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Abstract. In this paper, we study the geometry of various Hessenberg varieties in type A, as well as families thereof, with the additional goal of laying the groundwork for future computations of Newton-Okounkov bodies of Hessenberg varieties. Our main results are as follows. We find explicit and computationally convenient generators for the local defining ideals of indecomposable regular nilpotent Hessenberg varieties, and then show that all regular nilpotent Hessenberg varieties are local complete intersections. We also show that certain families of Hessenberg varieties, whose generic fibers are regular semisimple Hessenberg varieties and whose special fiber is a regular nilpotent Hessenberg variety, are flat and have reduced fibres. This result further allows us to give a computationally effective formula for the degree of a regular nilpotent Hessenberg variety with respect to a Plücker embedding. Furthermore, we construct certain flags of subvarieties of a regular nilpotent Hessenberg variety, obtained by intersecting with Schubert varieties, which are suitable for computing Newton-Okounkov bodies. As an application of our results, we explicitly compute many Newton-Okounkov bodies of the two-dimensional Peterson variety with respect to Plücker embeddings.

Contents

1. Introduction

In this paper we study Hessenberg varieties of various types and families thereof, with a view towards applications to the theory of Newton-Okounkov bodies and, more generally, the rather new connections between the theory of Hessenberg varieties and combinatorics (e.g. [10, 4]). We first provide some background before listing our concrete results.

Throughout this paper, for simplicity we restrict to Lie type A although we suspect that our discussion generalizes to other Lie types.

Hessenberg varieties in type A are subvarieties of the full flag variety $\mathrm{Flags}(\mathbb{C}^n)$ of nested sequences of linear subspaces in $\mathbb{C}^n$. Their geometry and (equivariant) topology have been studied extensively since the late 1980s [13, 15, 14]. This subject lies at the intersection of, and makes connections between, many research areas such as geometric representation theory (see for example [42, 20]), combinatorics (see e.g. [18, 31]), and algebraic geometry and topology (see e.g. [29, 9, 43, 25, 37, 38, 2]). A special case of Hessenberg varieties

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called the Peterson variety Pet_n arises in the study of the quantum cohomology of the flag variety [29, 39], and more generally, geometric properties and invariants of many different types of Hessenberg varieties (including in Lie types other than A) have been widely studied. In addition, very recent developments provide further evidence of deep connections between Hessenberg varieties and combinatorics. Specifically, Shareshian and Wachs formulated in 2011 a conjecture [41] relating the chromatic quasisymmetric function of the incomparability graph of a natural unit interval order to an $\mathfrak{S}_n$-representation on the cohomology of the associated regular semisimple Hessenberg variety as defined by Tymoczko [44]. The Shareshian-Wachs conjecture represents a significant step towards a solution of the famous Stanley-Stembridge conjecture in combinatorics (concerning e-positivity of certain chromatic polynomials). Recently Brosnan and Chow [10] proved the Shareshian-Wachs conjecture by showing a remarkable relationship between the Betti numbers of different Hessenberg varieties; a key ingredient in their approach is a certain family of Hessenberg varieties.

We next briefly introduce the theory of Newton-Okounkov bodies, which was a significant motivation for the current manuscript. Briefly, this relatively recent theory gives a new method of associating combinatorial data to geometric objects. Recall that the famous Atiyah-Guillemin-Sternberg and Kirwan convexity theorems link equivariant symplectic and algebraic geometry to the combinatorics of polytopes. In the case of a toric variety X, the combinatorics of its moment map polytope Δ fully encodes the geometry of X, but this fails in the general case. Building on the work of Okounkov [34, 35], Kaveh-Khovanskii [27] and Lazarsfeld-Mustaţă [30] construct a convex body Δ in $\mathbb{R}^n$ associated to X equipped with the auxiliary data of a divisor D and a choice of valuation ν on the space of rational functions $\mathbb{C}(X)$. The theory of Newton-Okounkov bodies is powerful for several reasons. Firstly, it applies to an arbitrary projective algebraic variety, and secondly, under a mild hypothesis on the auxiliary data, the construction guarantees that the associated convex body Δ is maximal-dimensional, as in the classical setting of toric varieties. Hence one interpretation of the results of Lazarsfeld-Mustata and Kaveh-Khovanskii is that there *is* a combinatorial object of 'maximal' dimension associated to X, even when X is not a toric variety. This represents a vast expansion of the possible settings in which combinatorial methods may be used to analyze the geometry of algebraic varieties. There is promise of a rich theory which interacts with a wide range of inter-related areas: for instance, Kaveh showed in [26] that the Littelmann-Berenstein-Zelevinsky string polytopes from representation theory, which generalize the well-known Gelfand-Cetlin polytopes, are examples of Δ. Nevertheless, there are many open questions in Newton-Okounkov body theory; in particular, relatively few explicit examples of Newton-Okounkov bodies have been computed thus far. Therefore, it is an interesting problem to compute new concrete examples, and one of our motivations for this paper was to compute Newton-Okounkov bodies of Hessenberg varieties and to analyze the relation between the combinatorics of these Newton-Okounkov bodies with the existing results which relate geometric invariants of Hessenbergs to combinatorics.

We now turn to a description of the main results of this paper. In the first part, we study Hessenberg varieties and families thereof by studying their local equations. More specifically, we do the following. (For definitions we refer to Section 2.)

(1) We determine an explicit list of generators for the local defining ideals of indecomposable regular nilpotent Hessenberg varieties (Proposition 3.5).
(2) We prove that regular nilpotent Hessenberg varieties are local complete intersections (Theorem 4.1).
(3) We prove that certain families of Hessenberg varieties are irreducible, flat over $\mathbb{A}^1$ (or $\mathbb{P}^1$), and have reduced fibers (Theorem 5.1).

By exploiting the above, in the second part of the paper we begin to develop a theory of the Newton-Okounkov bodies of Hessenberg varieties.

(4) We construct families of flags $Y_\bullet = \{Y_0 = \mathrm{Hess}(N, h) \supset Y_1 \supset \cdots \supset Y_n\}$ of subvarieties in regular nilpotent Hessenberg varieties arising from intersections with (dual) Schubert varieties; the intersections are smooth at $Y_n = \{\mathrm{pt}\}$, where $n = \dim_{\mathbb{C}} \mathrm{Hess}(N, h)$ (Theorem 7.4).
(5) We compute the degree of an arbitrary indecomposable regular nilpotent Hessenberg variety with respect to a Plücker embedding associated to a weight $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ as a polynomial in the λ_i (Theorem 8.3).
(6) We explicitly compute many Newton-Okounkov bodies associated to the Peterson variety in $\mathrm{Flags}(\mathbb{C}^3)$, a special case of regular nilpotent Hessenberg varieties (Theorems 9.6 and 9.10).

Some remarks are in order. Firstly, our results in (1) and (2) generalize a result of Insko-Yong [25] for the case of Peterson variety and also a result of Insko [24] where he proves that regular nilpotent Hessenberg varieties are local complete intersections under the hypothesis that the Hessenberg function is strictly increasing. Secondly, the family we consider in (3) is presumably the one which is meant in the discussion in [5], where it is also mentioned (without further discussion) that the family is flat. While we do not claim originality, we record in this paper a proof that the family is flat using standard techniques; the more important claim in (3) is that the fibres are reduced, and for this we analyze generators of the defining ideal of the family in a manner similar to (1). Thirdly, the reason for studying the flags of subvarieties in (4) is that well-behaved such flags are often a crucial ingredient in the construction of Newton-Okounkov bodies, as we explain further in Section 6. Fourthly, the polynomial mentioned in (5) is called a *volume polynomial* in [4], where the authors also show that the natural Poincaré duality algebra associated to this polynomial is in fact isomorphic to the ordinary cohomology ring of the regular nilpotent Hessenberg variety. Furhtermore, our computation of the degree in (5) proves useful for the computation of Newton-Okounkov bodies, as we explain in Section 6. Finally, we view the results of (6) as a first "test case" of a Newton-Okounkov-type computation for Hessenberg varieties, which illustrate the computational potential of the results of this paper.

The paper is organized as follows. We briefly recall definitions concerning Hessenberg varieties in Section 2. In Section 3 we produce a list of generators for the local defining ideals of regular nilpotent Hessenberg varieties. The results of Section 3 allow us to show in Section 4 that regular nilpotent Hessenberg varieties are local complete intersections. In Section 5 we study a family of Hessenberg varieties, give a set of generators for its defining ideals in an argument similar to Section 3, and use these generators to prove that the fibers of the family are reduced. A brief introduction to the theory of Newton-Okounkov bodies is in Section 6 where, in particular, we make an elementary but nevertheless crucial observation concerning the degree of a projective variety and its role in the computation of Newton-Okounkov bodies in Section 6.2. In Section 7 we construct a flag of subvarieties in a regular nilpotent Hessenberg variety which has convenient geometric properties and is very natural from the point of view of Schubert calculus. We give a semi-explicit formula for the degree of regular nilpotent Hessenberg varieties in Section 8 which links our work to that of [4]. Finally, in Section 9 we utilize everything we have shown thus far to give our first complete computation of a Newton-Okounkov body of a Hessenberg variety.

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2. Preliminaries: Hessenberg varieties

In this section we recall some basic definitions used in the study of Hessenberg varieties. Since detailed exposition is available in the literature [43, 14] we keep discussion brief.

Throughout this paper, for simplicity we restrict attention to Lie type A, i.e. to the case $G = \mathrm{GL}_n(\mathbb{C})$. We expect that analogous results will hold for more general Lie types but leave this open for future work.

By the **flag variety** we mean the homogeneous space $\mathrm{GL}_n(\mathbb{C})/B$, where B denotes the subgroup of upper-triangular matrices. This homogeneous space may also be identified with the space of nested sequences of linear subspaces of $\mathbb{C}^n$, i.e.

$$\mathrm{Flags}(\mathbb{C}^n) := \{V_\bullet = (\{0\} \subseteq V_1 \subseteq V_2 \subseteq \cdots V_{n-1} \subseteq V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}}(V_i) = i\} \cong \mathrm{GL}_n(\mathbb{C})/B; \tag{2.1}$$

the identification with $\mathrm{GL}_n(\mathbb{C})/B$ takes a coset MB, for $M \in \mathrm{GL}_n(\mathbb{C})$, to the flag $V_\bullet$ with V_i defined as the span of the leftmost i columns of M.

We use the notation

$$[n] := \{1, 2, \ldots, n\}.$$

A **Hessenberg function** is a function $h\colon [n] \to [n]$ satisfying $h(i) \geqslant i$ for all $1 \leqslant i \leqslant n$ and $h(i+1) \geqslant h(i)$ for all $1 \leqslant i < n$. We frequently denote a Hessenberg function by listing its values in sequence, $h = (h(1), h(2), \ldots, h(n) = n)$. To a Hessenberg function h we associate a subspace of $\mathfrak{gl}_n(\mathbb{C})$ (the vector space of $n \times n$ complex matrices) defined as

$$H(h) := \{(a_{i,j})_{i,j \in [n]} \in \mathfrak{gl}_n(\mathbb{C}) \mid a_{i,j} = 0 \text{ if } i > h(j)\}, \tag{2.2}$$

which we call the **Hessenberg subspace** $H(h)$. It is sometime useful to visualize this space as a configuration of boxes on a square grid of size $n \times n$ whose shaded boxes correspond to the $a_{i,j}$ with no condition imposed in the right-hand side of (2.2) (see Figure 2.1).

