s-2}}{\Gamma_3(s)}\,\mathrm{d}u|_{s=\lambda_2-r}
\end{aligned}$$

by Lemma 4.5. Put

$$L = M-\lambda_1-\lambda_2, \qquad r_1 = r-\lambda_2.$$

Notice that $b\le c$ by assumption. The coefficient of $Y_2^{L-b}Y_3^{L-c}$ in the last integral is given by

$$\begin{aligned}
&\sum_{i=0}^b\frac{2^{L-b-c}(-1)^{b+c}\cdot L!}{(b-i)!(c-i)!(L-b-c)!(2i)!}\int_{\mathrm{Sym}_3^+(\mathbf{R})} e^{-\operatorname{tr}(u)}u_{33}^{b-i}u_{22}^{c-i}(2u_{23})^{2i}\frac{(\det u)^{s-2}}{\Gamma_3(s)}\,\mathrm{d}u|_{s=-r_1}\\
=&\sum_{i=0}^b\frac{2^{L-b-c}\cdot L!2^{2i}}{(b-i)!(c-i)!(L-b-c)!(2i)!}\partial_{33}^{b-i}\partial_{22}^{c-i}\partial_{23}^{2i}(T_{22}T_{33}-T_{23}^2)^{r_1}|_{T_{22}=T_{33}=1,T_{23}=0}\\
=&\sum_{i=0}^b\frac{2^{L-b-c}\cdot L!2^{2i}}{(b-i)!(c-i)!(L-b-c)!(2i)!}\sum_{j=0}^{r_1}\binom{r_1}{j}\partial_{33}^{b-i}\partial_{22}^{c-i}\partial_{23}^{2i}T_{22}^{r_1-j}T_{33}^{r_1-j}(-T_{23}^2)^j|_{T_{22}=T_{33}=1,T_{23}=0}\\
=&\sum_{i=0}^b\frac{2^{L-b-c}\cdot L!}{(b-i)!(c-i)!(L-b-c)!}\binom{r_1}{i}(-1)^i\partial_{33}^{b-i}\partial_{22}^{c-i}(T_{22}^{r_1-i}T_{33}^{r_1-i})|_{T_{22}=T_{33}=1}\\
=&\frac{2^{L-b-c}\cdot L!}{(L-b-c)!}\sum_{i=0}^b\binom{r_1}{i}\binom{r_1-i}{b-i}\binom{r_1-i}{c-i}(-1)^i
\end{aligned}$$

in view of Lemma 4.6. The last summation equals

$$\frac{r_1!}{(r_1-b)!b!}\sum_{i=0}^b\binom{b}{i}\binom{r_1-i}{r_1-c}(-1)^i = \frac{r_1!}{(r_1-b)!b!}\cdot\binom{r_1-b}{r_1-b-c} = \frac{r_1!}{b!c!(r_1-b-c)!},$$

where we can deduce this equality by equating the terms of degree r_1-c of the identity

$$\sum_{i=0}^b\binom{b}{i}(1+X)^{r_1-i}(-1)^i = (1+X)^{r_1}\left(1-\frac{1}{1+X}\right)^b = (1+X)^{r_1-b}X^b,$$

Finally, we see that $w_{0,b,c}$ equals

$$(4\pi)^{3M-b-c-2\lambda_1-\lambda_2}C_2^{[k,r,\lambda]}\frac{2^{M-\lambda_1-\lambda_2-b-c}\cdot(M-\lambda_1-\lambda_2)!}{(M-\lambda_1-\lambda_2-b-c)!}\frac{(r-\lambda_2)!}{b!c!(r-\lambda_2-b-c)!}$$

by putting together the above computations, which completes our proof. □

4.4. **The archimedean zeta integral.** Let $V_\pm$ be the weight raising/lowering operator given by

$$V_\pm := \frac{1}{(-8\pi)}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes 1 \pm \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\otimes\sqrt{-1}\right) \in \mathrm{Lie}(\mathrm{GL}_2(\mathbf{R}))\otimes_{\mathbf{R}}\mathbf{C}.$$

For each integer k we denote by σ_k the (limit of) discrete series of $\mathrm{GL}_2(\mathbf{R})$ of the minimal weight $\pm k$ and by W_k the Whittaker function of σ_k characterized by

$$W_k(\mathrm{diag}(y,1)) = y^{k/2}e^{-2\pi y}\mathbb{I}_{\mathbf{R}_+}(y).$$

Set $W_k^{[t]} = V_+^t W_k$. It follows from (6.2) below that

$$W_k^{[t]}(\mathrm{diag}(y,1)) = \sum_{j=0}^{t}(-4\pi)^{j-t}\binom{t}{j}\frac{\Gamma(t+k)}{\Gamma(j+k)}\cdot y^{\frac{k}{2}+j}e^{-2\pi y}\mathbb{I}_{\mathbf{R}_+}(y). \tag{4.5}$$

Fix a triplet (k,l,m) of positive integers such that $k \geq l \geq m$ and $k < l+m$. Put $\mathcal{J}_\infty = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Define

$$Z_\infty(s) := Z\left(\rho(\mathcal{J}_\infty)W_k, \rho(\mathcal{J}_\infty)W_l^{\left[\frac{k-l-\lambda_2}{2}\right]}, \rho(\mathcal{J}_\infty)W_m^{\left[\frac{k-m-\lambda_3}{2}\right]}, f_{s,\infty}^{[k,\lambda]}\right),$$

where λ is the parity type of (k,l,m). Recall that

$$\begin{aligned}L(s,\sigma_k\times\sigma_l\times\sigma_m) =& \Gamma_{\mathbf{C}}\left(s+\frac{k+l+m-3}{2}\right)\Gamma_{\mathbf{C}}\left(s+\frac{k-l+m-1}{2}\right)\\ &\times\Gamma_{\mathbf{C}}\left(s+\frac{k+l-m-1}{2}\right)\Gamma_{\mathbf{C}}\left(s+\frac{m+l-k-1}{2}\right).\end{aligned}$$

Put

$$\gamma^\star_{(k,m,l)}(s) = (\sqrt{-1})^{k+2\lambda_2+\lambda_1}\frac{\Gamma\big(s+\frac{k-m-l}{2}+1\big)}{\Gamma\big(s-\frac{k-\lambda_1}{2}+\lambda_2+1\big)}\cdot\frac{\Gamma\big(s+\frac{k+\lambda_1}{2}\big)}{\Gamma\big(s+\frac{k+\lambda_1}{2}+1\big)}\cdot\frac{\pi^{3s+1}(4\pi)^{l+m-\frac{k-\lambda_1}{2}+\lambda_2}}{4\Gamma\big(s+\frac{m+l-k}{2}\big)\Gamma(2s+k)}.$$

Lemma 4.8. *If λ is the parity type of (k,l,m), then*

$$Z_\infty(s) = (\chi_\infty\hat{\omega}_\infty)(-1)\,\mathrm{vol}(\mathrm{SO}(2))^3\cdot\frac{\gamma^\star_{(k,m,l)}(s)}{2^{5+(k+m+l)}}L\left(s+\frac{1}{2},\sigma_k\times\sigma_l\times\sigma_m\right).$$

PROOF. For $a=(a_1,a_2,a_3)\in\mathbf{R}_+^3$ and $x\in\mathbf{R}$, we set

$$\begin{aligned}z &= x+\sqrt{-1}(a_1^2+a_2^2+a_3^2),\\ t(a) &= \mathrm{diag}\left(\begin{pmatrix}a_1 & 0\\ 0 & a_1^{-1}\end{pmatrix},\begin{pmatrix}a_2 & 0\\ 0 & a_2^{-1}\end{pmatrix},\begin{pmatrix}a_3 & 0\\ 0 & a_3^{-1}\end{pmatrix}\right),\\ u(x) &= \mathrm{diag}\left(\begin{pmatrix}1 & \frac{x}{3}\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & \frac{x}{3}\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & \frac{x}{3}\\ 0 & 1\end{pmatrix}\right).\end{aligned}$$

When $x\neq 0$, the Iwasawa decomposition of $\boldsymbol{\eta}\iota(u(x)t(a))$ can described as follows: Put

$$P = \begin{pmatrix}a_1^2 & a_1a_2 & a_1a_3\\ a_1a_2 & a_2^2 & a_2a_3\\ a_1a_3 & a_2a_3 & a_3^2\end{pmatrix}.$$

We write $\boldsymbol{\eta}\iota(u(x)t(a)) = \mathbf{n}(z)\mathbf{m}(A)\mathbf{u}$ with $z\in\mathrm{Sym}_3(\mathbf{R})$, $A\in\mathrm{GL}_3(\mathbf{R})$ and $\mathbf{u}=\begin{pmatrix}D & -C\\ C & D\end{pmatrix}\in\mathrm{U}(3)$. Since $D^{-1}C = x^{-1}P$, we can choose $U\in\mathrm{GL}_3(\mathbf{R})$ so that

$$U^{\mathrm{t}}U = (x^2\mathbf{1}_3+P^2)^{-1}, \qquad \mathbf{u} = \begin{pmatrix}Ux & -UP\\ UP & Ux\end{pmatrix}\in\mathrm{U}(3).$$

Put $u = Ux-\sqrt{-1}UP$. Then $u^{\mathrm{t}}u = (x\mathbf{1}_3-P\sqrt{-1})(x\mathbf{1}_3+P\sqrt{-1})^{-1}$. By direct computations we get

$$\det A = a_1a_2a_3\,|z|^{-1}, \qquad \det u = \frac{\bar{z}}{|z|}, \qquad H_{23}(u) = -2\sqrt{-1}\frac{a_2a_3}{z}, \qquad H_{12}(u) = -2\sqrt{-1}\frac{a_1a_2}{z}$$

(see [GK92, (6.7), (6.8)]). Put

$$b=\frac{1}{2}(k-l-\lambda_2), \qquad c=\frac{1}{2}(k-m-\lambda_3), \qquad (\mathbf{s},\mathbf{k},\mathbf{l},\mathbf{m})=\left(s+\frac{\lambda_3}{2},k-\lambda_2,l,m-\lambda_1\right).$$

It follows that

$$\begin{aligned}f^{[k,\lambda]}_{s,\infty}(\boldsymbol{\eta}\iota(u(x)t(a)\mathbf{d}(-1)))&=\chi_\infty(-1)(-1)^{k-\lambda_1}\left(2\sqrt{-1}\frac{a_1a_2}{\bar{z}}\right)^{\lambda_1}\left(-2\sqrt{-1}\frac{a_2a_3}{z}\right)^{\lambda_2}\left(\frac{a_1a_2a_3}{|z|}\right)^{2s+2}\left(\frac{z}{|z|}\right)^{k-\lambda_1}\\&=\chi_\infty(-1)2^{\lambda_1+\lambda_2}\sqrt{-1}^{\lambda_2-\lambda_1}(a_1a_2a_3)^{2\mathbf{s}+2}a_1^{-\lambda_2}a_3^{-\lambda_1}|z|^{-2\mathbf{s}-2-\mathbf{k}}(-z)^{\mathbf{k}}.\end{aligned}$$

From (4.2) and (4.5) $\frac{2Z_\infty(s)}{\mathrm{vol}(\mathrm{SO}(2))^3}$ equals

$$\begin{aligned}&\int_{\mathbf{R}}\int_{\mathbf{R}_+^3}W_k(\mathbf{n}(x/3)\mathbf{m}(a_1))W_l^{[b]}(\mathbf{n}(x/3)\mathbf{m}(a_2))W_m^{[c]}(\mathbf{n}(x/3)\mathbf{m}(a_3))f^{[k,\lambda]}_{s,\infty}(\boldsymbol{\eta}\iota(u(x)t(a)\mathbf{d}(-1)))\,\mathrm{d}x\prod_{j=1}^3\frac{\mathrm{d}^\times a_j}{|a_j|^2}\\=&\chi_\infty(-1)2^{\lambda_1+\lambda_2}\sqrt{-1}^{\lambda_2-\lambda_1}(-4\pi)^{-b-c}\sum_{A=0}^\infty\sum_{B=0}^\infty(-4\pi)^{A+B}\binom{b}{A}\binom{c}{B}\frac{\Gamma(l+b)}{\Gamma(l+A)}\frac{\Gamma(m+c)}{\Gamma(m+B)}\\&\times\int_{\mathbf{R}}\int_{\mathbf{R}_+^3}a_1^{2\mathbf{s}+\mathbf{k}}a_2^{2\mathbf{s}+\mathbf{l}+2A}a_3^{2\mathbf{s}+\mathbf{m}+2B}|z|^{-2\mathbf{s}-2-\mathbf{k}}(-z)^{\mathbf{k}}e^{2\pi\sqrt{-1}x}e^{-2\pi(a_1^2+a_2^2+a_3^2)}\,\mathrm{d}x\prod_{j=1}^3\mathrm{d}^\times a_j.\end{aligned}$$

Put $\alpha=\mathbf{s}+1+\frac{\mathbf{k}}{2}$ and $\beta=\mathbf{s}+1-\frac{\mathbf{k}}{2}$. The last integral equals

$$\frac{(-2\pi\sqrt{-1})^\alpha(2\pi\sqrt{-1})^\beta}{\Gamma(\alpha)\Gamma(\beta)}\int_{\mathbf{R}_+^4}a_1^{\mathbf{k}+2\mathbf{s}}a_2^{\mathbf{l}+2A+2\mathbf{s}}a_3^{\mathbf{m}+2B+2\mathbf{s}}\frac{(1+t)^{\alpha-1}t^{\beta-1}}{e^{4\pi(a_1^2+a_2^2+a_3^2)(1+t)}}\mathrm{d}t\prod_{j=1}^3\mathrm{d}^\times a_j.$$

We here use the identity

$$\int_{\mathbf{R}}\frac{e^{-2\pi\sqrt{-1}x}\,\mathrm{d}x}{(x+\sqrt{-1}y)^\alpha(x-\sqrt{-1}y)^\beta}=\frac{(-2\pi\sqrt{-1})^\alpha(2\pi\sqrt{-1})^\beta}{\Gamma(\alpha)\Gamma(\beta)}\int_{\mathbf{R}_+}\frac{(t+1)^{\alpha-1}t^{\beta-1}}{e^{2\pi y(1+2t)}}\,\mathrm{d}t$$

(see [GK92, (6.11)]). The quadruple integral above equals

$$\begin{aligned}&\frac{1}{(4\pi)^{\frac{\mathbf{k}+\mathbf{l}+\mathbf{m}}{2}+3\mathbf{s}+A+B}}\int_{\mathbf{R}_+^4}\frac{a_1^{\mathbf{k}+2\mathbf{s}}a_2^{\mathbf{l}+2A+2\mathbf{s}}a_3^{\mathbf{m}+2B+2\mathbf{s}}(1+t)^{\alpha-1}t^{\beta-1}}{e^{a_1^2+a_2^2+a_3^2}(1+t)^{\frac{\mathbf{k}+\mathbf{l}+\mathbf{m}}{2}+3\mathbf{s}+A+B}}\mathrm{d}t\prod_{j=1}^3\mathrm{d}^\times a_j\\=&\frac{\Gamma\big(\frac{\mathbf{k}}{2}+\mathbf{s}\big)\Gamma\big(\frac{\mathbf{l}}{2}+\mathbf{s}+A\big)\Gamma\big(\frac{\mathbf{m}}{2}+\mathbf{s}+B\big)}{2^3(4\pi)^{\frac{\mathbf{k}+\mathbf{l}+\mathbf{m}}{2}+3\mathbf{s}+A+B}}\int_{\mathbf{R}_+}\frac{(1+t)^{\alpha-1}t^{\beta-1}}{(1+t)^{\frac{\mathbf{k}+\mathbf{l}+\mathbf{m}}{2}+3\mathbf{s}+A+B}}\mathrm{d}t.\end{aligned}$$

Recall that

$$\begin{aligned}\int_0^\infty\frac{(1+t)^{\alpha-1}t^{\beta-1}}{(1+t)^{\frac{\mathbf{k}+\mathbf{l}+\mathbf{m}}{2}+3\mathbf{s}+A+B}}\mathrm{d}t=&B\left(\beta,1-\alpha-\beta+\frac{\mathbf{k}+\mathbf{l}+\mathbf{m}}{2}+3\mathbf{s}+A+B\right)\\=&\frac{\Gamma(\beta)\Gamma\left(1-\alpha-\beta+\frac{\mathbf{k}+\mathbf{l}+\mathbf{m}}{2}+3\mathbf{s}+A+B\right)}{\Gamma\left(1-\alpha+\frac{\mathbf{k}+\mathbf{l}+\mathbf{m}}{2}+3\mathbf{s}+A+B\right)}.\end{aligned}$$

We finally get

$$\begin{aligned}Z_\infty(s)=&\mathrm{vol}(\mathrm{SO}(2))^3\chi_\infty(-1)2^{\lambda_1+2\lambda_2-2-4s-3k}\pi^{2-s+\frac{\lambda_2+\lambda_3-3k}{2}}(-\sqrt{-1})^{k+\lambda_1}(-1)^{\lambda_2+b+c}\\&\times\frac{\Gamma\big(\mathbf{s}+\frac{\mathbf{k}}{2}\big)}{\Gamma\big(\mathbf{s}+\frac{\mathbf{k}}{2}+1\big)}\Gamma(l+b)\Gamma(m+c)\sum_{A,B}(-1)^{A+B}\binom{b}{A}\binom{c}{B}\frac{\Gamma_\infty(s;A,B)}{\Gamma(l+A)\Gamma(m+B)},\end{aligned}$$

where

$$\Gamma_\infty(\mathbf{s};A,B)=\frac{\Gamma\big(\mathbf{s}+\frac{\mathbf{l}}{2}+A\big)\Gamma\big(\mathbf{s}+\frac{\mathbf{m}}{2}+B\big)\Gamma\big(\mathbf{s}+\frac{\mathbf{k}+\mathbf{l}+\mathbf{m}}{2}-1+A+B\big)}{\Gamma(2\mathbf{s}+\frac{\mathbf{l}+\mathbf{m}}{2}+A+B)}.$$

Lemma 3 of [Orl87] with $\alpha = l = \mathbf{l}$, $t = \mathbf{s} + \frac{l}{2}$, $\beta = B + \frac{\mathbf{m}-l}{2}$ and $N = b$ gives

$$\Gamma(l+b)\sum_{A=0}^{b}(-1)^A\binom{b}{A}\frac{\Gamma(\mathbf{s}+\frac{l}{2}+A)\Gamma(\mathbf{s}+\frac{\mathbf{k}+l+\mathbf{m}}{2}-1+A+B)}{\Gamma(l+A)\Gamma(2\mathbf{s}+\frac{l+\mathbf{m}}{2}+A+B)}$$
$$=(-1)^b\frac{\Gamma(\mathbf{s}+\frac{l}{2})\Gamma(\mathbf{s}+B+\frac{\mathbf{m}}{2}+l+b-1)\Gamma(\mathbf{s}+B+\frac{\mathbf{m}}{2}+b)\Gamma(\mathbf{s}-\frac{l}{2}+1)}{\Gamma(2\mathbf{s}+B+\frac{\mathbf{m}+l}{2}+b)\Gamma(\mathbf{s}+B+\frac{\mathbf{m}}{2})\Gamma(\mathbf{s}-\frac{l}{2}-b+1)}.$$

It follows that

$$\Gamma(l+b)\Gamma(m+c)\sum_{A,B}(-1)^{A+B}\binom{b}{A}\binom{c}{B}\frac{\Gamma_\infty(s;A,B)}{\Gamma(l+A)\Gamma(m+B)}$$
$$=(-1)^b\frac{\Gamma(\mathbf{s}+\frac{l}{2})\Gamma(\mathbf{s}-\frac{l}{2}+1)}{\Gamma(\mathbf{s}-\frac{l}{2}-b+1)}\Gamma(m+c)\sum_{B}(-1)^B\binom{c}{B}\frac{\Gamma(\mathbf{s}+B+\frac{\mathbf{m}}{2}+l+b-1)\Gamma(\mathbf{s}+B+\frac{\mathbf{m}}{2}+b)}{\Gamma(2\mathbf{s}+B+\frac{\mathbf{m}+l}{2}+b)\Gamma(m+B)}.$$

Again we apply Lemma 3 of [Orl87] with $\alpha = m$, $t = \mathbf{s} + \frac{\mathbf{m}}{2} + b$, $\beta = \frac{l-\mathbf{m}}{2} - b$ and $N = c$ to obtain

$$\Gamma(m+c)\sum_{B}(-1)^B\binom{c}{B}\frac{\Gamma(\mathbf{s}+B+\frac{\mathbf{m}}{2}+l+b-1)\Gamma(\mathbf{s}+B+\frac{\mathbf{m}}{2}+b)}{\Gamma(2\mathbf{s}+B+\frac{\mathbf{m}+l}{2}+b)\Gamma(m+B)}$$
$$=(-1)^c\frac{\Gamma(\mathbf{s}+\frac{\mathbf{m}}{2}+b)\Gamma(\mathbf{s}+\frac{l}{2}+m+c-1)\Gamma(\mathbf{s}+\frac{l}{2}+c)\Gamma(\mathbf{s}+\frac{\mathbf{m}}{2}+b-m+1)}{\Gamma(2\mathbf{s}+\frac{\mathbf{m}+l}{2}+b+c)\Gamma(\mathbf{s}+\frac{l}{2})\Gamma(\mathbf{s}+\frac{\mathbf{m}}{2}-m+b-c+1)}.$$

Then we can see that the double summation equals

$$(-1)^{b+c}\frac{\Gamma(\mathbf{s}+\frac{\mathbf{m}}{2}+b)\Gamma(\mathbf{s}+\frac{l}{2}+m+c-1)\Gamma(\mathbf{s}+\frac{l}{2}+c)\Gamma(\mathbf{s}+c-\frac{l}{2}+1)}{\Gamma(\mathbf{s}-\frac{l}{2}-b+1)\Gamma(2\mathbf{s}+\frac{\mathbf{m}+l}{2}+b+c)}$$
$$=(-1)^{b+c}\frac{\Gamma(s+\frac{k-l+m}{2})\Gamma(s+\frac{k+l+m}{2}-1)\Gamma(s+\frac{k-l-m}{2}+1)\Gamma(s+\frac{k-m+l}{2})}{\Gamma(2s+k)}\cdot\frac{1}{\Gamma(\mathbf{s}-\frac{\mathbf{k}}{2}+1)}.$$

The last equality uses $b = \frac{\mathbf{k}-\mathbf{l}}{2}$, $m + c = \frac{\mathbf{k}+\mathbf{m}}{2}$, $\mathbf{s} + c = s + \frac{k-m}{2}$ and $2\mathbf{s} + \mathbf{k} + \mathbf{m} = 2s + k + m$. □

5. Classical and p-adic modular forms

5.1. **Conventions.** Besides the standard symbols $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Z}_\ell$, $\mathbf{Q}_\ell$ we denote by $\mathbf{R}_+$ the group of strictly positive real numbers. Fix algebraic closures of $\mathbf{Q}$ and $\mathbf{Q}_p$, denoting them by $\overline{\mathbf{Q}}$ and $\overline{\mathbf{Q}}_p$. Let $\mathbf{A}$ be the ring of adèles of $\mathbf{Q}$ and μ_n the group of n-th roots of unity in $\overline{\mathbf{Q}}$. Given a place v of $\mathbf{Q}$, we write $\mathbf{Q}_v$ for the completion of $\mathbf{Q}$ with respect to v. We shall regard $\mathbf{Q}_v$ and $\mathbf{Q}_v^\times$ as subgroups of $\mathbf{A}$ and $\mathbf{A}^\times$ in a natural way. We denote by the formal symbol ∞ the real place of $\mathbf{Q}$ and do not use ℓ for the infinite place. Let $\boldsymbol{\psi}_{\mathbf{Q}} : \mathbf{A}/\mathbf{Q} \to \mathbf{C}^\times$ be the additive character with the archimedean component $\boldsymbol{\psi}_\infty(x) = e^{2\pi\sqrt{-1}x}$ and $\boldsymbol{\psi}_\ell : \mathbf{Q}_\ell \to \mathbf{C}^\times$ the local component of $\boldsymbol{\psi}_{\mathbf{Q}}$ at ℓ.

Denote by $\boldsymbol{\alpha}_{\mathbf{Q}_v} = |\cdot|_v$ the absolute value on $\mathbf{Q}_v$ normalized so that $\boldsymbol{\alpha}_{\mathbf{R}}$ is the usual absolute value on $\mathbf{R}$, and $|\ell|_\ell = \ell^{-1}$ if $v = \ell$ is finite. For $a \in \mathbf{A}^\times$, let $a_v \in \mathbf{Q}_v^\times$ denote the v-component of a. Define the character $\boldsymbol{\alpha}_{\mathbf{A}} = \mathbf{Q}^\times\backslash\mathbf{A}^\times \to \mathbf{R}_+$ by $\boldsymbol{\alpha}_{\mathbf{A}}(a) = |a|_{\mathbf{A}} = \prod_v |a_v|_v$. Recall the local Riemann zeta functions

$$\zeta_\infty(s) = \pi^{-s/2}\Gamma(s/2), \qquad \zeta_\ell(s) = (1-\ell^{-s})^{-1}.$$

Define the completed Riemann zeta function $\zeta_{\mathbf{Q}}(s)$ by $\zeta_{\mathbf{Q}}(s) = \prod_v \zeta_v(s)$. In particular, $\zeta_{\mathbf{Q}}(2) = \frac{\pi}{6}$. For each rational prime ℓ, let $v_\ell : \mathbf{Q}_\ell^\times \to \mathbf{Z}$ denote the valuation normalized so that $v_\ell(\ell) = 1$. To avoid possible confusion, denote by $\varpi_\ell = (\varpi_{\ell,v}) \in \mathbf{A}^\times$ the idèle defined by $\varpi_{\ell,\ell} = \ell$ and $\varpi_{\ell,v} = 1$ if $v \neq \ell$.

