

GENERIC TORUS ORBIT CLOSURES IN SCHUBERT VARIETIES

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EUNJEONG LEE AND MIKIYA MASUDA

ABSTRACT. The closure of a generic torus orbit in the flag variety G/B of type A_{n-1} is known to be a permutohedral variety and well studied. In this paper we introduce the notion of a generic torus orbit in the Schubert variety X_w ($w \in \mathfrak{S}_n$) and study its closure Y_w . We identify the maximal cone in the fan of Y_w corresponding to a fixed point uB ($u \leq w$), associate a graph $\Gamma_w(u)$ to each $u \leq w$, and show that Y_w is smooth at uB if and only if $\Gamma_w(u)$ is a forest. We also introduce a polynomial $A_w(t)$ for each w , which agrees with the Eulerian polynomial when w is the longest element of \mathfrak{S}_n , and show that the Poincaré polynomial of Y_w agrees with $A_w(t^2)$ when Y_w is smooth.

1. INTRODUCTION

Let $G = \mathrm{GL}_n(\mathbb{C})$, $B \subset G$ the Borel subgroup of upper triangular matrices, and $T \subset B$ the torus subgroup of diagonal matrices. The left multiplication by T on G induces the T -action on the flag variety G/B . The set of T -fixed points in G/B bijectively corresponds to the symmetric group \mathfrak{S}_n on the set $\{1, 2, \dots, n\}$.

Let \mathcal{O} be a T -orbit and $\overline{\mathcal{O}}$ its closure. It is known that $\overline{\mathcal{O}}$ is normal ([10, Proposition 4.8]), so $\overline{\mathcal{O}}$ is a toric variety. When \mathcal{O} is *generic*, which means that the closure $\overline{\mathcal{O}}$ contains all the T -fixed points in G/B , $\overline{\mathcal{O}}$ is known to be the permutohedral variety of complex dimension $n - 1$. The maximal cones in the fan of the permutohedral variety are Weyl chambers in type A_{n-1} . The permutohedral variety appears in many mathematics and is well studied (see, for instance, [1], [13], [16], [18], and [20]). It is smooth and its Poincaré polynomial is given by $A_n(t^2)$ where $A_n(t)$ denotes the Eulerian polynomial associated to \mathfrak{S}_n .

For an element $w \in \mathfrak{S}_n$ we denote by X_w the *Schubert variety* $\overline{BwB/B}$ in G/B . The T -action on G/B leaves X_w invariant. We say that a T -orbit \mathcal{O} in X_w is *generic in X_w* if the closure $\overline{\mathcal{O}}$ contains all the T -fixed points in X_w . Here the T -fixed points in X_w are uB 's for $u \leq w$ in Bruhat order. When w is the longest element w_0 of \mathfrak{S}_n , our notion of *generic in X_w* agrees with the *generic* mentioned in the above paragraph but otherwise different.

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In this paper we study the closure, denoted by Y_w , of a generic T -orbit in X_w . First we identify maximal cones in the fan of Y_w . To each $v \in \mathfrak{S}_n$ we associate a cone in \mathbb{R}^n :

$$C(v) := \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid a_{v(1)} \leq a_{v(2)} \leq \dots \leq a_{v(n)}\}.$$

These cones projected on the quotient vector space $\mathbb{R}^n/\mathbb{R}(1, \dots, 1)$ are exactly the maximal cones in the fan of the permutohedral variety Y_{w_0} . To describe the maximal cones in the fan of Y_w , we introduce an operation $v \rightarrow v'$ on \mathfrak{S}_n (depending on w), which satisfies the following properties:

$$v' \leq w \text{ for any } v, \quad v' = v \text{ if } v \leq w, \quad \text{and} \quad w'_0 = w.$$

(See Section 3 for the definition of the operation.) With this understood we have

Theorem 1.1 (Corollary 3.7). *The maximal cone in the fan of Y_w corresponding to a fixed point uB in Y_w is the union of $C(v)$'s with $v' = u$ projected on the quotient vector space \mathbb{R}^n/F_w , where the linear subspace F_w of \mathbb{R}^n is determined by the subtorus of T which fixes Y_w pointwise (see Remark 3.1 for F_w).*

Using this description of the maximal cones (in fact, considering their dual cones), we obtain a criterion of smoothness of Y_w in terms of graphs. Indeed, we associate a graph $\Gamma_w(u)$ to each $u \leq w$ and prove the following.

Theorem 1.2 (Corollary 7.12). *The generic torus orbit closure Y_w in the Schubert variety X_w is smooth at a fixed point uB in Y_w if and only if the graph $\Gamma_w(u)$ is a forest. Therefore, Y_w is smooth if and only if $\Gamma_w(u)$ is a forest for every $u \leq w$.*

To our surprise, the graph $\Gamma_w(u)$ with $u = w$, abbreviated as Γ_w , has been studied in [7] (see also [24]) and the following three conditions are shown to be equivalent:

- (1) Γ_w is a forest.
- (2) w avoids the patterns 4231 and 45 $\bar{3}$ 12 (see Section 6).
- (3) X_w is factorial.

Looking at examples of $\Gamma_w(u)$, we conjecture that if Γ_w is a forest, then so is $\Gamma_w(u)$ for any $u \leq w$, in other words, Y_w is smooth if Y_w is smooth at wB (Conjecture 7.16). One can also see that the fixed point idB in Y_w , where id denotes the identity permutation in \mathfrak{S}_n , is smooth in Y_w for any $w \in \mathfrak{S}_n$ (Corollary 7.12). These are in sharp contrast with the Schubert variety X_w because the fixed point wB is smooth in X_w for any $w \in \mathfrak{S}_n$ and X_w is smooth if and only if idB is smooth in X_w (see [5, p. 208]). This sharp contrast shows that if X_w is a toric variety, i.e. $X_w = Y_w$, then X_w is smooth because it is smooth at idB . Moreover, the smoothness at idB implies the result in [17] that X_w is a toric variety if and only if w is a product of distinct simple reflections (Corollary 7.14).

There is a moment map $\mu: G/B \rightarrow \mathbb{R}^n$ whose image is the convex hull of points $(u^{-1}(1), \dots, u^{-1}(n))$ in \mathbb{R}^n for all $u \in \mathfrak{S}_n$, that is, the permutohedron of dimension $n - 1$. It follows from [2, Theorem 2] that $\mu(Y_w)$ is the convex hull of points $(u^{-1}(1), \dots, u^{-1}(n))$ in \mathbb{R}^n for all $u \leq w$, which is the Bruhat interval polytope $Q_{id, w^{-1}}$ in [22]. One can see that the graph $\Gamma_w(u)$ is a forest for every $u \leq w$ if and only if the polytope $Q_{id, w^{-1}}$ is simple. Our conjecture is equivalent to saying that the polytope $Q_{id, w^{-1}}$ is simple if $\mu(wB) = (w^{-1}(1), \dots, w^{-1}(n))$ is a simple vertex.

We also study the Poincaré polynomial of Y_w . We introduce a polynomial $A_w(t)$ for each w in a purely combinatorial way. The polynomial $A_w(t)$ agrees with the

Eulerian polynomial $A_n(t)$ when $w = w_0$, so the following theorem generalizes the known result on the Poincaré polynomial of the permutohedral variety Y_{w_0} .

Theorem 1.3 (Theorem 8.3). *If Y_w is smooth, then the Poincaré polynomial of Y_w agrees with $A_w(t^2)$ and hence the polynomial $A_w(t)$ is palindromic and unimodal.*

When $w = 4231$ or 3412 , Y_w is singular and $A_w(t)$ is not palindromic but $A_w(t^2)$ still agrees with the Poincaré polynomial of Y_w , so it would be interesting to ask whether $A_w(t^2)$ agrees with the Poincaré polynomial of Y_w for any $w \in \mathfrak{S}_n$ and palindromicity of $A_w(t)$ implies smoothness of Y_w .

This paper is organized as follows. In Section 2, we discuss about generic orbits and generic points in the Schubert variety X_w . In Section 3 we introduce the operation on \mathfrak{S}_n and prove Theorem 1.1. In Section 4 we show that the set $\{v \in \mathfrak{S}_n \mid v' = u\}$ for $u \leq w$ is an interval of the right weak Bruhat order. In Section 5 we identify the cone dual to the maximal cone in the fan of Y_w corresponding to wB . In Section 6 we associate the graph Γ_w to the dual cone and discuss simpliciality of the dual cone. In addition, we prove that simpliciality and smoothness are same in our case. In Section 7 we discuss the smoothness of Y_w at the other fixed points uB ($u < w$). Indeed, we identify the cone dual to the maximal cone in the fan of Y_w corresponding to uB , introduce the graph $\Gamma_w(u)$, and prove Theorem 1.2. We also discuss when X_w is a toric variety. In Section 8 we introduce the polynomial $A_w(t)$ and prove Theorem 1.3. In Appendix A we give an alternative proof to the pattern avoidance criterion of when Γ_w is a forest. In Appendix B we compute Poincaré polynomials of Y_w when $w = 4231$ and 3412 using retraction sequences of polytopes.

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2. GENERIC POINTS IN SCHUBERT VARIETIES

Let $G = \mathrm{GL}_n(\mathbb{C})$, $B \subset G$ the Borel subgroup of upper triangular matrices and $T \subset B$ the torus subgroup of diagonal matrices. Let \mathfrak{S}_n be the symmetric group on the set $\{1, 2, \dots, n\}$. An element w of \mathfrak{S}_n defines a permutation matrix $[e_{w(1)} \ \cdots \ e_{w(n)}]$ where e_1, \dots, e_n denote the standard column vectors in \mathbb{R}^n . Through this correspondence, we think of an element of \mathfrak{S}_n as an element of G . For an element $w \in \mathfrak{S}_n$ we denote the *Schubert variety* $\overline{BwB/B}$ in the flag variety G/B by X_w . The left multiplication by T on G induces the T -action on G/B which leaves X_w invariant. The set of T -fixed points in G/B bijectively corresponds to the symmetric group \mathfrak{S}_n through the map $v \in \mathfrak{S}_n \rightarrow vB \in G/B$ and vB lies in X_w if and only if $v \leq w$ in Bruhat order (see [14, §10.5]).

Definition 2.1. We call a T -orbit in X_w *generic* if its closure contains all the T -fixed points in X_w and call a point in X_w *generic* if it is in a generic T -orbit. We will denote the closure of a generic T -orbit in X_w by Y_w .

We describe some generic points in the Schubert variety X_w using the Plücker coordinates. To introduce the Plücker coordinate, we define the set

$$I_{d,n} = \{\underline{i} = (i_1, \dots, i_d) \in \mathbb{Z}^d \mid 1 \leq i_1 < \dots < i_d \leq n\}.$$

For an element $x = (x_{ij}) \in G = \mathrm{GL}_n(\mathbb{C})$, the \underline{i} th Plücker coordinate $p_{\underline{i}}(x)$ of x is given by the $d \times d$ minor of x , with row indices i_1, \dots, i_d and the column indices $1, \dots, d$ for $\underline{i} = (i_1, \dots, i_d) \in I_{d,n}$. Then the Plücker embedding ψ is defined to be

$$(2.1) \quad \psi: G/B \rightarrow \prod_{d=1}^{n-1} \mathbb{C}P^{\binom{n}{d}-1}, \quad xB \mapsto \prod_{d=1}^{n-1} (p_{\underline{i}}(x))_{\underline{i} \in I_{d,n}}.$$

The map ψ is T -equivariant with respect to the action of T on $\prod_{d=1}^{n-1} \mathbb{C}P^{\binom{n}{d}-1}$ given by

$$(t_1, \dots, t_n) \cdot (p_{\underline{i}})_{\underline{i} \in I_{d,n}} := (t_{i_1} \cdots t_{i_d} \cdot p_{\underline{i}})_{\underline{i} \in I_{d,n}}$$

for $(t_1, \dots, t_n) \in T$ and $\underline{i} = (i_1, \dots, i_d)$.

Example 2.2. Suppose that $G = \mathrm{GL}_3(\mathbb{C})$. Then the Plücker embedding $\psi: G/B \rightarrow \mathbb{C}P^{\binom{3}{1}-1} \times \mathbb{C}P^{\binom{3}{2}-1}$ maps an element $x = (x_{ij}) \in \mathrm{GL}_3(\mathbb{C})$ to

$$\begin{aligned} & ([p_1(x), p_2(x), p_3(x)], [p_{1,2}(x), p_{1,3}(x), p_{2,3}(x)]) \\ & = ([x_{11}, x_{21}, x_{31}], [x_{11}x_{22} - x_{21}x_{12}, x_{11}x_{32} - x_{31}x_{12}, x_{21}x_{32} - x_{31}x_{22}]). \end{aligned}$$

Since the action of T on $\mathrm{GL}_3(\mathbb{C})$ is given by

$$(t_1, t_2, t_3) \cdot \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} t_1x_{11} & t_1x_{12} & t_1x_{13} \\ t_2x_{21} & t_2x_{22} & t_2x_{23} \\ t_3x_{31} & t_3x_{32} & t_3x_{33} \end{pmatrix},$$

one can easily check that the map ψ is T -equivariant.

Given $\underline{i} = (i_1, \dots, i_d), \underline{j} = (j_1, \dots, j_d) \in I_{d,n}$, define a partial order \geq on $I_{d,n}$ by

$$(2.2) \quad \underline{i} \geq \underline{j} \iff i_t \geq j_t \text{ for all } 1 \leq t \leq d.$$

With this partial order, it is known from [5, Theorem 3.2.10] that the ideal sheaf of the Schubert variety X_w is generated by $\{p_{\underline{i}} \mid \underline{i} \in I_w\}$ where

$$I_w := \bigcup_{1 \leq d \leq n-1} \{\underline{i} \in I_{d,n} \mid w^{(d)} \not\geq \underline{i}\}.$$

Here $w^{(d)}$ denotes the ordered d -tuple obtained from $\{w(1), \dots, w(d)\}$ by arranging its elements in ascending order. It follows that

$$(2.3) \quad \underline{i} \in I_w \implies x_{\underline{i}} := p_{\underline{i}}(x) = 0 \quad \text{for any } x \in X_w.$$

Now we introduce

$$(2.4) \quad J_w := \bigcup_{1 \leq d \leq n-1} \{\underline{j} \in I_{d,n} \mid \underline{j} = v^{(d)} \text{ for some } v \leq w\}.$$

Then $p_{\underline{j}}$ for $\underline{j} \in J_w$ is not identically zero on X_w because if $\underline{j} = v^{(d)}$ for some $v \leq w$, then $p_{\underline{j}}(vB) = \pm 1$. Since the set of points $x \in X_w$ with $p_{\underline{j}}(x) = 0$ is of codimension one in X_w and $|J_w|$ is finite, there exists a point $x \in X_w$ such that $x_{\underline{j}} \neq 0$ for any $\underline{j} \in J_w$. We will see in Proposition 3.8 that such a point x is generic in X_w .