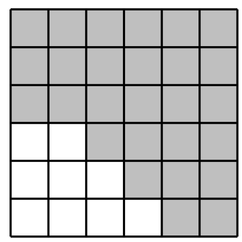


FIGURE 2.1. The picture of $H(h)$ for $h = (3, 3, 4, 5, 6, 6)$.

We can now define the central object of study.

Definition 2.1. Let $A\colon \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator and $h\colon [n] \to [n]$ a Hessenberg function. The **Hessenberg variety** associated to A and h is defined to be

$$\mathrm{Hess}(A, h) := \{V_\bullet \in \mathrm{Flags}(\mathbb{C}^n) \mid AV_i \subseteq V_{h(i)},\ \forall i\}.$$

Equivalently, under the identification (2.1) and viewing A as an element in $\mathfrak{gl}_n(\mathbb{C})$,

$$\mathrm{Hess}(A, h) = \{MB \in \mathrm{GL}_n(\mathbb{C})/B \mid M^{-1}AM \in H(h)\}. \tag{2.3}$$

In particular, any Hessenberg variety $\mathrm{Hess}(A, h)$ is, by definition, an algebraic subset of the flag variety $\mathrm{Flags}(\mathbb{C}^n)$. It is straightforward to see that $\mathrm{Hess}(A, h)$ and $\mathrm{Hess}(gAg^{-1}, h)$ are isomorphic $\forall g \in \mathrm{GL}_n(\mathbb{C})$, so we frequently assume without loss of generality that A is in standard Jordan canonical form with respect to the standard basis on $\mathbb{C}^n$.

In this manuscript we discuss two important special cases of Hessenberg varieties: the regular nilpotent Hessenberg varieties and the regular semisimple Hessenberg varieties.

Definition 2.2. A Hessenberg variety $\mathrm{Hess}(A, h)$ is called **regular nilpotent** if A is a principal nilpotent operator. Equivalently, the Jordan canonical form of A has a single Jordan block with eigenvalue zero, i.e., up to a change of basis A is of the form:

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

For the remainder of this paper we let N denote the matrix (operator) above.

Regular nilpotent Hessenberg varieties are known to be irreducible [5, Lemma 7.1], and they are the subject of Sections 3 and 4 of this paper. When we study families of Hessenberg varieties in Section 5, the following type will also become relevant.

Definition 2.3. A Hessenberg variety $\mathrm{Hess}(A, h)$ is called **regular semisimple** if A is a semisimple operator with distinct eigenvalues. Equivalently, there is a basis of $\mathbb{C}^n$ with respect to which A is diagonal with pairwise distinct entries along the diagonal.

We will need the following terminology from [16, Definition 4.4].

Definition 2.4. Let $h\colon [n] \to [n]$ be a Hessenberg function. If $h(j) \geqslant j+1$ for $j \in \{1, 2, \ldots, n-1\}$, then we say that h is **indecomposable**.

Finally, we give the definition of a special case of a regular nilpotent Hessenberg variety which is studied in more detail in Section 9.

Definition 2.5. When h is of the form $h(j) = j+1$ for $j \in \{1, 2, \ldots, n-1\}$, the corresponding regular nilpotent Hessenberg variety is called a **Peterson variety**.

The regular semisimple Hessenberg variety for the same Hessenberg function $h(j) = j + 1$ for $j \in \{1, 2, \ldots, n-1\}$ is isomorphic to the toric variety associated to the root system of type A_{n-1} [14].

3. The defining ideals of regular nilpotent Hessenberg varieties

As mentioned in the Introduction, in this paper we are interested in the geometry and associated invariants of regular nilpotent Hessenberg varieties $\mathrm{Hess}(N, h)$. In order to make the arguments in the later sections, it will be convenient for us to have explicit lists of (local) generators for the defining ideals of $\mathrm{Hess}(N, h)$, considered as subvarieties of $\mathrm{Flags}(\mathbb{C}^n)$. It is quite easy, as we shall see below, to produce an explicit list of polynomials which cut out $\mathrm{Hess}(N, h)$ set-theoretically; the issue which we must address is whether the ideal that these polynomials generate is radical, or whether the relevant quotient ring is reduced. The main content of this section, recorded in Proposition 3.5, is to show that in fact the quotient rings associated to our lists of polynomials are reduced and thus we have found generators for the defining ideals of our varieties.

To prove Proposition 3.5 we proceed in steps. A regular nilpotent Hessenberg variety $\mathrm{Hess}(N, h)$ is defined as a subvariety of $\mathrm{Flags}(\mathbb{C}^n)$. As a scheme, it is well-known that $\mathrm{Flags}(\mathbb{C}^n)$ can be covered by affine coordinate patches, each isomorphic to $\mathbb{A}^{n(n-1)/2}$, as we now very briefly recount. Let

$$U := \left\{ M = \begin{pmatrix} 1 & & & & \\ \star & 1 & & & \\ \vdots & \vdots & \ddots & & \\ \star & \star & \ldots & 1 & \\ \star & \star & \cdots & \star & 1 \end{pmatrix} \;\middle|\; \begin{matrix} M \text{ is lower-triangular} \\ \text{with 1's along the diagonal} \end{matrix} \right\} \cong \mathbb{A}^{n(n-1)/2} \subseteq \mathrm{Mat}(n \times n, \mathbb{C}).$$

Then the map $U \to \mathrm{Flags}(\mathbb{C}^n) \cong \mathrm{GL}_n(\mathbb{C})/B$ given by $M \in U \mapsto MB \in \mathrm{GL}_n(\mathbb{C})/B$, is an open embedding. By slight abuse of notation we denote also by U its image in $\mathrm{Flags}(\mathbb{C}^n)$; again it is well-known that the set of translates $\{\mathcal{N}_w := wU\}$ of U by the permutations $w \in \mathfrak{S}_n$ (identified with their associated permutation matrices), along with the open embeddings $\Psi_w\colon U \cong \mathbb{A}^{n(n-1)/2} \xrightarrow{\cong} \mathcal{N}_w \subseteq \mathrm{Flags}(\mathbb{C}^n)$ sending $M \mapsto wMB$, form an open cover of $\mathrm{Flags}(\mathbb{C}^n)$. The transition functions $\varphi_{v,w}$ between these coordinate patches corresponding to a non-empty intersection $\mathcal{N}_v \cap \mathcal{N}_w \neq \emptyset$ are also well-known (and straightforward to check directly) to be realized by right multiplication by an upper-triangular matrix.

Some facts about the transition functions $\varphi_{v,w}$ will be used in the technical arguments in what follows, so we record them in Lemma 3.2 below. To state it precisely we need some terminology. Using the bijection $U \cong \mathbb{A}^{n(n-1)/2} \xrightarrow{\cong} \mathcal{N}_w$, a point in $\mathcal{N}_w$ is uniquely identified with the w-translate of a lower-triangular matrix with 1's along the diagonal. Therefore a point in $\mathcal{N}_w$ is uniquely determined by a matrix $(x_{i,j})$ whose entries are subject to the following relations

$$\begin{aligned} x_{w(j),j} &= 1, \quad \forall j \in [n], \\ x_{w(i),j} &= 0, \quad \forall i, j \in [n] : j > i. \end{aligned} \tag{3.1}$$

Thus the coordinate ring of $\mathcal{N}_w$, denoted by $\mathbb{C}[\mathbf{x}_w]$ from now on, is isomorphic to the quotient of the polynomial ring $\mathbb{C}[x_{i,j}]$ by the relations (3.1). Observe that $\mathbb{C}[\mathbf{x}_w]$ is isomorphic to a polynomial ring in the $n(n-1)/2$ variables $x_{i,j}$ not covered by the relations (3.1).

Example 3.1. Let $n = 4$ and $w = (2, 4, 1, 3) \in \mathfrak{S}_4$ in the standard one-line notation. An element M of $\mathcal{N}_w = wU$ can be written as

$$wM = \begin{pmatrix} x_{1,1} & x_{1,2} & 1 & 0 \\ 1 & 0 & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 \\ x_{4,1} & 1 & 0 & 0 \end{pmatrix}.$$

Also let $\psi_w^v \in \mathbb{C}[\mathbf{x}_w]$ denote the polynomial obtained by taking the product of the leading principal minors (i.e. the upper-left-$(k \times k)$ determinants) of $v^{-1}(wM)$ for $M \in U$ where the (i, j)-th matrix entries of M for $i > j$ are interpreted as the variables in $\mathbb{C}[\mathbf{x}_w]$. It is not hard to see that $\Psi_w^{-1}(\mathcal{N}_v \cap \mathcal{N}_w) \subseteq U \cong \mathcal{N}_w$ is the non-vanishing locus of ψ_w^v. Therefore the coordinate ring of $\mathcal{N}_v \cap \mathcal{N}_w$ is isomorphic to the localization $\mathbb{C}[\mathbf{x}_w]_{\psi_w^v}$.

Lemma 3.2. Let $v, w \in \mathfrak{S}_n$ be such that $\mathcal{N}_v \cap \mathcal{N}_w \neq \emptyset$. Then the transition function $\varphi_{v,w} \colon \Psi_w^{-1}(\mathcal{N}_v \cap \mathcal{N}_w) \subseteq U \to \Psi_v^{-1}(\mathcal{N}_v \cap \mathcal{N}_w) \subseteq U$ is of the form

$$M \mapsto MC$$

where C is an upper-triangular $n \times n$ matrix depending on v and w with entries in $\mathbb{C}[\mathbf{x}_w]_{\psi_w^v}$.

Now recall that the definition of the regular nilpotent Hessenberg variety, thought of as a subvariety of the flag variety, is

$$\mathrm{Hess}(N, h) := \{MB \in \mathrm{Flags}(\mathbb{C}^n) \mid M^{-1}NM \in H(h)\}. \tag{3.2}$$

As we have just seen, the affine coordinate charts $\{\mathcal{N}_w\}_{w \in \mathfrak{S}_n}$ form an open cover of $\mathrm{Flags}(\mathbb{C}^n)$, so we immediately obtain an open cover $\{\mathcal{N}_{w,h}\}_{w \in \mathfrak{S}_n}$ of $\mathrm{Hess}(N, h)$ by defining

$$\mathcal{N}_{w,h} := \mathcal{N}_w \cap \mathrm{Hess}(N, h) \subset \mathcal{N}_w \cong \mathbb{A}^{n(n-1)/2}. \tag{3.3}$$

Thus we now turn our attention to a study of these subvarieties of $\mathbb{A}^{n(n-1)/2}$. Set-theoretically, it is easy to identify a defining set of equations. From (3.2) it follows that the coset MB associated to a matrix $M = (x_{i,j})$ lies in $\mathcal{N}_{w,h}$ if and only if it lies in $\mathcal{N}_w$ and, in addition,

$$(M^{-1}NM)_{i,j} = 0 \tag{3.4}$$

for all $i, j \in [n]$ with $i > h(j)$. Observe that any matrix of the form wM for $M \in U$ satisfies $\det(wM) = \pm 1$. This implies that for any $w \in \mathfrak{S}_n$ and any $M \in U \cong \mathbb{A}^{n(n-1)/2}$, the matrix entries of the inverse $(wM)^{-1}$ and hence also of the matrix $(wM)^{-1}N(wM)$ are polynomial expressions in the affine coordinates on $U \cong \mathbb{A}^{n(n-1)/2}$. We can then make the following definition.