If $\omega : \mathbf{A}^\times \to \mathbf{C}^\times$ is a quasi-character, then we denote by $\omega_v : \mathbf{Q}_v^\times \to \mathbf{C}^\times$ the local component of ω at v. If $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \overline{\mathbf{Q}}^\times$ is a Dirichlet character modulo N, then we denote the ℓ-exponent of the conductor of χ by $c_\ell(\chi) \le v_\ell(N)$. We can associate to a Dirichlet character χ of conductor N a Hecke character $\chi_{\mathbf{A}}$, called the *adèlic lift* of χ, which is the unique finite order Hecke character $\chi_{\mathbf{A}} : \mathbf{Q}^\times\mathbf{R}_+\backslash\mathbf{A}^\times/(1+N\widehat{\mathbf{Z}}) \cap \widehat{\mathbf{Z}}^\times \to \overline{\mathbf{Q}}^\times$ of conductor N such that $\chi_{\mathbf{A}}(\varpi_\ell) = \chi(\ell)^{-1}$ for any prime number $\ell \nmid N$.

Fix an odd prime number p and an isomorphism $\iota_p : \overline{\mathbf{Q}}_p \simeq \mathbf{C}$ once and for all.

Definition 5.1 (Teichmüller and cyclotomic characters)**.** The action of $G_{\mathbf{Q}}$ on $\mu_{p^\infty} := \varinjlim_n \mu_{p^n}$ gives rise to a continuous homomorphism $\boldsymbol{\varepsilon}_{\mathrm{cyc}} : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$, called the p-adic cyclotomic character, defined by $\sigma(\zeta) = \zeta^{\boldsymbol{\varepsilon}_{\mathrm{cyc}}(\sigma)}$ for every $\zeta \in \mu_{p^\infty}$. The character $\boldsymbol{\varepsilon}_{\mathrm{cyc}}$ splits into the p-adic Teichmüller character $\boldsymbol{\omega} : G_{\mathbf{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q}) \to \mathbf{Z}_p^\times$ and $\langle\cdot\rangle : G_{\mathbf{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \xrightarrow{\sim} 1 + p\mathbf{Z}_p$. The character $\boldsymbol{\omega}$ sends σ to the unique solution in $\mathbf{Z}_p^\times$ of $\boldsymbol{\omega}(\sigma)^p = \boldsymbol{\omega}(\sigma) \equiv \boldsymbol{\varepsilon}_{\mathrm{cyc}}(\sigma) \pmod p$. We often regard $\boldsymbol{\omega}$ and $\langle\cdot\rangle^s$ with $s \in \mathbf{Z}_p$ as characters of $\mathbf{Z}_p^\times$. We sometimes identify $\boldsymbol{\omega}$ with the Dirichlet character $\iota_p \circ \boldsymbol{\omega} : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{C}^\times$.

Remark 5.2.
(1) Let χ be a Dirichlet character. If χ_v stands for the restriction of $\chi_{\mathbf{A}}$ to $\mathbf{Q}_v^\times$, then $\chi_\ell(\ell) = \chi(\ell)^{-1}$ for each prime number $\ell \nmid N$. Furthermore, if N is a power of p and b is not divisible by p, then $\chi_p(b) = \chi(b)$.
(2) Let χ be a character of $\mathbf{Z}_p^\times$ of finite order, which can be regard as either a complex character or a p-adic character via composition with ι_p. We view χ as a character of $G_{\mathbf{Q}}$ via composition with the cyclotomic character $\boldsymbol{\varepsilon}_{\mathrm{cyc}}$. Let $\mathbf{Q}^{\mathrm{ab}} = \bigcup_{N=1}^\infty \mathbf{Q}(\mu_N)$ be the maximal abelian extension of $\mathbf{Q}$ and
$$\mathrm{rec}_{\mathbf{Q}} : \mathbf{Q}^\times \mathbf{R}_+ \backslash \mathbf{A}^\times \xrightarrow{\sim} \mathrm{Gal}(\mathbf{Q}^{\mathrm{ab}}/\mathbf{Q})$$
the geometrically normalized reciprocity law map, i.e., $\mathrm{rec}_{\mathbf{Q}}(\varpi_\ell)|_{\mathbf{Q}(\mu_{p^\infty})} = \mathrm{Frob}_\ell$ for $\ell \neq p$. Since χ factors through the quotient $\mathbf{Z}_p^\times \twoheadrightarrow (\mathbf{Z}/p^{c(\chi)}\mathbf{Z})^\times$, we can identify χ with a Dirichlet character of p-power conductor. Then since $\chi_{\mathbf{A}}(\varpi_\ell) = \chi(\ell)^{-1} = \chi(\boldsymbol{\varepsilon}_{\mathrm{cyc}}(\mathrm{Frob}_\ell))$ for $\ell \neq p$,
$$\chi_{\mathbf{A}} = \chi \circ \boldsymbol{\varepsilon}_{\mathrm{cyc}} \circ \mathrm{rec}_{\mathbf{Q}}, \qquad \chi_p|_{\mathbf{Z}_p^\times} = \chi.$$

5.2. **Differential operators and nearly holomorphic modular forms.** Let $\mathrm{GL}_2^+(\mathbf{R})$ be the subgroup of $\mathrm{GL}_2(\mathbf{R})$ consisting of matrices with positive determinant and $\mathfrak{H}_1$ the upper half plane on which $\mathrm{GL}_2^+(\mathbf{R})$ acts via fractional transformation. Define a subgroup of $\mathrm{SL}_2(\mathbf{Z})$ of finite index
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}) \;\middle|\; N|c \right\}.$$
The Lie group $\mathrm{GL}_2^+(\mathbf{R})$ acts on the complex vector space of complex valued functions f on $\mathfrak{H}_1$ as in (4.1).

The Maass-Shimura differential operators δ_k and λ_z on $C^\infty(\mathfrak{H}_1)$ are given by
$$\delta_k = \frac{1}{2\pi\sqrt{-1}} \left(\frac{\partial}{\partial z} + \frac{k}{2\sqrt{-1}y} \right), \qquad \lambda_z = -\frac{1}{2\pi\sqrt{-1}} y^2 \frac{\partial}{\partial \bar{z}}$$
with $y = \mathrm{Im}\, z \in \mathbf{R}_+$. Let $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ be a Dirichlet character, which we extend to a character $\chi^\downarrow : \Gamma_0(N) \to \mathbf{C}^\times$ by $\chi^\downarrow\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \chi(d)$. For a non-negative integer m the space $\mathcal{N}_k^{[m]}(N,\chi)$ of nearly holomorphic modular forms of weight k, level N and character χ consists of slowly increasing functions $f \in C^\infty(\mathfrak{H}_1)$ such that $\lambda_z^{m+1} f = 0$ and $f|_k\gamma = \chi^\downarrow(\gamma) f$ for $\gamma \in \Gamma_0(N)$ (*cf.* [Hid93, page 314]). Put $\mathcal{N}_k(N,\chi) = \bigcup_{m=0}^\infty \mathcal{N}_k^{[m]}(N,\chi)$ (*cf.* [Hid93, (1a), page 310]). By definition $\mathcal{N}_k^{[0]}(N,\chi) = \mathcal{M}_k(N,\chi)$ is the space of elliptic modular forms of weight k, level N and character χ. Denote the space of elliptic cusp forms in $\mathcal{M}_k(N,\chi)$ by $\mathcal{S}_k(N,\chi)$. Put $\delta_k^m = \delta_{k+2m-2} \cdots \delta_{k+2}\delta_k$. If $f \in \mathcal{N}_k(N,\chi)$, then $\delta_k^m f \in \mathcal{N}_{k+2m}(N,\chi)$ (see [Hid93, page 312]).

Define an open compact subgroup of $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ by
$$U_0(N) = \left\{ g \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) \;\middle|\; g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N\widehat{\mathbf{Z}}} \right\}.$$
We extend $\chi_{\mathbf{A}}$ to a character $\chi_{\mathbf{A}}^\downarrow$ of $U_0(N)$ by $\chi_{\mathbf{A}}^\downarrow(g) = \prod_{\ell|N} \chi_\ell^\downarrow(g_\ell)$ (see (2.4) for the definition of $\chi_\ell^\downarrow$). Let $\mathcal{A}_k(N,\chi_{\mathbf{A}}^{-1})$ be the space of functions $\Phi : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ such that $V_-^m \Phi = 0$ for some m and such that
$$\Phi(z\gamma g \kappa_\theta u) = \chi_{\mathbf{A}}(z)^{-1} \Phi(g) e^{\sqrt{-1}k\theta} \chi_{\mathbf{A}}^\downarrow(u)^{-1} \qquad (z \in \mathbf{A}^\times,\ \gamma \in \mathrm{GL}_2(\mathbf{Q}),\ \theta \in \mathbf{R},\ u \in U_0(N)).$$

Definition 5.3 (The adèlic lift)**.** With each nearly holomorphic modular form $f \in \mathcal{N}_k(N,\chi)$ we can associate a unique automorphic form $\Phi(f) \in \mathcal{A}_k(N,\chi_{\mathbf{A}}^{-1})$ defined by the equation
$$\Phi(f)(\gamma g_\infty u) := (f|_k g_\infty)(\sqrt{-1}) \cdot \chi_{\mathbf{A}}^\downarrow(u)^{-1}$$

for $\gamma \in \mathrm{GL}_2(\mathbf{Q})$, $g_\infty \in \mathrm{GL}_2^+(\mathbf{R})$ and $u \in U_0(N)$ (*cf.* [Cas73, §3]). We call $\Phi(f)$ the *adèlic lift* of f. Conversely, we can recover f from $\Phi(f)$ by

$$f(x+\sqrt{-1}y) = y^{-k/2}\,\Phi(f)\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right).$$

Recall that $V_\pm$ are the operators as defined in §4.4. By definition we have

$$\Phi(\delta_k f) = V_+\Phi(f), \qquad \Phi(\lambda_z f) = V_-\Phi(f).$$

We define the Whittaker coefficient and the constant term of $\Phi \in \mathcal{A}_k(N,\chi_{\mathbf{A}}^{-1})$ by

$$W(g,\Phi) = \int_{\mathbf{Q}\backslash\mathbf{A}} \Phi(\mathbf{n}(x)g)\overline{\psi_{\mathbf{Q}}(x)}\,\mathrm{d}x, \qquad \mathbf{a}_0(g,\Phi) = \int_{\mathbf{Q}\backslash\mathbf{A}} \Phi(\mathbf{n}(x)g)\,\mathrm{d}x.$$

5.3. **Ordinary I-adic modular forms.** For any subring $A \subset \mathbf{C}$ the space $\mathcal{S}_k(N,\chi;A)$ consists of elliptic cusp forms $f = \sum_{n=1}^\infty \mathbf{a}(n,f)q^n \in \mathcal{S}_k(N,\chi)$ such that $\mathbf{a}(n,f) \in A$ for all n. For every subring $A \subset \overline{\mathbf{Q}}_p$ containing $\mathbf{Z}[\chi]$ we define the space of cusp forms over A by

$$\mathcal{S}_k(N,\chi;A) = \mathcal{S}_k(N,\chi;\mathbf{Z}[\chi]) \otimes_{\mathbf{Z}[\chi]} A.$$

Here we view χ as a p-adic Dirichlet character via ι_p^{-1}.

Definition 5.4 (p-stabilized newforms)**.** We say that a normalized Hecke eigenform $f \in \mathcal{S}_k(Np,\chi)$ is an (ordinary) p-stabilized newform (with respect to $\iota_p : \mathbf{C} \simeq \overline{\mathbf{Q}}_p$) if f is new outside p and the eigenvalue of U_p, i.e. the p-th Fourier coefficient $\iota_p(\mathbf{a}(p,f))$, is a p-adic unit. The prime-to-p part N' of the conductor of f is called the tame conductor of f. There is a unique decomposition $\chi = \chi'\boldsymbol{\omega}^a\epsilon$ with $a \in \mathbf{Z}/(p-1)\mathbf{Z}$, where χ' is a Dirichlet character modulo N' and ϵ is a character of $1+p\mathbf{Z}_p$. We call $\chi'\boldsymbol{\omega}^a$ the tame nebentypus of f.

Let $f^\circ = \sum_{n=1}^\infty \mathbf{a}(n,f^\circ)q^n \in \mathcal{S}_k(Np,\chi)$ be a primitive Hecke eigenform of conductor N_{f°. We call f° ordinary if $\iota_p^{-1}(\mathbf{a}(p,f^\circ))$ is a p-adic unit. If this is the case, then precisely one of the roots of the polynomial $X^2 - \mathbf{a}(p,f^\circ)X + \chi(p)p^{k-1}$ (call it $\alpha_p(f)$) satisfies $|\iota_p(\alpha_p(f))|_p = 1$. We associate to an ordinary primitive form f° the p-stabilized newform by

$$f(\tau) = f^\circ(\tau) - \frac{\chi(p)p^{k-1}}{\alpha_p(f)} f^\circ(p\tau) \in \mathcal{S}_k(N_{f^\circ}p,\chi), \tag{5.1}$$

if N_{f° and p are coprime, and $f = f^\circ$ if p divides N_{f°.

Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbf{Q}_p$ and $\mathbf{I}$ a normal domain finite flat over $\Lambda = \mathcal{O}[\![1+p\mathbf{Z}_p]\!]$. A point $Q \in \operatorname{Spec}\mathbf{I}(\overline{\mathbf{Q}}_p)$, a ring homomorphism $Q : \mathbf{I} \to \overline{\mathbf{Q}}_p$, is said to be locally algebraic if the restriction of Q to $1+p\mathbf{Z}_p$ is of the form $Q(z) = z^{k_Q}\epsilon_Q(z)$ with k_Q an integer and $\epsilon_Q(z) \in \mu_{p^\infty}$. We shall call k_Q the *weight* of Q and ϵ_Q the *finite part* of Q. Let $\mathfrak{X}_{\mathbf{I}}$ be the set of locally algebraic points $Q \in \operatorname{Spec}\mathbf{I}(\overline{\mathbf{Q}}_p)$ of weight $k_Q \geq 1$. A point $Q \in \mathfrak{X}_{\mathbf{I}}$ is said to be *arithmetic* if $k_Q \geq 2$. Let $\mathfrak{X}_{\mathbf{I}}^+$ be the set of arithmetic points, $\wp_Q = \operatorname{Ker} Q$ the prime ideal of $\mathbf{I}$ corresponding to Q and $\mathcal{O}(Q)$ the image of $\mathbf{I}$ under Q.

Let N be a positive integer prime to p and $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$ a Dirichlet character modulo Np. An $\mathbf{I}$-adic cusp form is a formal power series $\boldsymbol{f}(q) = \sum_{n=1}^\infty \mathbf{a}(n,\boldsymbol{f})q^n \in \mathbf{I}[\![q]\!]$ with the following property: there exists an integer $a_{\boldsymbol{f}}$ such that for arithmetic points $Q \in \mathfrak{X}_{\mathbf{I}}^+$ with $k_Q \geq a_{\boldsymbol{f}}$, the specialization $\boldsymbol{f}_Q(q) = \sum_{n=1}^\infty Q(\mathbf{a}(n,\boldsymbol{f}))q^n$ is the Fourier expansion of a cusp form $\boldsymbol{f}_Q \in \mathcal{S}_{k_Q}(Np^e, \chi\boldsymbol{\omega}^{-k_Q}\epsilon_Q; \mathcal{O}(Q))$. Denote by $\mathbf{S}(N,\chi,\mathbf{I})$ the space of $\mathbf{I}$-adic cusp forms of tame level N and (even) branch character χ. This space $\mathbf{S}(N,\chi,\mathbf{I})$ is equipped with the action of the Hecke operators T_ℓ for $\ell \nmid Np$ as in [Wil88, page 537] and the operators $\mathbf{U}_\ell$ for $\ell \mid pN$ given by $\mathbf{U}_\ell(\sum_n \mathbf{a}(n,\boldsymbol{f})q^n) = \sum_n \mathbf{a}(n\ell,\boldsymbol{f})q^n$.

Hida's ordinary projector e_{ord} is defined by

$$e_{\mathrm{ord}} := \lim_{n\to\infty} \mathbf{U}_p^{n!}.$$

It has a well-defined action on the space of classical modular forms preserving the cuspidal part as well as on the space $\mathbf{S}(N,\chi,\mathbf{I})$ (*cf.* [Wil88, page 537 and Proposition 1.2.1]). The space $\mathbf{S}^{\mathrm{ord}}(N,\chi,\mathbf{I}) := e_{\mathrm{ord}}\mathbf{S}(N,\chi,\mathbf{I})$ is called the space of ordinary $\mathbf{I}$-adic forms with respect to χ. Put

$$\mathcal{M}_k^{\mathrm{ord}}(N,\chi;A) = e_{\mathrm{ord}}\mathcal{M}_k(Np^e,\chi;A), \qquad \mathcal{S}_k^{\mathrm{ord}}(N,\chi;A) = e_{\mathrm{ord}}\mathcal{S}_k(Np^e,\chi;A)$$

where e is any integer that is greater than the exponent of the p-primary part of the conductor of χ. A key result in Hida's theory for ordinary $\mathbf{I}$-adic cusp forms is that if $\boldsymbol{f} \in \mathbf{S}^{\mathrm{ord}}(N, \chi, \mathbf{I})$, then for *every* arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}}^+$, we have $\boldsymbol{f}_Q \in \mathcal{S}_{k_Q}^{\mathrm{ord}}(N, \chi\boldsymbol{\omega}^{-k_Q}\epsilon_Q; \mathcal{O}(Q))$. We call $\boldsymbol{f} \in \mathbf{S}^{\mathrm{ord}}(N, \chi, \mathbf{I})$ a *primitive Hida family* if $\boldsymbol{f}_Q$ is a cuspidal p-stabilized newform of tame level N for every arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}}^+$.

6. A p-adic family of pull-backs of Siegel Eisenstein series

6.1. **Siegel Eisenstein series.** We work in adèlic form, which allows us to assemble Eisenstein series out of local data. Put $K_n = \mathrm{U}(n)\,\mathrm{GSp}_{2n}(\widehat{\mathbf{Z}})$. Fix characters $\chi, \hat{\omega}$ of $\mathbf{Z}_p^\times$ of finite order and extend them to Hecke characters $\chi_{\mathbf{A}}, \hat{\omega}_{\mathbf{A}} : \mathbf{Q}^\times\backslash\mathbf{A}^\times \to \mathbf{C}^\times$ by composition with the quotient map $\mathbf{Q}^\times\mathbf{R}_+\backslash\mathbf{A}^\times \simeq \widehat{\mathbf{Z}}^\times \twoheadrightarrow \mathbf{Z}_p^\times$. We regard χ as either a p-adic character or a complex character via composition with ι_p. For each place v we write χ_v for the restriction of $\chi_{\mathbf{A}}$ to $\mathbf{Q}_v^\times$. Our setting means that $\chi_p = \chi$ and $\chi_\ell(\ell) = \chi(\ell)^{-1}$ for $\ell \neq p$. Let

$$I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}}\hat{\omega}_{\mathbf{A}}\boldsymbol{\alpha}_{\mathbf{A}}^s) = \mathrm{Ind}_{\mathcal{P}_3(\mathbf{A})}^{\mathrm{GSp}_6(\mathbf{A})}(\chi_{\mathbf{A}}^2\hat{\omega}_{\mathbf{A}} \boxtimes \chi_{\mathbf{A}}^{-3}\hat{\omega}_{\mathbf{A}}^{-1}\boldsymbol{\alpha}_{\mathbf{A}}^s) \simeq \otimes_v' I_3(\hat{\omega}_v^{-1}, \chi_v\hat{\omega}_v\boldsymbol{\alpha}_{\mathbf{Q}_v}^s)$$

be the global degenerate principal series representation of $\mathrm{GSp}_6(\mathbf{A})$ on the space of right K_3-finite functions $f : \mathrm{GSp}_6(\mathbf{A}) \to \mathbf{C}$ satisfying the transformation laws

$$f(\mathbf{n}(z)\mathbf{m}(A,\nu)g) = \hat{\omega}_{\mathbf{A}}(\nu^{-1}\det A)\chi_{\mathbf{A}}(\nu^{-3}(\det A)^2)\left|\nu^{-3}(\det A)^2\right|_{\mathbf{A}}^{1+s} f(g)$$

for $A \in \mathrm{GL}_3(\mathbf{A})$, $\nu \in \mathbf{A}^\times$, $z \in \mathrm{Sym}_3(\mathbf{A})$ and $g \in \mathrm{GSp}_6(\mathbf{A})$. We define global holomorphic sections of $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}}\hat{\omega}_{\mathbf{A}}\boldsymbol{\alpha}_{\mathbf{A}}^s)$ similarly. The Eisenstein series associated to a holomorphic section f_s of $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}}\hat{\omega}_{\mathbf{A}}\boldsymbol{\alpha}_{\mathbf{A}}^s)$ is defined by

$$E_{\mathbf{A}}(g, f_s) = \sum_{\gamma \in \mathcal{P}_3(\mathbf{Q})\backslash\,\mathrm{GSp}_6(\mathbf{Q})} f_s(\gamma g).$$

Such series is absolutely convergent for $\mathrm{Re}\, s > 1$ and can be continued to a meromorphic function in s on the whole plane.

Let k be an integer and λ a parity type. Fix a square-free integer N which is not divisible by p. We write $\hat{\omega} = \omega_1\omega_2\omega_3$ as a product of three characters $\omega_1, \omega_2, \omega_3$ of $\mathbf{Z}_p^\times$. Set

$$\mathcal{D} = (\chi, \omega_1, \omega_2, \omega_3).$$

Assume that $\hat{\omega}_\infty = \mathrm{sgn}^{k-\lambda_1}$. Now we define a holomorphic section of $I_3(\hat{\omega}_v^{-1}, \chi_v\hat{\omega}_v\alpha_{\mathbf{Q}_v}^s)$ for $v \nmid N$:

- In the archimedean case we consider the section $f_{s,\infty}^{[k,\lambda]}$ defined in §4.1;
- In the p-adic case we consider $f_{\mathcal{D},s,p}$, where the section $f_{\mathcal{D},s,p}$ of $I_3(\hat{\omega}_p^{-1}, \chi_p\hat{\omega}_p\boldsymbol{\alpha}_{\mathbf{Q}_p}^s)$ is attached to the quadruplet $\mathcal{D}$ in Definition 2.5;
- If ℓ and Np are coprime, then $f_{s,\ell}^0$ is the section with $f_{s,\ell}^0(\mathrm{GSp}_6(\mathbf{Z}_\ell)) = 1$.