Example 2.3. Let $w = 312 \in \mathfrak{S}_3$ in the one-line notation. Then we have $w^{(1)} = (3)$, $w^{(2)} = (1, 3)$, $w^{(3)} = (1, 2, 3)$. Hence $p_{\underline{i}} = 0$ on X_w if $\underline{i} = (2, 3)$, and the following points

$$x = \begin{pmatrix} \alpha & 1 & 0 \\ \beta & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B \quad (\alpha, \beta \in \mathbb{C} \setminus \{0\})$$

satisfy that $x_{\underline{j}} \neq 0$ for any $\underline{j} \in J_w = \{(1), (2), (3), (1, 2), (1, 3)\}$.

3. FAN OF THE GENERIC TORUS ORBIT CLOSURE Y_w

The set $\text{Hom}(\mathbb{C}^*, T)$ of algebraic homomorphisms from $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ to T forms an abelian group under the multiplication of T and $\text{Hom}(\mathbb{C}^*, T)$ is isomorphic to \mathbb{Z}^n through the correspondence

$$(3.1) \quad (a_1, \dots, a_n) \in \mathbb{Z}^n \rightarrow (t \mapsto (t^{a_1}, \dots, t^{a_n})) \in \text{Hom}(\mathbb{C}^*, T).$$

In the following we identify $\text{Hom}(\mathbb{C}^*, T)$ with \mathbb{Z}^n through (3.1) and $\text{Hom}(\mathbb{C}^*, T) \otimes \mathbb{R}$ with \mathbb{R}^n but we often denote the element of $\text{Hom}(\mathbb{C}^*, T)$ corresponding to $\mathbf{a} \in \mathbb{Z}^n$ by $\lambda^{\mathbf{a}}$.

Remark 3.1. Since the action of T on Y_w is not effective, the ambient space of the fan of Y_w is the quotient of $\text{Hom}(\mathbb{C}^*, T) \otimes \mathbb{R}$ by the subspace $\text{Hom}(\mathbb{C}^*, T_w) \otimes \mathbb{R}$, where T_w is the toral subgroup of T which fixes Y_w pointwise (F_w in Theorem 1.1 is $\text{Hom}(\mathbb{C}^*, T_w) \otimes \mathbb{R}$). However, for simplicity, we will think of \mathbb{R}^n as the ambient space of the fan of Y_w throughout the paper.

We will find the maximal cones in the fan of Y_w using the Orbit-Cone correspondence, see [12, Proposition 3.2.2.]. Namely, we observe the limit point $\lim_{t \rightarrow 0} \lambda^{\mathbf{a}}(t) \cdot x$ for a generic point $x \in X_w$ and $\mathbf{a} \in \mathbb{Z}^n$. It is known and not difficult to see that Y_{w_0} , where w_0 is the longest element of \mathfrak{S}_n , is the permutohedral variety of complex dimension $n - 1$ and the cones

$$(3.2) \quad C(v) := \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid a_{v(1)} \leq a_{v(2)} \leq \dots \leq a_{v(n)}\}$$

for $v \in \mathfrak{S}_n$ are the maximal cones in the fan of Y_{w_0} . Unless $w = w_0$, we will see that a maximal cone in the fan of Y_w is the union of some of $C(v)$'s (Corollary 3.7). Here is an example which shows how to find the maximal cones in the fan of Y_w .

Example 3.2. Suppose that $G = \text{GL}_3(\mathbb{C})$ and $w = 312 \in \mathfrak{S}_3$. Take a (generic) point x in Example 2.3. Then Example 2.2 shows that for $\mathbf{a} = [a_1, a_2, a_3] \in \mathbb{Z}^3$, the corresponding curve in the product $\mathbb{C}P^2 \times \mathbb{C}P^2$ of projective spaces is given by

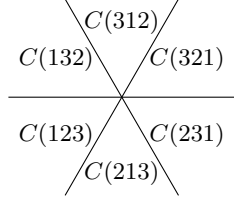
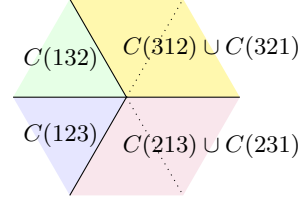
$$\begin{aligned} \lambda^{\mathbf{a}}(t) \cdot \psi(x) &= (t^{a_1}, t^{a_2}, t^{a_3}) \cdot ([\alpha, \beta, 1], [-\beta, -1, 0]) \\ &= ([t^{a_1}\alpha, t^{a_2}\beta, t^{a_3}], [-t^{a_1+a_2}\beta, -t^{a_1+a_3}, 0]). \end{aligned}$$

Take $v = 321$. Then for any $\mathbf{a} = (a_1, a_2, a_3) \in \text{Int}(C(321)) \cap \mathbb{Z}^3$ (so that $a_3 < a_2 < a_1$), we have that

$$\begin{aligned} \lim_{t \rightarrow 0} \lambda^{\mathbf{a}}(t) \cdot \psi(x) &= \lim_{t \rightarrow 0} ([t^{a_1}\alpha, t^{a_2}\beta, t^{a_3}], [-t^{a_1+a_2}\beta, -t^{a_1+a_3}, 0]) \\ &= \lim_{t \rightarrow 0} ([t^{a_1-a_3}\alpha, t^{a_2-a_3}\beta, 1], [t^{a_2-a_3}\beta, 1, 0]) \\ &= ([0, 0, 1], [0, 1, 0]) \\ &= \psi(312B). \end{aligned}$$

A similar argument shows that the limit point corresponding to each $C(v)$ is as follows:

$$\begin{aligned} C(123): & ([1, 0, 0], [1, 0, 0]), & C(132): & ([1, 0, 0], [0, 1, 0]), & C(213): & ([0, 1, 0], [1, 0, 0]), \\ C(231): & ([0, 1, 0], [1, 0, 0]), & C(312): & ([0, 0, 1], [0, 1, 0]), & C(321): & ([0, 0, 1], [0, 1, 0]). \end{aligned}$$

FIGURE 1. Cones $C(v)$ for $v \in \mathfrak{S}_3$.FIGURE 2. The fan of Y_{312} .

Hence there are four limit points and accordingly the fan of Y_w consists of four maximal cones:

$$C(123), C(132), C(213) \cup C(231), C(312) \cup C(321).$$

See Figure 2. We note that Y_w is a Hirzebruch surface $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. In this case, the Schubert variety X_{312} itself is a toric variety, i.e. $X_{312} = Y_{312}$. We will discuss when X_w is a toric variety in general in Section 7.

Motivated by the above observation, we introduce an operation on \mathfrak{S}_n with respect to w .

Definition 3.3. Fix $w \in \mathfrak{S}_n$. For $v \in \mathfrak{S}_n$, we inductively choose

$$\begin{aligned} i_1 &:= \min\{i \in [n] \mid v(i) \leq w(1) = w^{(1)}\}, \\ i_2 &:= \min\{i \in [n] \setminus \{i_1\} \mid \{v(i_1), v(i)\} \uparrow \leq w^{(2)}\}, \\ &\vdots \\ i_n &:= \min\{i \in [n] \setminus \{i_1, \dots, i_{n-1}\} \mid \{v(i_1), \dots, v(i_{n-1}), v(i)\} \uparrow \leq w^{(n)}\}, \end{aligned}$$

and define

$$v' := v(i_1)v(i_2) \cdots v(i_n) \in \mathfrak{S}_n.$$

Here, $\{a_1, \dots, a_d\} \uparrow$ denotes the ordered d -tuple obtained from $\{a_1, \dots, a_d\}$ by arranging its elements in ascending order and $w^{(d)} = \{w(1), \dots, w(d)\} \uparrow$.

The operation v' for $v \in \mathfrak{S}_n$ will play an important role in our argument. It depends on w but since w is fixed throughout the paper, we will not specify w in the notation of the operation for simplicity. As is well-known,

$$(3.3) \quad v \leq w \iff v^{(d)} \leq w^{(d)} \quad \text{for all } 1 \leq d \leq n-1$$

(see [5, (3.2.5)]), so the following lemma immediately follows from the definition of the operation.

Lemma 3.4. *The operation v' for $v \in \mathfrak{S}_n$ has the following properties:*

- (1) $v' \leq w$ for any $v \in \mathfrak{S}_n$,
- (2) $v' = v$ if $v \leq w$,
- (3) $w'_0 = w$ where w_0 is the longest element of \mathfrak{S}_n as before.

Example 3.5. Let $w = 3412$ and $v = 4123$. Then i_1, \dots, i_4 are given as follows:

$$\begin{aligned} i_1 &= \min\{i \in [4] \mid v(i) \leq w(1) = 3\} = 2, \\ i_2 &= \min\{i \in [4] \setminus \{2\} \mid \{1, v(i)\} \uparrow \leq (3, 4)\} = 1, \\ i_3 &= \min\{i \in [4] \setminus \{1, 2\} \mid \{1, 4, v(i)\} \uparrow \leq (1, 3, 4)\} = 3, \\ i_4 &= 4. \end{aligned}$$

Hence $v' = v(2)v(1)v(3)v(4) = 1423$.

The following proposition is a key observation.

Proposition 3.6. *Let x be a point in X_w with $x_{\underline{j}} \neq 0$ for any $\underline{j} \in J_w$ (see (2.4) for J_w). Then for any $v \in \mathfrak{S}_n$ and any $\mathbf{a} \in \text{Int}(C(v)) \cap \mathbb{Z}^n$, we have*

$$\lim_{t \rightarrow 0} \lambda^{\mathbf{a}}(t) \cdot x = v'B.$$

Proof. Suppose that $v \leq w$. Then, since $v^{(d)} \leq w^{(d)}$ for any $1 \leq d \leq n-1$ by (3.3), we have $v^{(d)} \in J_w$ and hence

$$x_{v^{(d)}} \neq 0 \text{ for all } 1 \leq d \leq n-1.$$

On the other hand, since $\mathbf{a} = (a_1, \dots, a_n) \in \text{Int}(C(v)) \cap \mathbb{Z}^n$, we have $a_{v(1)} < a_{v(2)} < \dots < a_{v(n)}$. Therefore the sum $\sum_{k=1}^d a_{v(k)}$ is smallest among the sum of arbitrary d elements in a_1, \dots, a_n . Then, the same argument as in Example 3.2 shows that

$$\lim_{t \rightarrow 0} \lambda^{\mathbf{a}}(t) \cdot x = vB.$$

Here $v' = v$ by Lemma 3.4 because $v \leq w$. This proves the proposition when $v \leq w$.

Suppose that $v \not\leq w$. Since $v' \leq w$ by Lemma 3.4, $v'^{(d)} \in J_w$ and hence

$$x_{v'^{(d)}} \neq 0 \text{ for all } 1 \leq d \leq n-1.$$

By (2.3) we have $x_{\underline{i}} = 0$ for $\underline{i} \in I_{d,n}$ with $\underline{i} \not\leq w^{(d)}$, so the construction of v' shows that the sum $\sum_{k=1}^d a_{v'(k)}$ is smallest among the sum $\sum_{k \in \underline{i}} a_k$ for $\underline{i} \in I_{d,n}$ with $x_{\underline{i}} \neq 0$. This implies the desired identity in the proposition as before. \square

The Orbit-Cone correspondence and Proposition 3.6 imply the following.

Corollary 3.7. *The maximal cone in the fan of Y_w corresponding to a fixed point uB ($u \leq w$) is of the form*

$$C_w(u) := \bigcup_{v \in \mathfrak{S}_n \text{ s.t. } v'=u} C(v).$$

Using Proposition 3.6, we can characterize generic points in X_w as follows.

Proposition 3.8. *A point x in X_w is generic in X_w if and only if $x_{\underline{j}} \neq 0$ for any $\underline{j} \in J_w$. (Therefore, there exists a generic point in X_w as explained after (2.4).)*

Proof. Suppose that $x_{\underline{j}} \neq 0$ for any $\underline{j} \in J_w$. Then Proposition 3.6 shows that the orbit closure $\overline{T \cdot x}$ contains the fixed point $v'B$ for any v but $v' = v$ if $v \leq w$ by Lemma 3.4 (2). Hence $\overline{T \cdot x}$ contains all the T -fixed points in X_w , which means that x is generic in X_w , proving the “if” part in the proposition.

Suppose that x is a generic point in X_w but $x_{\underline{j}} = 0$ for some $\underline{j} \in J_w$. Then the \underline{j} -th Plücker coordinate of the limit point $\lim_{t \rightarrow 0} \lambda^{\mathbf{a}}(t) \cdot x$ vanishes for any $\mathbf{a} \in \mathbb{Z}^n$, which means that the \underline{j} -th coordinate of any T -fixed point in $\overline{T \cdot x}$ vanishes. On

the other hand, there is some $v \leq w$ with $v^{(d)} = \underline{j}$ by definition of J_w where d is the length of \underline{j} . Since $v^{(k)}$ -th Plücker coordinate of the T -fixed point vB does not vanish for any $1 \leq k \leq n$, this shows that $vB \notin \overline{T \cdot x}$ although $v \leq w$. This contradicts x being generic in X_w . Therefore $x_{\underline{j}} \neq 0$ for any $\underline{j} \in J_w$, proving the “only if” part in the proposition. \square

4. RIGHT WEAK ORDER

In this section, we describe the set $\{v \in \mathfrak{S}_n \mid v' = u\}$ in Corollary 3.7 using the right weak (Bruhat) order on \mathfrak{S}_n (see [6, §3.1]). For $u_1, u_2 \in \mathfrak{S}_n$, $u_1 \leq_R u_2$ means that $u_2 = u_1 s_{i_1} \cdots s_{i_k}$ for some simple reflections s_{i_1}, \dots, s_{i_k} such that $\ell(u_1 s_{i_1} \cdots s_{i_j}) = \ell(u_1) + j$ for $0 \leq j \leq k$, where $\ell(v)$ denotes the length of $v \in \mathfrak{S}_n$. This order is called the *right weak order*. The right weak order interval $[u_1, u_2]_R$ is defined to be the set $[u_1, u_2]_R := \{x \in \mathfrak{S}_n \mid u_1 \leq_R x \leq_R u_2\}$.