Definition 3.3. Let $w \in \mathfrak{S}_n$ and let $i, j \in [n]$ with $i > h(j)$. We define the polynomial $f_{i,j}^w \in \mathbb{C}[\mathbf{x}_w]$ by

$$f_{i,j}^w := \left((wM)^{-1}N(wM)\right)_{i,j}$$

where here the (k, ℓ)-th matrix entries of M for $k > \ell$ are viewed as variables. We also define the ideal

$$J_{w,h} := \langle f_{i,j}^w \mid i > h(j)\rangle \subseteq \mathbb{C}[\mathbf{x}_w]$$

to be the ideal in $\mathbb{C}[\mathbf{x}_w]$ generated by the $f_{i,j}^w$.

Example 3.4. Let $n = 4$ and $w = (2, 4, 1, 3) \in \mathfrak{S}_4$, continued from Example 3.1. Then it is straightforward to check that

$$(wM)^{-1} = \begin{pmatrix} x_{1,1} & x_{1,2} & 1 & 0 \\ 1 & 0 & 0 & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 1 \\ x_{4,1} & 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -x_{4,1} & 0 & 1 \\ 1 & -x_{1,1} + x_{1,2}x_{4,1} & 0 & -x_{1,2} \\ -x_{3,3} & y & 1 & -x_{3,2} + x_{1,2}x_{3,3} \end{pmatrix}$$

where $y = -x_{3,1} + x_{1,1}x_{3,3} + x_{4,1}(x_{3,2} - x_{1,2}x_{3,3})$. So, for example, we have

$$\begin{aligned} f_{4,1}^w &= \left((wM)^{-1}N(wM)\right)_{4,1} = -x_{3,3} + x_{3,1}(-x_{3,1} + x_{1,1}x_{3,3} + x_{4,1}(x_{3,2} - x_{1,2}x_{3,3})) + x_{4,1}, \\ f_{4,2}^w &= \left((wM)^{-1}N(wM)\right)_{4,1} = -x_{3,1} + x_{1,1}x_{3,3} + x_{4,1}(x_{3,2} - x_{1,2}x_{3,3}) + 1. \end{aligned} \tag{3.5}$$

So if $h = (3, 3, 4, 4)$, then we have $J_{w,h} = \langle f_{4,1}^w, f_{4,2}^w\rangle$ with these polynomials.

We now state the main result of this section.

Proposition 3.5. Let $h\colon [n] \to [n]$ be an indecomposable Hessenberg function. For every $w \in \mathfrak{S}_n$, the ring $\mathbb{C}[\mathbf{x}_w]/J_{w,h}$ is the coordinate ring of the subvariety $\mathcal{N}_{w,h} = \mathrm{Hess}(N,h) \cap \mathcal{N}_w$ of $\mathcal{N}_w$. In particular, the ideal $J_{w,h}$ is radical and is the defining ideal of $\mathcal{N}_{w,h}$.

The necessity of the indecomposability hypothesis can be seen from a small example.

Example 3.6. Let $n = 2$ and $h = (1,2)$. We have $J_{\mathrm{id},h} = \langle f^{\mathrm{id}}_{2,1}\rangle \subseteq \mathbb{C}[x_{2,1}]$ where $f^{\mathrm{id}}_{2,1} = -x_{2,1}^2$. Clearly the ring $\mathbb{C}[x_{2,1}]/J_{\mathrm{id},h}$ is not reduced, so it is not the coordinate ring of $\mathcal{N}_{\mathrm{id},h}$.

The proof of Proposition 3.5 requires a number of steps, so we first describe the basic ideas. First, it is immediate from the definition (3.2) that $\mathcal{N}_{w,h} = \mathrm{Hess}(N,h) \cap \mathcal{N}_w$ is precisely the vanishing locus of $J_{w,h}$. What is not immediately clear is that $J_{w,h}$ is radical, or that $\mathbb{C}[\mathbf{x}_w]/J_{w,h}$ is reduced. In order to see this on every affine chart $\mathcal{N}_w$, we construct in Lemma 3.7 below a scheme $\mathrm{Hess}'(N,h)$ by gluing together the affine schemes $\mathbb{C}[\mathbf{x}_w]/J_{w,h}$ in the obvious way, and our goal is then to prove that $\mathrm{Hess}'(N,h)$ is the same as $\mathrm{Hess}(N,h)$. The second idea is to focus on a particular affine patch. Specifically, let w_0 be the longest element in $\mathfrak{S}_n$, i.e. the full inversion given by $w_0(i) = n+1-i$ for all $i \in [n]$. In order to prove Proposition 3.5 we will first prove, in Lemma 3.8 below, the analogous result on $\mathcal{N}_{w_0,h}$ via a direct analysis of the ideal $J_{w_0,h}$. Once we know the result for a neighborhood around w_0B, the irreducibility of $\mathrm{Hess}(N,h)$ and a simple algebraic argument yields the global result on all of $\mathrm{Hess}'(N,h)$ (and hence shows $\mathrm{Hess}'(N,h) \cong \mathrm{Hess}(N,h)$).

We now proceed to implement this overall plan. In order to proceed we need first to complete the definition of the scheme $\mathrm{Hess}'(N,h)$. This is just the usual process of gluing so we shall be brief. The only technical point to check is that the ideals $J_{w,h}$ for varying w behave well with respect to the transition functions discussed in Lemma 3.2 above. This is the content of the next lemma. Note that the transition function $\varphi_{v,w}\colon \Psi_w^{-1}(\mathcal{N}_v \cap \mathcal{N}_w) \to \Psi_v^{-1}(\mathcal{N}_v \cap \mathcal{N}_w)$ is algebraic by Lemma 3.2, therefore it corresponds to a ring homomorphism $\varphi^*_{v,w}\colon \mathbb{C}[\mathbf{x}_v]_{\psi^w_v} \to \mathbb{C}[\mathbf{x}_w]_{\psi^v_w}$.

Lemma 3.7. For $w, v \in \mathfrak{S}_n$, we have $\varphi^*_{v,w}((J_{v,h})_{\psi^w_v}) = (J_{w,h})_{\psi^v_w}$ in the localized ring $\mathbb{C}[\mathbf{x}_w]_{\psi^v_w}$.

Proof. Let $wM \in \mathcal{N}_{w,h}$ where $M \in U$. Although not strictly necessary we find it useful to think of $f^v_{i,j}$ as a function on $\mathcal{N}_v \cap \mathcal{N}_w \subseteq \mathcal{N}_v$ and $\varphi^*_{v,w}(f^v_{i,j})$ as a function on $\mathcal{N}_w \cap \mathcal{N}_v \subseteq \mathcal{N}_w$ so the notation below reflects this. For $i > h(j)$, we have

$$\begin{aligned}
\varphi^*_{v,w}(f^v_{i,j})(vM) &= f^v_{i,j}(\varphi_{v,w}(wM))\\
&= f^v_{i,j}(wMC)\\
&= (C^{-1}M^{-1}w^{-1}NwMC)_{i,j}\\
&= \sum_{k=1}^{n}\sum_{\ell=1}^{n}(C^{-1})_{\ell,k}(M^{-1}w^{-1}NwM)_{k,\ell}C_{\ell,j}\\
&= \sum_{k\geqslant i}\sum_{\ell\leqslant j}(C^{-1})_{i,k}C_{\ell,j}(M^{-1}w^{-1}NwM)_{k,\ell}.
\end{aligned} \tag{3.6}$$

Note that the last equality follows from C being upper triangular. For indices k and ℓ appearing in the last expression in (3.6) we therefore conclude $k \geqslant i > h(j) \geqslant h(\ell)$, so $k > h(\ell)$, and therefore $(M^{-1}w^{-1}NwM)_{k,\ell} = f^w_{k,\ell}(wM)$. We deduce that $\varphi^*_{v,w}((J_{v,h})_{\psi^w_v}) \subseteq (J_{w,h})_{\psi^v_w}$. Exchanging the role of w and v, we get $\varphi^*_{w,v}((J_{w,h})_{\psi^v_w}) \subseteq (J_{v,h})_{\psi^w_v}$. Since $\varphi^{-1}_{w,v} = \varphi_{v,w}$, we obtain $(J_{w,h})_{\psi^v_w} = \varphi^*_{v,w}((J_{v,h})_{\psi^w_v})$. □

We now analyze the affine patch near w_0B. This is the most important computation in our argument.

Lemma 3.8. Let $h\colon [n] \to [n]$ be an indecomposable Hessenberg function. Then the ring $\mathbb{C}[\mathbf{x}_{w_0}]/J_{w_0,h}$ is isomorphic to a polynomial ring, hence it is reduced.

Remark 3.9. It is already known that the intersection $\mathrm{Hess}(N,h) \cap \mathcal{N}_{w_0}$ of the variety $\mathrm{Hess}(N,h)$ with the affine coordinate patch around w_0 is isomorphic as a variety to a complex affine space ([43] and [37]). The point of Lemma 3.8 is that $J_{w_0,h}$ is its defining ideal, and that its generators take a particular form.

Before proving the lemma, we give some concrete examples.