Let $f_{s,N}$ be an arbitrary holomorphic section of $\bigotimes_{\ell|N} I_3(\hat{\omega}_\ell^{-1}, \chi_\ell\hat{\omega}_\ell\alpha_{\mathbf{Q}_\ell}^s)$ for the moment. We define the normalized Siegel Eisenstein series

$$E_{\mathbf{A}}^\star(g, f_{\mathcal{D},s,N}^{[k,\lambda]}) = L^{(\infty pN)}(2s+2, \chi_{\mathbf{A}}^2\hat{\omega}_{\mathbf{A}})L^{(\infty pN)}(4s+2, \chi_{\mathbf{A}}^4\hat{\omega}_{\mathbf{A}}^2)\gamma_{(k,l,m)}^\star(s)^{-1} \cdot E_{\mathbf{A}}(g, f_{\mathcal{D},s,N}^{[k,\lambda]}),$$

where $\gamma_{(k,l,m)}^\star(s)$ is defined in §4.4 and $f_{\mathcal{D},s,N}^{[k,\lambda]}$ is a global holomorphic section of $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}}\hat{\omega}_{\mathbf{A}}\boldsymbol{\alpha}_{\mathbf{A}}^s)$ defined by

$$f_{\mathcal{D},s,N}^{[k,\lambda]}(g) = f_{s,\infty}^{[k,\lambda]}(g_\infty)f_{s,N}((g_\ell)_{\ell|N})f_{\mathcal{D},s,p}(g_p)\prod_{\ell\nmid Np} f_{s,\ell}^0(g_\ell).$$

Since $f_{\mathcal{D},s,p}$ is supported in the big cell $\mathcal{P}_3(\mathbf{Q}_p)J_3\mathcal{P}_3(\mathbf{Q}_p)$, we have the Fourier expansion

$$E_{\mathbf{A}}(g, f_{\mathcal{D},s,N}^{[k,\lambda]}) = \sum_{B \in \mathrm{Sym}_3(\mathbf{Q})} W_B(g, f_{\mathcal{D},s,N}^{[k,\lambda]}) \tag{6.1}$$

if $g_p \in \mathcal{P}_3(\mathbf{Q}_p)$. Recall that for a holomorphic section f_s of $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}}\hat{\omega}_{\mathbf{A}}\boldsymbol{\alpha}_{\mathbf{A}}^s)$

$$\mathcal{W}_B(g, f_s) = \int_{\mathrm{Sym}_3(\mathbf{A})} f_s(J_3\mathbf{n}(z)g)\boldsymbol{\psi}_{\mathbf{Q}}(-\mathrm{tr}(Bz))\,\mathrm{d}z.$$

6.2. **The Fourier expansion of the pull-back of Eisenstein series.** Recall that

$$\Xi_p = \{(b_{ij}) \in \mathrm{Sym}_3(\mathbf{Z}_p) \mid b_{11}, b_{22}, b_{33} \in p\mathbf{Z}_p \text{ and } b_{12}, b_{23}, b_{31} \in \mathbf{Z}_p^\times\}.$$

Now we evaluate its pull-back at $s_0 = \frac{k-\lambda_1}{2} - r - 1$. Let $\mathbf{E}^{[k,r,\lambda]}_{\mathcal{D},N}(f_{s_0,N})\colon \mathfrak{H}_1^3 \to \mathbf{C}$ be the modular form of weight $(k, k-\lambda_2, k-\lambda_3)$ defined by

$$\mathbf{E}^{[k,r,\lambda]}_{\mathcal{D},N}(x+y\sqrt{-1}, f_{s_0,N}) := \lim_{s\to s_0} \frac{E^\star_{\mathbf{A}}\big(\iota(\mathbf{n}(x_1)\mathbf{m}(\sqrt{y_1}), \mathbf{n}(x_2)\mathbf{m}(\sqrt{y_2}), \mathbf{n}(x_3)\mathbf{m}(\sqrt{y_3})), f^{[k,\lambda]}_{\mathcal{D},s,N}\big)}{\sqrt{y_1}^{k}\sqrt{y_2}^{k-\lambda_2}\sqrt{y_3}^{k-\lambda_3}}$$

for $y = (y_1, y_2, y_3) \in \mathbf{R}^3_+$ and $x = (x_1, x_2, x_3) \in \mathbf{R}^3$.

Since ω_i factors through the quotient $\mathbf{Z}_p^\times \to (\mathbf{Z}/p^{c(\omega_i)}\mathbf{Z})^\times$, we can view ω_i as a Dirichlet character. The polynomial $F_{B,\ell}$ is defined in §2.2. We here set $\mathbf{Q}_N = \prod_{\ell|N} \mathbf{Q}_\ell$. Let Sym_3^+ denote the set of positive definite rational symmetric matrices of rank 3.

Proposition 6.1. *Put $n = \max\{1, c(\chi), c(\omega_i)\}$. The pull-back $\mathbf{E}^{[k,r,\lambda]}_{\mathcal{D},N}(f_{s_0,N})$ is a nearly holomorphic cusp form on $\mathfrak{H}_1^3$ of level $\Gamma_0(Np^{2n})^3$ and nebentypus $(\omega_1^{-1}, \omega_2^{-1}, \omega_3^{-1})$ with Fourier expansion given by*

$$\mathbf{E}^{[k,r,\lambda]}_{\mathcal{D},N}(f_{s_0,N}) = \frac{C_1^{[k,r,\lambda]}}{\gamma^\star_{(k,l,m)}\big(\frac{k-\lambda_1}{2} - r - 1\big)} \sum_{B\in \mathrm{Sym}_3^+\cap \Xi_p} \mathbf{W}_B^{[k,r,\lambda]}(y)\cdot \mathcal{Q}_B(\mathcal{D}) a_B(\chi^2\hat{\omega}, k-2r-\lambda_1) b^{[k,r,\lambda]}_{B,N} q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}},$$

where

$$a_B(\chi^2\hat{\omega}, k-2r-\lambda_1) := \prod_{\ell\nmid Np} F_{B,\ell}(\chi_\ell(\ell)^2\hat{\omega}_\ell(\ell)\ell^{2r+\lambda_1-k}),$$

$$b^{[k,r,\lambda]}_{B,N} := \lim_{s\to \frac{k-\lambda_1}{2}-r-1} \int_{\mathrm{Sym}_3(\mathbf{Q}_N)} f_{s,N}(J_3\mathbf{n}(z))\boldsymbol{\psi}_{\mathbf{Q}}(-\mathrm{tr}(Bz))\,\mathrm{d}z.$$

Proof. The level and nebentypus are determined by Proposition 2.6. Note that $\det B \in \mathbf{Z}_p^\times$ for $B \in \Xi_p$. In particular, $W_B(g, f^{[k,\lambda]}_{\mathcal{D},s,N}) = 0$ unless $\det B \neq 0$. Lemma 4.4 says that $W_B(g, f^{[k,\lambda]}_{\mathcal{D},s,N}) = 0$ unless $B \in T_3^+$. We can derive the Fourier expansion formula from (6.1), recalling that local Whittaker functions

$$\lim_{s\mapsto \frac{k-\lambda_1}{2}-r-1} \mathcal{W}_B(\mathbf{m}(A), f^{[k,\lambda]}_{s,\infty}), \qquad \mathcal{W}_B(\mathbf{1}_6, f^0_{s,\ell}), \qquad \mathcal{W}_B(\mathbf{1}_6, f_{\mathcal{D},s,p})$$

are computed in (2.3), Proposition 2.6 and Lemma 4.4, respectively. □

6.3. **Holomorphic and ordinary projections of $\mathbf{E}^{[k,r,\lambda]}_{\mathcal{D},N}$.** Recall that λ_z is the weight-lowering operator defined in §5.2. We write Hol for the holomorphic projection. Let T_3^+ denote the set of positive definite symmetric half-integral matrices of rank 3.

Definition 6.2. Define a holomorphic section $f_{s,\ell}$ of $I_3(\hat{\omega}_\ell^{-1}, \chi_\ell\hat{\omega}_\ell\alpha^s_{\mathbf{Q}_\ell})$ by letting $f_{s,\ell} = f_{\Phi_\ell,s}$ with $\Phi_\ell = \mathbb{I}_{\mathrm{Sym}_3(\mathbf{Z}_\ell)}$. When $f_{s,N} = \bigotimes_{\ell|N} f_{s,\ell}$, we write $\mathbf{E}^{[k,r,\lambda]}_{\mathcal{D},N} = \mathbf{E}^{[k,r,\lambda]}_{\mathcal{D},N}(f_{s_0,N})$. If $B \in \mathrm{Sym}_3^+$, then $b^{[k,r,\lambda]}_{B,N} = 1$ by (2.12). Proposition 6.1 gives

$$\mathbf{E}^{[k,r,\lambda]}_{\mathcal{D},N} = \frac{C_1^{[k,r,\lambda]}}{\gamma^\star_{(k,l,m)}\big(\frac{k-\lambda_1}{2} - r - 1\big)} \sum_{B\in T_3^+\cap \Xi_p} \mathbf{W}_B^{[k,r,\lambda]}(y)\cdot \mathcal{Q}_B(\mathcal{D}) a_B(\chi^2\hat{\omega}, k-2r-\lambda_1) q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}}.$$

Proposition 6.3. *Let λ be the parity type of (k,l,m) and r an integer which satisfies*

$$k - \frac{l+m+\lambda_1}{2} \le r \le \frac{l+m}{2} - 2.$$

Put $n = k - r - 2 + \frac{l+m-\lambda_1}{2}$. Then $e_{\mathrm{ord}}\,\mathrm{Hol}\left(\lambda_{z_2}^{\frac{k-l-\lambda_2}{2}}\lambda_{z_3}^{\frac{k-m-\lambda_3}{2}}\mathbf{E}^{[k,r,\lambda]}_{\mathcal{D},N}\right)$ has the q-expansion

$$(-1)^{k+\frac{m+l+\lambda_1}{2}+\lambda_2} \sum_{B=(b_{ij})\in T_3^+\cap \Xi_p} \mathcal{Q}_B(\chi\varepsilon^n_{\mathrm{cyc}}, \omega_1\varepsilon^{-k}_{\mathrm{cyc}}, \omega_2\varepsilon^{-l}_{\mathrm{cyc}}, \omega_3\varepsilon^{-m}_{\mathrm{cyc}}) a_B(\chi^2\hat{\omega}, k-2r-\lambda_1) q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}}.$$

PROOF. Put $b=\frac{k-l-\lambda_2}{2}$ and $c=\frac{k-m-\lambda_3}{2}$. If f is a holomorphic function on $\mathfrak{H}_1$, then

$$\lambda_z^n(y^{-a}f)=\begin{cases}(4\pi)^{-n}n!\binom{a}{n}\cdot y^{n-a}f & \text{if } n\leq a,\\ 0 & \text{if } n>a.\end{cases}$$

By (4.4) the difference

$$\lambda_{z_2}^b\lambda_{z_3}^c\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(q)-C_1^{[k,r,\lambda]}\frac{b!c!}{(4\pi)^{b+c}}\sum_B Q_{0,b,c}^{[k,\lambda]}(B,r)\mathcal{Q}_B(\mathcal{D})a_B(\chi^2\hat{\omega},k-2r-\lambda_1)q_1^{b_{11}}q_2^{b_{22}}q_3^{b_{33}}$$

belongs to $(y_1^{-1},y_2^{-1},y_3^{-1})\mathbf{C}[y_1^{-1},y_2^{-1},y_3^{-1}][\![q_1,q_2,q_3]\!]$. On the other hand, we can write

$$\lambda_{z_2}^b\lambda_{z_3}^c\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(q)=\mathrm{Hol}(\lambda_{z_2}^b\lambda_{z_3}^c\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]})(q)+\sum_{i+j+t\geq 1}\delta_{k-i}^i f_i(q_1)\delta_{l-j}^j g_j(q_2)\delta_{m-t}^t h_t(q_3),$$

where f_i, g_j and h_t are holomorphic modular forms. Equating the constant terms of this identity as a polynomial in $y_1^{-1},y_2^{-1},y_3^{-1}$ and employing the relation

$$\delta_k^t=\sum_{a=0}^t\binom{t}{a}\frac{\Gamma(t+k)}{\Gamma(a+k)}(-4\pi y)^{a-t}\left(\frac{1}{2\pi\sqrt{-1}}\frac{\partial}{\partial z}\right)^a \tag{6.2}$$

(see [Hid93, (3), page 311]), we see that the holomorphic projection $\mathrm{Hol}(\lambda_{z_2}^b\lambda_{z_3}^c\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]})(q)$ equals

$$C_1^{[k,r,\lambda]}\frac{b!\,c!}{(4\pi)^{b+c}}\sum_B Q_{0,b,c}^{[k,\lambda]}(B,r)\mathcal{Q}_B(\mathcal{D})a_B(\chi^2\hat{\omega},k-2r-\lambda_1)q_1^{b_{11}}q_2^{b_{22}}q_3^{b_{33}}-\sum_{i+j+t\geq 1}\theta^i f_i(q_1)\theta^j g_j(q_2)\theta^t h_t(q_3).$$

Here θ stands for the Serre's operator $\theta(\sum_i a_iq^i)=\sum_i ia_iq^i$. Since $e_{\mathrm{ord}}\theta=0$, the q-expansion of the ordinary projection $e_{\mathrm{ord}}\,\mathrm{Hol}(\lambda_{z_2}^b\lambda_{z_3}^c\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]})(q)$ equals

$$C_1^{[k,r,\lambda]}\frac{b!\,c!}{(4\pi)^{b+c}}\sum_B \mathbf{c}_B\cdot q_1^{b_{11}}q_2^{b_{22}}q_3^{b_{33}},$$

where

$$\mathbf{c}_B=\lim_{j\to\infty}Q_{0,b,c}^{[k,\lambda]}(B_j,r)\mathcal{Q}_{B_j}(\mathcal{D})a_{B_j}(\chi^2\hat{\omega},k-2r-\lambda_1),\qquad B_j:=\begin{pmatrix}p^{j!}b_{11} & b_{12} & b_{13}\\ b_{12} & p^{j!}b_{22} & b_{23}\\ b_{13} & b_{23} & p^{j!}b_{33}\end{pmatrix}.$$

Since $p^{j!}\to 1$ in $\mathbf{Z}_\ell$ as $j\to\infty$ for any rational prime $\ell\neq p$, we get

$$\mathcal{Q}_{B_j}(\mathcal{D})=\mathcal{Q}_B(\mathcal{D}),\qquad \lim_{j\to\infty}a_{B_j}(\chi^2\hat{\omega},k-2r-\lambda_1)=a_B(\chi^2\hat{\omega},k-2r-\lambda_1).$$

Since $Q_{0,b,c}^{[k,\lambda]}(B,r)$ is a polynomial in B, we find that

$$\mathbf{c}_B=Q_{0,b,c}^{[k,\lambda]}(B_\infty,r)\mathcal{Q}_B(\mathcal{D})a_B(\chi^2\hat{\omega},k-2r-\lambda_1),\qquad B_\infty=\begin{pmatrix}0 & b_{12} & b_{13}\\ b_{12} & 0 & b_{23}\\ b_{13} & b_{23} & 0\end{pmatrix}.$$

Applying Lemma 4.7 to $Q_{0,b,c}^{[k,\lambda]}(B_\infty,r)$, we obtain

$$\mathbf{c}_B=w_{0,b,c}\cdot 2^{-3n+k+l+m}\mathcal{Q}_B(\chi\varepsilon_{\mathrm{cyc}}^n,\omega_1\varepsilon_{\mathrm{cyc}}^{-k},\omega_2\varepsilon_{\mathrm{cyc}}^{-l},\omega_3\varepsilon_{\mathrm{cyc}}^{-m})a_B(\chi^2\hat{\omega},k-2r-\lambda_1)$$

in view of Definition 2.5 of $\mathcal{Q}_B$. We thus obtain the lemma by noting the equality

$$(-1)^{k+\frac{m+l+\lambda_1}{2}+\lambda_2}\gamma_{(k,l,m)}^\star\left(\frac{k-\lambda_1}{2}-r-1\right)=C_1^{[k,r,\lambda]}b!\,c!(4\pi)^{-b-c}2^{-3n+k+l+m}\cdot w_{0,b,c}.$$

The constant $C_1^{[k,r,\lambda]}$ is defined in Lemma 4.4. The equality can be checked by the following items:

- The power of 2:

$$\begin{aligned}&3(3+2r-k-\lambda_2)+\{2(k-r)-3\}-2b-2c+(k+l+m-3n)\\&+(7M-3b-3c-3\lambda_1)=-2-k+2(l+m)+\lambda_1+2\lambda_2.\end{aligned}$$

- The power of π: $(6-2)-b-c+(3M-b-c-2\lambda_1-\lambda_2)=-3r+k+l+m+\lambda_2-\lambda_1-2$.

□

6.4. **The modular forms $G^{[n]}_{k_1,k_2,k_3}(\mathscr{D})$.**

Definition 6.4. Let (k_1,k_2,k_3) be a triplet of positive integers. Put $k^*=\max\{k_1,k_2,k_3\}$. We say that (k_1,k_2,k_3) is *balanced* if $2k^*<k_1+k_2+k_3$. An integer n is said to be *critical* for (k_1,k_2,k_3) if

$$k^*\le n\le k_1+k_2+k_3-k^*-2.$$

Definition 6.5. Fix a balanced triplet (k_1,k_2,k_3) of positive integers. Take a permutation σ of $\{1,2,3\}$ so that $k^*=k_{\sigma(1)}\ge k_{\sigma(2)}\ge k_{\sigma(3)}$. Denote the parity type of $(k_{\sigma(1)},k_{\sigma(2)},k_{\sigma(3)})$ by $\delta=(\delta_1,\delta_2,\delta_3)$. For each critical integer n for (k_1,k_2,k_3) and quadruplet $\mathscr{D}=(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)$ of finite-order p-adic characters of $\mathbf{Z}_p^\times$ we define the modular form $G^{[n]}_{k_1,k_2,k_3}(\mathscr{D})$ by

$$G^{[n]}_{k_1,k_2,k_3}(\mathscr{D}):=(-1)^{k+\frac{m+l+\lambda_1}{2}+\lambda_2}e_{\mathrm{ord}}\,\mathrm{Hol}\left(\lambda_{z_{\sigma(2)}}^{\frac{k^*-k_{\sigma(2)}-\delta_2}{2}}\lambda_{z_{\sigma(3)}}^{\frac{k^*-k_{\sigma(3)}-\delta_3}{2}}\mathbf{E}_{\mathcal{D}}^{[k^*,r,\delta]}\right),$$

where $r=\left\lceil\frac{k^*+k_1+k_2+k_3}{2}\right\rceil-n-2$ and $\mathcal{D}=(\iota_p\circ\epsilon_0,\iota_p\circ\epsilon_1,\iota_p\circ\epsilon_2,\iota_p\circ\epsilon_3)$.

Corollary 6.6. *With notation in Definition 6.5, $G^{[n]}_{k_1,k_2,k_3}(\mathscr{D})$ is an ordinary cusp form of weight (k_1,k_2,k_3), level $\Gamma_0(Np^\infty)^3$ and nebentypus $(\epsilon_1^{-1},\epsilon_2^{-1},\epsilon_3^{-1})$ whose q-expansion at the infinity cusp is given by*

$$\sum_{B=(b_{ij})\in T_3^+\cap\Xi_p}\mathcal{Q}_B(\epsilon_0\varepsilon_{\mathrm{cyc}}^n,\epsilon_1\varepsilon_{\mathrm{cyc}}^{-k_1},\epsilon_2\varepsilon_{\mathrm{cyc}}^{-k_2},\epsilon_3\varepsilon_{\mathrm{cyc}}^{-k_3})a_B(\epsilon_0^2\epsilon_1\epsilon_2\epsilon_3,2n-(k_1+k_2+k_3)+4)\cdot q_1^{b_{11}}q_2^{b_{22}}q_3^{b_{33}}.$$

Proof. The assertion for the Fourier expansion is a direct consequence of Proposition 6.3 by symmetry. Lemma 6.7 below implies the cuspidality of $G^{[n]}_{k_1,k_2,k_3}(\mathscr{D})$. □

Lemma 6.7. *Let $f\in\mathcal{M}_k^{\mathrm{ord}}(N,\chi;A)$. Assume that $\mathbf{a}_0(g,\Phi(f))=0$ whenever $g_p\in B_2(\mathbf{Q}_p)$. Then $f\in\mathcal{S}_k^{\mathrm{ord}}(N,\chi;A)$.*

Proof. Out task is to prove that $\mathbf{a}_0(g,\Phi(f))=0$ for all $g\in\mathrm{GL}_2(\mathbf{A})$. Since

$$\mathbf{a}_0(\gamma\mathbf{n}(x)\mathrm{diag}(a,d)g\kappa_\theta,\Phi(f))=(ad^{-1})^{k/2}e^{\sqrt{-1}k\theta}\mathbf{a}_0(g_{\mathbf{f}},\Phi(f))$$

for $\gamma\in B_2(\mathbf{Q})$, $x\in\mathbf{A}$, $a,d\in\mathbf{R}_+$ and $\theta\in\mathbf{R}$, it suffices to show that $\mathbf{a}_0(g,\Phi(f))=0$ for all $g\in\mathrm{GL}_2(\widehat{\mathbf{Z}})$. Since

$$\mathrm{GL}_2(\mathbf{Z}_p)=\mathbf{n}^-(p\mathbf{Z}_p)B_2(\mathbf{Z}_p)\sqcup\mathbf{n}(\mathbf{Z}_p)J_1B_2(\mathbf{Z}_p),$$

where $J_1=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, we have only to show that $\mathbf{a}_0(h\mathbf{n}^-(y),\Phi(f))=\mathbf{a}_0(hJ_1,\Phi(f))=0$ for all $h\in\mathrm{GL}_2(\widehat{\mathbf{Z}}^{(p)})$ and $y\in p\mathbf{Z}_p$. Recall that the operator $\mathbf{U}_p$ is defined by

$$[\mathbf{U}_p\Phi](g,f)=p^{(k-2)/2}\sum_{x\in\mathbf{Z}_p/p\mathbf{Z}_p}\Phi\left(g\begin{pmatrix}\varpi_p&x\\0&1\end{pmatrix},f\right).$$

Recall that $\varpi_p\in\widehat{\mathbf{Q}}^\times$ is defined by $\varpi_{p,p}=p$ and $\varpi_{p,\ell}=1$ for $\ell\ne p$. Since

$$\begin{pmatrix}1&0\\y&1\end{pmatrix}\begin{pmatrix}\varpi_p^m&x\\0&1\end{pmatrix}=\begin{pmatrix}\frac{\varpi_p^m}{1+xy}&\frac{x}{1+xy}\\0&1\end{pmatrix}\begin{pmatrix}1&0\\\varpi_p^my&1+xy\end{pmatrix}\in B_2(\mathbf{Q}_p)U_0(N)$$

for $y\in p\mathbf{Z}_p$, $x\in\mathbf{Z}_p$ and sufficiently large m, we get

$$\mathbf{a}_0(h\mathbf{n}^-(y),\mathbf{U}_p^mf)=p^{(k-2)m/2}\sum_{x\in\mathbf{Z}_p/p\mathbf{Z}_p}\Phi\left(h\begin{pmatrix}\frac{\varpi_p^m}{1+xy}&\frac{x}{1+xy}\\0&1\end{pmatrix},f\right)=0$$

by assumption. It follows that $\mathbf{a}_0(h\mathbf{n}^-(y),f)=\lim_{n\to\infty}\mathbf{a}_0(h\mathbf{n}^-(y),\mathbf{U}_p^{n!}f)=0$. If $x\in p^n\mathbf{Z}_p^\times$ with $n<m$, then

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\varpi_p^m&x\\0&1\end{pmatrix}=\begin{pmatrix}\varpi_p^{m-n}&-\varpi_p^nx^{-1}\\0&\varpi_p^n\end{pmatrix}\begin{pmatrix}\varpi_p^nx^{-1}&0\\\varpi_p^{m-n}&\varpi_p^{-n}x\end{pmatrix}\in B_2(\mathbf{Q}_p)\mathbf{n}^-(p\mathbf{Z}_p).$$

One can therefore see that

$$\mathbf{a}_0(hJ_1, \mathbf{U}_p^m f) = p^{(k-2)m/2}\mathbf{a}_0(hJ_1 \mathrm{diag}(\varpi_p^m, 1), f) = p^{(k-1)m/2}\mathbf{a}_0(\mathrm{diag}(1, \varpi_{(p)}^{-m})hJ_1, f),$$

from which we conclude that

$$\mathbf{a}_0(hJ_1, f) = \lim_{n\to\infty} p^{(k-1)n!/2}\mathbf{a}_0(\mathrm{diag}(1, \varpi_{(p)}^{-n!})hJ_1, f) = 0.$$

Here $\varpi_{(p)} \in \widehat{\mathbf{Z}}_p^\times$ is defined by $\varpi_{(p),p} = 1$ and $\varpi_{(p),\ell} = p$ for $\ell \neq p$. □

6.5. **The p-adic interpolation of $G^{[n]}_{k_1,k_2,k_3}(\mathscr{D})$.** We give the construction the p-adic triple L-function in this subsection. Let $\mathbf{u} = 1 + p \in 1 + p\mathbf{Z}_p$ be a topological generator. We identify $\mathcal{O}[\![\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]\!]$ with $\mathcal{O}[\![X]\!]$ where $X = [\mathbf{u}] - 1$ with the group-like element $[\mathbf{u}]$ in Λ. Put

$$\Lambda = \mathcal{O}[\![\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]\!], \qquad \Lambda_3 = \mathcal{O}[\![X_1, X_2, X_3]\!], \qquad \Lambda_4 = \Lambda_3[\![T]\!].$$

For each $\ell \nmid Np$ and $B \in T_3^+$, let $F_{B,\ell}(X) \in \mathbf{Z}[X]$ be as defined in (2.3). Let $\alpha_X : \mathbf{Z}_p^\times \to \mathbf{Z}_p[\![X]\!]^\times$ be the character $\alpha_X(z) = \langle z\rangle_X = (1+X)^{\log_p z/\log_p \mathbf{u}}$. Let $\underline{\chi} = (\chi_1, \chi_2, \chi_3)$ be a triplet of $\mathcal{O}$-valued finite-order characters of $\mathbf{Z}_p^\times$. For each $a \in \mathbf{Z}/(p-1)\mathbf{Z}$ we define the formal power series $\mathcal{G}^{(a)}_{\underline{\chi}} \in \Lambda_4[\![q_1, q_2, q_3]\!]$ by

$$\mathcal{G}^{(a)}_{\underline{\chi}}(X_1, X_2, X_3, T) = \sum_{B=(b_{ij})\in T_3^+\cap \Xi_p} \mathcal{Q}^{(a)}_B(X_1, X_2, X_3, T)\cdot \mathcal{F}^{(a)}_B(X_1, X_2, X_3, T)\cdot q_1^{b_{11}}q_2^{b_{22}}q_3^{b_{33}},$$

where $\mathcal{Q}^{(a)}_B$ and $\mathcal{F}^{(a)}_B \in \Lambda_3[\![T]\!]$ are power series given by

$$\mathcal{Q}^{(a)}_B(X_1, X_2, X_3, T) = \boldsymbol{\omega}(8b_{23}b_{31}b_{12})^a \langle 8b_{23}b_{31}b_{12}\rangle_T\, \chi_1(2b_{23})^{-1}\langle 2b_{23}\rangle^{-1}_{X_1}\chi_2(2b_{31})^{-1}\langle 2b_{31}\rangle^{-1}_{X_2}\chi_3(2b_{12})^{-1}\langle 2b_{12}\rangle^{-1}_{X_3},$$

$$\mathcal{F}^{(a)}_B(X_1, X_2, X_3, T) = \prod_{\ell\nmid pN} F_{B,\ell}(\langle\ell\rangle^{(a)}_{X_1,X_2,X_3,T}\,\ell^{-2}),$$

where

$$\langle\ell\rangle^{(a)}_{X_1,X_2,X_3,T} := (\boldsymbol{\omega}^{-2a}\chi_1\chi_2\chi_3)(\ell)\ell^{-2}\cdot\langle\ell\rangle_{X_1}\langle\ell\rangle_{X_2}\langle\ell\rangle_{X_3}\langle\ell\rangle_T^{-2} \in \Lambda_4^\times.$$

Here the set $\mathfrak{X}^{\mathrm{bal}}_{\Lambda_4}$ consists of $(\underline{Q}, P) = (Q_1, Q_2, Q_3, P) \in (\mathfrak{X}^+_\Lambda)^3 \times \mathfrak{X}_\Lambda \subset \mathrm{Spec}\,\Lambda_4(\overline{\mathbf{Q}}_p)$ such that $(k_{Q_1}, k_{Q_2}, k_{Q_3})$ is balanced and k_P is critical for $(k_{Q_1}, k_{Q_2}, k_{Q_3})$.