Lemma 4.1. $v \geq_R v'$ for any $v \in \mathfrak{S}_n$.

Proof. There is $0 \leq d \leq n$ such that $v(i) = v'(i)$ for $1 \leq i \leq d$. We shall prove the lemma by downward induction on d starting from n . If $d = n$, then $v = v'$ and the lemma trivially holds in this case. We assume that the lemma holds for v with $v(i) = v'(i)$ for $1 \leq i \leq d$ and prove the lemma for v with $v(i) = v'(i)$ for $1 \leq i \leq d-1$. We may assume $v(d) \neq v'(d)$. Define $j \in [n]$ by $v(j) = v'(d)$. Since $v(i) = v'(i)$ for $1 \leq i \leq d-1$, we have $d < j$ and it follows from the construction of v' that

$$(4.1) \quad \begin{aligned} \{v(1), \dots, v(d-1), v(i)\} &\not\uparrow \leq w^{(d)} && \text{for } d \leq i < j, \\ \{v(1), \dots, v(d-1), v(j)\} &\uparrow \leq w^{(d)}. \end{aligned}$$

In particular

$$(4.2) \quad v(i) > v(j) \text{ for } d \leq i < j.$$

We consider $u \in \mathfrak{S}_n$ defined by

$$u = v(1) \cdots v(d-1)v(j)v(d) \cdots v(j-1)v(j+1) \cdots v(n).$$

Since $v(i) = v'(i)$ for $1 \leq i \leq d-1$, it follows from (4.1) and (4.2) that we have

$$\begin{aligned} u(i) &= v'(i) && \text{for } 1 \leq i \leq d, \\ u' &= v', \\ v &= u s_d s_{d+1} \cdots s_{j-1} && \text{with } \ell(v) = \ell(u) + (j-d), \text{ so } v \geq_R u. \end{aligned}$$

Because of the first and second identities above, the induction assumption can be applied to u , so that $u \geq_R u'$. This together with the second and last identities above shows $v \geq_R v'$. \square

Proposition 4.2. Let $u \leq w$. Then there exists a (unique) $u_w \in \mathfrak{S}_n$ such that

$$\{v \in \mathfrak{S}_n \mid v' = u\} = [u, u_w]_R.$$

Proof. We denote the left hand side in the proposition by $S(u)$. It suffices to prove that if

- (1) $\ell(v s_p) = \ell(v s_q) = \ell(v) + 1$ for some $p < q$, and,
- (2) $v' = (v s_p)' = (v s_q)' = u$,

then there exists $\tilde{v} \in \mathfrak{S}_n$ such that $\{vs_p, vs_q\} \subset [v, \tilde{v}]_R \subset S(u)$. We note that

$$(4.3) \quad v(p) < v(p+1), \quad v(q) < v(q+1)$$

by (1) above. We define $i, j \in [n]$ by

$$v(p) = u(i), \quad v(q) = u(j).$$

Then it follows from (2) above that

$$(4.4) \quad \begin{aligned} \{u(1), \dots, u(i-1), v(p)\} \uparrow \leq w^{(i)}, & \quad \{u(1), \dots, u(i-1), v(p+1)\} \uparrow \not\leq w^{(i)}, \\ \{u(1), \dots, u(j-1), v(q)\} \uparrow \leq w^{(j)}, & \quad \{u(1), \dots, u(j-1), v(q+1)\} \uparrow \not\leq w^{(j)}. \end{aligned}$$

We consider two cases.

Case 1. The case where $q - p \geq 2$. In this case, s_p and s_q commute and we take $\tilde{v} = vs_p s_q$. Then $[v, \tilde{v}]_R = \{v, vs_p, vs_q, \tilde{v} = vs_p s_q\}$, and (2) above implies $\tilde{v}' = u$. Therefore $\{vs_p, vs_q\} \subset [v, \tilde{v}]_R \subset S(u)$.

Case 2. The case where $q - p = 1$, i.e. $q = p + 1$. In this case, it follows from (4.3) that

$$(4.5) \quad v(p) < v(p+1) < v(p+2).$$

Note that $i < j$ since $v' = u$ and $u(i) = v(p) < v(p+1) = u(j)$. We take $\tilde{v} = vs_p s_{p+1} s_p$, i.e.

$$(4.6) \quad \tilde{v} = v(1) \cdots v(p-1)v(p+2)v(p+1)v(p)v(p+3) \cdots v(n).$$

Then

$$[v, \tilde{v}]_R = \{v, vs_p, vs_{p+1}, vs_p s_{p+1}, vs_{p+1} s_p, \tilde{v} = vs_p s_{p+1} s_p\}.$$

Since $s_p s_{p+1} s_p = s_{p+1} s_p s_{p+1}$ and $q = p + 1$, we have $\tilde{v} \geq_R vs_p$ and $\tilde{v} \geq_R vs_q$ by (4.5). By (4.4) and (4.5) we have

$$\{u(1), \dots, u(i-1), v(p+1)\} \uparrow \not\leq w^{(i)}, \quad \{u(1), \dots, u(i-1), v(p+2)\} \uparrow \not\leq w^{(i)}.$$

This together with the first inequality in (4.4), (4.6) and the assumption $v' = u$ show that $\tilde{v}'(i) = v(p) = u(i)$. Then, the conclusion $\tilde{v}' = u$ follows from the second line in (4.4) and the assumption $v' = u$. Since

$$\begin{aligned} vs_p s_{p+1} &= v(1) \cdots v(p-1)v(p+1)v(p+2)v(p)v(p+3) \cdots v(n), \\ vs_{p+1} s_p &= v(1) \cdots v(p-1)v(p+2)v(p)v(p+1)v(p+3) \cdots v(n), \end{aligned}$$

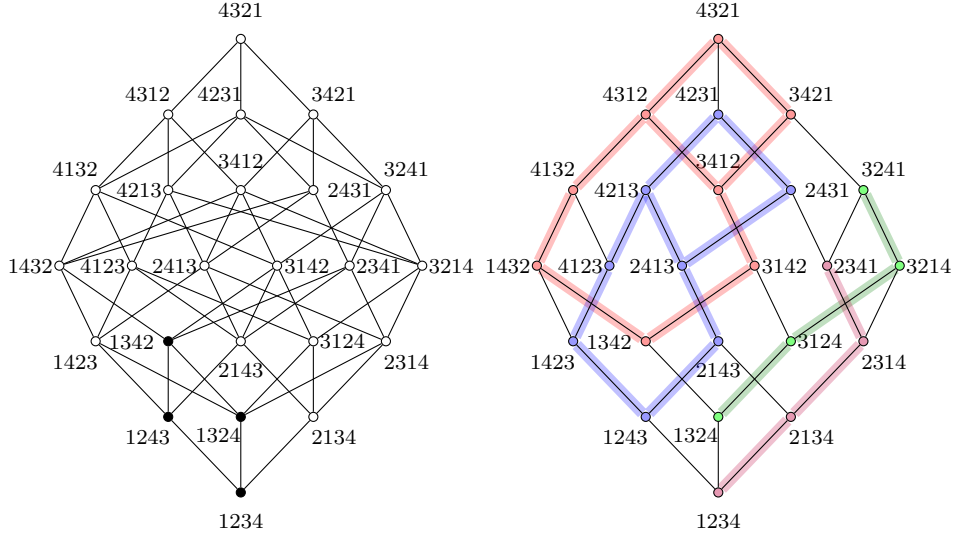
the same observation for \tilde{v} shows that $(vs_p s_{p+1})' = u = (vs_{p+1} s_p)'$. Therefore $\{vs_p, vs_q\} \subset [v, \tilde{v}]_R \subset S(u)$ in this case, too. \square

Example 4.3. Take $w = 1342$. Then there are four elements $u \in \mathfrak{S}_4$ such that $u \leq w$ (see Figure 3(1)) and one can check the following (see Figure 3(2)):

u	1342	1243	1324	1234
u_w	4321	4231	3241	2341

TABLE 1. u_w when $w = 1342$.

$$\begin{aligned} [1342, 4321]_R &= \{1342, 3142, 1432, 4132, 3412, 4312, 3421, 4321\}, \\ [1243, 4231]_R &= \{1243, 2143, 1423, 4123, 2413, 4213, 2431, 4231\}, \\ [1324, 3241]_R &= \{1324, 3124, 3214, 3241\}, \\ [1234, 2341]_R &= \{1234, 2134, 2314, 2341\}. \end{aligned}$$



(1) Bruhat order of \mathfrak{S}_4 and elements smaller than or equal to 1342 are marked by \bullet . (2) Right weak order of \mathfrak{S}_4 and right weak order intervals in Table 1.

FIGURE 3. Bruhat order and right weak order of \mathfrak{S}_4 .

5. DUAL CONE OF $C_w(w)$

By Corollary 3.7 and Proposition 4.2, the maximal cone in the fan of Y_w corresponding to the fixed point uB ($u \leq w$) is of the form

$$C_w(u) = \bigcup_{v \in [u, u_w]_R} C(v).$$

As noted in Lemma 3.4 $w'_0 = w$, so $u_w = w_0$ when $u = w$. Hence the maximal cone corresponding to wB is of the form

$$C_w := C_w(w) = \bigcup_{v \in [w, w_0]_R} C(v).$$

Our purpose of this section is to identify the dual of the maximal cone C_w (Proposition 5.3).

Definition 5.1. For $w \in \mathfrak{S}_n$, we define

$$E_w := \{(w(i), w(j)) \mid 1 \leq i < j \leq n, \ell(w) - \ell(t_{w(i), w(j)}w) = 1\}$$

where $t_{a,b}$ denotes the transposition of a and b and $\ell(v)$ denotes the length of a permutation v as before. (Note. The condition $\ell(w) - \ell(t_{w(i), w(j)}w) = 1$ above is equivalent to $w(i) > w(j)$ and $w(k) \notin [w(j), w(i)]$ for $i < \forall k < j$.)

Example 5.2. (1) If $w = 3152674$, then

$$E_w = \{(3, 1), (3, 2), (5, 2), (5, 4), (6, 4), (7, 4)\}.$$

(2) If $w = 3715264$, then

$$E_w = \{(3, 1), (3, 2), (7, 1), (7, 5), (7, 6), (5, 2), (5, 4), (6, 4)\}.$$

We define

$$(5.1) \quad D_w := \text{the cone in } \mathbb{R}^n \text{ spanned by } \{e_b - e_a \mid (a, b) \in E_w\}.$$

Since $C(v)$'s are the closed regions divided by the hyperplanes defined by $e_i - e_j$ ($1 \leq i < j \leq n$), it follows from (5.1) that the dual cone of D_w in \mathbb{R}^n , denoted by D_w^\vee , is the union of some of $C(v)$'s.

Proposition 5.3. $D_w^\vee = C_w$.

This proposition follows from the following two lemmas. Remember that $u_w = w_0$ when $u = w$ and hence $v \in [w, w_0]_R$ if and only if $v' = w$ by Proposition 4.2.

Lemma 5.4. *If $v \in [w, w_0]_R$, i.e. $v' = w$, then $C(v)$ is contained in D_w^\vee .*

Proof. Since $v \in [w, w_0]_R$, one can write

$$v = ws_{i_1} \cdots s_{i_k}$$

where $\ell(v) = \ell(w) + k$. We prove the lemma by induction on k . If $k = 0$, then $v = w$ and the lemma is obvious. Suppose that $k \geq 1$ and the lemma holds for $k - 1$. We set

$$u = ws_{i_1} \cdots s_{i_{k-1}} \quad \text{and} \quad i_k = p.$$

Then, since $v = us_p$, we have

$$(5.2) \quad \begin{aligned} v &= v(1) \cdots v(p-1)v(p)v(p+1)v(p+2) \cdots v(n), \\ u &= v(1) \cdots v(p-1)v(p+1)v(p)v(p+2) \cdots v(n), \quad \text{and} \\ \ell(v) &= \ell(u) + 1. \end{aligned}$$

Take any element (a, b) of E_w . By induction assumption we have

$$\langle e_b - e_a, x \rangle \geq 0 \quad (\forall x \in C(u))$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbb{R}^n . The above inequality is equivalent to a being ahead of b in the one-line notation for u . What we have to prove is that a is still ahead of b in the one-line notation for v . We take two cases.

Case 1. The case where $\{v(p), v(p+1)\} \neq \{a, b\}$. In this case, it is easy to see from (5.2) that a is still ahead of b in the one-line notation for v since so is for u .

Case 2. The case where $\{v(p), v(p+1)\} = \{a, b\}$. Since a is ahead of b in the one-line notation for u , we have $a = v(p+1)$ and $b = v(p)$ by (5.2) in this case. We note that $a > b$ since $(a, b) \in E_w$. This together with $v = us_p$ shows that $\ell(v) = \ell(u) - 1$ but this contradicts the last identity in (5.2). Therefore Case 2 does not occur. \square

Lemma 5.5. *If $v \notin [w, w_0]_R$, i.e. $v' \neq w$, then $C(v)$ is not contained in D_w^\vee .*

Proof. Since $v' < w$ by assumption and Lemma 3.4 (1), there exists $i \in [n]$ such that

$$(5.3) \quad v'(j) = w(j) \quad (1 \leq j \leq i-1), \quad v'(i) < w(i).$$

Define q and m by

$$(5.4) \quad v'(i) = v(q) = w(m).$$

It follows from (5.3) that

$$(5.5) \quad i < m$$

and from the construction of v' , (5.3) and (5.4) that

$$(5.6) \quad \text{any element in } \{v(1), \dots, v(q-1)\} \setminus \{w(1), \dots, w(i-1)\} > w(i).$$

Since $i < m$ by (5.5) but $w(i) > w(m)$ by (5.3) and (5.4), there exists ℓ between i and $m-1$ such that

$$(5.7) \quad w(i) \geq w(\ell) > w(m) \quad \text{and} \quad (w(\ell), w(m)) \in E_w.$$

Now we define p by $v(p) = w(\ell)$. Then, since $i \leq \ell$, we have

$$(5.8) \quad v(p) = w(\ell) \notin \{w(1), \dots, w(i-1)\}.$$

It follows from (5.4), (5.6), (5.7) and (5.8) that

$$q < p \quad \text{and} \quad v(q) = w(m) < w(\ell) = v(p).$$

Since $q < p$, $e_{v(q)} - e_{v(p)}$ takes a negative value on $C(v)$ (through the inner product) but $(v(p), v(q)) = (w(\ell), w(m)) \in E_w$ by (5.7). This shows that $C(v)$ is not contained in D_w^\vee , proving the lemma. \square

The two lemmas above imply Proposition 5.3.