Example 3.10. Let $n = 4$ and $h = (3, 3, 4, 4)$. The coordinate ring of $\mathcal{N}_{w_0}$ is

$$\mathbb{C}[\mathbf{x}_{w_0}] \cong \mathbb{C}[x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{3,1}],$$

and a point in $\mathcal{N}_{w_0}$ is determined by a matrix

$$M = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & 1 \\ x_{2,1} & x_{2,2} & 1 & 0 \\ x_{3,1} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Given the form of M, it is easy to see that its inverse must have the form

$$M^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & y_{3,1} \\ 0 & 1 & y_{2,2} & y_{2,1} \\ 1 & y_{1,3} & y_{1,2} & y_{1,1} \end{pmatrix}. \tag{3.7}$$

Starting from the matrix equality $M^{-1}M = (\delta_{i,j})$, and comparing entries we can obtain expressions for the $y_{i,j}$ in terms of the $x_{i,j}$. For example,

$$\begin{aligned} y_{1,3} &= -x_{1,3}, \\ y_{1,2} &= -x_{1,2} - y_{1,3}x_{2,2} = -x_{1,2} + x_{1,3}x_{2,2}. \end{aligned}$$

It is also straightforward to see that each $y_{i,j}$ depends only on the variables $x_{k,\ell}$ with $k \geqslant i$ and $\ell \geqslant j$. Graphically, this says that $y_{i,j}$ depends only on $x_{i,j}$ and variables located to the right or below $x_{i,j}$ in the matrix M; for example, $y_{1,2}$ depends only on the variables contained in the bounded region depicted in Figure 3.1.

$$\begin{array}{c|cc|c} \cline{2-3} x_{1,1} & x_{1,2} & x_{1,3} & 1 \\ \cline{3-3} x_{2,1} & \multicolumn{1}{c|}{x_{2,2}} & 1 & 0 \\ \cline{2-2} x_{3,1} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}$$

Figure 3.1. Variables appearing in the expression of $y_{1,2}$

Now we describe the generators of $J_{w_0,h} = \langle f_{4,1}^{w_0}, f_{4,2}^{w_0} \rangle$. We have

$$M^{-1}NM = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & y_{3,1} \\ 0 & 1 & y_{2,2} & y_{2,1} \\ 1 & y_{1,3} & y_{1,2} & y_{1,1} \end{pmatrix} \begin{pmatrix} x_{2,1} & x_{2,2} & 1 & 0 \\ x_{3,1} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and from this we get

$$\begin{aligned} f_{4,1}^{w_0} &= (M^{-1}NM)_{4,1} = x_{2,1} + y_{1,3}x_{3,1} + y_{1,2} = x_{2,1} - x_{1,3}x_{3,1} - x_{1,2} + x_{1,3}x_{2,2}, \\ f_{4,2}^{w_0} &= (M^{-1}NM)_{4,2} = x_{2,2} + y_{1,3} = x_{2,2} - x_{1,3}. \end{aligned}$$

We deduce that $x_{2,1}$ and $x_{2,2}$ are determined by the other variables and conclude that $\mathbb{C}[\mathbf{x}_{w_0}]/J_{w_0,h} \cong \mathbb{C}[x_{1,1}, x_{1,2}, x_{1,3}, x_{3,1}]$ is a polynomial ring and in particular is reduced. It is possible to easily visualize, using the Hessenberg diagram, the variables which turn out to be dependent on other variables and hence "vanish" in the quotient $\mathbb{C}[\mathbf{x}_{w_0}]/J_{w_0,h}$, as illustrated in Figure 3.2 for this example. Specifically, we can first cross out any box which is *not* contained in the Hessenberg diagram for $h = (3344)$; see the left diagram in Figure 3.2. We then flip the picture upside down (so that, in this case, the boxes in positions $(1, 1)$ and $(1, 2)$ are now crossed out), and finally shift the entire picture downwards by one row. In this case we end up with a picture, as in the right-hand diagram in Figure 3.2, with the boxes in positions $(2, 1)$ and $(2, 2)$ crossed out. Then the variables corresponding to the crossed-out boxes are the ones which vanish in the quotient,

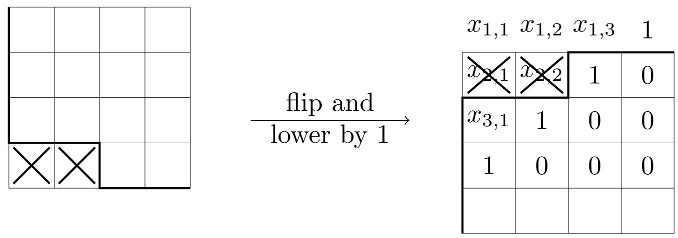


FIGURE 3.2. Variables killed in $\mathbb{C}[\mathbf{x}_{w_0}]/J_{w_0,h}$

and in fact (by the computation above) they are dependent on the (non-crossed-out) variables appearing either below it within the same column, or in a column to its right in a row at most one above it.

Example 3.11. Let $n = 5$ and $h = (3, 4, 4, 5, 5)$. The diagram in Figure 3.3 predicts that $\mathbb{C}[\mathbf{x}_{w_0}]/J_{w_0,h} \cong \mathbb{C}[x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, x_{3,2}, x_{4,1}]$. Indeed the generators of $J_{w_0,h}$ are

$$\begin{aligned} f^{w_0}_{5,1} &= x_{2,1} - x_{1,2} - x_{1,3}x_{4,1} + x_{1,3}x_{3,2} - x_{1,4}x_{3,1} \\ &\quad + x_{1,4}x_{2,2} + x_{1,4}x_{2,3}x_{4,1}, -x_{1,4}x_{2,3}x_{3,2} \\ f^{w_0}_{5,2} &= x_{2,2} - x_{1,3} - x_{1,4}x_{3,2} + x_{1,4}x_{2,3} \\ f^{w_0}_{5,3} &= x_{2,3} - x_{1,4} \\ f^{w_0}_{4,1} &= x_{3,1} - x_{2,2} - x_{2,3}x_{4,1} + x_{2,3}x_{3,2}. \end{aligned}$$

Again, we see that $\mathbb{C}[\mathbf{x}_{w_0}]/J_{w_0,h}$ is reduced. Following the method outlined in the previous example, we see that the variables which vanish in the quotient are $x_{2,1}, x_{2,2}, x_{2,3}$ and $x_{3,1}$. See Figure 3.3.

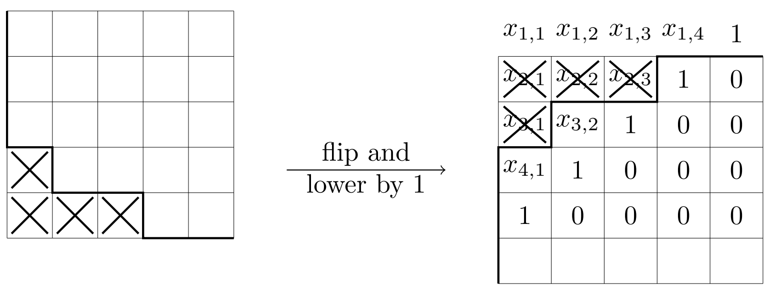


FIGURE 3.3. Variables killed in $\mathbb{C}[\mathbf{x}_{w_0}]/J_{w_0,h}$

Proof of Lemma 3.8. Let $M = (x_{i,j})$ determine a point in $\mathcal{N}_{w_0,h}$. Recall that, as elements of $\mathbb{C}[\mathbf{x}_{w_0}]$, the variables $x_{i,j}$ are subject to the following relations:

- $x_{i,n+1-i} = 1, \quad \forall i \in [n]$;
- $x_{i,j} = 0, \quad \forall i, j \in [n] : i > n + 1 - j$.

For all $i, j \in [n]$, we have $(M^{-1}M)_{n+1-i,j} = \delta_{n+1-i,j}$. This equality can be written more explicitly as

$$y_{i,j} + \sum_{k=1}^{n-j} y_{i,n+1-k}x_{k,j} = \delta_{n+1-i,j}, \tag{3.8}$$

where $y_{i,j} := (M^{-1})_{n+1-i,n+1-j}$ (see (3.7) or (3.9) below for visualizations of this indexing).

For all $i, j \in [n]$, the polynomials $y_{i,j}$ have the following properties:

(i) $y_{i,n+1-i} = 1$;
(ii) $y_{i,j} = 0$, whenever $i > n + 1 - j$;
(iii) $y_{i,j}$ is a polynomial in the variables $x_{k,l}$ with $k \geqslant i$ and $l \geqslant j$.

These properties follow from equation (3.8) using an elementary inductive argument. Using properties (i) and (ii), we deduce that

$$M^{-1} = \begin{pmatrix} & & & & 1 \\ & & & 1 & y_{n-1,1} \\ & & \iddots & \vdots & \vdots \\ & 1 & \dots & y_{2,2} & y_{2,1} \\ 1 & y_{1,n-1} & \dots & y_{1,2} & y_{1,1} \end{pmatrix}. \tag{3.9}$$

Let us compute the polynomial $f^{w_0}_{n+1-i,j}$. We have

$$NM = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n-1} & 1 \\ x_{2,1} & x_{2,2} & \dots & 1 & \\ \vdots & \vdots & \iddots & & \\ x_{n-1,1} & 1 & & & \\ 1 & & & & \end{pmatrix} = \begin{pmatrix} x_{2,1} & x_{2,2} & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \iddots & \\ x_{n-1,1} & 1 & \iddots & & \\ 1 & 0 & & & \\ 0 & & & & \end{pmatrix}.$$

The ideal $J_{w_0,h}$ is generated by the polynomials $f^{w_0}_{n+1-i,j}$ having $n+1-i > h(j)$. With this choice of indices, we obtain

$$f^{w_0}_{n+1-i,j} = (M^{-1}NM)_{n+1-i,j} = \sum_{k=i}^{n-j} x_{k+1,j} y_{i,n+1-k}.$$

Since we are dealing with $f^{w_0}_{n+1-i,j}$ for $n+1-i > h(j)$, we have $n+1-i > j$ by combining with $h(j) \geqslant j$. Therefore a generator of $J_{w_0,h}$ has the form

$$f^{w_0}_{n+1-i,j} = x_{i+1,j} + \sum_{k=i+1}^{n-j} x_{k+1,j} y_{i,n+1-k}. \tag{3.10}$$

Namely, the first summand $x_{i+1,j}$ always appears with $y_{i,n+1-i} = 1$. Now, since h is indecomposable, we have $h(j) \geqslant j+1$. In fact we have $i < n-j$ from the same reasoning as above, so that $x_{i+1,j}$ is a coordinate function on $\mathcal{N}_{w_0}$ (cf. (3.1)). The variable $x_{k+1,j}$ appearing in the summation has row index $k+1 \geqslant i+2$. As for $y_{i,n+1-k}$, it depends only on variables $x_{p,q}$ with row index $p \geqslant i$ and column index $q \geqslant j+1$. This follows from property (iii) combined with the observation that $k \leqslant n-j$ implies $n+1-k \geqslant j+1$. We conclude that the summation appearing in equation (3.10) depends only on variables $x_{p,q}$ with $q \geqslant j+1$ and $p \geqslant i$, or $q = j$ and $p \geqslant i+2$.

Finally, the above discussion and a simple inductive argument implies that setting $f^{w_0}_{n+1-i,j}$ equal to 0 has the effect of eliminating the variables $x_{i+1,j}$ from the quotient $\mathbb{C}[\mathbf{x}_{w_0}]/J_{w_0,h}$ and there are no further relations on the remaining variables. Namely, $\mathbb{C}[\mathbf{x}_{w_0}]/J_{w_0,h}$ is isomorphic to the polynomial ring

$$\mathbb{C}[x_{i,j} \mid 1 \leqslant i,j \leqslant n-1,\ i \notin \{2,3,\dots,n+1-h(j)\}],$$

which in particular is reduced, as was to be shown. It also follows that $J_{w_0,h}$ is radical and is the defining ideal of $\mathcal{N}_{w_0,h}$ in $\mathcal{N}_{w_0}$. □

Motivated by the above proof of Lemma 3.8, we introduce the following terminology which will be useful in Section 7: the set $\{x_{i,j} \mid 1 \leqslant i,j \leqslant n-1, i \in \{2,3,\dots,n+1-j\}\}$ consists of the **non-free** variables and the indices (i,j) for $1 \leqslant i,j \leqslant n-1, i \in \{2,3,\dots,n+1-j\}$ give the **positions of the non-free variables**. The other variables are the **free variables**. In particular, observe that $x_{1,1}$ is always a free variable.