Proposition 6.8. *For every $(\underline{Q}, P) \in \mathfrak{X}^{\mathrm{bal}}_{\Lambda_4}$, we have*

$$\mathcal{G}^{(a)}_{\underline{\chi}}(\underline{Q}, P) = G^{[k_P]}_{k_{Q_1},k_{Q_2},k_{Q_3}}(\epsilon_P\boldsymbol{\omega}^{a-k_P}, \chi_1^{-1}\epsilon_{Q_1}^{-1}\boldsymbol{\omega}^{k_{Q_1}}, \chi_2^{-1}\epsilon_{Q_2}^{-1}\boldsymbol{\omega}^{k_{Q_2}}, \chi_3^{-1}\epsilon_{Q_3}^{-1}\boldsymbol{\omega}^{k_{Q_3}}).$$

In particular, this implies that

$$\mathcal{G}^{(a)}_{\underline{\chi}} \in \mathbf{S}^{\mathrm{ord}}(N, \chi_1, \mathcal{O}[\![X_1]\!])\widehat{\otimes}_{\mathcal{O}}\mathbf{S}^{\mathrm{ord}}(N, \chi_2, \mathcal{O}[\![X_2]\!])\widehat{\otimes}_{\mathcal{O}}\mathbf{S}^{\mathrm{ord}}(N, \chi_3, \mathcal{O}[\![X_3]\!])\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[\![T]\!].$$

Proof. Set $\chi := \epsilon_P\boldsymbol{\omega}^{a-k_P}$, $\omega_i = \chi_i^{-1}\epsilon_{Q_i}^{-1}\boldsymbol{\omega}^{k_{Q_i}}$ and $\hat{\omega} = \omega_1\omega_2\omega_3$. One can check that

$$\begin{aligned}
\mathcal{Q}^{(a)}_B(\underline{Q}, P) &= \frac{(\epsilon_P\boldsymbol{\omega}^a)(8b_{12}b_{23}b_{13})\langle 8b_{12}b_{23}b_{13}\rangle^{k_P}}{(\chi_1\epsilon_{Q_1})(2b_{23})(\chi_2\epsilon_{Q_2})(2b_{13})(\chi_3\epsilon_{Q_3})(2b_{12})\langle 2b_{23}\rangle^{k_{Q_1}}\langle 2b_{31}\rangle^{k_{Q_2}}\langle 2b_{12}\rangle^{k_{Q_3}}}\\
&= \mathcal{Q}_B(\chi\boldsymbol{\varepsilon}_{\mathrm{cyc}}^{k_P}, \omega_1\boldsymbol{\varepsilon}_{\mathrm{cyc}}^{-k_{Q_1}}, \omega_2\boldsymbol{\varepsilon}_{\mathrm{cyc}}^{-k_{Q_2}}, \omega_3\boldsymbol{\varepsilon}_{\mathrm{cyc}}^{-k_{Q_3}}),\\
\langle\ell\rangle^{(a)}_{X_1,X_2,X_3,T}(\underline{Q}, P) &= (\boldsymbol{\omega}^{-2a}\chi_1\chi_2\chi_3)(\ell)\ell^{-2}\cdot(\epsilon_{Q_1}\epsilon_{Q_2}\epsilon_{Q_3}\epsilon_P^{-2}\boldsymbol{\omega}^{2k_P-k_{Q_1}-k_{Q_2}-k_{Q_3}})(\ell)^{-1}\ell^{k_{Q_1}+k_{Q_2}+k_{Q_3}-2k_P}\\
&= \chi^2_\ell(\ell)\,|\ell|^{2k_P+2}\,\hat{\omega}_\ell(\ell)\,|\ell|^{-(k_{Q_1}+k_{Q_2}+k_{Q_3})},\\
\mathcal{F}^{(a)}_B(\underline{Q}, P) &= a_B(\chi^2\hat{\omega}, 2k_P - (k_{Q_1}+k_{Q_2}+k_{Q_3}) + 4)
\end{aligned}$$

(see Definition 2.5 of $\mathcal{Q}_B$). Recall the convention that $\chi_\ell(\ell) = \iota_p(\chi(\ell))^{-1}$ and $\hat{\omega}_\ell(\ell) = \iota_p(\hat{\omega}(\ell))^{-1}$ (see Remark 5.2). From Corollary 6.6, we deduce the interpolation formula and that

$$\mathcal{G}^{(a)}_{\underline{\chi}}(\underline{Q}, P) \in \mathcal{S}^{\mathrm{ord}}_{k_{Q_1}}(N, \omega_1^{-1}; \mathcal{O}(Q_1))\widehat{\otimes}_{\mathcal{O}}\mathcal{S}^{\mathrm{ord}}_{k_{Q_2}}(N, \omega_2^{-1}; \mathcal{O}(Q_2))\widehat{\otimes}_{\mathcal{O}}\mathcal{S}^{\mathrm{ord}}_{k_{Q_3}}(N, \omega_3^{-1}; \mathcal{O}(Q_3))\widehat{\otimes}_{\mathcal{O}}\mathcal{O}(P). \tag{6.3}$$

By the control theorem for ordinary Λ-adic forms [Hid93, Theorem 3, p.215], for any arithmetic point Q, the specialization map $X \mapsto \mathbf{u}^{k_Q}\epsilon_Q(\mathbf{u}) - 1$ yields an isomorphism

$$\mathbf{S}^{\mathrm{ord}}(N, \chi, \mathcal{O}[\![X]\!])/(1 + X - \mathbf{u}^{k_Q}\epsilon_Q(\mathbf{u})) \simeq \mathcal{S}^{\mathrm{ord}}_{k_Q}(N, \chi\boldsymbol{\omega}^{-k_Q}\epsilon_Q; \mathcal{O}(Q)).$$

Hence, from (6.3) we find that for all P with $k_P = 2$

$$\mathcal{G}^{(a)}_{\underline{\chi}}(X_1, X_2, X_3, P) \in \mathbf{S}^{\mathrm{ord}}(N, \chi_1, \mathcal{O}[\![X_1]\!])\widehat{\otimes}_{\mathcal{O}}\mathbf{S}^{\mathrm{ord}}(N, \chi_2, \mathcal{O}[\![X_2]\!])\widehat{\otimes}_{\mathcal{O}}\mathbf{S}^{\mathrm{ord}}(N, \chi_3, \mathcal{O}[\![X_3]\!]) \otimes_{\mathcal{O}} \mathcal{O}(P).$$

Now we can deduce the second statement from the above equation combined with the argument in [Hid93, Lemma 1, page 328]. □

7. Four-variable p-adic triple product L-functions

7.1. **Measures.** We shall normalize the Haar measures $\mathrm{d}x_v$ on $\mathbf{Q}_v$ and $\mathrm{d}^\times x_v$ on $\mathbf{Q}_v^\times$ as follows: Let $\mathrm{d}x_\infty$ denote the usual Lebesgue measure on $\mathbf{R}$ and $\mathrm{d}^\times x_\infty = \frac{\mathrm{d}x_\infty}{|x_\infty|_\infty}$. If $v = \ell$ is finite, then $\mathrm{vol}(\mathbf{Z}_\ell, \mathrm{d}x_\ell) = \mathrm{vol}(\mathbf{Z}_\ell^\times, \mathrm{d}^\times x_\ell) = 1$. Define the compact subgroups $\mathbf{K}_v$ of $\mathrm{GL}_2(\mathbf{Q}_v)$ and $\mathbf{K}'_v$ of $\mathrm{SL}_2(\mathbf{Q}_v)$ by

$$\mathbf{K}_\infty = \mathrm{O}(2, \mathbf{R}), \qquad \mathbf{K}_\ell = \mathrm{GL}_2(\mathbf{Z}_\ell), \qquad \mathbf{K}'_\infty = \mathrm{SO}(2, \mathbf{R}), \qquad \mathbf{K}'_\ell = \mathrm{SL}_2(\mathbf{Z}_\ell).$$

Let $\mathrm{d}k_v$ and $\mathrm{d}k'_v$ be the Haar measures on $\mathbf{K}_v$ and $\mathbf{K}'_v$ which have total volume 1.

The Haar measure $\mathrm{d}g_v$ on $\mathrm{PGL}_2(\mathbf{Q}_v)$ is given by $\mathrm{d}g_v = |y_v|_v^{-1}\mathrm{d}x_v\mathrm{d}^\times y_v\mathrm{d}k_v$ for $g_v = \begin{pmatrix} y_v & x_v \\ 0 & 1 \end{pmatrix}$ with $y_v \in \mathbf{Q}_v^\times$, $x_v \in \mathbf{Q}_v$ and $k_v \in \mathbf{K}_v$. Define the Haar measure $\mathrm{d}g'_v$ on $\mathrm{SL}_2(\mathbf{Q}_v)$ by $\mathrm{d}g'_v = |y_v|_v^{-2}\mathrm{d}x_v\mathrm{d}^\times y_v\mathrm{d}k'_v$ for $g_v = \mathbf{n}(x_v)\mathbf{m}(y_v)k'_v$ with $y_v \in \mathbf{Q}_v^\times$, $x_v \in \mathbf{Q}_v$ and $k'_v \in \mathbf{K}'_v$. The Tamagawa measures $\mathrm{d}g$ on $\mathrm{PGL}_2(\mathbf{A})$ and $\mathrm{d}g'$ on $\mathrm{SL}_2(\mathbf{A})$ are given by $\mathrm{d}g = \zeta_{\mathbf{Q}}(2)^{-1}\prod_v \mathrm{d}g_v$ and $\mathrm{d}g' = \zeta_{\mathbf{Q}}(2)^{-1}\prod_v \mathrm{d}g'_v$. Since $Z\backslash H \simeq \mathrm{PGL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$, we can define the Tamagawa measure on $Z\backslash H$ by $\mathrm{d}g_1\mathrm{d}g'_2\mathrm{d}g'_3$, where $\mathrm{d}g_1$ is the Tamagawa measure on $\mathrm{PGL}_2(\mathbf{A})$ and $\mathrm{d}g'_2 = \mathrm{d}g'_3$ are that on $\mathrm{SL}_2(\mathbf{A})$. The Tamagawa numbers of PGL_2, SL_2 and $Z\backslash H$ are 2, 1 and 2, respectively.

7.2. **Garrett's integral representation.** Let π_i $(i = 1, 2, 3)$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ generated by an elliptic cusp form of weight k_i and nebentypus ω_i^{-1}. Put $\hat{\omega} = \omega_1\omega_2\omega_3$ and $\breve{\pi}_i = \pi_i \otimes \omega_{i,\mathbf{A}}^{-1}$ for $i = 1, 2, 3$. Fix a character $\chi_{\mathbf{A}}$ of $\mathbf{A}^\times/\mathbf{Q}^\times\mathbf{R}_+$. For each triplet of cusp forms $\varphi_i \in \breve{\pi}_i$ and a holomorphic section f_s of $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}}\hat{\omega}_{\mathbf{A}}\boldsymbol{\alpha}_{\mathbf{A}}^s)$ we consider the global zeta integral defined by

$$Z(\varphi_1, \varphi_2, \varphi_3, E_{\mathbf{A}}(f_s)) = \int_{Z(\mathbf{A})H(\mathbf{Q})\backslash H(\mathbf{A})} \varphi_1(g_1)\varphi_2(g_2)\varphi_3(g_3)E_{\mathbf{A}}(\iota(g_1, g_2, g_3), f_s)\,\mathrm{d}g_1\mathrm{d}g_2\mathrm{d}g_3.$$

The integral converges absolutely for all s away from the poles of the Eisenstein series and is hence meromorphic in s. Unfolding the Eisenstein series as in [PSR87], we get

$$Z(\varphi_1, \varphi_2, \varphi_3, E_{\mathbf{A}}(f_s)) = \int_{Z(\mathbf{A})U^0(\mathbf{A})\backslash H(\mathbf{A})} W(g_1, \varphi_1)W(g_2, \varphi_2)W(g_3, \varphi_3)f_s(\delta\iota(g_1, g_2, g_3))\,\mathrm{d}g_1\mathrm{d}g_2\mathrm{d}g_3.$$

If $W(g, \varphi_i) = \prod_v W_{i,v}(g_v)$ and $f_s(g) = \prod_v f_{s,v}(g_v)$ are factorizable, then the integral factors into a product of local integrals and so by §2.2

$$Z(\varphi_1, \varphi_2, \varphi_3, E_{\mathbf{A}}(f_s)) = \frac{\zeta_{\mathbf{Q}}(2)^{-3}L\left(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}}\right)}{L^S(2s + 2, \chi_{\mathbf{A}}^2\hat{\omega}_{\mathbf{A}})L^S(4s + 2, \chi_{\mathbf{A}}^4\hat{\omega}_{\mathbf{A}}^2)} \prod_{v \in S} \frac{Z(W_{1,v}, W_{2,v}, W_{3,v}, f_{s,v})}{L\left(s + \frac{1}{2}, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v} \otimes \chi_v\right)},$$

where S is a large enough set of places such that $\pi_{i,\ell}$, $W_{i,\ell}$, χ_ℓ and $f_{s,\ell}$ are unramified for all $\ell \notin S$. The complete L-function $L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})$ admits meromorphic continuation and a functional equation

$$L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}}) = \varepsilon(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})L(1 - s, \pi_1 \times \pi_2 \times \pi_3 \otimes \hat{\omega}_{\mathbf{A}}^{-1}\chi_{\mathbf{A}}^{-1}).$$

By Theorem 2.7 of [Ike92] the L-function $L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})$ has a pole if and only if there exists an imaginary quadratic field E and characters of χ_i of $\mathbf{A}_E^\times/E^\times$ such that $\chi_1\chi_2\chi_3\chi^E = 1$ and such that π_i is induced automorphically from χ_i, where χ^E denotes the base change of χ to E. Recall that $k^* = \max\{k_1, k_2, k_3\}$. In particular, if $k_1 + k_2 + k_3 \geq 2k^* + 2$, then $L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})$ is holomorphic everywhere. Let us put

$$\mathscr{J}_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{R}), \qquad t_n = \begin{pmatrix} 0 & p^{-n} \\ -p^n & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_p) \hookrightarrow \mathrm{GL}_2(\mathbf{A}).$$

Let $E_{\mathbf{A}}^\star\big(f_{\mathcal{D},s,N}^{[k_1,\lambda]}\big)$ be the Eisenstein series associated with a section $f_{s,N} = \bigotimes_{\ell|N} f_{s,\ell}$ of $\bigotimes_{\ell|N} I_3(\hat{\omega}_\ell^{-1}, \chi_\ell\hat{\omega}_\ell\boldsymbol{\alpha}_{\mathbf{Q}_\ell}^s)$ as in §6.1.

Lemma 7.1. *Let $f_i \in \mathcal{S}_{k_i}(N_i, \omega_i^{-1})$ be an ordinary p-stabilized newform. Put*

$$\varphi_i = \Phi(f_i), \qquad \breve{\varphi}_i = \varphi_i \otimes \omega_{i,\mathbf{A}}^{-1}, \qquad W(\varphi_i) = \prod_v W_{i,v}, \qquad \breve{W}_{i,v} = W_{i,v} \otimes \omega_{i,v}^{-1},$$

Let χ be a character of $\mathbf{Z}_p^\times$ of finite order. Put $n = \{1, c(\chi), c(\omega_i)\}$. If $k_1 \geq k_2 \geq k_3$ and λ is the parity type of (k_1, k_2, k_3), then

$$\begin{aligned} & Z\Big(\rho(\mathcal{J}_\infty t_n)\breve{\varphi}_1, \rho(\mathcal{J}_\infty t_n)V_+^{\frac{k_1-k_2-\lambda_2}{2}}\breve{\varphi}_2, \rho(\mathcal{J}_\infty t_n)V_+^{\frac{k_1-k_3-\lambda_3}{2}}\breve{\varphi}_3, E_{\mathbf{A}}^\star\big(f_{\mathcal{D},s,N}^{[k_1,\lambda]}\big)\Big) \\ =& L^{(N)}\left(s+\frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}}\right) E_p\left(s+\frac{1}{2}, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \chi_p\right) \\ & \times \prod_{i=1}^3 \frac{\zeta_p(2)}{\zeta_p(1)} \left(\frac{\alpha_p(f_i)^2}{p^{k_i}\omega_{i,p}(p)}\right)^n \frac{\prod_{\ell|N} Z(\breve{W}_{1,\ell}, \breve{W}_{2,\ell}, \breve{W}_{3,\ell}, f_{s,\ell})}{\zeta_{\mathbf{Q}}(2)^3 2^{5+(k_1+k_2+k_3)}} \end{aligned}$$

Proof. By Garrett's integral representation of triple L-functions the left hand side equals

$$\begin{aligned} & \zeta_{\mathbf{Q}}(2)^{-3} L^{(\infty p N)}\left(s+\frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}}\right) \\ & \times \gamma^\star_{(k_1,k_2,k_3)}(s)^{-1} Z\Big(\rho(\mathcal{J}_\infty)\breve{W}_{1,\infty}, \rho(\mathcal{J}_\infty)V_+^{\frac{k_1-k_2-\lambda_2}{2}}\breve{W}_{2,\infty}, \rho(\mathcal{J}_\infty)V_+^{\frac{k_1-k_3-\lambda_3}{2}}\breve{W}_{3,\infty}, f_{s,\infty}^{[k_1,\lambda]}\Big) \\ & \times Z(\rho(t_n)\breve{W}_{1,p}, \rho(t_n)\breve{W}_{2,p}, \rho(t_n)\breve{W}_{3,p}, f_{\mathcal{D},s,p}) \prod_{\ell|N} Z(\breve{W}_{1,\ell}, \breve{W}_{2,\ell}, \breve{W}_{3,\ell}, f_{s,\ell}) \end{aligned}$$

in view of Definition 6.2 of $E_{\mathbf{A}}^\star\big(f_{\mathcal{D},s}^{[k_1,\lambda]}\big)$. Since $\rho(\mathcal{J}_\infty)\breve{W}_{i,\infty} = \omega_i(-1)\rho(\mathcal{J}_\infty)W_{i,\infty}$, Lemma 4.8 yields

$$\begin{aligned} & Z\Big(\rho(\mathcal{J}_\infty)\breve{W}_{1,\infty}, \rho(\mathcal{J}_\infty)V_+^{\frac{k_1-k_2-\lambda_2}{2}}\breve{W}_{2,\infty}, \rho(\mathcal{J}_\infty)V_+^{\frac{k_1-k_3-\lambda_3}{2}}\breve{W}_{3,\infty}, f_{s,\infty}^{[k_1,\lambda]}\Big) \\ =& \hat{\omega}_\infty(-1) Z_\infty(s) = \chi_\infty(-1)\frac{\mathrm{vol}(\mathrm{SO}(2))^3 \gamma^\star_{(k_1,k_2,k_3)}(s)}{2^{5+(k_1+k_2+k_3)}} L\left(s+\frac{1}{2}, \sigma_{k_1} \times \sigma_{k_2} \times \sigma_{k_3} \otimes \chi_\infty\right). \end{aligned}$$

Proposition 2.6 calculates the p-adic part:

$$\begin{aligned} & \frac{Z(\rho(t_n)\breve{W}_{1,p}, \rho(t_n)\breve{W}_{2,p}, \rho(t_n)\breve{W}_{3,p}, f_{\mathcal{D},s,p})}{L\left(s+\frac{1}{2}, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \chi_p\right)} \prod_{i=1}^3 \frac{\zeta_p(1)}{\zeta_p(2)} \left(\frac{p^{k_i}\omega_{i,p}(p)}{\alpha_p(f_i)^2}\right)^n \\ =& Z_p^*(f_{\mathcal{D},s,p}) = \chi_p(-1) E_p\left(s+\frac{1}{2}, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \chi_p\right). \end{aligned}$$

Since $\chi_{\mathbf{A}}$ is unramified outside p, we have $\chi_\infty(-1) = \chi_p(-1)$. □

7.3. **The congruence number.** Put $\Delta = (\mathbf{Z}/Np\mathbf{Z})^\times$. Let $\widehat{\Delta}$ be the group of Dirichlet characters modulo Np. Enlarging $\mathcal{O}$ if necessary, we assume that every $\chi \in \widehat{\Delta}$ takes value in $\mathcal{O}^\times$. Let

$$\mathbf{S}^{\mathrm{ord}}(N, \mathbf{I}) := \oplus_{\chi \in \widehat{\Delta}} \mathbf{S}^{\mathrm{ord}}(N, \chi, \mathbf{I})$$

be the space of ordinary $\mathbf{I}$-adic cusp forms of tame level $\Gamma_1(N)$. Let σ_d denote the usual diamond operator for $d \in \Delta$ acting on $\mathbf{S}^{\mathrm{ord}}(N, \mathbf{I})$ by $\sigma_d(\boldsymbol{f})_{\chi \in \widehat{\Delta}} = (\chi(d)\boldsymbol{f})_{\chi \in \widehat{\Delta}}$. The ordinary $\mathbf{I}$-adic cuspidal Hecke algebra $\mathbf{T}(N, \mathbf{I})$ is defined as the $\mathbf{I}$-subalgebra of $\mathrm{End}_{\mathbf{I}}\, \mathbf{S}^{\mathrm{ord}}(N, \mathbf{I})$ generated over $\mathbf{I}$ by the Hecke operators T_ℓ with $\ell \nmid Np$, the operators $\mathbf{U}_\ell$ with $\ell | Np$ and the diamond operators σ_d with $d \in \widehat{\Delta}$. Let $\mathrm{T}_k^{\mathrm{ord}}(N, \chi)$ denote the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathbf{C}}\, e_{\mathrm{ord}}\mathcal{S}_k(N, \chi)$ generated over $\mathcal{O}$ by the operators T_ℓ with $\ell \nmid Np$ and $\mathbf{U}_\ell$ with $\ell | Np$.

Let $\boldsymbol{f} \in \mathbf{S}^{\mathrm{ord}}(N, \chi, \mathbf{I})$ be a primitive Hida family of tame conductor N and character χ. The corresponding homomorphism $\lambda_{\boldsymbol{f}} : \mathbf{T}(N, \mathbf{I}) \to \mathbf{I}$ is defined by $\lambda_{\boldsymbol{f}}(T_\ell) = \mathbf{a}(\ell, \boldsymbol{f})$ for $\ell \nmid Np$, $\lambda_{\boldsymbol{f}}(\mathbf{U}_\ell) = \mathbf{a}(\ell, \boldsymbol{f})$ for $\ell \mid Np$ and $\lambda_{\boldsymbol{f}}(\sigma_d) = \chi(d)$ for $d \in \Delta$. We denote by $\mathfrak{m}_{\boldsymbol{f}}$ the maximal of $\mathbf{T}(N, \mathbf{I})$ containing $\mathrm{Ker}\, \lambda_{\boldsymbol{f}}$ and by $\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}}$ the localization of $\mathbf{T}(N, \mathbf{I})$ at $\mathfrak{m}_{\boldsymbol{f}}$. It is the local ring of $\mathbf{T}(N, \mathbf{I})$ through which $\lambda_{\boldsymbol{f}}$ factors. Recall that the congruence ideal $C(\boldsymbol{f})$ of the morphism $\lambda_{\boldsymbol{f}} : \mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}} \to \mathbf{I}$ is defined by

$$C(\boldsymbol{f}) := \lambda_{\boldsymbol{f}}(\mathrm{Ann}_{\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}}}(\mathrm{Ker}\, \lambda_{\boldsymbol{f}})) \subset \mathbf{I}.$$

It is well-known that $\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}}$ is a local finite flat Λ-algebra, and there is an algebra direct sum decomposition

$$\widetilde{\lambda}_{\boldsymbol{f}} : \mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}} \otimes_{\mathbf{I}} \operatorname{Frac}\mathbf{I} \simeq \operatorname{Frac}\mathbf{I} \oplus \mathscr{B}, \qquad t \mapsto \widetilde{\lambda}_{\boldsymbol{f}}(t) = (\lambda_{\boldsymbol{f}}(t), \lambda_{\mathscr{B}}(t)), \tag{7.1}$$

where $\mathscr{B}$ is some finite dimensional $(\operatorname{Frac}\mathbf{I})$-algebra ([Hid88b, Corollary 3.7]). Then we have

$$C(\boldsymbol{f}) = \lambda_{\boldsymbol{f}}(\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}} \cap \widetilde{\lambda}_{\boldsymbol{f}}^{-1}(\operatorname{Frac}\mathbf{I} \oplus \{0\}))$$

by definition. Now we impose the following hypothesis:

Hypothesis (CR)**.** The residual Galois representation $\overline{\rho}_{\boldsymbol{f}}$ of $\rho_{\boldsymbol{f}}$ is absolutely irreducible and p-distinguished.