6. SIMPLICIALITY OF D_w AND GRAPH Γ_w

Let Γ_w be the graph associated to E_w , i.e., the vertices of Γ_w are the positive integers appearing in E_w (so the vertices of Γ_w are elements in $[n]$) and elements in E_w are edges. We show that Y_w is smooth at wB if and only if Γ_w is a forest (Theorem 6.8).

Example 6.1. Following Example 5.2, we have graphs $\Gamma_{3152674}$ and $\Gamma_{3715264}$ as in Figures 4 and 5.

Lemma 6.2. *If $(a, b) \in E_w$, then $e_b - e_a$ is an edge vector of D_w .*

Proof. Suppose that

$$(6.1) \quad e_b - e_a = \sum_{i=1}^k c_i (e_{b_i} - e_{a_i}) \quad \text{with } c_i > 0$$

where $(a, b) \neq (a_i, b_i) \in E_w$ for $i = 1, \dots, k$. Suppose $k \geq 2$ and we deduce a contradiction. Since $(a_i, b_i) \in E_w$, we have $a_i > b_i$. Therefore, it follows from (6.1) that $a = \max_{1 \leq i \leq k} \{a_i\}$. We may assume $a_1 = a$ if necessary by changing indices. Since

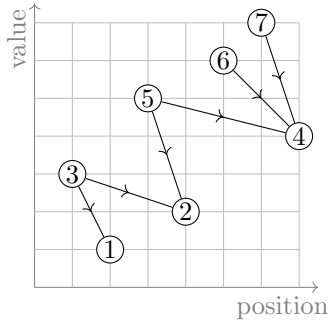


FIGURE 4. Graph $\Gamma_{3152674}$.

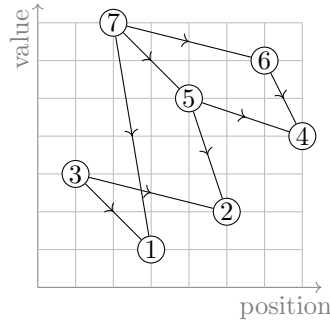


FIGURE 5. Graph $\Gamma_{3715264}$.

$(a_1, b_1) \neq (a, b)$ and $a_1 = a$, we have $b_1 \neq b$. In order for (6.1) to hold, e_{b_1} must be killed at the right hand side of (6.1). This means that there exists some $i \neq 1$ such that $a_i = b_1$. We may assume $a_2 = b_1$ if necessary by changing indices. If $b_2 \neq b$, then we repeat the same argument and may assume that $a_3 = b_2$. We repeat this argument. Then we reach $m \geq 2$ such that $b_m = b$, i.e. we obtain the following sequence of pairs in E_w :

$$(6.2) \quad \begin{aligned} & (a_1, b_1), (a_2, b_2), \dots, (a_m, b_m) \\ & \text{such that } a_1 = a, b_m = b, b_i = a_{i+1} \text{ for } 1 \leq i \leq m-1. \end{aligned}$$

Remember that since $(a_i, b_i) \in E_w$, a_i is ahead of b_i in the one-line notation for w . Since $m \geq 2$, this together with (6.2) shows that some positive integer x with $a > x > b$ (e.g. $x = b_1 = a_2$) appears between a and b in the one-line notation for w . This contradicts (a, b) being in E_w . \square

Corollary 6.3. *The dual cone D_w is simplicial if and only if $|E_w| = \dim D_w$.*

We note that unless the identity $|E_w| = \dim D_w$ is satisfied, Y_w is singular by Proposition 5.3. Here is a way to find $\dim D_w$. The idea is to project D_w successively along vectors $e_b - e_a$ for $(a, b) \in E_w$.

Example 6.4. (1) Take $w = 3152674$ in Example 5.2(1). The cone D_w is spanned by

$$\{e_1 - e_3, e_2 - e_3, e_2 - e_5, e_4 - e_5, e_4 - e_6, e_4 - e_7\}.$$

Take any element from the above, say $e_4 - e_5$, and set $e_4 = e_5$. Then the resulting vectors are

$$\{e_1 - e_3, e_2 - e_3, e_2 - e_4, e_4 - e_6, e_4 - e_7\}.$$

Next, take $e_4 - e_6$ for example and set $e_4 = e_6$. Then the resulting vectors are

$$\{e_1 - e_3, e_2 - e_3, e_2 - e_4, e_4 - e_7\}.$$

Continue this procedure. Then all the vectors above vanish after four more times. Therefore, $\dim D_w = 6$. Since $|E_w|$ is also 6, D_w is simplicial and hence the merged cone C_w in the fan is also simplicial by Proposition 5.3. The corresponding graph Γ_w is a path graph, see Figure 4.

- (2) Take $w = 3715264$ in Example 5.2(2). Then $|E_w| = 8$ but $\dim D_w = 6$. Therefore Y_w is singular. The corresponding graph Γ_w is connected and $b_1(\Gamma_w) = 2$ where b_1 denotes the first Betti number, see Figure 5.
- (3) Take $w = 3412$. Then $|E_w| = 4$ but $\dim D_w = 3$. Therefore Y_w is singular. The corresponding graph Γ_w is connected and $b_1(\Gamma_w) = 1$, see Figure 6.
- (4) Take $w = 341265$. Then $|E_w| = 5$ but $\dim D_w = 4$. Therefore Y_w is singular. The corresponding graph Γ_w has two connected components and $b_1(\Gamma_w) = 1$, see Figure 7.

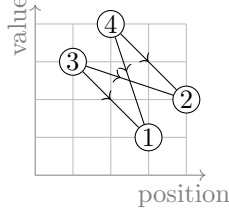
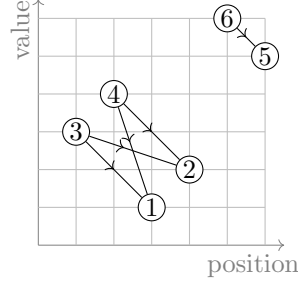
Since $|E_w|$ is nothing but the number of the edges in Γ_w , the same argument as in Example 6.4 proves the following.

Lemma 6.5. *We have*

$$\dim_{\mathbb{C}} Y_w = \dim D_w = |\text{vertices of } \Gamma_w| - |\text{connected components of } \Gamma_w|.$$

Therefore

$$|E_w| - \dim D_w = b_1(\Gamma_w) \geq 0,$$

FIGURE 6. Graph Γ_{3412} .FIGURE 7. Graph Γ_{341265} .

where $b_1(\Gamma_w)$ denotes the first Betti number of Γ_w , and hence D_w is simplicial if and only if Γ_w is a forest (as an undirected graph) by Corollary 6.3.

Remember that our cone D_w has $\{e_b - e_a \mid (a, b) \in E(\Gamma_w)\}$ as edge vectors (Lemma 6.2). They lie in the linear subspace, denoted by H^{n-1} , of \mathbb{R}^n with the sum of the coordinates equals to zero. In general, we have the following.

Lemma 6.6. *Let Γ be a directed graph with vertices in $\{1, \dots, n\}$ and let D be a cone in \mathbb{R}^n with $\{e_b - e_a \mid (a, b) \in E(\Gamma)\}$ as edge vectors, where $E(\Gamma)$ denotes the set of directed edges in Γ . Then the following are equivalent:*

- (1) D is simplicial.
- (2) Γ is a forest (as an undirected graph).
- (3) D is non-singular, i.e. the set $\{e_b - e_a \mid (a, b) \in E(\Gamma)\}$ is a part of a \mathbb{Z} -basis of $H^{n-1} \cap \mathbb{Z}^n$.

In particular, D_w is simplicial if and only if D_w is non-singular.

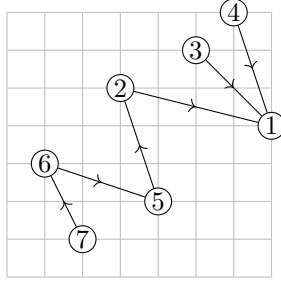
Proof. The equivalence of (1) and (2) can be seen by the same argument as above for D_w . It is obvious that (3) implies (1), so it is enough to show that (2) implies (3). We may assume that Γ is a tree. To prove (3), we may change the order of coordinates of \mathbb{R}^n and note that changing the order of the coordinates is nothing but relabeling the vertices of Γ .

We shall relabel the vertices of Γ . For any two vertices x and y of Γ , there is a unique path connecting them since Γ is a tree. We define the distance $d(x, y)$ to be the number of edges in the path. Choose any vertex of Γ and label it as 1. For any positive integer d , consider the set

$$A_d := \{v \in V(\Gamma) \mid d(v, 1) = d\}$$

where $V(\Gamma)$ denotes the set of vertices of Γ . Then we label the vertices in A_1 as $2, 3, \dots, |A_1| + 1$ and then label the vertices in A_2 as $|A_1| + 2, |A_1| + 3, \dots, |A_1| + |A_2| + 1$ and so on. We orient each edge (a, b) of this relabeled graph in such a way that $a > b$ and denote the relabeled directed graph by $\bar{\Gamma}$. Then it is not difficult to see that $\{e_b - e_a \mid (a, b) \in E(\bar{\Gamma})\}$ is a \mathbb{Z} -basis of the free abelian group generated by $e_1 - e_2, e_2 - e_3, \dots, e_{m-1} - e_m$, where m is the number of vertices of Γ , and hence a part of a \mathbb{Z} -basis of $H^{n-1} \cap \mathbb{Z}^n$, proving (3). \square

Example 6.7. Take $w = 3152674$ in Example 5.2(1). Then the graph Γ_w is in Figure 4. We can reorder the vertices and change directions on edges as follows.



Then the corresponding edge vectors are

$$\{e_1 - e_2, e_1 - e_3, e_1 - e_4, e_2 - e_5, e_5 - e_6, e_6 - e_7\},$$

which is a part of \mathbb{Z} -basis of $H^6 \cap \mathbb{Z}^7$.

Combining Corollary 6.3, Lemmas 6.5 and 6.6, we obtain

Theorem 6.8. *The generic torus orbit closure Y_w is smooth at the fixed point wB if and only if Γ_w is a forest.*

To our surprise, the graph Γ_w has been studied and the following is proven.

Proposition 6.9 ([7]). *The graph Γ_w is a forest if and only if w avoids the patterns 4231 and $45\bar{3}12$.*

In Appendix A, we will provide a proof different from the proof in [7]. Here, a permutation w avoids the pattern 4231 if one cannot find indices $i < j < k < \ell$ such that $w(\ell) < w(j) < w(k) < w(i)$. Similarly w avoids the pattern $45\bar{3}12$ if every occurrence of the pattern 4512 is a subsequence of an occurrence of $45\bar{3}12$.

Remark 6.10. In [7], they associate a graph G_π to a permutation π and prove that G_π is a forest if and only if π avoids the patterns 1324 and $21\bar{3}54$. In fact, our Γ_w is their G_{w_0w} , so we obtain the statement in Proposition 6.9.

We restate Theorem 6.8 in terms of pattern avoidance using Proposition 6.9.

Theorem 6.11. *The generic torus orbit closure Y_w is smooth at the fixed point wB if and only if w avoids the patterns 4231 and $45\bar{3}12$.*

Remark 6.12. According to [7] and [24], Γ_w is a forest if and only if our Schubert variety $X_w (= \overline{BwB/B})$ is factorial. (Note. X_w in [24] is different from our X_w , indeed their X_w is Ω_w in Fulton's book [14], which is the closure of the dual Schubert cell indexed by w .)

7. SMOOTHNESS OF Y_w AT OTHER FIXED POINTS

In the previous sections, we identified the cone dual to the maximal cone $C_w = C_w(w)$ and associated the graph Γ_w . In this section, we will do the same task to the other maximal cones $C_w(u)$ ($u \leq w$) generalizing the previous case, namely we identify the cone dual to $C_w(u)$ and associate a graph, denoted by $\Gamma_w(u)$, to each $u \leq w$. It seems that if the graph Γ_w is a forest, then so is $\Gamma_w(u)$ for any $u \leq w$. This means that Y_w is smooth if it is smooth at the fixed point wB . We propose this and a slightly more general statement as a conjecture at the end of this section.

We begin with the generalization of E_w introduced in Definition 5.1.

Definition 7.1. For $u \leq w$, we define

$\tilde{E}_w(u) := \{(u(i), u(j)) \mid 1 \leq i < j \leq n, t_{u(i), u(j)}u \leq w, |\ell(u) - \ell(t_{u(i), u(j)}u)| = 1\}$
 where $t_{a,b}$ denotes the transposition of a and b and $\ell(v)$ denotes the length of a permutation v as before.

Remark 7.2. (1) $\tilde{E}_w(w) = E_w$ because the condition $\ell(w) - \ell(t_{w(i), w(j)}w) = 1$ in Definition 5.1 is equivalent to

$$t_{w(i), w(j)}w \leq w \quad \text{and} \quad |\ell(w) - \ell(t_{w(i), w(j)}w)| = 1.$$

(2) The condition $|\ell(u) - \ell(t_{u(i), u(j)}u)| = 1$ in Definition 7.1 is equivalent to

$$\begin{aligned} u(k) &\notin [u(j), u(i)] && \text{when } u(i) > u(j), \\ u(k) &\notin [u(i), u(j)] && \text{when } u(i) < u(j), \end{aligned}$$

for $i < \forall k < j$.

Example 7.3. Take $w = 3412$.

- (1) If $u = 2143$, then $\tilde{E}_w(u) = \{(1, 4), (2, 3), (2, 1), (4, 3)\}$.
- (2) If $u = 2413$, then $\tilde{E}_w(u) = \{(2, 3), (2, 1), (4, 1), (4, 3)\}$.
- (3) If $u = 1432$, then $\tilde{E}_w(u) = \{(1, 3), (4, 3), (3, 2)\}$.