We also record the following fact which follows easily from the above analysis and which we use in Section 7.

Lemma 3.12. Let $h\colon [n] \to [n]$ be an indecomposable Hessenberg function. Then, for each pair (i,j) with $n-i \geqslant j$, we have

$$f^{w_0}_{n+1-i,j} = x_{i+1,j} - g,$$

where g is a polynomial contained in the ideal of $\mathbb{C}[\mathbf{x}_{w_0}]$ generated by $\{x_{i,\ell} \mid j+1 \leqslant \ell \leqslant n-i\}$.

Proof. Let us denote by $I_{i,j+1}$ the ideal mentioned in the claim. From the expression (3.10) of $f^{w_0}_{n+1-i,j}$, it suffices to show that $y_{i,\ell} \in I_{i,\ell}$ for $j+1 \leqslant \ell \leqslant n-i$. We fix arbitrary $1 \leqslant i < n$ and $j < n-i$, and prove this by induction on ℓ with $j+1 \leqslant \ell \leqslant n-i$. Recall from (3.8) with the properties (i) and (ii) that we have

$$y_{i,\ell} = -\sum_{k=i}^{n-\ell} y_{i,n+1-k} x_{k,\ell} = -x_{i,\ell} - \sum_{k=i+1}^{n-\ell} y_{i,n+1-k} x_{k,\ell}, \tag{3.11}$$

where the second equality follows from $i \leqslant n-\ell$ and $y_{i,n+1-i} = 1$. So when $\ell = n-i$, we have

$$y_{i,n-i} = -x_{i,n-i} \in I_{i,n-i}.$$

Now, by induction, we assume that $y_{i,p} \in I_{i,p}$ ($\ell+1 \leqslant p \leqslant n-i$), and we prove that $y_{i,\ell} \in I_{i,\ell}$. Our polynomial $y_{i,\ell}$ is described by the rightmost expression of (3.11). There we have $x_{i,\ell} \in I_{i,\ell}$, and also $y_{i,n+1-k} \in I_{i,n+1-k}$, by the inductive hypothesis, since $\ell+1 \leqslant n+1-k \leqslant n-i$. These inequalities also imply that we have $I_{i,n+1-k} \subset I_{i,\ell}$, and hence we obtain $y_{i,\ell} \in I_{i,\ell}$, as desired. ☐

Having just proved directly that $\mathbb{C}[\mathbf{x}_{w_0}]/J_{w_0,h}$ is reduced, the reader may wonder why we do not do the same for all $w \in \mathfrak{S}_n$. As the proof of Lemma 3.8 may suggest, the argument works out well for w_0 due to the particular form of the matrices $w_0 M$ for $M \in U$; for general $w \in \mathfrak{S}_n$, it seems to be more complicated to analyze these ideals directly, as the following simple example illustrates.

Example 3.13. Let $n = 4$ and $h = (3,3,4,4)$. Let $w = (2,4,1,3) \in \mathfrak{S}_4$ as in Example 3.1 and Example 3.4. The ideal $J_{w,h}$ is generated by $f^w_{4,1}$ and $f^w_{4,2}$ described in (3.5). Although one can check computationally (using, say, Macaulay2 [21]) that this ideal is reduced, it does not seem so straightforward to prove it directly.

Instead of proving reducedness for each $w \in \mathfrak{S}_n$ separately, we resort to a different strategy, the essence of which is summarized in the following simple and purely algebraic lemma. We will use this also in Section 5 when we deal with the family of Hessenberg varieties. Some readers may be familiar with the general fact from commutative algebra that a Cohen-Macaulay ring is reduced if and only if it is generically reduced (cf. [17, Exercise 18.9]). We could also use this fact here, but for our situation the argument is so simple that we choose to record it.

Lemma 3.14. Let R be a Cohen-Macaulay ring with a unique minimal prime $\mathfrak{p}$. Suppose there exists a prime ideal $\mathfrak{q} \in \operatorname{Spec} R$ such that the localization $R_\mathfrak{q}$ is reduced. Then R is reduced.

Proof. Since R is Cohen-Macaulay, $\operatorname{Spec} R$ has no embedded components and the unique minimal prime $\mathfrak{p}$ is also the unique associated prime of R [17, Corollary 18.10]. This in turn implies that $\mathfrak{p}$ is precisely the set of zero-divisors in R and so the natural map $R \to R_\mathfrak{p}$ is injective (cf. [7, Proposition 4.7]; see also the discussion in [45, § 5.5]). In particular it suffices to show that $R_\mathfrak{p}$ is reduced. Since $\mathfrak{p}$ is the unique minimal prime, any $\mathfrak{q} \in \operatorname{Spec} R$ must contain $\mathfrak{p}$, and so by transitivity of localization we have $R_\mathfrak{p} \cong (R_\mathfrak{q})_{\mathfrak{p}R_\mathfrak{q}}$. Thus if $R_\mathfrak{q}$ is reduced for some $\mathfrak{q}$, then so is $R_\mathfrak{p}$, and we are done. ☐

In order to apply Lemma 3.14 to our situation we need to know that regular nilpotent Hessenberg varieties are irreducible (so the relevant coordinate rings have unique minimal primes), but this is already known [5, Lemma 7.1]. We also need to know that the relevant rings are Cohen-Macaulay. This is the content of the next lemma. It is worth emphasizing that although we use this result in the case when h is indecomposable, this lemma is valid for any Hessenberg function h.

Lemma 3.15. Let $h\colon [n] \to [n]$ be a Hessenberg function and $w \in \mathfrak{S}_n$. Then the ring $\mathbb{C}[\mathbf{x}_w]/J_{w,h}$ is Cohen-Macaulay.

Proof. If $\mathbb{C}[\mathbf{x}_w]/J_{w,h} = 0$, the statement is obvious. Thus we may assume that $\mathbb{C}[\mathbf{x}_w]/J_{w,h} \neq 0$. Observe that the polynomial ring $\mathbb{C}[\mathbf{x}_w]$ is regular, hence Cohen-Macaulay. By definition, the ideal $J_{w,h}$ can be generated by $\sum_{i=1}^n (n-h(i))$ elements. If we can show that $\operatorname{codim}(J_{w,h}) = \sum_{i=1}^n (n-h(i))$, then [17, Proposition 18.13] will imply that $\mathbb{C}[\mathbf{x}_w]/J_{w,h}$ is Cohen-Macaulay.

By [17, Corollary 13.4], we have

$$\operatorname{codim}(J_{w,h}) = \dim(\mathbb{C}[\mathbf{x}_w]) - \dim(\mathbb{C}[\mathbf{x}_w]/J_{w,h}).$$

The definition of $\mathbb{C}[\mathbf{x}_w]$ gives

$$\dim(\mathbb{C}[\mathbf{x}_w]) = \sum_{i=1}^{n}(n-i).$$

On the other hand, we have

$$\dim(\mathbb{C}[\mathbf{x}_w]/J_{w,h}) = \dim(\operatorname{Spec}(\mathbb{C}[\mathbf{x}_w]/J_{w,h})) = \dim(\mathcal{N}_{w,h}).$$

Since we assumed $\mathbb{C}[\mathbf{x}_w]/J_{w,h} \neq 0$, $\operatorname{Spec}(\mathbb{C}[\mathbf{x}_w]/J_{w,h})$ is non-empty. Therefore $\mathcal{N}_{w,h}$ is a non-empty open subset of $\operatorname{Hess}(N,h)$. By [5, Lemma 7.1], $\operatorname{Hess}(N,h)$ is irreducible of dimension $\sum_{i=1}^{n}(h(i)-i)$, hence we have

$$\dim(\mathcal{N}_{w,h}) = \dim \operatorname{Hess}(N,h) = \sum_{i=1}^{n}(h(i)-i).$$

Altogether, we get

$$\operatorname{codim}(J_{w,h}) = \sum_{i=1}^{n}(n-i) - \sum_{i=1}^{n}(h(i)-i) = \sum_{i=1}^{n}(n-h(i)),$$

completing the proof of the proposition. □

We can now prove Proposition 3.5.

Proof of Proposition 3.5. Let $w \in \mathfrak{S}_n$ and $R = \mathbb{C}[\mathbf{x}_w]/J_{w,h}$. If R is the zero ring, then there is nothing to prove, so we may assume $R \neq 0$, or equivalently, $\operatorname{Spec}(\mathbb{C}[\mathbf{x}_w]/J_{w,h}) \neq \emptyset$. Note that the set of closed points in $\operatorname{Spec}(R)$ (i.e. the underlying variety of $\operatorname{Spec}(R)$) is homeomorphic to $\mathcal{N}_{w,h}$; thus $R \neq 0$ implies $\mathcal{N}_{w,h} \neq \emptyset$. Furthermore, $\mathcal{N}_{w,h}$ is open in $\operatorname{Hess}(N,h)$. Since $\operatorname{Hess}(N,h)$ is irreducible [5, Lemma 7.1], we deduce that $\mathcal{N}_{w,h}$ is also irreducible. Therefore $\operatorname{Spec}(R)$ is irreducible, hence R has a single minimal prime $\mathfrak{p}$ (namely, its nilradical). Moreover, R is Cohen-Macaulay by Lemma 3.15. Since $\operatorname{Hess}(N,h)$ is irreducible, any two nonempty open subsets intersect nontrivially, and in particular, $\mathcal{N}_{w,h} \cap \mathcal{N}_{w_0,h} \neq \emptyset$. Since we have seen in Lemma 3.8 that the coordinate ring of $\mathcal{N}_{w_0,h}$ is a polynomial ring, the local ring at any of its closed points is a regular local ring and in particular is reduced. In other words, there exists a maximal ideal $\mathfrak{m}$ in R such that $R_{\mathfrak{m}}$ is reduced. The result now follows from Lemma 3.14. □

4. Regular nilpotent Hessenberg varieties are local complete intersections

Having just determined in Section 3 a convenient list of generators for the (local) defining ideals $J_{w,h}$ of regular nilpotent Hessenberg varieties, we now apply our knowledge to prove that all regular nilpotent Hessenberg varieties are local complete intersections. This means that at any closed point of $\operatorname{Hess}(N,h)$ the local ring is a complete intersection (see [17, § 18.5]). We emphasize that, for this result, we do *not* need to require h to be indecomposable. Our discussion here generalizes a result of Insko and Yong [25, Corollary 7] for Peterson varieties, as well as a result of Insko [24, Lemma 4.6, Theorem 4.9] that holds for strictly increasing Hessenberg functions.

The main result of this section is the following.

Theorem 4.1. Let $h\colon [n] \to [n]$ be any Hessenberg function. Then the corresponding regular nilpotent Hessenberg variety $\operatorname{Hess}(N,h)$ is a local complete intersection.