Under the hypothesis above $\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}}$ is Gorenstein by [Wil95, Corollary 2, page 482]. With this property of $\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}}$ Hida in [Hid88a] proved that the congruence ideal $C(\boldsymbol{f})$ is generated by a non-zero element $\eta_{\boldsymbol{f}} \in \mathbf{I}$, called the congruence number for $\boldsymbol{f}$. Let $1^*_{\boldsymbol{f}}$ be the unique element in $\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}} \cap \widetilde{\lambda}_{\boldsymbol{f}}^{-1}(\operatorname{Frac}\mathbf{I} \oplus \{0\})$ such that $\lambda_{\boldsymbol{f}}(1^*_{\boldsymbol{f}}) = \eta_{\boldsymbol{f}}$. Then $1_{\boldsymbol{f}} := \eta_{\boldsymbol{f}}^{-1} 1^*_{\boldsymbol{f}}$ is the idempotent in $\mathbf{T}_{\mathfrak{m}_{\boldsymbol{f}}} \otimes_{\mathbf{I}} \operatorname{Frac}\mathbf{I}$ corresponding to the direct summand $\operatorname{Frac}\mathbf{I}$ of (7.1) and $1_{\boldsymbol{f}}$ does not depend on any choice of a generator of $C(\boldsymbol{f})$. For $d \in \widehat{\Delta}$ we write $\wp_{Q,\chi}$ for the ideal of $\mathbf{T}(N, \mathbf{I})$ generated by $\wp_Q = \operatorname{Ker} Q$ and $\{\sigma_d - \chi(d)\}_{d \in \Delta}$. A classical result in Hida theory asserts that

$$\mathbf{T}(N, \mathbf{I})/\wp_{Q,\chi} \simeq \mathrm{T}^{\mathrm{ord}}_{k_Q}(Np^e, \chi\boldsymbol{\omega}^{-k_Q}\epsilon_Q) \otimes_{\mathcal{O}} \mathcal{O}(Q)$$

(see Theorem 3.4 of [Hid88b]). Moreover, for each arithmetic point Q, it is also shown by Hida that the specialization $\eta_{\boldsymbol{f}}(Q) \in \mathcal{O}(Q)$ is the congruence number for $\boldsymbol{f}_Q$ and

$$1_f := \eta_{\boldsymbol{f}}^{-1} 1^*_{\boldsymbol{f}} \ (\mathrm{mod}\ \wp_{\chi,Q}) \in \mathrm{T}^{\mathrm{ord}}_{k_Q}(Np^e, \chi\boldsymbol{\omega}^{-k_Q}\epsilon_Q) \otimes_{\mathcal{O}} \operatorname{Frac}\mathcal{O}(Q)$$

is the idempotent with $\lambda_f(1_f) = 1$.

Definition 7.2. Let $\boldsymbol{f}$ be a primitive Hida family satisfying (CR). To each choice of the congruence number $\eta_{\boldsymbol{f}}$ we associate Hida's canonical period Ω_f of a p-ordinary newform f of weight k obtained by the specialization of $\boldsymbol{f}$ defined by

$$\Omega_f := \eta_f^{-1} \cdot (-2\sqrt{-1})^{k+1} \|f^\circ\|^2_{\Gamma_0(N_{f^\circ})} \cdot \mathcal{E}_p(f, \mathrm{Ad}),$$

where η_f is the specialization of $\eta_{\boldsymbol{f}}$, f° the primitive form associated with f, N_{f° its conductor and $\mathcal{E}_p(f, \mathrm{Ad})$ the modified p-Euler factor attached to the adjoint motive of f (*cf.* [Hsi19, (3.10)]).

7.4. **Hida's functional.** When $\varphi \in \mathcal{A}_k(N, \omega_{\mathbf{A}})$ and $\varphi' \in \mathcal{A}_k(N, \omega_{\mathbf{A}}^{-1})$ are cuspidal, we define the pairing by

$$\langle \rho(\mathcal{J}_\infty)\varphi, \varphi' \rangle = \int_{\mathbf{A}^\times \mathrm{GL}_2(\mathbf{Q}) \backslash \mathrm{GL}_2(\mathbf{A})} \varphi(g\mathcal{J}_\infty)\varphi'(g)\,\mathrm{d}g.$$

Let χ be a Dirichlet character and let $f \in \mathcal{S}_k(N_f, \chi)$ be an ordinary p-stabilized newform of level N_f, i.e., $\mathbf{U}_p f = \alpha_p(f) f$ with p-unit $\iota_p^{-1}(\alpha_p(f))$. Write $N_f = N_{\mathrm{t}} p^c$ with N_{t} prime to p. For $n \geq c$, we define Hida's functional L_f on $\mathcal{S}_k(N_{\mathrm{t}} p^{2n}, \chi; \mathcal{O})$ by

$$L_f(\mathcal{F}) = \left(\frac{\omega_{f,p}(p)p^k}{\alpha_p(f)^2}\right)^{n-1} \frac{\langle \rho(\mathcal{J}_\infty t_n)(\varphi \otimes \omega_{\mathbf{A}}^{-1}), \varPhi(\mathcal{F})\rangle}{\langle \rho(\mathcal{J}_\infty t_1)(\varphi \otimes \omega_{\mathbf{A}}^{-1}), \varphi\rangle},$$

where $\omega_{\mathbf{A}}$ denotes the central character of the adèlic lift $\varphi = \varPhi(f)$ of f. Note that for $\mathcal{F} \in \mathcal{S}_k(Np^{2n}, \chi)$ with $N_{\mathrm{t}} \mid N$,

$$L_f(\mathcal{F}) = [\Gamma_0(N) : \Gamma_0(N_{\mathrm{t}})]^{-1} L_f(\operatorname{Tr}_{N/N_{\mathrm{t}}} \mathcal{F}).$$

Lemma 7.3. *(1)* $L_f(f) = 1$.

(2) If $\mathcal{F}_0 \in \mathcal{N}_{k+2m}(Np^{2n}, \chi)$ *with* $N_{\mathrm{t}} \mid N$, *then*

$$\frac{L_f(\mathbf{1}^*_f \operatorname{Tr}_{N/N_{\mathrm{t}}} e_{\mathrm{ord}} \operatorname{Hol}(\lambda_z^m \mathcal{F}_0))}{\zeta_{\mathbf{Q}}(2)[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)]} = (-1)^{m+1}(2\sqrt{-1})^{k+1} \frac{\langle \rho(\mathcal{J}_\infty t_n)(V_+^m \varphi \otimes \omega_{\mathbf{A}}^{-1}), \varPhi(\mathcal{F}_0)\rangle}{\Omega_f \left(\frac{\alpha_p(f)^2}{p^k \omega_{f,p}(p)}\right)^n \frac{\zeta_p(2)}{\zeta_p(1)}}.$$

PROOF. The first assertion follows from the following formula stated in [Hsi19, Lemma 3.6]:

$$\begin{aligned}\langle\rho(\mathcal{J}_\infty t_n)(\varphi\otimes\omega_{\mathbf{A}}^{-1}),\varphi\rangle &= \omega_{f,\infty}(-1)\langle\rho(\mathcal{J}_\infty t_n)\varphi\otimes\omega_{\mathbf{A}}^{-1},\varphi\rangle\\ &= \frac{(-1)^k\zeta_{\mathbf{Q}}(2)^{-1}}{[\mathrm{SL}_2(\mathbf{Z}):\Gamma_0(N_{\mathrm{t}})]}\cdot\|f^\circ\|^2_{\Gamma_0(N_{f^\circ})}\cdot\mathcal{E}_p(f,\mathrm{Ad})\cdot\frac{\alpha_p(f)^{2n}\zeta_p(2)}{p^{kn}\omega_{f,p}(p)^n\zeta_p(1)}\\ &= -\frac{\zeta_{\mathbf{Q}}(2)^{-1}}{[\mathrm{SL}_2(\mathbf{Z}):\Gamma_0(N_{\mathrm{t}})]}\cdot\frac{\eta_f\Omega_f}{(2\sqrt{-1})^{k+1}}\cdot\left(\frac{\alpha_p(f)^2}{p^k\omega_{f,p}(p)}\right)^n\frac{\zeta_p(2)}{\zeta_p(1)}.\end{aligned}$$

We remark that $\breve{\varphi}=\varphi$ and $\omega_{(p)}=\omega_{\mathbf{A}}$ in the notation of [Hsi19].

To see the second part, we note that as a consequence of strong multiplicity one theorem for elliptic modular forms, the idempotent $1_f=\eta_f^{-1}1_f^*$ is generated by the Hecke operators T_ℓ with $\ell\nmid Np$, which implies that 1_f is the adjoint operator of $1_{\varphi\otimes\omega_{\mathbf{A}}^{-1}}$ with respect to the pairing. We are thus led to $L_f(\mathbf{1}_f^*\mathcal{F})=\eta_f L_f(\mathcal{F})$. Moreover, $L_f(\mathbf{U}_p\mathcal{F})=\alpha_p(f)L_f(\mathcal{F})$ (cf. the proof of Proposition 2.10 of [Kob13]) and hence

$$L_f(e_{\mathrm{ord}}\mathcal{F})=\lim_{j\to\infty}L_f(\mathbf{U}_p^{j!}\mathcal{F})=\lim_{j\to\infty}\alpha_p(f)^{j!}L_f(\mathcal{F})=L_f(\mathcal{F}).$$

One can easily verify that for $\phi\in\Phi(\mathcal{S}_k(M,\chi^{-1}))$, $\mathcal{F}_1\in\mathcal{N}_k(M,\chi)$ and $\mathcal{F}_2\in\mathcal{N}_{k+2}(M,\chi)$

$$\langle\rho(\mathcal{J}_\infty)\phi,\Phi(\mathrm{Hol}\,\mathcal{F}_1)\rangle=\langle\rho(\mathcal{J}_\infty)\phi,\Phi(\mathcal{F}_1)\rangle,\qquad \langle\rho(\mathcal{J}_\infty)\phi,\Phi(\lambda_z\mathcal{F}_2)\rangle=-\langle\rho(\mathcal{J}_\infty)V_+\phi,\Phi(\mathcal{F}_2)\rangle.$$

The second part is a consequence of these results. □

7.5. **The construction of p-adic triple product L-functions.** Let

$$\boldsymbol{F}=(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\in\mathbf{S}^{\mathrm{ord}}(N_1,\chi_1,\mathbf{I})\times\mathbf{S}^{\mathrm{ord}}(N_2,\chi_2,\mathbf{I})\times\mathbf{S}^{\mathrm{ord}}(N_3,\chi_3,\mathbf{I})$$

be a triplet of primitive $\mathbf{I}$-adic Hida families of tame square-free level (N_1,N_2,N_3) and tame characters (χ_1,χ_2,χ_3), where $\mathbf{I}$ is a finite flat domain over $\Lambda=\mathcal{O}[\![\Gamma]\!]$. Assuming that all $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ satisfy Hypothesis (CR), we fix a choice of the congruence numbers $(\eta_{\boldsymbol{f}},\eta_{\boldsymbol{g}},\eta_{\boldsymbol{h}})$. Let

$$\mathbf{1}_{\boldsymbol{f}}^*\in\mathbf{T}(N_1,\mathbf{I}),\qquad \mathbf{1}_{\boldsymbol{g}}^*\in\mathbf{T}(N_2,\mathbf{I}),\qquad \mathbf{1}_{\boldsymbol{h}}^*\in\mathbf{T}(N_3,\mathbf{I})$$

be the idempotents multiplied by a fixed choice of congruence numbers $(\eta_{\boldsymbol{f}},\eta_{\boldsymbol{g}},\eta_{\boldsymbol{h}})$ in the Hecke algebras attached to the newforms $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. Put

$$N:=\mathrm{lcm}(N_1,N_2,N_3),\qquad N^-:=\gcd(N_1,N_2,N_3),\qquad \mathbf{I}_3:=\mathbf{I}\widehat{\otimes}_{\mathcal{O}}\mathbf{I}\widehat{\otimes}_{\mathcal{O}}\mathbf{I}.$$

Definition 7.4. Define the p-adic triple product L-function $L_{\boldsymbol{F},(a)}$ in $\mathbf{I}_3[\![T]\!]$ by

$$L_{\boldsymbol{F},(a)}:=\text{ the first Fourier coefficient of }\mathbf{1}_{\boldsymbol{f}}^*\otimes\mathbf{1}_{\boldsymbol{g}}^*\otimes\mathbf{1}_{\boldsymbol{h}}^*(\mathrm{Tr}_{N/N_1}\otimes\mathrm{Tr}_{N/N_2}\otimes\mathrm{Tr}_{N/N_3}(\mathcal{G}^{(a)}_{\underline{\chi}}))\in\mathbf{I}_3[\![T]\!].$$

We denote by $V_{\boldsymbol{f}}$ the associated p-adic Galois representation, and by $\mathrm{WD}_\ell(V_{\boldsymbol{f}_Q})$ the representation of the Weil-Deligne group $W_{\mathbf{Q}_\ell}$ attached to $V_{\boldsymbol{f}_Q}$ for each prime ℓ. The epsilon factor of $\mathbf{V}_{(\underline{Q},P)}$ at ℓ is defined by

$$\varepsilon_\ell(\mathbf{V}_{(\underline{Q},P)},s)=\varepsilon(s+k_P-w_{\underline{Q}}/2,\mathrm{WD}_\ell(V_{\boldsymbol{f}_{Q_1}})\otimes\mathrm{WD}_\ell(V_{\boldsymbol{g}_{Q_2}})\otimes\mathrm{WD}_\ell(V_{\boldsymbol{h}_{Q_3}})\otimes\boldsymbol{\omega}^{a-k_P}\epsilon_P,\boldsymbol{\psi}_\ell).$$

By the assumption (sf) and the rigidity of automorphic types of Hida families $\mathrm{WD}_\ell(V_{\boldsymbol{f}_{Q_1}})$, $\mathrm{WD}_\ell(V_{\boldsymbol{g}_{Q_2}})$, $\mathrm{WD}_\ell(V_{\boldsymbol{h}_{Q_3}})$ are either unramified or the Steinberg representation twisted by an unramified character. Moreover, for $\ell|N_1$, there is an unramified finite order character $\xi_{\boldsymbol{f},\ell}:G_{\mathbf{Q}_\ell}\to\overline{\mathbf{Q}}^\times$ such that $\xi_{\boldsymbol{f},\ell}^2=\chi_{1,\ell}^{-1}$ and

$$V_{\boldsymbol{f}}|_{G_{\mathbf{Q}_\ell}}\simeq\begin{pmatrix}\xi_{\boldsymbol{f},\ell}\boldsymbol{\varepsilon}_{\mathrm{cyc}}\langle\boldsymbol{\varepsilon}_{\mathrm{cyc}}\rangle_I^{-1/2} & *\\ 0 & \xi_{\boldsymbol{f},\ell}\langle\boldsymbol{\varepsilon}_{\mathrm{cyc}}\rangle_I^{-1/2}\end{pmatrix}.$$

Define $\varepsilon_\ell(\boldsymbol{f})_{X_1}\in\mathbf{I}[\![X_1]\!]$ by $\varepsilon_\ell(\boldsymbol{f})_{X_1}=\xi_{\boldsymbol{f},\ell}(\ell)^{-1}\langle\ell\rangle_{X_1}^{1/2}$. Then

$$\varepsilon_\ell((1-k_{Q_1})/2,\mathrm{WD}_\ell(V_{\boldsymbol{f}_{Q_1}}),\psi_\ell)=\varepsilon_\ell(\boldsymbol{f}_{Q_1}).$$

Let $N_{\mathbf{V}}=N^-N^4$ be the tame level of $\mathbf{V}$, which independent of the choice of of arithmetic specializations by the rigidity. We define the $\mathbf{I}_4$-adic root number $\varepsilon^{(p\infty)}(\mathbf{V})\in\mathbf{I}_4^\times$ by

$$\varepsilon^{(p\infty)}(\mathbf{V})=\prod_{i=1}^3\langle N_{\mathbf{V}}\rangle_{X_i}\cdot\boldsymbol{\omega}(N_{\mathbf{V}})^{-a}N_{\mathbf{V}}^{-1}\langle N_{\mathbf{V}}\rangle_T^{-1}(\chi_1\chi_2\chi_3)(N^2)\prod_{\ell|N^-}\xi_{\boldsymbol{f},\ell}(\ell)^{-1}\xi_{\boldsymbol{g},\ell}(\ell)^{-1}\xi_{\boldsymbol{h},\ell}(\ell)^{-1}. \tag{7.2}$$

Lemma 7.5. *Notation being as above, we get*

$$\varepsilon^{(p\infty)}(\mathbf{V}_{(\underline{Q},P)}) = \prod_{\ell\neq p} \varepsilon_\ell(\mathbf{V}_{(\underline{Q},P)}, 0).$$

Proof. We retain the notation of the proof of Proposition 6.8. Remark 3.5 gives

$$\varepsilon_\ell(\mathbf{V}_{(\underline{Q},P)}, 0) = \chi_\ell(\ell)^4 \hat{\omega}(\ell)^2 \ell^{-4k_P+2(k_{Q_1}+k_{Q_2}+k_{Q_3})-4} = \left\langle \ell^2 \right\rangle^{(a)}_{X_1,X_2,X_3,T}(\underline{Q},P)$$

if ℓ divides N/N^-. Put $\xi_\ell = \xi_{\boldsymbol{f},\ell}\xi_{\boldsymbol{g},\ell}\xi_{\boldsymbol{h},\ell}$. If ℓ divides N^-, then

$$\begin{aligned}\varepsilon_\ell(\mathbf{V}_{(\underline{Q},P)}, 0) &= \chi_\ell(\ell)^5 \xi_\ell(\mathrm{Frob}_\ell)^5 \ell^{5(-2k_P+k_{Q_1}+k_{Q_2}+k_{Q_3}-2)/2} \\ &= \boldsymbol{\omega}(\ell)^{-5a}(\chi_1\chi_2\chi_3)(\ell)^2 \xi_{\boldsymbol{f},\ell}(\mathrm{Frob}_\ell)(\langle\ell\rangle_T^{-5} \langle\ell\rangle_{X_1}^{5/2} \langle\ell\rangle_{X_2}^{5/2} \langle\ell\rangle_{X_3}^{5/2})(\underline{Q},P)\ell^{-5}.\end{aligned}$$

We have thus completed our proof. □

7.6. **The interpolation formulae.** Let $\mathbf{V} = V_{\boldsymbol{f}} \widehat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \widehat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}} \widehat{\otimes}_{\mathcal{O}} \boldsymbol{\omega}^a \langle \boldsymbol{\varepsilon}_{\mathrm{cyc}} \rangle_T$ be the triple tensor product of $\mathbf{I}$-adic Galois representations associated with primitive Hida families $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{h}$ twisted by $\boldsymbol{\omega}^a \langle \boldsymbol{\varepsilon}_{\mathrm{cyc}} \rangle_T$. Define the rank four $G_{\mathbf{Q}_p}$-invariant subspace of $\mathbf{V}$ by

$$\mathrm{Fil}^+ \mathbf{V} = (\mathrm{Fil}^0 V_{\boldsymbol{f}} \otimes \mathrm{Fil}^0 V_{\boldsymbol{g}} \otimes V_{\boldsymbol{h}} + \mathrm{Fil}^0 V_{\boldsymbol{f}} \otimes V_{\boldsymbol{g}} \otimes \mathrm{Fil}^0 V_{\boldsymbol{h}} + V_{\boldsymbol{f}} \otimes \mathrm{Fil}^0 V_{\boldsymbol{g}} \otimes \mathrm{Fil}^0 V_{\boldsymbol{h}}) \otimes \boldsymbol{\omega}^a \langle \boldsymbol{\varepsilon}_{\mathrm{cyc}} \rangle_T .$$

Recall that $w_{\underline{Q}} = k_{Q_1} + k_{Q_2} + k_{Q_3} - 3$ and $\Gamma_{\mathbf{V}_{(\underline{Q},P)}}(s) = L_\infty\left(s + k_P - \frac{w_{\underline{Q}}}{2}, \pi_{\boldsymbol{f}_{Q_1}} \times \pi_{\boldsymbol{g}_{Q_2}} \times \pi_{\boldsymbol{h}_{Q_3}}\right)$. The modified p-Euler factor $\mathcal{E}_p(\mathrm{Fil}^+ \mathbf{V}_{(\underline{Q},P)})$ is defined in the introduction.