We define $(a, b) + (b, c) = (a, c)$. Then, in (1) in the example above, we have

$$(2, 3) = (2, 1) + (1, 4) + (4, 3).$$

We say that an element of $\tilde{E}_w(u)$ is *decomposable* if it is sum of some other elements in $\tilde{E}_w(u)$, and *indecomposable* otherwise.

Definition 7.4. We define

$$E_w(u) := \{\text{indecomposable elements in } \tilde{E}_w(u)\}$$

and

$$D_w(u) := \text{the cone in } \mathbb{R}^n \text{ spanned by } \{e_b - e_a \mid (a, b) \in E_w(u)\}.$$

Remark 7.5. (1) $E_w(w) = \tilde{E}_w(w) = E_w$, so $D_w(w) = D_w$,

(2) $E_{w_0}(u) = \{(u(i), u(i+1)) \mid i = 1, 2, \dots, n-1\}$.

(3) When $u = id$ (the identity permutation), $E_w(id) = \{(a, a+1) \mid t_{a, a+1}id \leq w\}$.

Example 7.6. For (1) in Example 7.3, $E_w(u) = \{(1, 4), (2, 1), (4, 3)\}$, and for (2) and (3), $E_w(u) = \tilde{E}_w(u)$. One can check that $C_w(u) = D_w(u)^\vee$ in these cases.

The purpose of this section is to prove the following proposition.

Proposition 7.7. $C_w(u) = D_w(u)^\vee$ for any $u \leq w$.

As a first step, we prove the following lemma.

Lemma 7.8. $C(u) = D_{w_0}(u)^\vee$ for any $u \in \mathfrak{S}_n$.

Proof. Let (a, b) be an arbitrary element in $E_{w_0}(u)$. Then a appears ahead of b in the one-line notation for u . Therefore $e_b - e_a$ is non-negative on the cone $C(u)$, which means that $C(u) \subset D_{w_0}(u)^\vee$. Suppose $C(u) \subsetneq D_{w_0}(u)^\vee$ (and we deduce a contradiction). Then there is some $v (\neq u) \in \mathfrak{S}_n$ such that $C(v) \subset D_{w_0}(u)^\vee$. Since $v \neq u$, there is a pair (a, b) such that a appears ahead of b in the one-line notation

for u while b appears ahead of a in the one-line notation for v . Suppose $a > b$. Then there is a sequence of pairs $(a_{i-1}, a_i) \in E_{w_0}(u)$ ($i = 1, \dots, k$) such that

$$(7.1) \quad a = a_0 > a_1 > \dots > a_k = b.$$

Since $C(v) \subset D_{w_0}(u)$ and $(a_{i-1}, a_i) \in E_{w_0}(u)$, $e_{a_i} - e_{a_{i-1}}$ is non-negative on $C(v)$. This means that in the one-line notation for v , a_{i-1} appears ahead of a_i for any $i = 1, \dots, k$ so that a appears ahead of b , which is a contradiction. The same argument works when $a < b$ if the inequalities in (7.1) are reversed. \square

Lemma 7.9. *Let $u \leq w$. If $v' = u$, then $C(v) \subset D_w(u)^\vee$. Therefore $C_w(u) \subset D_w(u)^\vee$ by Corollary 3.7.*

Proof. Since $v' = u$, we have $v \geq_R u$ by Lemma 4.1. Therefore v is of the form

$$v = u s_{i_1} \cdots s_{i_k} \quad \text{with } \ell(v) = \ell(u) + k.$$

We shall prove the lemma by induction on k . When $k = 0$, that is, $v = u$, $C(v) = C(u)$. By Lemma 7.8, $C(u) = D_{w_0}(u)^\vee$. Since $D_{w_0}(u) \supset D_w(u)$, we obtain $C(u) \subset D_w(u)^\vee$ by taking their dual. This proves the lemma when $k = 0$.

Suppose $k \geq 1$ and the lemma holds for v with $\ell(v) \leq \ell(u) + k - 1$. We set

$$\bar{v} = u s_{i_1} \cdots s_{i_{k-1}} \quad \text{and} \quad i_k = p.$$

Then $v = \bar{v} s_p$, i.e.

$$(7.2) \quad \begin{aligned} \bar{v} &= v(1) \cdots v(p-1) v(p+1) v(p) v(p+2) \cdots v(n), \\ v &= v(1) \cdots v(p-1) v(p) v(p+1) v(p+2) \cdots v(n), \end{aligned}$$

and $C(\bar{v}) \subset D_w(u)^\vee$ by induction assumption (note that $\bar{v}' = u$ since $\bar{v} \in [u, v]_R$ and $v' = u$). Let (a, b) be an arbitrary element of $E_w(u)$. Since $e_b - e_a$ is non-negative on $C(\bar{v})$, a appears ahead of b in the one-line notation for \bar{v} .

If $\{v(p), v(p+1)\} \neq \{a, b\}$, then the order of a and b in the one-line notation for v is same as that for \bar{v} . Therefore, $e_b - e_a$ is non-negative on $C(v)$ as well. Thus it suffices to show that the case where $\{v(p), v(p+1)\} = \{a, b\}$ does not occur. Suppose $\{v(p), v(p+1)\} = \{a, b\}$. Then we have

$$(7.3) \quad a = v(p+1) < v(p) = b$$

because a appears ahead of b in the one-line notation for \bar{v} and $v = \bar{v} s_p$ with $\ell(v) = \ell(\bar{v}) + 1$. On the other hand, since $(a, b) \in E_w(u)$, a appears ahead of b in the one-line notation for u ; so $a = u(s)$ and $b = u(t)$ with some $s < t$, i.e.

$$\begin{aligned} u &= u(1) \cdots u(s-1) a u(s+1) \cdots u(t-1) b u(t+1) \cdots u(n), \\ t_{a,b} u &= u(1) \cdots u(s-1) b u(s+1) \cdots u(t-1) a u(t+1) \cdots u(n). \end{aligned}$$

Since $(a, b) \in E_w(u)$, we have $t_{a,b} u \leq w$ by definition. This means that

$$(7.4) \quad \{u(1), \dots, u(s-1), b\} \uparrow \leq w^{(s)}.$$

Now remember that $\bar{v}' = u$. This together with (7.2), (7.3) and (7.4) shows that $v'(s) = b$ and hence $v' \neq u$, a contradiction. \square

Lemma 7.10. *Let $x \leq w$, $y \leq w$ and $x \neq y$. If the cones $C_w(x)$ and $C_w(y)$ share a common facet, then they are related by a transposition, i.e. $x = t_{a,b} y$ for some transposition $t_{a,b}$.*

Proof. x and y are fixed points in the flag variety G/B under the torus action and the assumption implies that they are in CP^1 fixed pointwise under the codimension one subtorus corresponding to the facet $C_w(x) \cap C_w(y)$. In other words, x and y are vertices of the GKM graph of G/B and they are joined by an edge. Therefore they are related by a transposition ([9], see also [23, Proposition 2.1]). \square

We prepare one more lemma.

Lemma 7.11. *Let $u \leq w$. If $t_{u(j),u(k)}u \leq w$ with $j < k$, then $e_{u(k)} - e_{u(j)} \in D_w(u)$.*

Proof. It suffices to show that

$$(*) \quad (u(j), u(k)) \text{ is a finite sum of elements in } \tilde{E}_w(u).$$

If there is no $j < m < k$ such that $u(m)$ is in between $u(j)$ and $u(k)$ (we say that $u(j)$ and $u(k)$ are *adjacent* in this case), then $(u(j), u(k)) \in \tilde{E}_w(u)$ by definition of $\tilde{E}_w(u)$ and hence $e_{u(k)} - e_{u(j)} \in D_w(u)$ by definition of $D_w(u)$. Thus we may assume that $u(j)$ and $u(k)$ are not adjacent. We consider two cases.

Case 1. The case where $u(j) > u(k)$. In this case, there is a sequence $j < m_1 < m_2 < \cdots < m_p < k$ such that

$$u(j) > u(m_1) > u(m_2) > \cdots > u(m_p) > u(k) \quad \text{and}$$

$$u(m_\ell) \text{ and } u(m_{\ell+1}) \text{ are adjacent for } \ell = 0, 1, \dots, p, \text{ where } m_0 = j, m_{p+1} = k.$$

Since $(u(m_\ell), u(m_{\ell+1}))$ is an inversion of u and $u \leq w$, we have $t_{u(m_\ell),u(m_{\ell+1})}u \leq w$ and hence $(u(m_\ell), u(m_{\ell+1})) \in \tilde{E}_w(u)$ for $\ell = 0, 1, \dots, p$. This shows that

$$(7.5) \quad (u(j), u(k)) = \sum_{\ell=0}^p (u(m_\ell), u(m_{\ell+1}))$$

proving the assertion (*).

Case 2. The case where $u(j) < u(k)$. Similarly to Case 1 above, there is a sequence $j < m_1 < m_2 < \cdots < m_p < k$ such that

$$u(j) < u(m_1) < u(m_2) < \cdots < u(m_p) < u(k) \quad \text{and}$$

$$u(m_\ell) \text{ and } u(m_{\ell+1}) \text{ are adjacent for } \ell = 0, 1, \dots, p, \text{ where } m_0 = j, m_{p+1} = k.$$

In this case, since $(u(m_\ell), u(m_{\ell+1}))$ is not an inversion of u , it is not immediate that $t_{u(m_\ell),u(m_{\ell+1})}u \leq w$. However, $t_{u(j),u(k)}u \leq w$ by assumption and this will imply $t_{u(m_\ell),u(m_{\ell+1})}u \leq w$. Indeed, setting $u(m_\ell) = a_\ell$ for simplicity, one can see that

$$(7.6) \quad \begin{aligned} t_{a_0, a_1} u &= t_{a_{p+1}, a_1} t_{a_1, a_0} (t_{a_0, a_{p+1}} u), \\ t_{a_\ell, a_{\ell+1}} u &= t_{a_{p+1}, a_\ell} t_{a_{\ell+1}, a_0} t_{a_\ell, a_0} t_{a_{p+1}, a_{\ell+1}} (t_{a_0, a_{p+1}} u) \quad \text{for } \ell = 1, \dots, p. \end{aligned}$$

Here $t_{a_0, a_{p+1}} u = t_{u(j), u(k)} u \leq w$ by assumption and all the transpositions $t_{a,b}$ in (7.6) (except $t_{a_0, a_{p+1}}$) have $a > b$ and (a, b) is an inversion of the permutation, say v , which $t_{a,b}$ is applied to; so if $v \leq w$, then $t_{a,b} v \leq w$. Therefore (7.6) shows that $t_{a_\ell, a_{\ell+1}} u \leq w$ for $\ell = 0, 1, \dots, p$ and hence $(u(m_\ell), u(m_{\ell+1})) = (a_\ell, a_{\ell+1}) \in \tilde{E}_w(u)$, proving the assertion (*) by (7.5). \square

Now we are in a position to prove Proposition 7.7.

Proof of Proposition 7.7. We know $C_w(u) \subset D_w(u)^\vee$ by Lemma 7.9. Suppose that $C_w(u) \subsetneq D_w(u)^\vee$. Then there exist $v \in \mathfrak{S}_n$ and a simple reflection s_i such that

$$(7.7) \quad C(v) \subset C_w(u), \quad C(vs_i) \not\subset C_w(u), \quad C(vs_i) \subset D_w(u)^\vee.$$

Claim. $e_{v(i+1)} - e_{v(i)} \in D_w(u)$.

We admit the claim and complete the proof. Since $C(vs_i) \subset D_w(u)^\vee$ by assumption, the claim says that $e_{v(i+1)} - e_{v(i)}$ takes non-negative values on $C(vs_i)$. On the other hand, in the one-line notation of vs_i , $v(i+1)$ appears on the left of $v(i)$ and this means that $e_{v(i+1)} - e_{v(i)}$ takes non-positive values on $C(vs_i)$. Therefore $e_{v(i+1)} - e_{v(i)}$ must vanish on $C(vs_i)$ but this contradicts $C(vs_i)$ being of dimension n . Thus, it suffices to prove the claim above.

By (7.7), $C(vs_i) \subset C_w(x)$ for some $x(\neq u) \leq w$. The intersection $C(v) \cap C(vs_i)$ is a facet of $C(v)$ and $C(vs_i)$, and $C_w(u) \cap C_w(x)$ contains $C(v) \cap C(vs_i)$, so $C_w(u)$ and $C_w(x)$ are adjacent. Therefore $x = t_{a,b}u$ for some transposition $t_{a,b}$ by Lemma 7.10. Since $C(vs_i) \subset C_w(x)$ and $C(v) \subset C_w(u)$, it follows that

$$(7.8) \quad (vs_i)' = x = t_{a,b}u \quad \text{and} \quad v' = u.$$

We define j and k by

$$(7.9) \quad v(i) = u(j), \quad v(i+1) = u(k).$$

We consider two cases.

Case 1. The case where $v(i) < v(i+1)$. Since $v' = u$, we have $j < k$ in this case. This together with $(vs_i)' \neq u$ implies that

$$(7.10) \quad \begin{aligned} v' = u &= u(1) \cdots u(j-1)u(j) \cdots u(k) \cdots u(n), \\ (vs_i)' &= u(1) \cdots u(j-1)u(k) \cdots \cdots . \end{aligned}$$

On the other hand, we know $(vs_i)' = t_{a,b}u$ by (7.8). This together with (7.10) implies that

$$(vs_i)' = t_{u(j),u(k)}u.$$

Since $j < k$ and $(vs_i)' \leq w$, the above together with Lemma 7.11 shows that $e_{u(k)} - e_{u(j)} \in D_w(u)$. Then the claim follows from (7.9).

Case 2. The case where $v(i) > v(i+1)$. We claim $j < k$ in this case, too. Indeed, since $v(i+1)$ appears on the left of $v(i)$ in the one-line notation of vs_i and $v(i+1) < v(i)$, one sees that $v(i+1)(= u(k))$ appears ahead of $v(i)(= u(j))$ in the one line notation of $(vs_i)'$. Therefore, if $k < j$, then $(vs_i)'$ coincides with u which contradicts the assumption $(vs_i)' \neq u$. Therefore $j < k$. Then (7.10) holds in this case too and the claim follows from the same argument as in Case 1. This completes the proof of the claim and the proposition. \square

We associate a graph $\Gamma_w(u)$ to $E_w(u)$ as before. Note that $\Gamma_w(id)$ is the disjoint union of path graphs (in particular, a forest) by Remark 7.5(3). Then by Lemma 6.6, we obtain

Corollary 7.12. *The dual cone $D_w(u)$ is smooth if and only if $\Gamma_w(u)$ is a forest. Indeed, Y_w is smooth at the fixed point uB if and only if $\Gamma_w(u)$ is a forest. In particular, Y_w is smooth at the fixed point idB for any w .*

Remark 7.13. Since Y_w has at least one smooth T -fixed point idB , it follows from [11, Proposition 7.2 and Remark 7.3] that Y_w is smooth if and only if the Bruhat graph of Y_w is regular, where regular means that there are exactly $\dim_{\mathbb{C}} Y_w$ T -stable curves at each T -fixed point in Y_w .