To prove Theorem 4.1 we use a certain ordering on the generators of the ideal $J_{w,h}$ (see Definition 3.3), which we illustrate in the following example.

Example 4.2. In Example 3.11, we listed generators $f_{5,1}^{w_0}, f_{5,2}^{w_0}, f_{5,3}^{w_0}, f_{4,1}^{w_0}$ of the ideal $J_{w_0,h}$ for the case $n=5$ and $h=(3,4,4,5,5)$. These generators can be visualized in terms of the Hessenberg diagram, where $f_{i,j}^{w_0}$ corresponds to the (i,j)-th box; see the left picture in Figure 4.1. On the right-hand side of Figure 4.1 we impose an ordering on the relevant boxes, which therefore imposes an order on the generators. We have

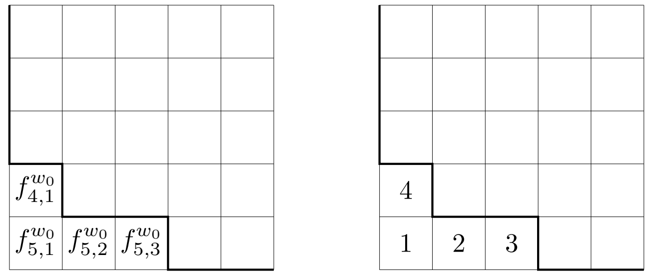


FIGURE 4.1. Order on the generators of $J_{w,h}$

chosen an order so that, for all $i \in [4]$, the generators in position 1 through i generate an ideal $J_{w_0,h'}$ for some Hessenberg function h'. Indeed, in this example, we have

$$\begin{aligned} J_{w_0,(4,5,5,5,5)} &= \langle f^{w_0}_{5,1} \rangle, \\ J_{w_0,(4,4,5,5,5)} &= \langle f^{w_0}_{5,1}, f^{w_0}_{5,2} \rangle, \\ J_{w_0,(4,4,4,5,5)} &= \langle f^{w_0}_{5,1}, f^{w_0}_{5,2}, f^{w_0}_{5,3} \rangle, \\ J_{w_0,(3,4,4,5,5)} &= \langle f^{w_0}_{5,1}, f^{w_0}_{5,2}, f^{w_0}_{5,3}, f^{w_0}_{4,1} \rangle. \end{aligned}$$

For the purposes of our argument below, in general we define an ordering on the generators $f^w_{i,j}$ of $J_{w,h}$ for any $w \in \mathfrak{S}_n$ and any h by defining $f^w_{i,j} < f^w_{k,\ell}$ if and only if $i > k$ or $i = k$ and $j < \ell$. Denote the generators of $J_{w,h}$ as $f_1, \ldots, f_c$, listed from the smallest to the largest in the total order just defined. As illustrated in the example above, it is straightforward to see that this order has the property that any subset of the generators of the form $f_1, \ldots, f_i$ for $1 \leqslant i \leqslant c$, generates the ideal $J_{w,h'}$ for some Hessenberg function h'.

Before launching into the proof we summarize the overall strategy. We treat the indecomposable and decomposable cases separately. If h is indecomposable, we use the ordering above and also exploit the fact that $\mathrm{Hess}(N, h)$ is irreducible for different values of h. If h is not indecomposable, we do not attempt to define or analyze its defining ideal using methods similar to the previous section; instead, we simply reduce to the indecomposable case by treating abstractly its indecomposable pieces.

Proof of Theorem 4.1. Since the claim is local, it is enough to prove the claim for each $\mathcal{N}_{w,h}$ in the affine cover. We saw in Proposition 3.5 that for any $w \in \mathfrak{S}_n$, the coordinate ring of $\mathcal{N}_{w,h}$ is isomorphic to $\mathbb{C}[\mathbf{x}_w]/J_{w,h}$. Thus it suffices to show that $\mathbb{C}[\mathbf{x}_w]/J_{w,h}$ is locally a complete intersection in the sense of [17, § 18.5]. Moreover, since localization preserves regular sequences [17, Lemma 18.1], it suffices to prove that the generators of $J_{w,h}$ form a regular sequence.

We first prove the claim for the case when the Hessenberg function h is indecomposable. Let $c = \mathrm{codim}(J_{w,h})$. As observed in the proof of Lemma 3.15, c is equal to the number of generators $f^w_{i,j}$ of $J_{w,h}$. Let $f_1, \ldots, f_c$ be the generators of $J_{w,h}$, totally ordered as explained above. We now prove that f_i is not a zero-divisor in the quotient ring $\mathbb{C}[\mathbf{x}_w]/\langle f_1, \ldots, f_{i-1} \rangle$ for all i with $1 \leqslant i \leqslant c$. As we have seen above, there exists a Hessenberg function $h' \colon [n] \to [n]$ such that $h'(j) \geqslant h(j)$ for $j \in [n]$, and $J_{w,h'} = \langle f_1, \ldots, f_{i-1} \rangle$. It follows that h' is also indecomposable. By Proposition 3.5, $\mathbb{C}[\mathbf{x}_w]/J_{w,h'}$ is the coordinate ring of $\mathcal{N}_{w,h'}$, which is irreducible (since $\mathrm{Hess}(N, h')$ is irreducible). Therefore the ring $\mathbb{C}[\mathbf{x}_w]/J_{w,h'}$ is an integral domain. So f_i in $\mathbb{C}[\mathbf{x}_w]/\langle f_1, \ldots, f_{i-1} \rangle = \mathbb{C}[\mathbf{x}_w]/J_{w,h'}$ is not a zero-divisor.

Now suppose that h is not indecomposable. Then by the definition of indecomposability we must have $h(j) = j$ for some $j \in \{2, 3, \ldots, n-1\}$. In this case, $\mathrm{Hess}(N, h)$ is isomorphic to a product of regular nilpotent Hessenberg varieties whose Hessenberg functions are indecomposable [16, Theorem 4.5]. By induction, we can reduce the argument to the case of two factors, i.e. suppose $\mathrm{Hess}(N, h) \cong \mathrm{Hess}(N', h') \times \mathrm{Hess}(N'', h'')$, where N' and N'' are regular nilpotent operators of the appropriate size in Jordan canonical form, and h' and h'' are indecomposable. Now $\mathrm{Hess}(N, h)$ is covered by affine schemes

$$\mathrm{Spec}\left((\mathbb{C}[\mathbf{x}_{w'}]/J_{w',h'}) \otimes_{\mathbb{C}} (\mathbb{C}[\mathbf{x}_{w''}]/J_{w'',h''})\right),$$

for permutations w' and w'' of the appropriate size. Moreover, we have an isomorphism of $\mathbb{C}$-algebras

$$(\mathbb{C}[\mathbf{x}_{w'}]/J_{w',h'}) \otimes_{\mathbb{C}} (\mathbb{C}[\mathbf{x}_{w''}]/J_{w'',h''}) \cong \mathbb{C}[\mathbf{x}_{w'}, \mathbf{x}_{w''}]/(J_{w',h'} + J_{w'',h''}).$$

By the first part of the proof, $J_{w',h'}$ and $J_{w'',h''}$ are generated by regular sequences. Since the two sequences are in independent sets of variables their union is again regular. Thus we conclude that $J_{w',h'} + J_{w'',h''}$ is generated by a regular sequence, as was to be shown. □

Recall that a Noetherian local ring which is a complete intersection is automatically Cohen-Macaulay and Gorenstein [11, Proposition 3.1.20]. In light of this fact, we obtain the following immediate consequence of Theorem 4.1.

Corollary 4.3. For any Hessenberg function h, the regular nilpotent Hessenberg variety $\mathrm{Hess}(N, h)$ is Cohen-Macaulay and Gorenstein.

5. Properties of a family of Hessenberg varieties

Let $h\colon [n] \to [n]$ be a Hessenberg function and let $H(h) \subseteq \mathfrak{gl}_n(\mathbb{C})$ be the corresponding Hessenberg space. The Hessenberg varieties (see Definition 2.1) with Hessenberg function h can be assembled into a family over $\mathfrak{gl}_n(\mathbb{C})$ defined as

$$\{(MB, \Gamma) \in \mathrm{GL}_n(\mathbb{C})/B \times \mathfrak{gl}_n(\mathbb{C}) \mid M^{-1}\Gamma M \in H(h)\} \subseteq \mathrm{Flags}(\mathbb{C}^n) \times \mathfrak{gl}_n(\mathbb{C}). \tag{5.1}$$

We are interested in a smaller family which we define as follows. Throughout the discussion we fix pairwise distinct complex numbers $\gamma_1, \gamma_2, \ldots, \gamma_n$. For $t \in \mathbb{C}$, we define

$$\Gamma_t := \begin{pmatrix} t\gamma_1 & 1 & & & \\ & t\gamma_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & t\gamma_{n-1} & 1 \\ & & & & t\gamma_n \end{pmatrix}.$$

Viewing $\mathbb{C}$ as the complex affine line $\mathbb{A}^1 = \mathbb{A}^1_{\mathbb{C}}$, we define a family over $\mathbb{A}^1$ by setting

$$\mathfrak{X}_h := \{(MB, t) \in \mathrm{Flags}(\mathbb{C}^n) \times \mathbb{A}^1 \mid M^{-1}\Gamma_t M \in H(h)\}$$

which can be viewed as a subfamily of (5.1) via the embedding $\mathbb{A}^1 \hookrightarrow \mathfrak{gl}_n(\mathbb{C})$ by $t \mapsto \Gamma_t$, and in particular there is a canonical projection

$$p\colon \mathfrak{X}_h \longrightarrow \mathbb{A}^1, \qquad (MB, t) \longmapsto t. \tag{5.2}$$

This family $\mathfrak{X}_h$ is presumably the family of Hessenberg varieties mentioned in [5]. In this section, we will prove the following geometric properties of $\mathfrak{X}_h$.

Theorem 5.1. Suppose that h is indecomposable. The morphism $p\colon \mathfrak{X}_h \to \mathbb{A}^1$ is flat, and its scheme-theoretic fibers over the closed points of $\mathbb{A}^1$ are reduced.

As mentioned in the Introduction, an application of this result to the study of Newton-Okounkov bodies is explained in Section 8.

Remark 5.2. Using the language and techniques of degeneracy loci, it is shown in [5] that regular nilpotent Hessenberg schemes are reduced, for the special case when the Hessenberg function is of the form $h = (k, n, \ldots, n)$ for some $2 \leqslant k \leqslant n$ [5, Theorem 7.6].

Our proof of Theorem 5.1 consists of two parts: flatness (Proposition 5.5) and reducedness (Proposition 5.10). We begin with flatness. Echoing the remarks in the introduction, we should emphasize that we do not claim originality for this result and we are recording the proof here only for completeness. In any event, as is well-known, flatness over $\mathbb{A}^1$ is a mild condition, and indeed the proof we give is a straightforward application of very standard results in e.g. the textbooks of Shafarevich [40] and Hartshorne [23]. There are other easy ways to prove this as well (see e.g. Remark 5.6).