Theorem 7.6. *Let $p > 3$. Assume that $N := \mathrm{lcm}(N_1, N_2, N_3)$ is square-free and that the conductor of tame nebentypus χ_i divides p. Let t denote the number of prime factors of N. If $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ satisfy Hypothesis (CR), then for each arithmetic point $(\underline{Q}, P) = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}^{\mathrm{bal}}_{\mathbf{I}_4}$ we have*

$$L_{\boldsymbol{F},(a)}(\underline{Q}, P) = \Gamma_{\mathbf{V}_{(\underline{Q},P)}}(0) \cdot \frac{L(\mathbf{V}_{(\underline{Q},P)}, 0)}{\Omega_{\boldsymbol{f}_{Q_1}} \Omega_{\boldsymbol{g}_{Q_2}} \Omega_{\boldsymbol{h}_{Q_3}}} \cdot (\sqrt{-1})^{k_{Q_1}+k_{Q_2}+k_{Q_3}-3} \cdot \mathcal{E}_p(\mathrm{Fil}^+ \mathbf{V}_{(\underline{Q},P)}) \cdot \mathfrak{f}_{\underline{\chi},a,N_1,N_2,N_3}(\underline{Q}, P),$$

where $\mathfrak{f}_{\underline{\chi},a,N_1,N_2,N_3} \in \mathbf{I}_4^\times$ is given by

$$\mathfrak{f}_{\underline{\chi},a,N_1,N_2,N_3} := \frac{(-1)^t}{N} \prod_{\ell | N} (\langle \ell \rangle^{(a)}_{X_1,X_2,X_3,T})^2 \varepsilon_\ell(\boldsymbol{f} \otimes \boldsymbol{g} \otimes \boldsymbol{h} \otimes \boldsymbol{\omega}^a \langle \boldsymbol{\varepsilon}_{\mathrm{cyc}} \rangle_T)^{-1}.$$

Proof. For brevity we write $(f_1, f_2, f_3) = (\boldsymbol{f}_{Q_1}, \boldsymbol{g}_{Q_2}, \boldsymbol{h}_{Q_3})$, $(k, l, m) = (k_{Q_1}, k_{Q_2}, k_{Q_3})$, $\pi_i = \pi_{f_i}$ and $N_i = N_{f_i}$. We may assume that $k \geq l \geq m$. Denote the parity type of (k, l, m) by λ. Put

$$\chi = \epsilon_P \boldsymbol{\omega}^{a-k_P}, \qquad \omega_i = \boldsymbol{\omega}^{k_{Q_i}} \chi_i^{-1} \epsilon_{Q_i}^{-1}, \qquad \mathcal{D} = (\chi, \omega_1^{-1}, \omega_2^{-1}, \omega_3^{-1}), \qquad n = \max\{1, c(\omega_i), c(\chi)\}.$$

We define the functional L_{f_1,f_2,f_3} on

$$\mathcal{S}_k(N_1 p^{2n}, \omega_1^{-1}; \mathcal{O}(Q_1)) \otimes_{\mathcal{O}} \mathcal{S}_l(N_2 p^{2n}, \omega_2^{-1}; \mathcal{O}(Q_2)) \otimes_{\mathcal{O}} \mathcal{S}_m(N_3 p^{2n}, \omega_3^{-1}; \mathcal{O}(Q_3))$$

by

$$L_{f_1,f_2,f_3}(\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3) = L_{f_1}(\mathcal{F}_1) L_{f_2}(\mathcal{F}_2) L_{f_3}(\mathcal{F}_3).$$

Let $1^*_{f_1}$ be the specialization of $\mathbf{1}^*_{\boldsymbol{f}}$ at Q_1. By definition and the theory of newforms

$$1^*_{f_1} \otimes 1^*_{f_2} \otimes 1^*_{f_3}(\mathrm{Tr}_{N/N_1} \otimes \mathrm{Tr}_{N/N_2} \otimes \mathrm{Tr}_{N/N_3}(\mathcal{G}^{(a)}_{\underline{\chi}}(\underline{Q}, P))) = L_{\boldsymbol{F},(a)}(\underline{Q}, P) \cdot f_1 \otimes f_2 \otimes f_3.$$

We apply the functional L_{f_1,f_2,f_3} to both the sides to get

$$L_{\boldsymbol{F},(a)}(\underline{Q}, P) = L_{f_1,f_2,f_3}(1^*_{f_1} \otimes 1^*_{f_2} \otimes 1^*_{f_3}(\mathrm{Tr}_{N/N_1} \otimes \mathrm{Tr}_{N/N_2} \otimes \mathrm{Tr}_{N/N_3}(\mathcal{G}^{(a)}_{\underline{\chi}}(\underline{Q}, P)))),$$

taking Lemma 7.3(1) into account. Let $\varphi_i = \Phi(f_i)$ and $G_{\mathbf{A}}(\mathcal{D}) = \Phi(\mathcal{G}^{(a)}_{\underline{\chi}}(\underline{Q}, P))$ be the adèlic lifts. Put $\breve{\varphi}_i = \varphi_i \otimes \omega_{i,\mathbf{A}}^{-1}$. In the previous section we verified that

$$G_{\mathbf{A}}(\mathcal{D}) = \lim_{s \to -r + \frac{k-\lambda_1}{2} - 1} (-1)^{k + \frac{l+m+\lambda_1}{2} + \lambda_2} e_{\mathrm{ord}} \,\mathrm{Hol}\left(\left(1 \otimes V_-^{\frac{k-l-\lambda_2}{2}} \otimes V_-^{\frac{k-m-\lambda_3}{2}}\right) \iota^* E^\star_{\mathbf{A}}\left(f^{[k,\lambda]}_{\mathcal{D},s,N}\right)\right),$$

where $r = k - k_P + \frac{l+m-\lambda_1}{2} - 2$ (see Proposition 6.8 and Definitions 6.2, 6.5). Lemma 7.3 (2) therefore gives

$$\frac{L_{\boldsymbol{F},(a)}(\underline{Q},P)}{\zeta_{\mathbf{Q}}(2)^3[\mathrm{SL}_2(\mathbf{Z}):\Gamma_0(N)]^3} = -(2\sqrt{-1})^{k+l+m+3}\frac{\zeta_p(1)^3}{\zeta_p(2)^3}\prod_{i=1}^{3}\Omega_{f_i}^{-1}\left(\frac{p^{k_{Q_i}}\omega_{i,p}(p)}{\alpha_p(f_i)^2}\right)^n$$
$$\times 4\lim_{s\to k_P-\frac{k+l+m}{2}+1} Z\Big(\rho(\mathcal{J}_\infty t_n)\breve{\varphi}_1, \rho(\mathcal{J}_\infty t_n)V_+^{\frac{k-l-\lambda_2}{2}}\breve{\varphi}_2, \rho(\mathcal{J}_\infty t_n)V_+^{\frac{k-m-\lambda_3}{2}}\breve{\varphi}_3, E_{\mathbf{A}}^\star\big(f_{\mathcal{D},s,N}^{[k,\lambda]}\big)\Big).$$

Let $W(\varphi_i)=\prod_v W_{i,v}$ be the Whittaker function of φ_i. Put $\breve{W}_{i,v} := W_{i,v}\otimes\omega_{i,v}^{-1}$. Let π_i be the automorphic representation generated by φ_i. Writing $N=\prod_{\ell|N}\ell$, we finally get

$$L_{\boldsymbol{F},(a)}(\underline{Q},P) = \frac{L\left(k_P-\frac{k+l+m-3}{2}, \pi_1\times\pi_2\times\pi_3\otimes\chi_{\mathbf{A}}\right)}{(\sqrt{-1})^{3-(k+l+m)}\Omega_{f_1}\Omega_{f_2}\Omega_{f_3}}\mathcal{E}_p(\mathrm{Fil}^+\mathbf{V}\otimes\epsilon_P\boldsymbol{\omega}^{a-k_P},k_P)\prod_{\ell|N}Z_\ell^*$$

by Lemma 7.1, where

$$Z_\ell^* = [\mathrm{SL}_2(\mathbf{Z}):\Gamma_0(\ell)]^3 \lim_{s\to k_P-\frac{k+l+m}{2}+1}\frac{Z(\breve{W}_{1,\ell},\breve{W}_{2,\ell},\breve{W}_{3,\ell},f_{\mathcal{D},s,\ell})}{L\left(s+\frac{1}{2},\pi_{1,\ell}\times\pi_{2,\ell}\times\pi_{3,\ell}\otimes\chi_\ell\right)}.$$

Proposition 3.4 gives

$$Z_\ell^* = -\ell(\hat{\omega}_\ell^2\chi_\ell^4)(\ell)|\ell|^{4k_P-2(k+l+m)+4}\varepsilon\left(k_P-\frac{k+l+m-3}{2},\pi_{1,\ell}\times\pi_{2,\ell}\times\pi_{3,\ell}\otimes\chi,\boldsymbol{\psi}_\ell\right)^{-1}.$$

By what we have seen in the proof of Proposition 6.8

$$\chi_\ell^2\hat{\omega}_\ell(\ell)\ell^{-2k_P+(k+l+m)-2} = \langle\ell\rangle_{X_1,X_2,X_3,T}(\underline{Q},P).$$

This completes the proof. □

Definition 7.7. We normalize p-adic triple product L-function by

$$L_{\boldsymbol{F},(a)}^* := L_{\boldsymbol{F},(a)}\cdot\mathfrak{f}_{\underline{\chi},a,N_1,N_2,N_3}^{-1}.$$

Remark 7.8. Provided that $p>3$, $\chi_1\chi_2\chi_3=\boldsymbol{\omega}^{2a}$ for some a, a three-variable p-adic L-function $\mathcal{L}_{\boldsymbol{F}}^{\mathrm{bal}}\in\mathbf{I}_3$ was constructed by a different approach in [Hsi19, Theorem B] such that for each balanced central point $\underline{Q}=(Q_1,Q_2,Q_3)\in\mathfrak{X}_{\mathbf{I}_3}^{\mathrm{bal}}$

$$\left(\mathcal{L}_{\boldsymbol{F}}^{\mathrm{bal}}(\underline{Q})\right)^2 = \Gamma_{\mathbf{V}_{\underline{Q}}}(0)\cdot\frac{L(\mathbf{V}_{\underline{Q}}^\dagger,0)}{\Omega_{\boldsymbol{f}_{Q_1}}\Omega_{\boldsymbol{g}_{Q_2}}\Omega_{\boldsymbol{h}_{Q_3}}}\cdot(\sqrt{-1})^{k_{Q_1}+k_{Q_2}+k_{Q_3}-3}\cdot\mathcal{E}_p(\mathrm{Fil}^+\mathbf{V}_{\underline{Q}}^\dagger),$$

where

$$\mathbf{V}^\dagger := \mathcal{V}\otimes\boldsymbol{\omega}^a\langle\boldsymbol{\varepsilon}_{\mathrm{cyc}}\rangle_{\mathbf{X}_1}^{1/2}\langle\boldsymbol{\varepsilon}_{\mathrm{cyc}}\rangle_{\mathbf{X}_2}^{1/2}\langle\boldsymbol{\varepsilon}_{\mathrm{cyc}}\rangle_{\mathbf{X}_3}^{1/2}\boldsymbol{\varepsilon}_{\mathrm{cyc}}^{-1},$$
$$\mathrm{Fil}^+\mathbf{V}^\dagger = \mathrm{Fil}^+\mathcal{V}\otimes\boldsymbol{\omega}^a\langle\boldsymbol{\varepsilon}_{\mathrm{cyc}}\rangle_{\mathbf{X}_1}^{1/2}\langle\boldsymbol{\varepsilon}_{\mathrm{cyc}}\rangle_{\mathbf{X}_2}^{1/2}\langle\boldsymbol{\varepsilon}_{\mathrm{cyc}}\rangle_{\mathbf{X}_3}^{1/2}\boldsymbol{\varepsilon}_{\mathrm{cyc}}^{-1}.$$

We remark that $\det V_{\boldsymbol{f}} = (\chi_1\circ\boldsymbol{\varepsilon}_{\mathrm{cyc}})^{-1}\langle\boldsymbol{\varepsilon}_{\mathrm{cyc}}\rangle_{\mathbf{I}}^{-1}\boldsymbol{\varepsilon}_{\mathrm{cyc}}$. By the interpolation formulae, we find that

$$L_{\boldsymbol{F},(a-1)}^*(X_1,X_2,X_3,\mathbf{u}^{-1}\{(1+X_1)(1+X_2)(1+X_3)\}^{1/2}-1) = \mathcal{L}_{\boldsymbol{F}}^{\mathrm{bal}}(X_1,X_2,X_3)^2.$$

This shows that the compatibility between p-adic L-functions constructed by different methods.

Without Hypothesis (CR) and the assumption $p>3$, our method yields the construction of the p-adic L-function with denominators. For each p-stabilized newform f of weight k, define the modified period by

$$\Omega_f^\flat := (-2\sqrt{-1})^{k+1}\cdot\|f^\circ\|_{\Gamma_0(N_{f^\circ})}^2\cdot\mathcal{E}_p(f,\mathrm{Ad}).$$

By definition, $\Omega_f^\flat\cdot\eta_f$ is equal to Hida's canonical period Ω_f up to p-adic units.

Corollary 7.9. *Let $p>2$. There exists an element*

$$L_{\boldsymbol{F},(a)}^{**}\in\mathbf{I}_4\otimes_{\mathbf{I}_3}(\mathrm{Frac}\,\mathbf{I}\otimes\mathrm{Frac}\,\mathbf{I}\otimes\mathrm{Frac}\,\mathbf{I})$$

such that

- *for any H_1, H_2 and H_3 in the congruence ideals of $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{h}$,*

$$H_1H_2H_3 \cdot L^{**}_{\boldsymbol{F},(a)} \in \mathbf{I}_4;$$

- *for each balanced critical $(\underline{Q}, P) = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}^{\mathrm{bal}}_{\mathbf{I}_4}$,*

$$L^{**}_{\boldsymbol{F},(a)}(\underline{Q}, P) = \frac{\Gamma_{\mathbf{V}_{(\underline{Q},P)}}(0)L(\mathbf{V}_{(\underline{Q},P)}, 0)}{\Omega^\flat_{\boldsymbol{f}_{Q_1}}\Omega^\flat_{\boldsymbol{g}_{Q_2}}\Omega^\flat_{\boldsymbol{h}_{Q_3}}} \cdot (\sqrt{-1})^{k_{Q_1}+k_{Q_2}+k_{Q_3}-3} \cdot \mathcal{E}_p(\mathrm{Fil}^+\,\mathbf{V}_{(\underline{Q},P)}),$$

Proof. For any H_1, H_2 and H_3 in the congruence ideals of $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{h}$, we let $L_H \in \mathbf{I}_3[\![T]\!]$ be the first Fourier coefficient of

$$H_1\mathbf{1}_{\boldsymbol{f}} \otimes H_2\mathbf{1}_{\boldsymbol{g}} \otimes H_3\mathbf{1}_{\boldsymbol{h}} \left(\mathrm{Tr}_{N/N_1} \otimes \mathrm{Tr}_{N/N_2} \otimes \mathrm{Tr}_{N/N_3}(\mathcal{G}^{(a)}_{\underline{\chi}})\right) \in \mathbf{I}_3[\![T]\!][\![q_1, q_2, q_3]\!].$$

Then $L^{**}_{\boldsymbol{F},(a)} := L_H \cdot (H_1H_2H_3)^{-1} \cdot \mathfrak{f}^{-1}_{\underline{\chi},a,N_1,N_2,N_3}$ enjoys the desired properties. □

This p-adic L-function $L^{**}_{\boldsymbol{F},(a)}$ is more canonical in the sense that it does not depend on any particular choice of generators of the congruence ideal of $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$.

7.7. **The functional equation.** Recall that we have fixed the topological generator $\mathbf{u} = 1+p$ of $\Gamma = 1 + p\mathbf{Z}_p$ as in §6.5.

Proposition 7.10. *Assume that $\chi_1\chi_2\chi_3 = \boldsymbol{\omega}^{a_0}$. Then*

$$L^*_{\boldsymbol{F},(a)}(X_1, X_2, X_3, T) = (-\varepsilon^{(p\infty)}(\mathbf{V})) \cdot L^*_{\boldsymbol{F},(a_0-a-2)}\left(X_1, X_2, X_3, \frac{(1+X_1)(1+X_2)(1+X_3)}{\mathbf{u}^2(1+T)} - 1\right).$$

Proof. Recall that $\chi = \epsilon_P\boldsymbol{\omega}^{a-k_P}$ and $\omega_i = \chi_i^{-1}\epsilon_{Q_i}^{-1}\boldsymbol{\omega}^{k_{Q_i}}$. Put

$$k_{\breve{P}} = k_{Q_1} + k_{Q_2} + k_{Q_3} - k_P - 2, \qquad \epsilon_{\breve{P}} = \epsilon_P^{-1}\epsilon_{Q_1}\epsilon_{Q_2}\epsilon_{Q_3}, \qquad \breve{\chi} = \epsilon_{\breve{P}}\boldsymbol{\omega}^{a_0-a-2-k_{\breve{P}}} = \chi^{-1}\omega_1^{-1}\omega_2^{-1}\omega_3^{-1}.$$

Thus the left hand side specialized at $(\underline{Q}, \breve{P})$ equals

$$L^*_{\boldsymbol{F},(a_0-a-2)}(\underline{Q}, \breve{P}) = \frac{L(1-s_0, \pi_1^\vee \times \pi_2^\vee \times \pi_3^\vee \otimes \chi_{\mathbf{A}}^{-1})}{(\sqrt{-1})^{3-(k_{Q_1}+k_{Q_2}+k_{Q_3})}\Omega_{f_1}\Omega_{f_2}\Omega_{f_3}} E_p\left(1-s_0, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \chi_p\right),$$

where $s_0 = k_P - \frac{k_{Q_1}+k_{Q_2}+k_{Q_3}-3}{2} = 1 - \left(k_{\breve{P}} - \frac{k_{Q_1}+k_{Q_2}+k_{Q_3}-3}{2}\right)$.

Since $(k_{Q_1}, k_{Q_2}, k_{Q_3})$ is balanced, we know that

$$\varepsilon(s, \pi_{1,\infty} \times \pi_{2,\infty} \times \pi_{3,\infty} \otimes \chi_\infty) = (-1)^{k_{Q_1}+k_{Q_2}+k_{Q_3}+1} = -\hat{\omega}_\infty(-1) = -\hat{\omega}_p(-1).$$

By the global functional equation we get

$$L^*_{\boldsymbol{F},(a_0-a-2)}(\underline{Q}, \breve{P}) = \frac{L(s_0, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})}{(\sqrt{-1})^{3-(k_{Q_1}+k_{Q_2}+k_{Q_3})}\Omega_{f_1}\Omega_{f_2}\Omega_{f_3}} \cdot \frac{-E_p\left(s_0, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \breve{\chi}_p\right)}{\prod_{\ell \neq p} \varepsilon(s_0, \pi_{1,\ell} \times \pi_{2,\ell} \times \pi_{3,\ell} \otimes \chi_\ell, \boldsymbol{\psi}_\ell)}$$

in view of Lemma 2.7. □

8. The trivial zero for the triple product of elliptic curves

8.1. **The cyclotomic p-adic triple product L-functions for elliptic curves.** Let $\boldsymbol{E} = E_1 \times E_2 \times E_3$ be the triple fiber product of rational elliptic curves E_i of square-free conductor M_i for $i = 1, 2, 3$. We denote the prime p-part of M_i by N_i. Recall the rank eight p-adic Galois representation $\mathbf{V}_{\boldsymbol{E}}$ defined in (1.1). We write $L(\boldsymbol{E} \otimes \chi, s)$ for the complex L-series attached to $\mathbf{V}_{\boldsymbol{E}}$ twisted by a Dirichlet character χ. Let M (resp. N) and M^- (resp. N^-) be the least common multiple and the greatest common divisor of M_1, M_2, M_3 (resp. N_1, N_2, N_3).

Remark 8.1. Let Σ^- be the set of prime factors ℓ of M^- such that $a_\ell(E_1)a_\ell(E_2)a_\ell(E_3) = 1$. From Remark 3.5, $\varepsilon(\boldsymbol{E}) = -(-1)^{\#\Sigma^-}$ is the sign in the functional equation for $L(s, \boldsymbol{E})$. From the formula (7.2) for the p-adic root number the p-adic sign $\varepsilon_p(\boldsymbol{E}) = -\varepsilon^{(p\infty)}(V_{\boldsymbol{E}}(2))$ differs from $\varepsilon(\boldsymbol{E})$ if and only if $p \in \Sigma^-$.

Let $f_i^\circ = \sum_{n=1}^\infty a_n(E_i)q^n \in \mathcal{S}_2(M_i, 1; \mathbf{Z})$ be the primitive Hecke eigenform associated with the p-adic Galois representation $\mathrm{H}^1_{\acute{e}t}(E_{i/\overline{\mathbf{Q}}}, \mathbf{Q}_p)$ by Wiles' modularity theorem. Hereafter, we assume that E_i has either good ordinary reduction or multiplicative reduction at p. Let $f_i \in \mathcal{S}_2(pM_i, 1; \mathbf{Z}_p)$ be the p-stabilization of f_i° (see (5.1)). If p and M_i are coprime, then $\alpha_i = \alpha_p(f_i) \in \mathbf{Z}_p^\times$ denotes the p-adic unit root of the Hecke polynomial $X^2 - a_p(E_i)X + p$ while if p divides M_i, then $\alpha_i = a_p(E_i)$. Define a period and a fudge factor by

$$\Omega(\boldsymbol{E}) = \prod_{i=1}^3 \Lambda(1, E_i, \mathrm{Ad}), \qquad c_p = \prod_{i=1}^3 \mathcal{E}_p(f_i, \mathrm{Ad}),$$

where $\Lambda(s, E_i, \mathrm{Ad})$ denotes the complete adjoint L-function for f_i

Let $\mathbf{T}_i = \mathbf{T}(N_i, \Lambda)$ be the big cuspidal ordinary Hecke algebra over $\Lambda = \mathbf{Z}_p[\![X]\!]$ with $X = [\mathbf{u}] - 1$. Each f_i induces a surjective homomorphism $\lambda_{f_i} : \mathbf{T}_i \twoheadrightarrow \mathbf{Z}_p$. Let $\mathfrak{m}_i$ be the maximal ideal of $\mathbf{T}_i$ containing $\ker \lambda_{f_i}$ and $\mathbf{I}_i = (\mathbf{T}_i)_{\mathfrak{m}_i}$ be the localization at $\mathfrak{m}_i$. Let $\boldsymbol{f}_i = \sum_{n=1}^\infty \mathbf{a}(n, \boldsymbol{f}_i)q^n \in \mathbf{S}(N_i, \boldsymbol{\omega}^2, \mathbf{I}_i)$ be the primitive Hida family of tame level N_i such that f_i is the specialization $\boldsymbol{f}_{i,Q_i^o}$ at some arithmetic point Q_i^o with $k_{Q_i^o} = 2$ and $\epsilon_{Q_i^o} = 1$. Now we consider the four-variable p-adic L-function $L^{**}_{\boldsymbol{F},(2)}$ in Corollary 7.9 with $\boldsymbol{F} = (\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$ and $a = 2$. Define the cyclotomic p-adic L-function by

$$L_p(\boldsymbol{E}, T) := c_p \cdot L^{**}_{\boldsymbol{F},(2)}(Q_1^o, Q_2^o, Q_3^o, \mathbf{u}^2(1+T) - 1) \in \mathbf{Z}_p[\![\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]\!] \otimes \mathbf{Q}_p.$$

Proposition 8.2. *The element $L_p(\boldsymbol{E}) \in \mathbf{Z}_p[\![\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]\!] \otimes \mathbf{Q}_p$ satisfies the following interpolation property*

$$\hat{\chi}(L_p(\boldsymbol{E})) = \frac{L(\boldsymbol{E} \otimes \hat{\chi}, 2)}{2^4 \pi^5 \Omega(\boldsymbol{E})} \mathcal{E}_p(\mathrm{Fil}^+ \mathbf{V}_{\boldsymbol{E}} \otimes \hat{\chi})$$

for all finite-order characters $\hat{\chi}$ of $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. Moreover, it satisfies the functional equation

$$L_p(\boldsymbol{E}, T) = \varepsilon_p(\boldsymbol{E}) \left\langle N^- N^4 \right\rangle_T^{-1} L_p(\boldsymbol{E}, (1+T)^{-1} - 1).$$

Proof. Define $(\underline{Q}^o, P) = (Q_1^o, Q_2^o, Q_3^o, P) \in \mathfrak{X}^{\mathrm{bal}}_{\mathbf{I}_4}$ with Q_i^o as above, $k_P = 2$ and $\epsilon_P = \hat{\chi}$. Then $\mathbf{V}_{(\underline{Q}^o, P)} = \mathbf{V}_{\boldsymbol{E}}(2) \otimes \hat{\chi}$ and $\hat{\chi}(L_p(\boldsymbol{E})) = c_p \cdot L^{**}_{\boldsymbol{F},(2)}(\underline{Q}^o, P)$. The assertions follows from Corollary 7.9, Proposition 7.10 and the equation $2^2 \|f_i^\circ\| = \Lambda(1, E_i, \mathrm{Ad})$ ([Hsi19, (2.18)]). □

8.2. **The trivial zero conjecture for the triple product of elliptic curves.** We prove the trivial zero conjecture for the cyclotomic p-adic triple product L-function. We define a function on $\mathbf{Z}_p$ by

$$L_p(\boldsymbol{E}, s) := L_p(\boldsymbol{E}, \mathbf{u}^s - 1).$$

We consider the case where $L_p(\boldsymbol{E}, s)$ has a trivial zero at the critical value $s = 2$. By Remark 8.3 below we essentially only need to consider the following two cases:

(i) all E_1, E_2 and E_3 have multiplicative reduction at p such that $\alpha_1\alpha_2\alpha_3 = 1$.
(ii) E_1 has multiplicative reduction at p; E_2 and E_3 have good ordinary reduction at p such that $\alpha_2 = \alpha_1\alpha_3$.

Remark 8.3. Let $\beta_i = p\alpha_i^{-1}$. Then $\mathcal{E}_p(\mathrm{Fil}^+ \mathbf{V}_{\boldsymbol{E}}(2)) = 0$ if and only if $L_p((\mathrm{Fil}^+ \mathbf{V}_{\boldsymbol{E}}(2))^\vee, 1)^{-1} = 0$ if and only if one of the following equations holds:

$$\beta_1\beta_2\beta_3 = p^2, \qquad \beta_1\beta_2\alpha_3 = p^2, \qquad \beta_1\alpha_2\beta_3 = p^2, \qquad \alpha_1\beta_2\beta_3 = p^2.$$

The ordinality hypothesis rules out the first equation. The Ramanujan conjecture forces one or all of E_i to have multiplicative reduction at p. When E_1 is multiplicative at p, we will have $\alpha_1 \in \{\pm 1\}$ and $\alpha_2 = \alpha_1\alpha_3$.