Using the fact that idB is a smooth point in Y_w , we can give an alternative proof to the following result which is indeed proved for any Lie type in [17].

Corollary 7.14 ([17]). *A Schubert variety X_w is a toric variety if and only if w is a product of distinct simple reflections.*

Proof. The Schubert variety X_w is a toric variety if and only if $X_w = Y_w$. This is equivalent to $\dim_{\mathbb{C}} X_w = \dim_{\mathbb{C}} Y_w$ since X_w is irreducible and $X_w \supset Y_w$. As is well-known $\dim_{\mathbb{C}} X_w = \ell(w)$. On the other hand, $\dim_{\mathbb{C}} Y_w$ is equal to the dimension of any maximal cone in the fan of Y_w . Since the fixed point idB is smooth in Y_w , the maximal cone corresponding to idB is simplicial and hence its dimension is equal to $|E_w(id)|$ by Proposition 7.7. Here

$$E_w(id) = \{v \in \mathfrak{S}_n \mid \ell(v) = 1, v \leq w\}$$

(see Remark 7.5(3)) and the condition $\ell(v) = 1$ and $v \leq w$ is equivalent to v being a simple reflection appearing in a reduced expression of w . This implies the desired result since $\ell(w)$ is the number of simple reflections appearing in a reduced expression of w . \square

Remark 7.15. A toric Schubert variety X_w is a Bott manifold (the top space of a Bott tower in [15]). Indeed, since w is a product of simple reflections, the Bruhat graph of X_w is combinatorially same as the 1-skeleton of a cube. This means that the underlying simplicial complex of the fan of X_w is the boundary complex of a cross-polytope. Therefore X_w is a Bott manifold (see [8, Corollary 7.8.11]).

We propose the following conjecture.

Conjecture 7.16. *The graph $\Gamma_w(u)$ is a forest for any $u \leq w$ if Γ_w is a forest. (This is equivalent to saying that the generic torus orbit closure Y_w is smooth if it is smooth at the fixed point wB .)*

Here is an example which supports the conjecture.

Example 7.17. Take $w = 32154$. Then $E_w = \{(3, 2), (2, 1), (5, 4)\}$. Using simple transpositions s_i for $1 \leq i \leq 4$, we have that $w = s_1 s_2 s_1 s_4$. There are twelve elements $u \in \mathfrak{S}_5$ such that $u \leq w$, and we have Table 2 for $E_w(u)$. We can see that each $\Gamma_w(u)$ is a forest for $u \leq w$.

u	$E_w(u)$	u	$E_w(u)$
$s_1 s_2 s_1 s_4 = 32154$	$(3, 2), (2, 1), (5, 4)$	$s_1 s_2 s_1 = 32145$	$(3, 2), (2, 1), (4, 5)$
$s_4 s_2 s_1 = 31254$	$(3, 1), (1, 2), (5, 4)$	$s_4 s_1 s_2 = 23154$	$(2, 3), (3, 1), (5, 4)$
$s_2 s_1 = 31245$	$(3, 1), (1, 2), (4, 5)$	$s_1 s_2 = 23145$	$(2, 3), (3, 1), (4, 5)$
$s_4 s_1 = 21354$	$(2, 1), (1, 3), (5, 4)$	$s_4 s_2 = 13254$	$(1, 3), (3, 2), (5, 4)$
$s_1 = 21345$	$(2, 1), (1, 3), (4, 5)$	$s_2 = 13245$	$(1, 3), (3, 2), (4, 5)$
$s_4 = 12354$	$(1, 2), (2, 3), (5, 4)$	$e = 12345$	$(1, 2), (2, 3), (4, 5)$

TABLE 2. $E_w(u)$ for $u \leq w$ when $w = 32154$.

One can check Conjecture 7.16 for $n \leq 5$ with the aid of computers. Indeed, the computer calculation suggests that $b_1(\Gamma_w(u)) \leq b_1(\Gamma_w)$ for any $u \leq w$, where $b_1(\Gamma)$ denotes the first Betti number of a graph Γ . More strongly, it suggests that $b_1(\Gamma_w(u)) \leq b_1(\Gamma_w(t_{a,b}u))$ for $(a, b) \in E_w(u)$ with $a < b$. The latter implies that

any two non-simple vertices in the Bruhat interval polytope $Q_{id,w^{-1}}$ (see Section 8 for $Q_{id,w^{-1}}$) would be joined by edges with non-simple vertices as endpoints (see Figures 24 and 26 in Appendix B).

8. POINCARÉ POLYNOMIAL OF Y_w

The Eulerian number $A(n, k)$ is the number of permutations in \mathfrak{S}_n with k ascents and the Eulerian polynomial $A_n(t)$ is the generating function of the Eulerian numbers, i.e. $A_n(t) = \sum_{k=0}^{n-1} A(n, k)t^k$. As is well-known, the Poincaré polynomial of the permutohedral variety Y_{w_0} agrees with $A_n(t^2)$. In this section we consider a generalization of Eulerian numbers, introduce a polynomial $A_w(t)$ for each $w \in \mathfrak{S}_n$, and show that the Poincaré polynomial of Y_w is given by $A_w(t^2)$ when Y_w is smooth.

We set

$$\begin{aligned} a_w(u) &:= \#\{(u(i), u(j)) \in E_w(u) \mid u(i) < u(j)\} \quad \text{for } u \leq w, \\ A_w(n, k) &:= \#\{u \leq w \mid a_w(u) = k\} \quad \text{for } k \geq 0, \end{aligned}$$

and define

$$A_w(t) := \sum_{u \leq w} t^{a_w(u)} = \sum_{k \geq 0} A_w(n, k)t^k.$$

Example 8.1. Using Figure 3(1) and Table 3, one can find

$$\begin{aligned} A_w(t) &= t^3 + 11t^2 + 7t + 1 \quad \text{when } w = 4231, \\ A_w(t) &= t^3 + 7t^2 + 5t + 1 \quad \text{when } w = 3412. \end{aligned}$$

In these cases, $A_w(t)$ are not palindromic and Y_w are not smooth. On the other hand, from Table 2 one can easily find

$$A_w(t) = t^3 + 5t^2 + 5t + 1 \quad \text{when } w = 32154.$$

In this case, $A_w(t)$ is palindromic and Y_w is smooth.

u	$E_w(u)$	$a_w(u)$
4231	(4, 2), (4, 3), (2, 1), (3, 1)	0
4132	(4, 1), (4, 3), (1, 2), (3, 2)	1
4213	(4, 2), (2, 1), (1, 3)	1
2431	(2, 4), (4, 3), (3, 1)	1
3241	(3, 2), (3, 4), (2, 1), (4, 1)	1
1432	(1, 4), (4, 3), (3, 2)	1
4123	(4, 1), (1, 2), (2, 3)	2
2413	(2, 4), (4, 1), (1, 3)	2
3142	(3, 1), (3, 4), (1, 2), (4, 2)	2
3214	(3, 2), (2, 1), (1, 4)	1
2341	(2, 3), (3, 4), (4, 1)	2
1423	(1, 4), (4, 2), (2, 3)	2
1342	(1, 3), (3, 4), (4, 2)	2
2143	(2, 1), (1, 4), (4, 3)	1
3124	(3, 1), (1, 2), (2, 4)	2
2314	(1, 4), (2, 3), (3, 1)	2
1243	(1, 2), (2, 4), (4, 3)	2
1324	(1, 3), (3, 2), (2, 4)	2
2134	(2, 1), (1, 3), (3, 4)	2
1234	(1, 2), (2, 3), (3, 4)	3

$w = 4231.$

u	$E_w(u)$	$a_w(u)$
3412	(3, 1), (3, 2), (4, 1), (4, 2)	0
1432	(1, 3), (4, 3), (3, 2)	1
2413	(2, 1), (2, 3), (4, 1), (4, 3)	1
3142	(3, 1), (1, 4), (4, 2)	1
3214	(3, 2), (2, 1), (2, 4)	1
1423	(1, 2), (4, 2), (2, 3)	2
1342	(1, 3), (3, 4), (4, 2)	2
2143	(2, 1), (1, 4), (4, 3)	1
3124	(3, 1), (1, 2), (2, 4)	2
2314	(2, 3), (3, 1), (3, 4)	2
1243	(1, 2), (2, 4), (4, 3)	2
1324	(1, 3), (3, 2), (2, 4)	2
2134	(2, 1), (1, 3), (3, 4)	2
1234	(1, 2), (2, 3), (3, 4)	3

$w = 3412.$

TABLE 3. $E_w(u)$ and $a_w(u)$.

Remark 8.2. Looking at descents

$$d_w(u) := \#\{(u(i), u(j)) \in E_w(u) \mid u(i) > u(j)\} \quad \text{for any } u \leq w,$$

one can define a polynomial

$$\bar{A}_w(t) := \sum_{u \leq w} t^{d_w(u)}.$$

When Y_w is smooth, we have $\bar{A}_w(t) = A_w(t)$ but otherwise they may differ. Indeed

$$\bar{A}_w(t) = t^4 + 2t^3 + 6t^2 + 10t + 1 \quad \text{when } w = 4231,$$

$$\bar{A}_w(t) = t^4 + t^3 + 4t^2 + 7t + 1 \quad \text{when } w = 3412.$$

As remarked in Remark 7.5,

$$E_{w_0}(u) = \{(u(i), u(i+1)) \mid i = 1, 2, \dots, n-1\},$$

so $a_{w_0}(u)$ is the number of ascents in u and hence $A_{w_0}(n, k)$ is the number of permutations in \mathfrak{S}_n with k ascents, which is the Eulerian number $A(n, k)$. Therefore, $A_{w_0}(t)$ is the Eulerian polynomial $A_n(t)$. As is well-known, the Poincaré polynomial of the permutohedral variety Y_{w_0} is given by $A_n(t^2) = A_{w_0}(t^2)$. The following theorem generalizes this fact.

Theorem 8.3. *If Y_w is smooth, then its Poincaré polynomial agrees with $A_w(t^2)$ and hence $A_w(t)$ is palindromic and unimodal.*

Remark 8.4. When $w = 4231$ or 3412 , Y_w is singular and $A_w(t)$ is not palindromic but one can see that $A_w(t^2)$ still agrees with the Poincaré polynomial of Y_w , see Appendix B.

Proof of Theorem 8.3. The maximal cone $C_w(u)$ in the fan of Y_w corresponds to the fixed point uB and its dual cone $D_w(u)$ is spanned by $e_{u(j)} - e_{u(i)}$'s for $(u(i), u(j)) \in E_w(u)$. Suppose that Y_w is smooth. Then $e_{u(j)} - e_{u(i)}$'s for $(u(i), u(j)) \in E_w(u)$ are weights of the tangential T -representation of Y_w at uB . We choose an element $\mathbf{a} = [a_1, \dots, a_n] \in \mathbb{Z}^n$ such that $a_1 < a_2 < \dots < a_n$ and consider the \mathbb{C}^* -action on Y_w through $\lambda^{\mathbf{a}}: \mathbb{C}^* \rightarrow T$. Then $Y_w^{\mathbb{C}^*} = Y_w^T$ and the integers $\langle e_{u(j)} - e_{u(i)}, \mathbf{a} \rangle = a_{u(j)} - a_{u(i)}$ are the weights of the \mathbb{C}^* -action and the number of positive weights at uB is $a_w(u)$. Since $Y_w^{\mathbb{C}^*} = Y_w^T = \{uB \mid u \leq w\}$, this shows that the Poincaré polynomial of Y_w is given by $\sum_{u \leq w} t^{2a_w(u)} = A_w(t^2)$.

Since Y_w is smooth and projective, the Poincaré duality and the hard Lefschetz theorem imply the palindromicity and unimodality of the Poincaré polynomial of Y_w , proving the theorem. \square

We shall interpret the proof of Theorem 8.3 from the viewpoint of symplectic geometry. There is a map called moment map

$$\mu: G/B \rightarrow \mathbb{R}^n$$

such that $\mu(uB) = (u^{-1}(1), \dots, u^{-1}(n))$ and $\mu(G/B)$ is the convex hull of $\mu(uB)$'s for all $u \in \mathfrak{S}_n$, that is the permutohedron of dimension $n-1$. The image $\mu(Y_w)$ is the convex hull of $\mu(uB)$'s for all $u \leq w$ by [2, Theorem 2]. The vectors $e_{u(j)} - e_{u(i)}$'s for $(u(i), u(j)) \in E_w(u)$ are exactly the edge vectors of $\mu(Y_w)$ emanating from $\mu(uB)$. If Y_w is smooth, then the polytope $\mu(Y_w)$ is simple.

A vector $\mathbf{a} \in \mathbb{Z}^n$ gives a linear map $L^{\mathbf{a}}: \mathbb{R}^n \rightarrow \mathbb{R}$ defined by the pairing $\langle \mathbf{a}, \cdot \rangle$. If Y_w is smooth and \mathbf{a} is generic as in the proof of Theorem 8.3, then the composition $L^{\mathbf{a}} \circ \mu: Y_w \rightarrow \mathbb{R}$ is a moment map with respect to the action of the circle in \mathbb{C}^* through $\lambda^{\mathbf{a}}: \mathbb{C}^* \rightarrow T$, so that $L^{\mathbf{a}} \circ \mu$ is a Morse function with uB 's as critical points and $2d_w(u)$ is the Morse index at uB . The function $-L^{\mathbf{a}} \circ \mu: Y_w \rightarrow \mathbb{R}$ is also a Morse function but the Morse index at uB is $2a_w(u)$. This is the interpretation of the proof of Theorem 8.3 from the viewpoint of symplectic geometry.

Now, since Y_w is smooth if and only if $\Gamma_w(u)$ is a forest for any $u \leq w$ (Corollary 7.12), we obtain the following corollary from Theorem 8.3.