As introduced above, $\mathfrak{X}_h$ is a Zariski-closed subset of the algebraic variety $\mathrm{Flags}(\mathbb{C}^n) \times \mathbb{A}^1$. We can think of $\mathrm{Flags}(\mathbb{C}^n) \times \mathbb{A}^1$ as an integral scheme in a standard way (cf. for example [23, II, Proposition 2.6]), and we can also consider $\mathfrak{X}_h$ as a subscheme of $\mathrm{Flags}(\mathbb{C}^n) \times \mathbb{A}^1$ with the reduced induced scheme structure. Moreover, the morphism of algebraic varieties $p\colon \mathfrak{X}_h \to \mathbb{A}^1$ defined above induces a morphism of schemes which by slight abuse of notation we also denote by p. We start with the following elementary observation.

Lemma 5.3. The morphism of schemes $p\colon \mathfrak{X}_h \to \mathbb{A}^1$ is surjective.

Proof. The map of varieties $p\colon \mathfrak{X}_h \to \mathbb{A}^1$ of (5.2) is clearly surjective. The functor from the category of varieties over $\mathbb{C}$ to the category of schemes over $\mathbb{C}$ preserves surjectivity, so the statement follows. $\square$

Lemma 5.4. Suppose that $h\colon [n] \to [n]$ is indecomposable. Then the scheme $\mathfrak{X}_h$ is irreducible.

Proof. It suffices to show that $\mathfrak{X}_h$ is irreducible as a variety, as this implies that $\mathfrak{X}_h$ is irreducible as a scheme. Thus, all spaces appearing in this proof should be interpreted as varieties.

We begin by constructing a projective version of the family $\mathfrak{X}_h$. For $[t:s] \in \mathbb{P}^1$, set

$$\Gamma_{t,s} := \begin{pmatrix} t\gamma_1 & s & & & \\ & t\gamma_2 & s & & \\ & & \ddots & \ddots & \\ & & & t\gamma_{n-1} & s \\ & & & & t\gamma_n \end{pmatrix}.$$

For $[t:s] \in \mathbb{P}^1$, the location of the zero entries in the matrix $M^{-1}\Gamma_{t,s}M$ is well-defined. Thus we can define

$$\widetilde{\mathfrak{X}}_h := \{(MB, [t:s]) \in \mathrm{Flags}(\mathbb{C}^n) \times \mathbb{P}^1 \mid M^{-1}\Gamma_{t,s}M \in H(h)\}.$$

This is a family over $\mathbb{P}^1$ via the projection

$$\tilde{p}\colon \widetilde{\mathfrak{X}}_h \longrightarrow \mathbb{P}^1, \qquad (MB, [t:s]) \longmapsto [t:s].$$

Clearly, $\tilde{p}$ is surjective.

We examine the fibres of $\tilde{p}$. The fibre $\tilde{p}^{-1}([0:1])$ is the regular nilpotent Hessenberg variety $\mathrm{Hess}(N,h)$, which is irreducible of dimension $\sum_{i=1}^n (h(i)-i)$ (see [5, Lemma 7.1]). For $[t:s] \neq [0:1]$, the matrix $\Gamma_{t,s}$ has n distinct eigenvalues, hence the fibre $\tilde{p}^{-1}([t:s])$ is a regular semisimple Hessenberg variety. It is known that regular semisimple Hessenberg varieties are smooth of dimension $\sum_{i=1}^n (h(i)-i)$ [14, Theorem 6]. In addition, if h is indecomposable, the regular semisimple Hessenberg variety is connected [14, Corollary 9]. Since $\tilde{p}^{-1}([t:s])$ is smooth and connected, we deduce that it is irreducible.

So far we have shown that $\tilde{p}$ is surjective and that its fibres are all irreducible and of the same dimension. Since $\mathbb{P}^1$ is irreducible, we can apply [40, I, § 6.3, Theorem 8] to deduce that $\widetilde{\mathfrak{X}}_h$ is irreducible. Finally, observe that the map

$$\mathfrak{X}_h \to \widetilde{\mathfrak{X}}_h, \qquad (MB, t) \mapsto (MB, [t:1])$$

embeds $\mathfrak{X}_h$ as a nonempty open subset in $\widetilde{\mathfrak{X}}_h$. Since $\widetilde{\mathfrak{X}}_h$ is irreducible, we conclude that $\mathfrak{X}_h$ is irreducible. $\square$

Proposition 5.5. Suppose that h is indecomposable. The morphism of schemes $p\colon \mathfrak{X}_h \to \mathbb{A}^1$ is flat.

Proof. The scheme $\mathfrak{X}_h$ is reduced by definition, and it is irreducible by Lemma 5.4. The morphism $p\colon \mathfrak{X}_h \to \mathbb{A}^1$ is surjective by Lemma 5.3. Moreover, the codomain $\mathbb{A}^1$ is an integral regular scheme of dimension 1. Therefore p is flat by [23, III, Proposition 9.7]. $\square$

Remark 5.6. We sketch a different approach to the proof of Proposition 5.5, which was communicated to us by P. Brosnan. As shown in [10, §8.2], the family of all regular Hessenberg varieties over regular matrices is smooth, and has equidimensional fibers. Since a morphism between non-singular varieties is flat if and only if its fibers are equidimensional (see e.g. [33]), it follows that this family is flat. Finally, since flatness is preserved under base change, restricting to our smaller family yields the desired statement.

To complete the proof of Theorem 5.1, we still need to show that the scheme-theoretic fibers of $p\colon \mathfrak{X}_h \to \mathbb{A}^1$ over closed points of $\mathbb{A}^1$ are reduced. In order to do this, we need to know the local defining ideals of $\mathfrak{X}_h$ as a subvariety (subscheme) of $\mathrm{Flags}(\mathbb{C}^n) \times \mathbb{A}^1$. This discussion almost exactly mirrors that of Section 3 and therefore we keep the exposition very brief.

The product $\mathrm{Flags}(\mathbb{C}^n) \times \mathbb{A}^1$ is covered by the affine varieties $\mathcal{N}_w \times \mathbb{A}^1$, for $w \in \mathfrak{S}_n$ with coordinate ring $\mathbb{C}[\mathbf{x}_w, t]$. The family $\mathfrak{X}_h$ is covered by $\mathfrak{X}_h \cap (\mathcal{N}_w \times \mathbb{A}^1)$, for $w \in \mathfrak{S}_n$, and if we define $F^w_{i,j} := (M^{-1}\Gamma_t M)_{i,j} \in \mathbb{C}[\mathbf{x}_w, t]$, then $\mathfrak{X}_h \cap (\mathcal{N}_w \times \mathbb{A}^1)$ is set-theoretically cut out by the equations $F^w_{i,j} = 0$, for all $i,j \in [n]$ with $i > h(j)$. Let $\mathcal{J}_{w,h} \subseteq \mathbb{C}[\mathbf{x}_w, t]$ denote the ideal generated by the $F^w_{i,j}$, for all $i,j \in [n]$ with $i > h(j)$. One can easily prove that the affine schemes $\operatorname{Spec}\mathbb{C}[\mathbf{x}_w,t]/\mathcal{J}_{w,h}$ glue together along the same lines of Section 3 and we denote this scheme as $\mathfrak{X}'_h$.

To prove that the scheme $\mathfrak{X}'_h$ is reduced and hence may be identified with $\mathfrak{X}_h$, we essentially follow the same strategy used in Section 3 although in a slightly different order. Namely, we show that for every $w \in \mathfrak{S}_n$, $\mathbb{C}[\mathbf{x}_w,t]/\mathcal{J}_{w,h}$ is Cohen-Macaulay, and $\mathbb{C}[\mathbf{x}_{w_0},t]/\mathcal{J}_{w_0,h}$ is reduced. The overall reducedness of $\mathfrak{X}_h$ will then follow.

Lemma 5.7. For any $w \in \mathfrak{S}_n$, the ring $\mathbb{C}[\mathbf{x}_w,t]/\mathcal{J}_{w,h}$ is Cohen-Macaulay.

Proof. The argument is the same as that for Proposition 3.15. The only difference is that the parameter t in $\mathbb{C}[\mathbf{x}_w,t]$, corresponding to the affine line $\mathbb{A}^1$, increases the dimension by one. However, the dimension of $\mathbb{C}[\mathbf{x}_w,t]/\mathcal{J}_{w,h}$ is also increased by one for the same reason. Therefore, we still have that

$$\operatorname{codim}(\mathcal{J}_{w,h}) = \sum_{i=1}^{n}(n-h(i)),$$

which equals the number of generators of $\mathcal{J}_{w,h}$. □

Lemma 5.8. Suppose that h is indecomposable. Then the localization $(\mathbb{C}[\mathbf{x}_{w_0},t]/\mathcal{J}_{w_0,h})_t$ is reduced.

Proof. The ideal $\mathcal{J}_{w_0,h}$ is generated by the polynomials $F^{w_0}_{i,j} = (M^{-1}\Gamma_t M)_{i,j}$ with $i > h(j)$. Recall that we have $M^{-1} = (y_{i,j})$, with the $y_{i,j}$ satisfying equation (3.8) and enjoying the properties ((i)), ((ii)), and ((iii)) recorded in the proof of Proposition 3.8. For $i < n+1-j$, equation (3.8) together with properties ((i)) and ((ii)) and ((iii)) imply that

$$y_{i,j} = -\sum_{k=i+1}^{n-j} y_{i,n+1-k}x_{k,j} - x_{i,j}.$$

Hence, by property ((iii)), the polynomial

$$\widetilde{y}_{i,j} := y_{i,j} + x_{i,j} \tag{5.3}$$

does not depend on the variable $x_{i,j}$.

From the definition of $F^{w_0}_{n+1-i,j}$ it follows that

$$F^{w_0}_{n+1-i,j} = \begin{pmatrix} 0 & \dots & 0 & 1 & y_{i,n-i} & \dots & y_{i,1}\end{pmatrix}\begin{pmatrix} t\gamma_1 x_{1,j} + x_{2,j} \\ \vdots \\ t\gamma_{n-j-1}x_{n-j-1,j} + x_{n-j,j} \\ t\gamma_{n-j}x_{n-j,j} + 1 \\ t\gamma_{n+1-j} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= (t\gamma_i x_{i,j} + x_{i+1,j}) + \sum_{k=i+1}^{n-j}(t\gamma_k x_{k,j} + x_{k+1,j})y_{i,n+1-k} + t\gamma_{n+1-j}y_{i,j}.$$

Note that the first and last summand always appear because the indecomposability of h implies that $i < n+1-h(j) \leqslant n+1-j$, hence $i < n+1-j$. The condition $i < n+1-j$ also guarantees that the variable $x_{i,j}$ appearing in the expression above is not 0 or 1 (cf. (3.1)). Using equation (5.3), we obtain

$$F^{w_0}_{n+1-i,j} = t(\gamma_i - \gamma_{n+1-j})x_{i,j} + x_{i+1,j} + \sum_{k=i+1}^{n-j}(t\gamma_k x_{k,j} + x_{k+1,j})y_{i,n+1-k} + t\gamma_{n+1-j}\widetilde{y}_{i,j}. \tag{5.4}$$

The coefficient of $x_{i,j}$ in equation (5.4) contains the factor $\gamma_i - \gamma_{n+1-j}$, which is nonzero since we assume the γ_k are pairwise distinct. As for the t factor, it will become invertible after passing to the localization $\mathbb{C}[\mathbf{x}_{w_0},t]_t$. With the exception of the first term, all the terms in equation (5.4) depend only on variables $x_{k,\ell}$ with $k > i$ and $\ell \geqslant j$, or $k \geqslant i$ and $\ell > j$.