In the above cases (i) and (ii), the trivial zero conjecture predicts that the leading coefficient of the Taylor expansion of $L_p(\boldsymbol{E}, s)$ at $s = 2$ should be essentially the product of Greenberg's $\mathscr{L}$-invariant for $\boldsymbol{E}$ and the central value $L(\boldsymbol{E}, 2)$. Note that the localization of $\mathbf{I}_i$ at Q_i^o is that of Λ at P_2, where P_2 is the principal ideal generated by $(1+X)\mathbf{u}^{-2} - 1$, so $\mathbf{I}_i$ is contained in $\Lambda[\frac{1}{t_i}]$ with some $t_i(\mathbf{u}^2 - 1) \neq 0$. In what follows, we shall replace $\mathbf{I}_i$ by $\Lambda[t_i^{-1}]$ with some $t_i(\mathbf{u}^2 - 1) \neq 0$. Let $\mathcal{U} \subset \mathbf{Z}_p$ be a neighborhood around 0 such that $(t_1t_2t_3)(\mathbf{u}^{s+2} - 1) \neq 0$ for any $s \in \mathcal{U}$. To introduce Greenberg's $\mathscr{L}$-invariants, we let

$$\mathbf{a}_i(s) := \mathbf{a}(p, \boldsymbol{f}_i)(\mathbf{u}^{s+2} - 1); \qquad \ell_i := \alpha_i^{-1} \cdot \left.\frac{\mathrm{d}\mathbf{a}_i(s)}{\mathrm{d}s}\right|_{s=0} \quad (s \in \mathcal{U}).$$

Note that $\mathbf{a}_i(0) = \alpha_i$ by definition. If $\alpha_i = 1$, then $-2\ell_i = \frac{\log_p q_{E_i}}{\mathrm{ord}_p q_{E_i}}$ by [GS93, Theorem 3.18]. According to the discussion in [Gre94b, §3], Greenberg's $\mathscr{L}$-invariant for the Galois representation (1.1) is given by

$$\mathscr{L}_p(\boldsymbol{E}) := \begin{cases} -8\ell_1\ell_2\ell_3 & \text{in Case (i);} \\ 4\ell_1^2 & \text{in Case (ii).} \end{cases}$$

The non-vanishing of these $\mathscr{L}$-invariants is known, thanks to the work [BSDGP96]. The aim of this section is to prove the following:

Theorem 8.4 (Trivial zero conjecture)**.** *(1) In Case (i),* $\mathrm{ord}_{s=2} L_p(\boldsymbol{E}, s) \geq 3$*, and*

$$\frac{L_p(\boldsymbol{E}, s)}{(s-2)^3}\Big|_{s=2} = \mathscr{L}_p(\boldsymbol{E}) \cdot \frac{L(\boldsymbol{E}, 2)}{2^4\pi^5\Omega(\boldsymbol{E})}.$$

(2) In Case (ii), $\mathrm{ord}_{s=2} L_p(\boldsymbol{E}, s) \geq 2$ *and*

$$\frac{L_p(\boldsymbol{E}, s)}{(s-2)^2}\Big|_{s=2} = \mathscr{L}_p(\boldsymbol{E})(-p\alpha_2^{-2})(1-\alpha_2^{-2})^2 \cdot \frac{L(\boldsymbol{E}, 2)}{2^4\pi^5\Omega(\boldsymbol{E})}.$$

8.3. **Improved p-adic L-functions.** We define an analytic function on $\mathcal{U}^3 \times \mathbf{Z}_p \subset \mathbf{Z}_p^4$ by

$$L_p(x, y, z, s) := c_p \cdot \left\langle N^- N^4 \right\rangle^{\frac{2s-(x+y+z)}{4}} L^{**}_{\boldsymbol{F},(2)}(\mathbf{u}^{x+2}-1, \mathbf{u}^{y+2}-1, \mathbf{u}^{z+2}-1, \mathbf{u}^{s+2}-1),$$

which satisfies

$$(8.1) \qquad L_p(0,0,0,s) = \left\langle N^- N^4 \right\rangle^{s/2} L_p(\boldsymbol{E}, s+2), \qquad L_p(x,y,z,s) = \varepsilon_p(\boldsymbol{E}) \cdot L_p(x,y,z,x+y+z-s).$$

To follow the method used in [GS93] (*cf.* [BDJ17]), we introduce p-adic L-functions which have only less variables but have better interpolation properties.

Lemma 8.5 (Improved p-adic L-functions)**.** *Suppose that f_1° is special at p, i.e.* $\alpha_1 = \mathbf{a}_1(0) = \pm 1$.

(1) There exist a two-variable improved p-adic L-function $L_p^\dagger(x, s)$ and a one-variable improved p-adic L-function $L_p^{\dagger\dagger}(s)$ such that

$$L_p(x,s,s,s) = \Big(1 - \frac{\mathbf{a}_2(s)}{\mathbf{a}_1(x)\mathbf{a}_3(s)}\Big)\Big(1 - \frac{\mathbf{a}_3(s)}{\mathbf{a}_1(x)\mathbf{a}_2(s)}\Big)L_p^\dagger(x,s), \qquad L_p^\dagger(s,s) = \Big(1 - \frac{\mathbf{a}_1(s)}{\mathbf{a}_2(s)\mathbf{a}_3(s)}\Big)L_p^{\dagger\dagger}(s).$$

(2) For any positive integer k with $k \equiv 2\,(\mathrm{mod}\; p-1)$ and $k-2 \in \mathcal{U}$, we have the interpolation formula

$$L_p^\dagger(0, k-2) = \mathcal{E}^\dagger(k-2) \cdot \frac{\Gamma(k-1)\Gamma(k)}{2^{2k-3}(\pi\sqrt{-1})^{2k+1}} \cdot \frac{L\big(\frac{1}{2}, \pi_{f_1} \times \pi_{\boldsymbol{f}_{2,k}} \times \pi_{\boldsymbol{f}_{3,k}}\big)}{c_p^{-1}\Omega^\flat_{f_1}\Omega^\flat_{\boldsymbol{f}_{2,k}}\Omega^\flat_{\boldsymbol{f}_{3,k}}},$$

where $\pi_{\boldsymbol{f}_{i,k}}$ is the automorphic representation generated by $\boldsymbol{f}_{i,k} = \boldsymbol{f}_i(\mathbf{u}^k - 1) \in \mathcal{S}_k(N_i p, 1; \overline{\mathbf{Q}})$, and

$$\mathcal{E}^\dagger(s) = (-\alpha_1)\mathbf{a}_2(s)^{-1}\mathbf{a}_3(s)^{-1}p^{s+1}(1 - \alpha_1 \cdot \mathbf{a}_2(s)^{-1}\mathbf{a}_3(s)^{-1}p^s)^2.$$

(3) If $\varepsilon_p(\boldsymbol{E}) = -1$, then

$$L_p^\dagger(0,s) = 0, \qquad \frac{\partial L_p^\dagger}{\partial x}(0,0) = (\ell_2 + \ell_3 - \ell_1)L_p^{\dagger\dagger}(0), \qquad \mathrm{ord}_{s=2} L_p(\boldsymbol{E}, s) \geq 3.$$

(4) In Case (i), $L_p^{\dagger\dagger}(0) = \frac{L(\boldsymbol{E},2)}{2^4\pi^5\Omega(\boldsymbol{E})}$.

Proof. The construction of these improved L-functions are similar to that of $L_{\boldsymbol{F},(a)}$ except that we need to replace the Λ_4-adic modular form $\mathcal{G}^{(a)}_{\underline{\chi}}$ in §6.5 with *improved* ones. To do so, we have to go back to §6.1 and modify the p-adic section $f_{\mathcal{D},s,p}$ used in the construction of the Siegel Eisenstein series $E_{\mathbf{A}}(g, f^{[k,\lambda]}_{\mathcal{D},s,N})$. In the notation of Definition 2.5, for a datum $\mathcal{D} = (\chi, \omega_1, \omega_2, \omega_3)$ of characters of $\mathbf{Z}_p^\times$ and a Bruhat-Schwartz function $\varphi_3 \in \mathcal{S}(\mathbf{Q}_p)$, we modify the definition of Bruhat-Schwartz functions in (2.8) by

$$\Phi_{\mathcal{D}}(\varphi_1)\left(\begin{pmatrix} u_1 & x_3 & x_2 \\ x_3 & u_2 & x_1 \\ x_2 & x_1 & u_3 \end{pmatrix}\right) = \prod_{i=1}^{3} \phi_i(u_i)\varphi_i(x_i),$$

where

$$\phi_1 = \phi_2 = \phi_3 = \widehat{\mathbb{I}}_{p\mathbf{Z}_p}, \qquad \varphi_2 = \varphi_3 = \mathbb{I}_{\mathbf{Z}_p}.$$

Define the modified Bruhat-Schwartz functions by

$$\Phi^{\dagger}_{\mathcal{D}} = \Phi_{\mathcal{D}}(\widehat{\varphi}_{\chi\omega_1}), \qquad \Phi^{\dagger\dagger}_{\mathcal{D}} = \Phi_{\mathcal{D}}(\mathbb{I}_{\mathbf{Z}_p}).$$

Following (2.5), we define the modified p-adic section $f^{\bullet}_{\mathcal{D},s} := f_{\Phi^{\bullet}_{\mathcal{D}}}(\chi\hat{\omega}\boldsymbol{\alpha}^s_{\mathbf{Q}_p})$ for $\bullet \in \{\dagger, \dagger\dagger\}$. Then the local degenerate Whittaker functions for these modified p-adic sections are given by

$$\mathcal{W}_B(f^{\dagger}_{\mathcal{D},s}) = (\chi\omega_1)(2b_{23})\mathbb{I}_{\Xi^{\dagger}_p}(B), \qquad \mathcal{W}_B(f^{\dagger\dagger}_{\mathcal{D},s}) = \mathbb{I}_{\Xi^{\dagger\dagger}_{\mathfrak{p}}}(B),$$

for $B = (b_{ij}) \in \mathrm{Sym}_3(\mathbf{Q}_p)$, where

$$\Xi^{\dagger\dagger}_p := \{(b_{ij}) \in \mathrm{Sym}_3(\mathbf{Z}_p) \mid b_{11}, b_{22}, b_{33} \in p\mathbf{Z}_p\}, \qquad \Xi^{\dagger}_p := \left\{(b_{ij}) \in \Xi^{\dagger\dagger}_p \mid 2b_{23} \in \mathbf{Z}^{\times}_p\right\}.$$

With the preparation above we define the power series

$$\mathcal{G}^{\dagger}(T,X) = \sum_{B=(b_{ij})\in T_3^+\cap\Xi^{\dagger}_p} \langle 2b_{23}\rangle_T \langle 2b_{23}\rangle_X^{-1} \mathcal{F}^{(2)}_B(X,T,T,T)\cdot q_1^{b_{11}}q_2^{b_{22}}q_3^{b_{33}} \in \mathbf{Z}_p[\![T,X]\!][\![q_1,q_2,q_3]\!],$$

$$\mathcal{G}^{\dagger\dagger}(T) = \sum_{B\in T_3^+\cap\Xi^{\dagger\dagger}_p} \mathcal{F}^{(2)}_B(T,T,T,T)\cdot q_1^{b_{11}}q_2^{b_{22}}q_3^{b_{33}} \in \mathbf{Z}_p[\![T]\!][\![q_1,q_2,q_3]\!].$$

Notation is as in §6.1. For arithmetic points (Q,P) with $k_Q = 2$ we have

$$\mathcal{G}^{\dagger}(Q,P) = e_{\mathrm{ord}}\mathbf{E}^{[k_P,r,\lambda]}_{\mathcal{D}^{\dagger},N}(\tau, f^{\dagger}_{\mathcal{D}^{\dagger},s,N})|_{s=0}, \qquad \mathcal{G}^{\dagger\dagger}(P) = e_{\mathrm{ord}}\mathbf{E}^{[k_P,r,\lambda]}_{\mathcal{D}^{\dagger\dagger},N}(\tau, f^{\dagger\dagger}_{\mathcal{D}^{\dagger\dagger},s,N})|_{s=0}$$

with $\lambda = (0,0,0)$ and $r = \frac{k_P}{2} - 1$, where we have written

$$\mathcal{D}^{\dagger} := (\epsilon_P\boldsymbol{\omega}^{2-k_P}, \epsilon_Q^{-1}\boldsymbol{\omega}^{k_Q-2}, \epsilon_P^{-1}\boldsymbol{\omega}^{k_P-2}, \epsilon_P^{-1}\boldsymbol{\omega}^{k_P-2}),$$
$$\mathcal{D}^{\dagger\dagger} := (\epsilon_P\boldsymbol{\omega}^{2-k_P}, \epsilon_P^{-1}\boldsymbol{\omega}^{k_P-2}, \epsilon_P^{-1}\boldsymbol{\omega}^{k_P-2}, \epsilon_P^{-1}\boldsymbol{\omega}^{k_P-2}),$$
$$f^{\bullet}_{\mathcal{D}^{\bullet},s,N} := f^{[k_P,\lambda]}_{s,\infty} \otimes f^{\bullet}_{\mathcal{D}^{\bullet},s} \otimes f_{s,N} \otimes_{\ell\nmid Np} f^0_{s,\ell}.$$

As in Proposition 6.8 we can show that

$$\mathcal{G}^{\dagger}(T,X) \in \mathbf{S}^{\mathrm{ord}}(N,\boldsymbol{\omega}^2,\mathbf{Z}_p[\![X]\!])\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{S}^{\mathrm{ord}}(N,\boldsymbol{\omega}^2,\mathbf{Z}_p[\![T]\!]) \otimes_{\mathbf{Z}_p[\![T]\!]} \mathbf{S}^{\mathrm{ord}}(N,\boldsymbol{\omega}^2,\mathbf{Z}_p[\![T]\!]);$$
$$\mathcal{G}^{\dagger\dagger}(T) \in \mathbf{S}^{\mathrm{ord}}(N,\boldsymbol{\omega}^2,\mathbf{Z}_p[\![T]\!]) \otimes_{\mathbf{Z}_p[\![T]\!]} \mathbf{S}^{\mathrm{ord}}(N,\boldsymbol{\omega}^2,\mathbf{Z}_p[\![T]\!]) \otimes_{\mathbf{Z}_p[\![T]\!]} \mathbf{S}^{\mathrm{ord}}(N,\boldsymbol{\omega}^2,\mathbf{Z}_p[\![T]\!]).$$

Choose an element H_i in the congruence ideal of $\boldsymbol{f}_i$ with $H_i(\mathbf{u}^2-1) \neq 0$. We define the improved p-adic L-functions $L^{\dagger}_{\boldsymbol{F},(2)}(X,T)$ and $L^{\dagger\dagger}_{\boldsymbol{F},(2)}(T)$ as the first Fourier coefficients of

$$\mathbf{1}_{\boldsymbol{f}_1} \otimes \mathbf{1}_{\boldsymbol{f}_2} \otimes \mathbf{1}_{\boldsymbol{f}_3}(\mathrm{Tr}_{N/N_1}\otimes\mathrm{Tr}_{N/N_2}\otimes\mathrm{Tr}_{N/N_3}(\mathcal{G}^{\dagger})) \in \mathbf{Z}_p[\![X,T]\!][\frac{1}{H^{\dagger}}];$$
$$\mathbf{1}_{\boldsymbol{f}_1} \otimes \mathbf{1}_{\boldsymbol{f}_2} \otimes \mathbf{1}_{\boldsymbol{f}_3}(\mathrm{Tr}_{N/N_1}\otimes\mathrm{Tr}_{N/N_2}\otimes\mathrm{Tr}_{N/N_3}(\mathcal{G}^{\dagger\dagger})) \in \mathbf{Z}_p[\![T]\!][\frac{1}{H^{\dagger\dagger}}]$$

respectively, where $H^{\dagger} = t_1H_1(X)t_2H_2t_3H_3(T)$ and $H^{\dagger\dagger} = t_1H_1t_2H_2t_3H_3(T)$. Define

$$L^{\dagger}_p(x,s) := c_p\cdot\left\langle N^-N^4\right\rangle^{\frac{-x}{4}} L^{\dagger}_{\boldsymbol{F},(2)}(\mathbf{u}^{x+2}-1,\mathbf{u}^{s+2}-1), \qquad L^{\dagger\dagger}_p(s) := c_p\cdot\left\langle N^-N^4\right\rangle^{\frac{-s}{4}} L^{\dagger\dagger}_{\boldsymbol{F},(2)}(\mathbf{u}^{s+2}-1).$$

In view of the proof of Lemma 7.1, to prove the interpolation formulae for $L^{\dagger}_p(x,s)$ and $L^{\dagger\dagger}_p(s)$, we need to compute the quantity $Z^*_p(f^{\bullet}_{\mathcal{D},s})$ defined in (2.13) attached to our modified p-adic sections $f^{\bullet}_{\mathcal{D},s}$ as well as a subrepresentation π_i of the induced representation $I(\mu_i,\nu_i)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ with μ_i unramified for $i = 1,2,3$. Applying the computation in Proposition 2.3, we find that whenever $\chi\omega_2$ and $\chi\omega_3$ are unramified,

$$Z^*_p(f^{\dagger}_{\mathcal{D},s}) = Z^*_p(f_{\mathcal{D},s})\prod_{i=2,3} L\Big(\frac{1}{2}-s, \chi^{-1}\mu_1^{-1}\mu_i^{-1}\nu_{5-i}^{-1}\Big)$$

and that when $\chi\omega_i$ are unramified for $i = 1,2,3$,

$$Z^*_p(f^{\dagger\dagger}_{\mathcal{D},s}) = Z^*_p(f^{\dagger}_{\mathcal{D},s})L\Big(\frac{1}{2}-s, \chi^{-1}\nu_1^{-1}\mu_2^{-1}\mu_3^{-1}\Big).$$

From the proof of Theorem 7.6 we can deduce the interpolation formulae for the improved L-functions. The formula for $\mathcal{E}^+(s)$ follows from that for $Z_p^*(f_{\mathcal{D},s})$ proved in Proposition 2.6 and Remark 3.5.

Whenever $k > 2$, the central sign for $L(s, \pi_{f_1} \times \pi_{\boldsymbol{f}_{2,k}} \times \pi_{\boldsymbol{f}_{3,k}})$ is $\varepsilon_p(\boldsymbol{E})$. Therefore if $\varepsilon_p(\boldsymbol{E}) = -1$, then $L_p^\dagger(0, s) = 0$ by (2), which implies that $\frac{\partial L_p^\dagger}{\partial x}(0,0) = \lim_{s\to 0} \frac{L_p^\dagger(s,s)}{s}$. The second equality of (1) gives the expression of $\lim_{s\to 0} \frac{L_p^\dagger(s,s)}{s}$. We write

$$L_p(x,y,z,s) = \sum_{j=0}^\infty A_j(x,y,z)\Big(s - \frac{x+y+z}{2}\Big)^j.$$

If $i \le r := \mathrm{ord}_{s=2} L_p(\boldsymbol{E}, s)$, then

$$r = \min\{j \mid A_j(0,0,0) \neq 0\}, \qquad \lim_{s\to 2} \frac{L_p(\boldsymbol{E}, s)}{(s-2)^i} = A_i(0,0,0). \tag{8.2}$$

Letting $y = z = s = 0$, we see by (1) that the power series

$$\sum_{j=0}^\infty A_j(x,0,0)\Big(-\frac{x}{2}\Big)^j = (1 - \alpha_1 \mathbf{a}_1(x)^{-1})^2 L_p^\dagger(x,0)$$

has at least a double zero at $x = 0$. If $\varepsilon_p(\boldsymbol{E}) = -1$, then since $A_{2n}(x,y,z) = 0$ for all non-negative integers n by the functional equation (8.1), we get $A_1(0,0,0) = 0$ and $r \ge 3$. □

8.4. **The proof of Theorem 8.4(1).** We discuss Case (i). Then $\varepsilon_p(\boldsymbol{E}) = -\varepsilon(\boldsymbol{E})$ by Remark 8.1. First suppose that $\varepsilon(\boldsymbol{E}) = 1$. The functional equation (8.1) allows us to write

$$L_p(x,y,z,s) = A_1(x,y,z)\Big(s - \frac{x+y+z}{2}\Big) + A_3(x,y,z)\Big(s - \frac{x+y+z}{2}\Big)^3 + \cdots$$

The proof of Lemma 8.5(3) gives $A_1(0,0,0) = 0$. From (8.2) and Lemma 8.5(4) the formula boils down to

$$A_3(0,0,0) = -8\ell_1\ell_2\ell_3 L_p^{\dagger\dagger}(0).$$

If we denote the degree two term of $A_1(x,y,z)$ by $ax^2 + by^2 + cz^2 + dxy + eyz + fxz$, then the degree three term of $L_p(x,s,s,s)$ is given by

$$L^{(3)}(x,s) = \{ax^2 + (b+c+e)s^2 + (d+f)xs\}(-x/2) + A_3(0,0,0)(-x/2)^3.$$

On the other hand, from Lemma 8.5(1), (3) we find that

$$\begin{aligned} L^{(3)}(x,s) &= (\ell_1 x + (\ell_3 - \ell_2)s)\cdot(\ell_1 x + (\ell_2 - \ell_3)s)x \cdot \lim_{x\to 0} x^{-1} L_p^\dagger(x,0) \\ &= (\ell_1^2 x^2 - (\ell_2 - \ell_3)^2 s^2)x \cdot (\ell_2 + \ell_3 - \ell_1) L_p^{\dagger\dagger}(0). \end{aligned}$$

Comparing the coefficients of $x^2 s$, xs^2 and x^3, we obtain the relations

$$d + f = 0, \quad b + c + e = 2(\ell_2 - \ell_3)^2(\ell_2 + \ell_3 - \ell_1) L_p^{\dagger\dagger}(0), \quad 4a + A_3(0,0,0) = -8\ell_1^2(\ell_2 + \ell_3 - \ell_1) L_p^{\dagger\dagger}(0).$$

By symmetry we get

$$\begin{aligned} &d + e = 0, && e + f = 0; \\ &a + c + f = 2(\ell_1 - \ell_3)^2(\ell_1 + \ell_3 - \ell_2) L_p^{\dagger\dagger}(0), && a + b + d = 2(\ell_1 - \ell_2)^2(\ell_1 + \ell_2 - \ell_3) L_p^{\dagger\dagger}(0). \end{aligned}$$

From these equations we conclude that $d = e = f = 0$ and

$$\begin{aligned} &a = \{(\ell_1 - \ell_2)^2(\ell_1 + \ell_2 - \ell_3) + (\ell_1 - \ell_3)^2(\ell_1 + \ell_3 - \ell_2) - (\ell_2 - \ell_3)^2(\ell_2 + \ell_3 - \ell_1)\} L_p^{\dagger\dagger}(0), \\ &A_3(0,0,0) = -8\ell_1^2(\ell_2 + \ell_3 - \ell_1) L_p^{\dagger\dagger}(0) - 4a = -8\ell_1\ell_2\ell_3 L_p^{\dagger\dagger}(0). \end{aligned}$$

Next assume that $\varepsilon(\boldsymbol{E}) = -1$. Then $\varepsilon_p(\boldsymbol{E}) = 1$. By (8.1) and Lemma 8.5(1)

$$\sum_{n=0}^\infty A_{2n}(x,s,s)\Big(\frac{s}{2}\Big)^{2n} = \bigg(1 - \frac{\mathbf{a}_2(s)}{\mathbf{a}_1(x)\mathbf{a}_3(s)}\bigg)\bigg(1 - \frac{\mathbf{a}_3(s)}{\mathbf{a}_1(x)\mathbf{a}_2(s)}\bigg) L_p^\dagger(x,s).$$

Since $L_p^\ddagger(0,0)=0$, every term in the right hand side has degree at least three. In particular, the constant term $A_0(0,0,0)$ of the left hand side is zero. If we denote the degree two term of $A_0(x,y,z)$ by $\alpha x^2+\beta y^2+\gamma z^2+\xi xy+\eta yz+\zeta xz$, then the degree two term of the left hand side is

$$\alpha x^2+(\beta+\gamma+\eta)s^2+(\xi+\zeta)xs+A_2(0,0,0)(x/2)^2.$$

It is zero, and so by symmetry we get

$$\begin{aligned} &A_2(0,0,0)=-4\alpha, && \beta+\gamma+\eta=0, && \xi+\zeta=0;\\ &A_2(0,0,0)=-4\beta, && \alpha+\gamma+\zeta=0, && \xi+\eta=0;\\ &A_2(0,0,0)=-4\gamma, && \alpha+\beta+\xi=0, && \eta+\zeta=0. \end{aligned}$$

We arrive at $\xi=\eta=\zeta=\alpha=\beta=\gamma=A_2(0,0,0)=0$. Hence $\mathrm{ord}_{s=2}L_p(\boldsymbol{E},s)\geq 4$.

8.5. **The proof of Theorem 8.4(2).** We discuss Case (ii). Then $\varepsilon_p(\boldsymbol{E})=\varepsilon(\boldsymbol{E})$ by Remark 8.1. If $\varepsilon(\boldsymbol{E})=-1$, then $\mathrm{ord}_{s=2}L_p(\boldsymbol{E},s)\geq 3$ by Lemma 8.5(3), and both sides of the declared identity are zero. We will consider the case $\varepsilon(\boldsymbol{E})=1$, i.e. Σ^- has odd cardinality. Unlike Case (i) we cannot apply Lemma 8.5(3). Our proof relies on the three-variable p-adic triple product L-function in the balanced case constructed in [Hsi19].

Let D be the definite quaternion algebra over $\mathbf{Q}$ of discriminant N^- and $\mathbf{S}^D(N,\Lambda)$ the space of Λ-adic modular forms on $D^\times$ defined in [Hsi19, Definition 4.1]. Let $\boldsymbol{f}_i^D\in\mathbf{S}^D(N,\Lambda)[t_i^{-1}]$ be a Jacquet-Langlands lift of $\boldsymbol{f}_i$ in the sense of [Hsi19, §4.5]. Since we do not assume Hypothesis (CR,Σ^-) of [Hsi19, §1.4], we cannot choose $\boldsymbol{f}_i^D$ to be a primitive Jacquet-Langlands lift as in [Hsi19, Theorem 4.5]. Nonetheless, $\boldsymbol{f}_i^D$ can be chosen so that $\boldsymbol{f}_i^D(\mathbf{u}^2-1)$ is a non-zero Jacquet-Langlnads lift of f_i. Replacing the triple $\boldsymbol{F}^D=(\boldsymbol{f}_1^D,\boldsymbol{f}_2^D,\boldsymbol{f}_3^D)$ with the well-chosen test vectors in [Hsi19, Definition 4.8], we can associate to $\boldsymbol{F}^D$ the three-variable *theta element* $\Theta_{\boldsymbol{F}^D}(X_1,X_2,X_3)$ in *loc.cit.* Define an analytic function on $\mathcal{U}^3\subset\mathbf{Z}_p^3$ by

$$\Theta(x,y,z)=\Theta_{\boldsymbol{F}^D}(\mathbf{u}^{x+2}-1,\mathbf{u}^{y+2}-1,\mathbf{u}^{z+2}-1).$$

By the interpolation formula for $\Theta_{\boldsymbol{F}^D}$ in [Hsi19, Theorem 7.1] (see Remark 7.8), we can find an analytic function $H(x,y,z)$ with $H(0,0,0)\neq 0$ such that

$$H(x,y,z)\cdot\Theta(x,y,z)^2=L_p\Big(x,y,z,\frac{x+y+z}{2}\Big).$$

To proceed, we introduce two-variable *improved* theta elements.