Corollary 8.5. *The polynomial $A_w(t)$ is palindromic if the graph $\Gamma_w(u)$ is a forest for any $u \leq w$.*

We conclude with remarks and questions about criterion of smoothness of Y_w . The polytope $\mu(Y_w)$ is the Bruhat interval polytope $Q_{id, w^{-1}}$ in [22] for any $w \in \mathfrak{S}_n$, where id denotes the identity permutation, and the vectors $e_{u(j)} - e_{u(i)}$'s for $(u(i), u(j)) \in E_w(u)$ are the edge vectors of the polytope emanating from the vertex $\mu(uB)$. Note that the 1-skeleton of the polytope $Q_{id, w^{-1}}$ is the Bruhat graph of Y_w (where the label of the vertex $\mu(uB)$ in the Bruhat graph is u although $\mu(uB) = (u^{-1}(1), \dots, u^{-1}(n))$). Therefore the following are equivalent:

- (1) Y_w is smooth.
- (2) The Bruhat interval polytope $Q_{id, w^{-1}}$ is simple, equivalently the Bruhat graph of Y_w is regular.
- (3) The graph $\Gamma_w(u)$ is a forest for any $u \leq w$.

The former part in Conjecture 7.16 says that the following (4) would imply (3) above:

- (4) The graph $\Gamma_w = \Gamma_w(w)$ is a forest.

Therefore, if the conjecture is true, then all four statements above are equivalent.

It is known that the Schubert variety X_w is smooth if and only if its Poincaré polynomial is palindromic (see [5, Theorem 6.0.4, Corollary 6.1.13, Theorem 6.2.4]). We may ask whether the same holds for Y_w , in other words, whether the following (5) is equivalent to (1) above:

(5) The Poincaré polynomial of Y_w is palindromic.

If Y_w is smooth, then $A_w(t)$ is palindromic (Theorem 8.3) but otherwise $A_w(t)$ may not be palindromic. As in Remark 8.4, $A_w(t^2)$ might agree with the Poincaré polynomial of Y_w for any $w \in \mathfrak{S}_n$. Related to (5), we may ask whether the following (6) is equivalent to (1) above:

(6) $A_w(t)$ is palindromic.

When $n \leq 3$, Y_w is smooth for all w . When $n = 4$, Y_w is smooth if and only if w is different from 4231 and 3412. As in Example 8.1, $A_w(t)$ is not palindromic when $w = 4231$ or 3412. Therefore (1) and (6) above are equivalent for $n \leq 4$.

APPENDIX A. ACYCLICITY OF GRAPH AND PATTERN AVOIDANCE OF PERMUTATION

In the appendix, we will give a proof of Proposition 6.9 which is different from the proof in [7]. If (a, b) is an inversion in w , namely $a > b$ and $w^{-1}(a) < w^{-1}(b)$, then there is a sequence of pairs $(a_i, b_i) \in E_w$ ($i = 1, \dots, k$) such that

$$a = a_1 > b_1 = a_2 > b_2 = a_3 > \dots > b_{k-1} = a_k > b_k = b.$$

This produces a path from a to b in Γ_w and we call such a path a *descending path* from a to b . There is a descending path from a to b if and only if (a, b) is an inversion of w . We denote by $\text{Inv}(w)$ the set of inversions of w .

Proof of Proposition 6.9. The proposition is equivalent to the statement that w has a pattern 4231 or 4512 which is not a subsequence of 45312 if and only if Γ_w has a cycle, and we shall prove this equivalent statement.

The only if part (\implies). **Case 1.** The case where w has the pattern 4231, see Figure 8(1). In this case there are $a, b, c, d \in [n]$ such that

$$w^{-1}(a) < w^{-1}(b) < w^{-1}(c) < w^{-1}(d) \quad \text{and} \quad a > c > b > d.$$

Since $(a, b), (b, d) \in \text{Inv}(w)$, there is a descending path P in Γ_w from a to d via b . Note that P does not contain c because $b \in P$ and $(b, c) \notin \text{Inv}(w)$. Similarly, since $(a, c), (c, d) \in \text{Inv}(w)$, there is a descending path Q in Γ_w from a to d via c and Q does not contain b because $c \in Q$ and $(b, c) \notin \text{Inv}(w)$. Therefore the union $P \cup Q$ contains a cycle.

Case 2. The case where w has the pattern 4512 which is not a subsequence of 45312, see Figure 8(2). The argument is similar to Case 1 but rather subtle. In this case there are $a, b, c, d \in [n]$ such that

$$(A.1) \quad w^{-1}(a) < w^{-1}(b) < w^{-1}(c) < w^{-1}(d) \quad \text{and} \quad b > a > d > c$$

and we may assume that

$$(A.2) \quad \begin{aligned} &\text{there is no } p \text{ such that } a > p > d \text{ and } w^{-1}(a) < w^{-1}(p) < w^{-1}(d), \text{ and} \\ &\text{there is no } q \text{ such that } b > q > c \text{ and } w^{-1}(b) < w^{-1}(q) < w^{-1}(c). \end{aligned}$$



(1) w has pattern 4231.

(2) w has pattern 4512 which is not a subsequence of 45312.

FIGURE 8. Graph Γ_w .

Since $(a, d) \in \text{Inv}(w)$, there is a descending path P_1 from a to d and since $(a, c) \in \text{Inv}(w)$, there is a descending path P_2 from a to c . Both P_1 and P_2 do not contain the vertex b because $b > a$. Note that

$P_1 \cap P_2 = \{a\}$, so $P_1 \cup P_2$ is a path which joins c and d and does not contain b .

Indeed, the vertex a is an end point of P_1 and P_2 and if P_1 and P_2 has another common vertex, say p , then $a > p > d$ and $w^{-1}(a) < w^{-1}(p) < w^{-1}(c)$ but this contradicts (A.2) because $w^{-1}(c) < w^{-1}(d)$ by (A.1). Similarly, since $(b, c) \in \text{Inv}(w)$, there is a descending path Q_1 from b to c , and since $(b, d) \in \text{Inv}(w)$, there is a descending path Q_2 from b to d . Both Q_1 and Q_2 do not contain the vertex a because they are descending paths and $w^{-1}(a) < w^{-1}(b)$. Note also that

$Q_1 \cap Q_2 = \{b\}$, so $Q_1 \cup Q_2$ is a path which joins c and d and does not contain a .

Indeed, the vertex b is an end point of Q_1 and Q_2 and if Q_1 and Q_2 have another common vertex, say q , then $b > q > d$ and $w^{-1}(b) < w^{-1}(q) < w^{-1}(c)$ but this contradicts (A.2) because $d > c$ by (A.1).

Both $P := P_1 \cup P_2$ and $Q := Q_1 \cup Q_2$ are *paths* joining c and d , and $a \in P$ but $a \notin Q$. Therefore, the union $P \cup Q$ contains a cycle.

The if part (\Leftarrow). Suppose that Γ_w contains a cycle S . We may assume that S is a circle.

Claim. The cycle S contains at least four vertices.

Indeed, if it has only three vertices, say x, y, z and $w^{-1}(x) < w^{-1}(y) < w^{-1}(z)$, then since S is a triangle, we must have $(x, y), (y, z), (x, z) \in E_w$ (Figure 9) and the belonging of the first two elements to E_w implies $x > y > z$ but this contradicts (x, z) being in E_w . Therefore the claim follows.

Let a (resp. d) be the vertex of S such that $w^{-1}(a) \leq w^{-1}(x)$ (resp. $w^{-1}(x) \leq w^{-1}(d)$) for all vertices x of S , in other words,

a is leftmost while d is rightmost among vertices of S
in the one-line notation for w .

We call the interval $[w^{-1}(a), w^{-1}(d)]$ the *width* of S . We may assume that the width of S is minimal, i.e. there is no cycle S' in Γ_w such that the width of S' is properly contained in the width of S . We say that a descending path P from x to

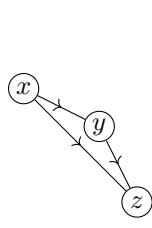


FIGURE 9

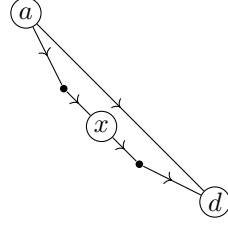


FIGURE 10

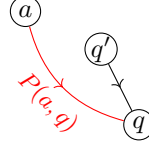


FIGURE 11

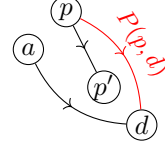


FIGURE 12

y is *emanating* from x or is *terminating* at y . Note that since a (resp. d) is the leftmost (resp. rightmost) vertex of S , the two edges at the vertex a (resp. d) in S are emanating from a (resp. terminating at d).

We consider two cases.

Case I. The case where a and d are joined by an edge, i.e. $(a, d) \in E_w$. In this case we will find vertices b and c of S such that $abcd$ is the pattern 4512 which is not a subsequence of 45312.

Since $(a, d) \in E_w$, we have

$$(A.3) \quad a > d.$$

We note that the path from a to d (outside of the edge (a, d)) is not a descending path because if so, any vertex x in the middle of the path (Figure 10) satisfies $w^{-1}(a) < w^{-1}(x) < w^{-1}(d)$ and $a > x > d$ but this contradicts $(a, d) \in E_w$. Therefore, there is a vertex q ($q \neq a, d$) of S such that the descending path from a to q is maximal, which means that if q' is the vertex of S next to q but outside of the descending path, then $(q', q) \in E_w$ (Figure 11). We denote this maximal descending path by

$$(A.4) \quad P(a, q).$$

Similarly, there is a vertex p ($p \neq a, d, q$) of S such that the path

$$(A.5) \quad P(p, d)$$

from p to d which does not contain a is a maximal descending path terminating at d , which means that if p' is the vertex of S next to p and outside of $P(p, d)$, then $(p, p') \in E_w$ (Figure 12). It follows that

$$(A.6) \quad \begin{aligned} a &> q, & p &> d \\ p &> p', & w^{-1}(p) &< w^{-1}(p'), \\ q' &> q, & w^{-1}(q') &< w^{-1}(q). \end{aligned}$$

(Although we will not use the last two inequalities above for q and q' , the argument below will work if we use q and q' instead of p and p' .)

We claim

$$(A.7) \quad p > a > d > q.$$

Indeed, if $a > p$ (Figure 13), then $a > p > d$ by (A.6) and $w^{-1}(a) < w^{-1}(p) < w^{-1}(d)$ since a is the leftmost vertex of S and d is the rightmost vertex of S . This contradicts $(a, d) \in E_w$. Therefore $p > a$. Similarly, if $q > d$ (Figure 14),

then $a > q > d$ by (A.6) and $w^{-1}(a) < w^{-1}(q) < w^{-1}(d)$ but this contradicts (a, d) being in E_w . Therefore $d > q$. These together with (A.3) prove (A.7).

Now we take two cases.

(1) The case where $w^{-1}(p) < w^{-1}(q)$ (Figure 15). In this case, we take $b = p$ and $c = q$. Then $w^{-1}(a) < w^{-1}(b) < w^{-1}(c) < w^{-1}(d)$ by assumption and $b > a > d > c$ by (A.7). Therefore $abcd$ is the pattern 4512. Moreover, since $(a, d) \in E_w$, there is no r such that $a > r > d$ and $w^{-1}(a) < w^{-1}(r) < w^{-1}(d)$, which means that $abcd$ is not a subsequence of 45312.

(2) The case where $w^{-1}(q) < w^{-1}(p)$ (Figure 16). We look at p' defined above (a similar argument will work if we look at q'). We claim

$$(A.8) \quad d > p'.$$

Indeed, if $p' > d$ (like Figure 16), then $(p', d) \in \text{Inv}(w)$ and hence there is a descending path

$$P(p', d)$$

from p' to d and $P(p', d)$ does not contain p because $w^{-1}(p) < w^{-1}(p')$. Therefore the union

$$(p, p') \cup P(p', d)$$

is a descending path from p to d . On the other hand, $P(p, d)$ in (A.5) is also a descending path from p to d but does not contain p' . Therefore the union

$$((p, p') \cup P(p', d)) \cup P(p, d)$$

contains a cycle and its width is $[w^{-1}(p), w^{-1}(d)]$. Since this width is properly contained in the width $[w^{-1}(a), w^{-1}(d)]$ of S , this contradicts the minimality of the width of S . Therefore $d > p'$.

By (A.7) and (A.8) we have

$$(A.9) \quad p > a > d > p' \quad (\text{Figure 17}).$$

Moreover, since a is the leftmost vertex of the cycle S and d is the rightmost vertex of S , it follows from (A.6) that

$$w^{-1}(a) < w^{-1}(p) < w^{-1}(p') < w^{-1}(d).$$

This together with (A.9) shows that if we take $b = p$ and $c = p'$, then $abcd$ is the pattern 4512. Moreover, since $(a, d) \in E_w$, there is no r such that $a > r > d$ and $w^{-1}(a) < w^{-1}(r) < w^{-1}(d)$, which means that $abcd$ is not a subsequence of 45312.

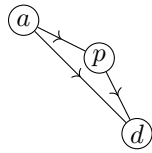


FIGURE 13

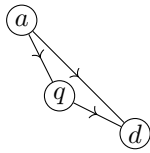


FIGURE 14

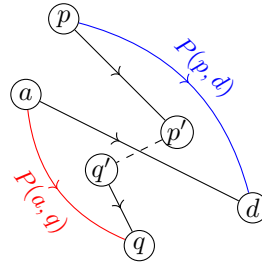


FIGURE 15

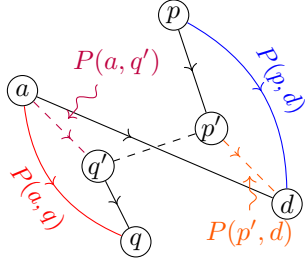


FIGURE 16

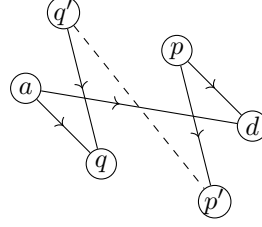


FIGURE 17

Case II. The case where a and d are not joined by an edge, i.e. $(a, d) \notin E_w$. In this case we will find vertices b and c of S such that $abcd$ is the pattern 4231. There are two edges of S emanating from a and we take vertices $p, q \in S$ such that $(a, p), (a, q) \in E_w$. We may assume

$$(A.10) \quad w^{-1}(p) < w^{-1}(q)$$

without loss of generality (Figures 18 and 19). Since $(a, p), (a, q) \in E_w$, we have

$$a > p, \quad a > q.$$

We claim $q > p$ so that

$$(A.11) \quad a > q > p \quad (\text{i.e. Figure 19 does not occur}).$$

Indeed, if $p > q$ (Figure 19), then $a > p > q$ and $w^{-1}(a) < w^{-1}(p) < w^{-1}(q)$ by (A.10). But this contradicts (a, q) being in E_w .