Lemma 8.6 (Improved theta elements)**.** *There exist analytic functions $\Theta_2^\ddagger(x,z)$, $\Theta_3^\ddagger(x,y)$ such that*

$$\Theta_2^\ddagger(0,0)=-\Theta_3^\ddagger(0,0),$$

$$\Theta(x,x+z,z)=\left(1-\frac{\mathbf{a}_2(x+z)}{\mathbf{a}_1(x)\mathbf{a}_3(z)}\right)\Theta_2^\ddagger(x,z),\qquad \Theta(x,y,x+y)=\left(1-\frac{\mathbf{a}_3(x+y)}{\mathbf{a}_1(x)\mathbf{a}_2(y)}\right)\Theta_3^\ddagger(x,y).$$

Proof. The idea of the proof is similar to [Hsi19, Proposition 8.3]. We give a sketch of the proof here. For every integer n, let R_n be the Eichler order of level p^nN/N^- in D and let $X_0(p^nN)=D^\times\backslash\widehat{D}^\times/\widehat{R}_n^\times$, where $\widehat{D}=D\otimes\widehat{\mathbf{Q}}$ and $\widehat{R}_n=R_n\otimes\widehat{\mathbf{Z}}$. Through an isomorphism $R_0\otimes\mathbf{Z}_p\simeq\mathrm{M}_2(\mathbf{Z}_p)$ we define

$$U_1(p^n):=\left\{g\in\widehat{R}_n\;\middle|\;g_p\equiv\begin{pmatrix}*&*\\0&1\end{pmatrix}\ (\mathrm{mod}\ p^n)\right\}.$$

Recall that $\mathbf{a}_i(Q)=\mathbf{a}(p,\boldsymbol{f}_{i,Q})$ and that $\varpi_p\in\widehat{\mathbf{Q}}^\times$ is the element with $\varpi_{p,p}=p$ and $\varpi_{p,\ell}=1$ for $\ell\neq p$. For all but finitely many arithmetic points Q with $k_Q=2$, the specialization $\boldsymbol{f}_{i,Q}^D:D^\times\backslash\widehat{D}^\times/U_1(p^n)\to\mathbf{C}_p$ is a p-stabilized form on $\widehat{D}^\times$ with the same Hecke eigenvalues with $\boldsymbol{f}_{i,Q}$ and the central character ϵ_Q^{-1} : $\mathbf{Q}^\times\backslash\widehat{\mathbf{Q}}^\times/(1+p^n\widehat{\mathbf{Z}})^\times\to\mu_{p^\infty}$ for any sufficiently large n. In particular, $\boldsymbol{f}_{i,Q}^D$ is a $\mathbf{U}_p$-eigenform with eigenvalue $\mathbf{a}_i(Q)$. Namely,

$$\mathbf{U}_p\boldsymbol{f}_{i,Q}^D(g):=\sum_{b\in\mathbf{Z}_p/p^n\mathbf{Z}_p}\boldsymbol{f}_{i,Q}^D\left(g\begin{pmatrix}\varpi_p^n&b\\0&1\end{pmatrix}\right)=\mathbf{a}_i(Q)^n\boldsymbol{f}_{i,Q}^D(g),\quad g\in\widehat{D}^\times. \tag{8.3}$$

In what follows, we shall write $(\boldsymbol{f}^D,\boldsymbol{g}^D,\boldsymbol{h}^D)=(\boldsymbol{f}_1^D,\boldsymbol{f}_2^D,\boldsymbol{f}_3^D)$. Let $\mathrm{N}_D : D\to\mathbf{Q}$ be the reduced norm. Put $\tau_{p^n}=\begin{pmatrix}0&1\\-\varpi_p^n&0\end{pmatrix}\in\mathrm{GL}_2(\mathbf{Q}_p)\subset\widehat{D}^\times$. By definition,

$$\Theta(Q_1,Q_2,Q_3)=\mathbf{a}_1(Q_1)^{-n}\mathbf{a}_2(Q_2)^{-n}\mathbf{a}_3(Q_3)^{-n} \tag{8.4}$$
$$\sum_{[a]\in X_0(p^nN)}\sum_{\substack{b\in\mathbf{Z}_p/p^n\mathbf{Z}_p\\ c\in(\mathbf{Z}_p/p^n\mathbf{Z}_p)^\times}}\boldsymbol{f}^D_{Q_1}\left(a\begin{pmatrix}\varpi_p^n&b\\0&1\end{pmatrix}\right)\boldsymbol{g}^D_{Q_2}\left(a\begin{pmatrix}\varpi_p^n&b+c\\0&1\end{pmatrix}\right)\boldsymbol{h}^D_{Q_3}(a\tau_{p^n})\epsilon^{\frac12}_{Q_1Q_2Q_3^{-1}}(c)\epsilon^{\frac12}_{Q_1Q_2Q_3}(\mathrm{N}_D(a)).$$

We replace the twisted diagonal cycle Δ_n in [Hsi19, Definition 4.6] by the *improved* diagonal cycle

$$\Delta_n^\ddagger:=\sum_{[a]\in X_0(Np^n)}\sum_{b\in\mathbf{Z}_p/p^n\mathbf{Z}_p}\left[\left(a\begin{pmatrix}\varpi_p^n&b\\0&1\end{pmatrix},a\tau_{p^n},a\right)\right].$$

We can define the regularized improved diagonal cycle by

$$\Delta_\infty^\ddagger:=\varprojlim_{n\to\infty}(\mathbf{U}_p^{-n}\otimes\mathbf{U}_p^{-n}\otimes 1)e_E(\Delta_n^\ddagger),$$

and the improved theta element

$$\Theta_2^\ddagger(X_1,X_3):=(\boldsymbol{F}^D)^*(\Delta_\infty^\ddagger)(X_1,(1+X_1)(1+X_3)-1,X_3)\in\mathbf{Z}_p[\![X_1,X_3]\!][t^{-1}].$$

for $t=t_1\cdot t_2((1+X_1)(1+X_3)-1)\cdot t_3$. Put $\Theta_2^\ddagger(x,z):=\Theta_2^\ddagger(\mathbf{u}^{x+2}-1,\mathbf{u}^{z+2}-1)$ for $(x,z)\in\mathcal{U}^2$. By definition and (8.3), for all but finitely many arithmetic points (Q_1,Q_3) with $k_{Q_1}=k_{Q_3}=2$

$$\Theta_2^\ddagger(Q_1,Q_3)=\mathbf{a}_2(Q_1Q_3)^{-n}\sum_{[a]\in X_0(Np^n)}\boldsymbol{f}^D_{Q_1}(a)\boldsymbol{g}^D_{Q_1Q_3}(a\tau_{p^n})\boldsymbol{h}^D_{Q_3}(a)\epsilon_{Q_1Q_3}(\mathrm{N}_D(a)).$$

The above expression holds for any n such that p^n is bigger than the conductors of ϵ_{Q_1} and ϵ_{Q_2}. Likewise we can define $\Theta_3^\ddagger\in\mathbf{Z}_p[\![X_1,X_2]\!]$ and $\Theta_3^\ddagger(x,y)$ with the interpolation property:

$$\Theta_3^\ddagger(Q_1,Q_2)=\mathbf{a}_3(Q_1Q_2)^{-n}\sum_{[a]\in X_0(Np^n)}\boldsymbol{f}^D_{Q_1}(a\tau_{p^n})\boldsymbol{g}^D_{Q_2}(a)\boldsymbol{h}^D_{Q_1Q_2}(a)\epsilon_{Q_1Q_2}(\mathrm{N}_D(a)).$$

To see the first relation, we note that

$$\Theta_2^\ddagger(0,0)=\alpha_2^{-1}\sum_{[a]\in X_0(Np)}\boldsymbol{f}_0^D(a)\boldsymbol{g}_0^D(a\tau_p)\boldsymbol{h}_0^D(a),\qquad \Theta_3^\ddagger(0,0)=\alpha_3^{-1}\sum_{[a]\in X_0(Np)}\boldsymbol{f}_0^D(a)\boldsymbol{g}_0^D(a)\boldsymbol{h}_0^D(a\tau_p).$$

Since $\boldsymbol{f}_0^D$ is a newform that is special at p, $\boldsymbol{f}_0^D(x\tau_p)=(-\alpha_1)\boldsymbol{f}_0^D(x)$, and hence $\Theta_2^\ddagger(0,0)=-\Theta_3^\ddagger(0,0)$.

To prove the last relation, it suffices to verify the following equation

$$\Theta(Q_1,Q_1Q_3,Q_3)=\left(1-\frac{\mathbf{a}_2(Q_1Q_3)}{\mathbf{a}_1(Q_1)\mathbf{a}_3(Q_3)}\right)\Theta_2^\ddagger(Q_1,Q_3) \tag{8.5}$$

for all but finitely many arithmetic poitns (Q_1,Q_3) with $k_{Q_1}=k_{Q_3}=2$. The formula for $\Theta_3^\ddagger$ can be done by a similar computation, so we leave it to the reader. Let n be a sufficiently large integer. From (8.4), we get

$$\begin{aligned}&\mathbf{a}_1(Q_1)^n\mathbf{a}_2(Q_1Q_3)^n\mathbf{a}_3(Q_3)^np^{-n}\operatorname{vol}(\widehat{R}_n^\times)\Theta(Q_1,Q_1Q_3,Q_3)\\ =&\int_{D^\times\backslash\widehat{D}^\times}\mathrm{d}^\times a\sum_{c\in(\mathbf{Z}_p/p^n\mathbf{Z}_p)^\times}\boldsymbol{f}^D_{Q_1}\left(a\begin{pmatrix}1&-c\varpi_p^{-n}\\0&1\end{pmatrix}\right)\boldsymbol{g}^D_{Q_1Q_3}(a)\boldsymbol{h}^D_{Q_3}\left(a\tau_{p^n}\begin{pmatrix}1&0\\0&\varpi_p^{-n}\end{pmatrix}\right)\epsilon_{Q_1}(c)\epsilon_{Q_1Q_3}(\mathrm{N}_D(a))\\ =&\int_{D^\times\backslash\widehat{D}^\times}\mathrm{d}^\times a\sum_{c\in(\mathbf{Z}_p/p^n\mathbf{Z}_p)^\times}\boldsymbol{f}^D_{Q_1}\left(a\tau_{p^n}\begin{pmatrix}1&-\varpi_p^{-n}\\0&c^{-1}\end{pmatrix}\right)\boldsymbol{g}^D_{Q_1Q_3}(a\tau_{p^n})\boldsymbol{h}^D_{Q_3}\left(a\begin{pmatrix}1&0\\0&\varpi_p^{-n}\end{pmatrix}\right)\epsilon_{Q_1Q_3}(\mathrm{N}_D(a))\end{aligned}$$

by change of variables. From the equations $\tau_{p^n}\begin{pmatrix}1 & -\varpi_p^{-n}\\ 0 & c^{-1}\end{pmatrix}=\begin{pmatrix}\varpi_p^n & c^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}c^{-1} & 0\\ -\varpi_p^n & 1\end{pmatrix}$, $\epsilon_{Q_1}(\varpi_p)=\epsilon_{Q_3}(\varpi_p)=1$ and (8.3), the last integral equals

$$\begin{aligned}
&\int_{D^\times\backslash\widehat{D}^\times}\mathrm{d}^\times a\sum_{c\in(\mathbf{Z}_p/p^n\mathbf{Z}_p)^\times}\boldsymbol{f}^D_{Q_1}\left(a\begin{pmatrix}\varpi_p^n & c\\ 0 & 1\end{pmatrix}\right)\boldsymbol{g}^D_{Q_1Q_3}(a\tau_{p^n})\boldsymbol{h}^D_{Q_3}\left(a\begin{pmatrix}\varpi_p^n & 0\\ 0 & 1\end{pmatrix}\right)\epsilon_{Q_1Q_3}(\mathrm{N}_D(a))\\
=&\int_{D^\times\backslash\widehat{D}^\times}\mathrm{d}^\times a\,\mathbf{a}_1(Q_1)^n\cdot\left\{\boldsymbol{f}^D_{Q_1}(a)-\mathbf{a}_1(Q_1)^{-1}\boldsymbol{f}^D_{Q_1}\left(a\begin{pmatrix}\varpi_p & 0\\ 0 & 1\end{pmatrix}\right)\right\}\boldsymbol{g}^D_{Q_1Q_3}(a\tau_{p^n})\boldsymbol{h}^D_{Q_3}\left(a\begin{pmatrix}\varpi_p^n & 0\\ 0 & 1\end{pmatrix}\right)\epsilon_{Q_1Q_3}(\mathrm{N}_D(a))\\
=&\mathbf{a}_1(Q_1)^n\int_{D^\times\backslash\widehat{D}^\times}\mathrm{d}^\times a\,\boldsymbol{f}^D_{Q_1}(a)\boldsymbol{g}^D_{Q_1Q_3}(a\tau_{p^n})p^{-n}\sum_{b\in\mathbf{Z}_p/p^n\mathbf{Z}_p}\boldsymbol{h}^D_{Q_3}\left(a\begin{pmatrix}\varpi_p^n & b\\ 0 & 1\end{pmatrix}\right)\epsilon_{Q_1Q_3}(\mathrm{N}_D(a))\\
&-\mathbf{a}_1(Q_1)^{n-1}\int_{D^\times\backslash\widehat{D}^\times}\mathrm{d}^\times a\,\boldsymbol{f}^D_{Q_1}(a)\boldsymbol{g}^D_{Q_1Q_3}(a\tau_{p^{n+1}})p^{-(n-1)}\sum_{b\in\mathbf{Z}_p/p^{n-1}\mathbf{Z}_p}\boldsymbol{h}^D_{Q_3}\left(a\begin{pmatrix}\varpi_p^{n-1} & b\\ 0 & 1\end{pmatrix}\right)\epsilon_{Q_1Q_3}(\mathrm{N}_D(a))\\
=&\{(\mathbf{a}_1(Q_1)\mathbf{a}_3(Q_3)\mathbf{a}_2(Q_1Q_3)/p)^n\operatorname{vol}(\widehat{R}_n^\times)-(\mathbf{a}_1(Q_1)\mathbf{a}_3(Q_3)/p)^{n-1}\mathbf{a}_2(Q_1Q_3)^{n+1}\operatorname{vol}(\widehat{R}_{n+1}^\times)\}\Theta_2^\ddagger(Q_1,Q_3)\\
=&\mathbf{a}_1(Q_1)^n\mathbf{a}_3(Q_3)^n\mathbf{a}_2(Q_1Q_3)^n\left(1-\frac{\mathbf{a}_2(Q_1Q_3)}{\mathbf{a}_1(Q_1)\mathbf{a}_3(Q_3)}\right)p^{-n}\operatorname{vol}(\widehat{R}_n^\times)\Theta_2^\ddagger(Q_1,Q_3).
\end{aligned}$$

This verifies (8.5). □

Now we return to the proof of Theorem 8.4(2). Write Θ_x for the partial derivative $\frac{\partial\Theta}{\partial x}$. Put

$$a=\Theta_x(0,0,0),\qquad b=\Theta_y(0,0,0),\qquad c=\Theta_z(0,0,0).$$

Taking derivatives $\Theta(x,y,x+y)$ with respect to x and y at $(0,0)$ in Lemma 8.6, we have

$$a+c=(\ell_1-\ell_3)\Theta_3^\ddagger(0,0),\qquad b+c=(\ell_2-\ell_3)\Theta_3^\ddagger(0,0).$$

Similarly, we have

$$a+b=(\ell_1-\ell_2)\Theta_2^\ddagger(0,0)=(\ell_2-\ell_1)\Theta_3^\ddagger(0,0).$$

These imply that

$$a=0,\qquad b=(\ell_2-\ell_1)\Theta_3^\ddagger(0,0),\qquad c=(\ell_1-\ell_3)\Theta_3^\ddagger(0,0).$$

On the other hand, by the functional equation (8.1) we obtain the Taylor expansion

$$L_p(x,y,z,s)=H(x,y,z)\Theta(x,y,z)^2+A_2(x,y,z)\cdot\left(s-\frac{x+y+z}{2}\right)^2+\cdots.$$

By Lemma 8.5(1), we find

$$(1-\alpha_1\mathbf{a}_1(x)^{-1})^2L_p^\dagger(x,0)=H(x,0,0)\Theta(x,0,0)^2+A_2(x,0,0)\cdot x^2/4.$$

From the vanishing of $\Theta_x(0,0,0)$ we deduce that

$$A_2(0,0,0)=4\ell_1^2L_p^\dagger(0,0).$$

Lemma 8.5(2) and (8.2) complete the proof of Theorem 8.4(2).

References

[BDJ17] Daniel Barrera, Mladen Dimitrov, and Andrei Jorza, *p-adic L-functions of Hilbert cusp forms and the trivial zero conjecture*, arXiv e-prints (2017), arXiv:1709.08105.

[Ben11] Denis Benois, *A generalization of Greenberg's $\mathscr{L}$-invariant*, Amer. J. Math. **133** (2011), no. 6, 1573–1632. MR 2863371

[BP06] Siegfried Böcherer and Alexei A. Panchishkin, *Admissible p-adic measures attached to triple products of elliptic cusp forms*, Doc. Math. (2006), no. Extra Vol., 77–132. MR 2290585

[BP09] ———, *p-adic interpolation for triple L-functions: analytic aspects*, Automorphic forms and L-functions II. Local aspects, Contemp. Math., vol. 489, Amer. Math. Soc., Providence, RI, 2009, pp. 1–39.

[BSDGP96] Katia Barré-Sirieix, Guy Diaz, François Gramain, and Georges Philibert, *Une preuve de la conjecture de Mahler-Manin*, Invent. Math. **124** (1996), no. 1-3, 1–9. MR 1369409

[Cas73] W. Casselman, *On some results of Atkin and Lehner*, Math. Ann. **201** (1973), 301–314.

[Coa89a] John Coates, *On p-adic L-functions*, Astérisque (1989), no. 177-178, Exp. No. 701, 33–59, Séminaire Bourbaki, Vol. 1988/89. MR 1040567

[Coa89b] ———, *On p-adic L-functions attached to motives over* **Q** *II*, Boletim da Sociedade Brasileira de Matemática - Bulletin/Brazilian Mathematical Society **20** (1989), no. 1, 101–112.

[CPR89] John Coates and Bernadette Perrin-Riou, *On p-adic L-functions attached to motives over* **Q**, Algebraic number theory, Adv. Stud. Pure Math., vol. 17, Academic Press, Boston, MA, 1989, pp. 23–54.

[Del79] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, With an appendix by N. Koblitz and A. Ogus, pp. 313–346.

[Fon94] Jean-Marc Fontaine, *Représentations l-adiques potentiellement semi-stables*, Astérisque (1994), no. 223, 321–347, Périodes p-adiques (Bures-sur-Yvette, 1988). MR 1293977

[Gar87] Paul B. Garrett, *Decomposition of Eisenstein series: Rankin triple products*, Ann. of Math. (2) **125** (1987), no. 2, 209–235.

[GH93] Paul B. Garrett and Michael Harris, *Special values of triple product L-functions*, Amer. J. Math. **115** (1993), no. 1, 161–240.

[GK92] Benedict H. Gross and Stephen S. Kudla, *Heights and the central critical values of triple product L-functions*, Compositio Math. **81** (1992), no. 2, 143–209.

[Gre94a] Ralph Greenberg, *Iwasawa theory and p-adic deformations of motives*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 193–223. MR 1265554

[Gre94b] ———, *Trivial zeros of p-adic L-functions*, p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), Contemp. Math., vol. 165, Amer. Math. Soc., Providence, RI, 1994, pp. 149–174. MR 1279608

[GS93] Ralph Greenberg and Glenn Stevens, *p-adic L-functions and p-adic periods of modular forms*, Invent. Math. **111** (1993), no. 2, 407–447.

[GS16] Matthew Greenberg and Marco Adamo Seveso, *Triple product p-adic L-functions for balanced weights*, Mathematische Annalen, to appear (2016), `arXiv`:1506.05681.

[Hid88a] Haruzo Hida, *Modules of congruence of Hecke algebras and L-functions associated with cusp forms*, Amer. J. Math. **110** (1988), no. 2, 323–382. MR 935010 (89i:11058)

[Hid88b] ———, *On p-adic hecke algebras for GL(2) over totally real fields*, Annals of Mathematics **128** (1988), 295–384.

[Hid93] ———, *Elementary theory of L-functions and Eisenstein series*, London Mathematical Society Student Texts, vol. 26, Cambridge University Press, Cambridge, 1993.

[Hid94] ———, *On the critical values of L-functions of* $\mathrm{GL}(2)$ *and* $\mathrm{GL}(2)\times\mathrm{GL}(2)$, Duke Math. J. **74** (1994), no. 2, 431–529.

[Hid16] ———, *Arithmetic of adjoint L-values*, p-adic aspects of modular forms, World Sci. Publ., Hackensack, NJ, 2016, pp. 185–236.

[Hsi19] Ming-Lun Hsieh, *Hida families and p-adic triple product L-functions*, American Journal of Mathematics, to appear (2019), `arXiv`:1705.02717.

[Ich08] Atsushi Ichino, *Trilinear forms and the central values of triple product L-functions*, Duke Math. J. **145** (2008), no. 2, 281–307.

[Ike89] Tamotsu Ikeda, *On the functional equations of the triple L-functions*, J. Math. Kyoto Univ. **29** (1989), no. 2, 175–219.

[Ike92] ———, *On the location of poles of the triple L-functions*, Compositio Math. **83** (1992), no. 2, 187–237.

[Kob13] Shinichi Kobayashi, *The p-adic Gross-Zagier formula for elliptic curves at supersingular primes*, Invent. Math. **191** (2013), 527–629.

[Miz90] Shin-ichiro Mizumoto, *Integrality of critical values of triple product L-functions*, Analytic number theory (Tokyo, 1988), Lecture Notes in Math., vol. 1434, Springer, Berlin, 1990, pp. 188–195. MR 1071755

[Mok09] Chung Pang Mok, *The exceptional zero conjecture for Hilbert modular forms*, Compos. Math. **145** (2009), no. 1, 1–55. MR 2480494

[MTT86] B. Mazur, J. Tate, and J. Teitelbaum, *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. Math. **84** (1986), no. 1, 1–48.

[MW86] B. Mazur and A. Wiles, *On p-adic analytic families of Galois representations*, Compositio Math. **59** (1986), no. 2, 231–264.

[Oht00] Masami Ohta, *Ordinary p-adic étale cohomology groups attached to towers of elliptic modular curves. II*, Math. Ann. **318** (2000), no. 3, 557–583. MR 1800769

[Orl87] Tobias Orloff, *Special values and mixed weight triple products (with an appendix by Don Blasius)*, Invent. Math. **90** (1987), no. 1, 169–188.

[PSR87] I. Piatetski-Shapiro and Stephen Rallis, *Rankin triple L functions*, Compositio Math. **64** (1987), no. 1, 31–115. MR 911357

[Ram00] Dinakar Ramakrishnan, *Modularity of the Rankin-Selberg L-series, and multiplicity one for* $\mathrm{SL}(2)$, Ann. of Math. (2) **152** (2000), no. 1, 45–111. MR 1792292

[Shi81] Goro Shimura, *Arithmetic of differential operators on symmetric domains*, Duke Math. J. **48** (1981), no. 4, 813–843.

[Shi82] ———, *Confluent hypergeometric functions on tube domains*, Math. Ann. **260** (1982), no. 3, 269–302.

[Shi97] ———, *Euler products and Eisenstein series*, CBMS Regional Conference Series in Mathematics, vol. 93, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1997.

[Spi14] Michael Spieß, *On special zeros of p-adic L-functions of Hilbert modular forms*, Invent. Math. **196** (2014), no. 1, 69–138. MR 3179573

[Tat79] J. Tate, *Number theoretic background*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 3–26. MR 546607

[Tay04] Richard Taylor, *Galois representations*, Ann. Fac. Sci. Toulouse Math. (6) **13** (2004), no. 1, 73–119. MR 2060030

[Wil88] A. Wiles, *On ordinary λ-adic representations associated to modular forms*, Invent. Math. **94** (1988), no. 3, 529–573.

[Wil95] ______, *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. (2) **141** (1995), no. 3, 443–551.

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