Now we take two cases.

(1) The case where $p > d$ (Figure 20). In this case $a > q > p > d$ by (A.11) and $w^{-1}(a) < w^{-1}(p) < w^{-1}(q) < w^{-1}(d)$ by (A.10). Therefore if we take $b = p$ and $c = q$, then $abcd$ is the pattern 4231.

(2) The case where $d > p$. We shall observe that this case does not occur. Let p' be the vertex of S such that the path

$$P(p, p')$$

in S joining p and p' is a maximal descending path from p to p' so that if p'' is the vertex next to p' and outside of $P(p, p')$, then $(p'', p') \in E_w$ (Figures 21 and 22). By the choice of p' and p'' we have

$$(A.12) \quad w^{-1}(p) \leq w^{-1}(p'), \quad p \geq p' \quad \text{and} \quad w^{-1}(p'') < w^{-1}(p'), \quad p'' > p'.$$

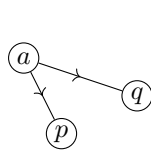


FIGURE 18

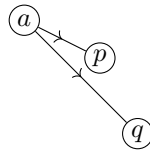


FIGURE 19

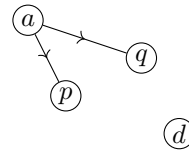


FIGURE 20

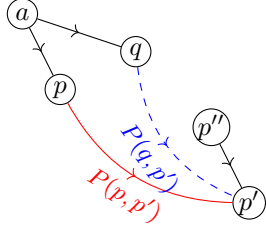


FIGURE 21

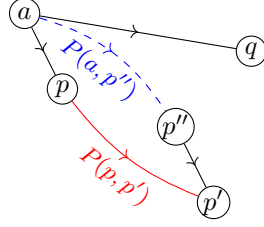


FIGURE 22

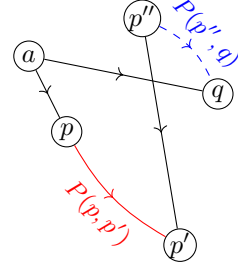


FIGURE 23

If $w^{-1}(q) < w^{-1}(p')$ (Figure 21), then there is a descending path $P(q, p')$ from q to p' in Γ_w because $q > p'$ by (A.11) and (A.12). Since $P(q, p')$ does not contain p by (A.10), the union $(a, q) \cup P(q, p')$ is a descending path from a to p' which does not contain p . On the other hand, the union $(a, p) \cup P(p, p')$ is also a descending path from a to p' but contains p . Therefore the union

$$((a, q) \cup P(q, p')) \cup ((a, p) \cup P(p, p'))$$

contains a cycle and its width is $[w^{-1}(a), w^{-1}(p')]$. But this contradicts the minimality of the width of S . Therefore

$$(A.13) \quad w^{-1}(p') < w^{-1}(q) \quad (\text{Figure 22}).$$

By (A.12) and (A.13), we have

$$w^{-1}(p'') < w^{-1}(q).$$

If $a > p''$ (like Figure 22), then $(a, p'') \in \text{Inv}(w)$ and hence there is a descending path $P(a, p'')$ from a to p'' in Γ_w . The union $P(a, p'') \cup (p'', p')$ is a descending path from a to p' . On the other hand, the union $(a, p) \cup P(p, p')$ is also a descending path from a to p' but does not contain p'' because p'' is the vertex of S next to p' but outside of $P(p, p')$. Therefore the union

$$(P(a, p'') \cup (p'', p')) \cup ((a, p) \cup P(p, p'))$$

contains a cycle and its width is $[w^{-1}(a), w^{-1}(p')]$. But this contradicts the minimality of the width of S . Therefore $p'' > a$ and hence

$$p'' > a > q \quad (\text{Figure 23})$$

by (A.11). Then since $w^{-1}(p'') < w^{-1}(q)$ by (A.12) and (A.13), we have $(p'', q) \in \text{Inv}(w)$ and hence there is a descending path $P(p'', q)$ from p'' to q in Γ_w . The union

$$(A.14) \quad P(p'', q) \cup (p'', p')$$

is a *path* joining p' and q (note that (p'', p') is an edge). Similarly, the union

$$(A.15) \quad (a, p) \cup P(p, p') \cup (a, q)$$

is also a *path* joining p' and q . However, the path in (A.14) does not contain the vertex a while the path in (A.15) does. Therefore the union of these two paths contains a cycle and its width is $[w^{-1}(a), w^{-1}(q)]$. This again contradicts the minimality of the width of S . Thus Case II (2) does not occur. \square

APPENDIX B. RETRACTION SEQUENCES OF A POLYTOPE AND
THE POINCARÉ POLYNOMIAL

In this appendix, we compute Poincaré polynomials of Y_w when $w = 4231$ and $w = 3412$. We recall the definition of *retraction sequences* of a polytope from [3, 4].

Definition B.1. A *polytopal complex* \mathcal{C} is a finite collection of polytopes in \mathbb{R}^n satisfying:

- (1) if E is a face of F and $F \in \mathcal{C}$ then $E \in \mathcal{C}$.
- (2) If $E, F \in \mathcal{C}$ then $E \cap F$ is a face of both E and F .

We denote the underlying set of a polytopal complex \mathcal{C} by $|\mathcal{C}| = \bigcup_{F \in \mathcal{C}} F$. Given an n -dimensional polytope Q , the polytopal complex $\mathcal{C}(Q)$ is a collection of all faces of Q . For a polytopal complex \mathcal{C} , we call a vertex $v \in |\mathcal{C}|$ is a *free vertex* if it has a neighborhood in $|\mathcal{C}|$ that is diffeomorphic to $\mathbb{R}_{\geq 0}^N$ as manifolds with corners for some integer N .

Definition B.2 ([4, §2]). A *retraction sequence* of a polytope Q is a sequence of triples $\{(B_k, E_k, b_k)\}_{1 \leq k \leq \ell}$ defined inductively:

- $B_1 = Q = E_1$ and b_1 is a free vertex in Q , i.e. a simple vertex.
- Given (B_k, E_k, b_k) , the next term $(B_{k+1}, E_{k+1}, b_{k+1})$ is defined to be:

$$B_{k+1} := |\{E \in \mathcal{C}(B_k) \mid b_k \notin V(E)\}|,$$

b_{k+1} is a free vertex in B_{k+1} , and E_{k+1} is the maximal face of B_{k+1} containing b_{k+1} .

Here, ℓ is the number of vertices of Q .

It is known that every simple polytope admits a retraction sequence, see [4, Proposition 2.3]. Suppose that X is a projective toric variety whose underlying polytope is Q . If Q admits a retraction sequence, then we can construct a \mathbf{q} -CW complex structure on X using the similar argument in the proof of [3, Proposition 4.4] (This observation will be deeply studied in [21]). We note that \mathbf{q} -CW complex structure is a generalization of CW complex structure using the quotient of a disc by the action of a finite group rather than ordinary cells, see [19]. Since a retraction sequence $\{(B_k, E_k, b_k)\}_{1 \leq k \leq \ell}$ produces only even dimensional “cells” of dimensions $\{2 \dim(E_k)\}_{1 \leq k \leq \ell}$, one can show the following:

Proposition B.3. *Suppose that X is a projective toric variety whose underlying polytope is Q . If Q admits a retraction sequence $\{(B_k, E_k, b_k)\}_{k=1}^{\ell}$ then the Poincaré polynomial of X agrees with*

$$\sum_{k=1}^{\ell} t^{2 \dim(E_k)}.$$

When $w = 4231$, the Bruhat interval polytope $Q_{id, w-1}$ is given in Figure 24. The vertices are labeled by the corresponding permutations, i.e. $\mu(uB) = (u^{-1}(1), \dots, u^{-1}(n))$ is labeled by u for $u \leq w$. The polytope $Q_{id, w-1}$ is not simple but it admits a retraction sequence as in Figure 25. By Proposition B.3, the Poincaré polynomial of Y_w is $t^6 + 11t^4 + 7t^2 + 1$, which agrees with $A_w(t^2)$ (see Table 4).

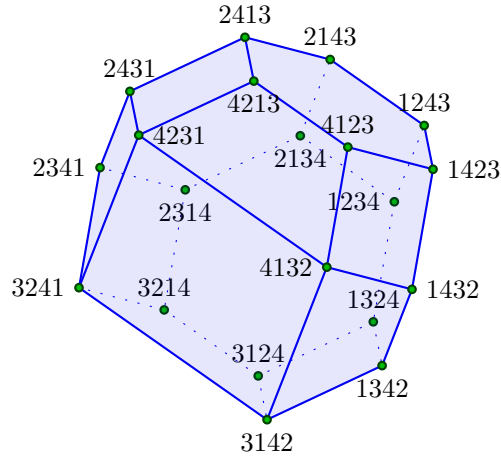


FIGURE 24. The Bruhat interval polytope $Q_{id,4231^{-1}} = Q_{id,4231}$.

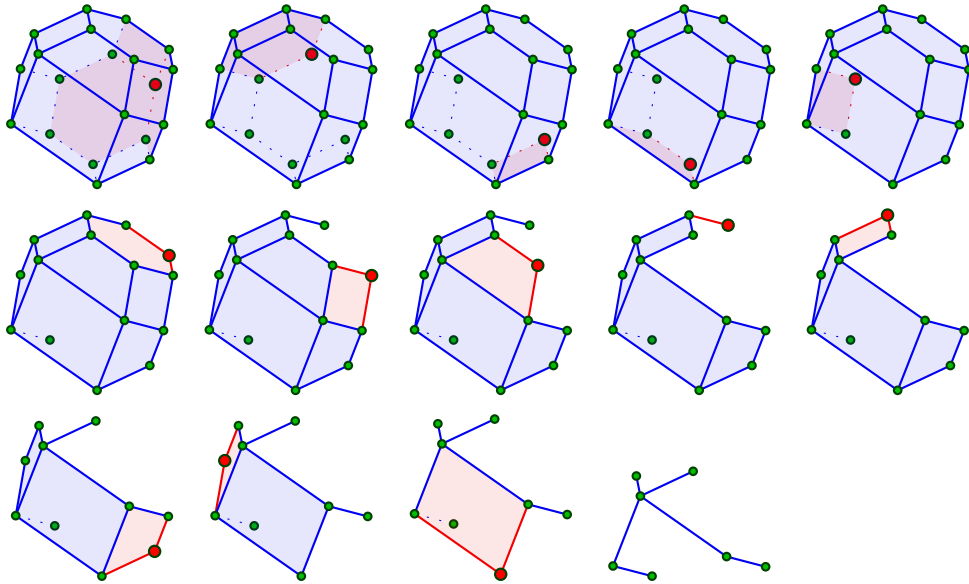


FIGURE 25. A retraction sequence $\{(B_k, E_k, b_k)\}$ of $Q_{id,4231}$.

b_k	1234	2134	1324	3124	2314	1243	1423	4123	2143	2413
$\dim(E_k)$	3	2	2	2	2	2	2	2	1	2
b_k	1342	2341	3142	1432	4132	3214	3241	4213	2431	4231
$\dim(E_k)$	2	2	2	1	1	1	1	1	1	0

TABLE 4. A sequence of vertices b_k and dimensions $\dim(E_k)$.

The Bruhat interval polytope $Q_{id,w^{-1}}$ for $w = 3412$ is given in Figure 26. The vertex $\mu(uB) = (u^{-1}(1), \dots, u^{-1}(n))$ is labeled by u for $u \leq w$. The polytope $Q_{id,w^{-1}}$ is not simple but it admits a retraction sequence as in Figure 27. By Proposition B.3, the Poincaré polynomial of Y_w is $t^6 + 7t^4 + 5t^2 + 1$, which agrees with $A_w(t^2)$ (see Table 5).

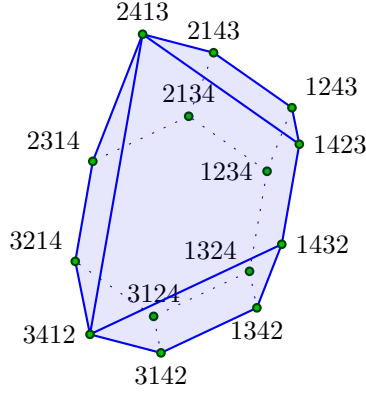


FIGURE 26. The Bruhat interval polytope $Q_{id,3412^{-1}} = Q_{id,3412}$.

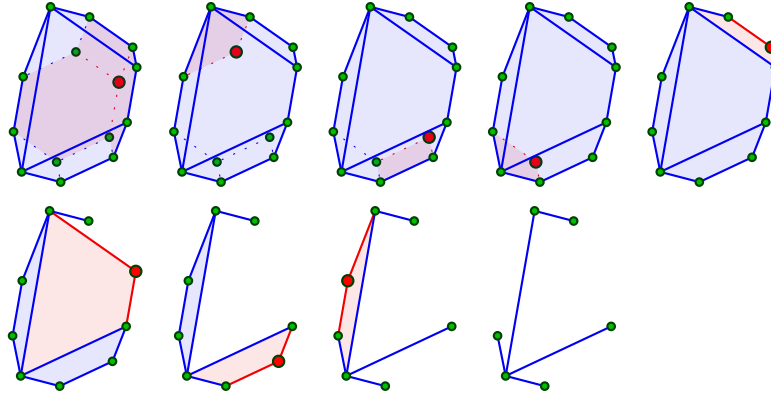


FIGURE 27. A retraction sequence $\{(B_k, E_k, b_k)\}$ of $Q_{id,3412}$.

b_k	1234	2134	1324	3124	1243	1423	1342
$\dim(E_k)$	3	2	2	2	2	2	2
b_k	2314	2143	2413	1432	3142	3214	3412
$\dim(E_k)$	2	1	1	1	1	1	0

TABLE 5. A sequence of vertices b_k and dimensions $\dim(E_k)$.

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