Now a simple inductive argument based on the above observations shows that in the localization $(\mathbb{C}[\mathbf{x}_{w_0}, t]/\mathcal{J}_{w_0,h})_t$ the variables $x_{i,j}$ with $1 \leqslant j \leqslant n-1$ and $1 \leqslant i \leqslant n-h(j)$ can be replaced by expressions involving the free variables $x_{i,j}$ with $1 \leqslant j \leqslant n-1$ and $i > n-h(j)$. More formally, we have the following ring isomorphisms

$$\begin{aligned}(\mathbb{C}[\mathbf{x}_{w_0}, t]/\mathcal{J}_{w_0,h})_t &\cong \mathbb{C}[\mathbf{x}_{w_0}, t]_t/(\mathcal{J}_{w_0,h})_t \\ &\cong \mathbb{C}[t^{\pm}] \otimes \mathbb{C}[x_{i,j} \mid 1 \leqslant j \leqslant n-1, i > n-h(j)].\end{aligned}$$

It follows that $(\mathbb{C}[\mathbf{x}_{w_0}, t]/\mathcal{J}_{w_0,h})_t$ is reduced.

In geometric terms, what we have shown may be phrased as follows. First observe that since $\mathfrak{X}_h$ (and hence $\mathfrak{X}'_h$) is irreducible, $\mathrm{Spec}(\mathbb{C}[\mathbf{x}_{w_0}, t]/\mathcal{J}_{w_0,h})$ is irreducible. The non-empty open subset $\mathrm{Spec}((\mathbb{C}[\mathbf{x}_{w_0}, t]/\mathcal{J}_{w_0,h})_t)$, the complement of the zero fiber $p^{-1}(0)$ in $\mathrm{Spec}(\mathbb{C}[\mathbf{x}_{w_0}, t]/\mathcal{J}_{w_0,h})$, is therefore dense. Since we have just seen that $\mathrm{Spec}((\mathbb{C}[\mathbf{x}_{w_0}, t]/\mathcal{J}_{w_0,h})_t)$ is reduced, we conclude $\mathrm{Spec}(\mathbb{C}[\mathbf{x}_{w_0}, t]/\mathcal{J}_{w_0,h})$ is generically reduced, and since it is also Cohen-Macaulay by Lemma 5.7 and irreducible, by Lemma 3.14 we conclude it is reduced, as desired. □

The fact that the polynomials $F^w_{i,j}$ generate the defining ideal of $\mathfrak{X}_h \cap (\mathcal{N}_w \times \mathbb{A}^1)$ now follows from an argument identical to that for Proposition 3.5 so we omit the proof.

Proposition 5.9. Let $h\colon [n] \to [n]$ be an indecomposable Hessenberg function. For every $w \in \mathfrak{S}_n$, the ring $\mathbb{C}[\mathbf{x}_w, t]/\mathcal{J}_{w,h}$ is the coordinate ring of the subvariety $\mathfrak{X}_h \cap (\mathcal{N}_w \times \mathbb{A}^1)$ of $\mathcal{N}_w \times \mathbb{A}^1$. In particular, the ideal $\mathcal{J}_{w,h}$ is radical and is the defining ideal of $\mathfrak{X}_h \cap (\mathcal{N}_w \times \mathbb{A}^1)$ in $\mathcal{N}_w \times \mathbb{A}^1$.

With the defining polynomials in hand, we are finally ready to prove that the scheme-theoretic fibers of $\mathfrak{X}_h$ are reduced. For this purpose, let $z \in \mathbb{C}$ be a complex number which we also think of as a closed point in $\mathbb{A}^1$. The local ring of $\mathbb{A}^1$ at z is the localization $\mathbb{C}[t]_{(t-z)}$, and let $k(z)$ denote its residue field. Recall that the scheme-theoretic fibre of the family $p\colon \mathfrak{X}_h \to \mathbb{A}^1$ over z is defined as

$$(\mathfrak{X}_h)_z := \mathfrak{X}_h \times_{\mathbb{A}^1} \mathrm{Spec}\,(k(z))\,.$$

Since $\mathfrak{X}_h$ is covered by the open affine schemes $\mathrm{Spec}(\mathbb{C}[\mathbf{x}_w, t]/\mathcal{J}_{w,h})$ for $w \in \mathfrak{S}_n$, the fibre $(\mathfrak{X}_h)_z$ has an open affine covering consisting of

$$\mathrm{Spec}(\mathbb{C}[\mathbf{x}_w, t]/\mathcal{J}_{w,h}) \times_{\mathbb{A}^1} \mathrm{Spec}\,(k(z)) \cong \mathrm{Spec}\left((\mathbb{C}[\mathbf{x}_w, t]/\mathcal{J}_{w,h}) \otimes_{\mathbb{C}[t]} k(z)\right).$$

Consider the ideal

$$\mathcal{J}_{w,h}|_{t=z} := \langle F^w_{i,j}|_{t=z} \mid i > h(j)\rangle$$

of $\mathbb{C}[\mathbf{x}_w]$ whose generators are obtained from the generators of $\mathcal{J}_{w,h}$ after setting the variable t equal to z. Since the functor $- \otimes_{\mathbb{C}[t]} k(z)$ has the effect of substituting t with z, we have an isomorphism of rings

$$(\mathbb{C}[\mathbf{x}_w, t]/\mathcal{J}_{w,h}) \otimes_{\mathbb{C}[t]} k(z) \cong \mathbb{C}[\mathbf{x}_w]/(\mathcal{J}_{w,h}|_{t=z}).$$

Thus,

$$(\mathfrak{X}_h)_z = \bigcup_{w \in \mathfrak{S}_n} \mathrm{Spec}\,(\mathbb{C}[\mathbf{x}_w]/(\mathcal{J}_{w,h}|_{t=z}))\,. \tag{5.5}$$

In order to show that the fibres $(\mathfrak{X}_h)_z$ are reduced, we will prove that the rings $\mathbb{C}[\mathbf{x}_w]/(\mathcal{J}_{w,h}|_{t=z})$ are reduced.

Proposition 5.10. Suppose that h is indecomposable. Let $z \in \mathbb{C}$ and $w \in \mathfrak{S}_n$. Then the ring $\mathbb{C}[\mathbf{x}_w]/(\mathcal{J}_{w,h}|_{t=z})$ is reduced.

Proof. First, let us consider the case $z \neq 0$. Focusing on the w_0 patch, we observe that the ideal $\mathcal{J}_{w_0,h}|_{t=z}$ is generated by the polynomials

$$F^{w_0}_{n+1-i,j}|_{t=z} = z(\gamma_i - \gamma_{n+1-j})x_{i,j} + x_{i+1,j} + \sum_{k=i+1}^{n-j} (z\gamma_k x_{k,j} + x_{k+1,j})y_{i,n+1-k} + z\gamma_{n+1-j}\widetilde{y}_{i,j}.$$

Proceeding as in Lemma 5.8, it easy to show that

$$\mathbb{C}[\mathbf{x}_{w_0}]/(\mathcal{J}_{w_0,h}|_{t=z}) \cong \mathbb{C}[x_{i,j} \mid 1 \leqslant j \leqslant n-1, i > n-h(j)].$$

In particular, $\mathbb{C}[\mathbf{x}_{w_0}]/(\mathcal{J}_{w_0,h}|_{t=z})$ is reduced. Using the same argument as in Lemma 5.7, we have that, for all $w \in \mathfrak{S}_n$, the rings $\mathbb{C}[\mathbf{x}_w]/(\mathcal{J}_{w,h}|_{t=z})$ are Cohen-Macaulay. Thus the rings $\mathbb{C}[\mathbf{x}_w]/(\mathcal{J}_{w,h}|_{t=z})$ are reduced by Lemma 3.14.

Next, consider the case $z = 0$. For any $w \in \mathfrak{S}_n$, we have

$$F^w_{i,j}|_{t=0} = (M^{-1}\Gamma_0 M)_{i,j} = (M^{-1}NM)_{i,j} = f^w_{i,j},$$

where $f^w_{i,j}$ is a generator of the ideal $J_{w,h}$ as introduced in Section 3. Then we have an equality of ideals $\mathcal{J}_{w,h}|_{t=0} = J_{w,h}$, for all $w \in \mathfrak{S}_n$. It follows that the ring $\mathbb{C}[\mathbf{x}_w]/(\mathcal{J}_{w,h}|_{t=0})$ is reduced by Proposition 3.5. □

Propositions 5.5 and 5.10 together conclude the proof of Theorem 5.1.

We end this section with an example showing that Theorem 5.1 does not hold when h is decomposable.

Example 5.11 (Non-reduced fiber when h is decomposable)**.** Let $n = 2$ and $h = (1, 2)$. We consider the open subset $\mathfrak{X}_h \cap (\mathcal{N}_{\mathrm{id}} \times \mathbb{A}^1)$ of our family $\mathfrak{X}_h$ and its fiber at $t = 0$. We have $\mathcal{J}_{\mathrm{id},h} = \langle F^{\mathrm{id}}_{2,1}\rangle \subseteq \mathbb{C}[x_{1,1}, t]$, where

$$F^{\mathrm{id}}_{2,1} = t(\gamma_2 - \gamma_1)x_{1,1} - x^2_{1,1}.$$

It is easy to see directly that the quotient ring $\mathbb{C}[\mathbf{x}_{\mathrm{id}}, t]/\mathcal{J}_{\mathrm{id},h}$ is reduced. However, we have $\mathcal{J}_{\mathrm{id},h}|_{t=0} = \langle x^2_{1,1}\rangle$. Thus the ring $\mathbb{C}[\mathbf{x}_{\mathrm{id}}]/\langle x^2_{1,1}\rangle$ is not reduced. We conclude that scheme-theoretic fiber $(\mathfrak{X}_h)_0$ is not reduced.

6. Preliminaries: Newton-Okounkov bodies and degrees

The results of the previous sections put us in a position to effectively study the Newton-Okounkov bodies of Hessenberg varieties, and our first results in this direction occupy the remaining sections of this manuscript. In particular, using (special cases of) the results of Section 7 and Section 8, we give in Section 9 some explicit computations of Newton-Okounkov bodies of Hessenberg varieties, which was in fact one of the primary motivations behind the present manuscript. Since we expect that some of the readers of this manuscript may not be familiar with the theory of Newton-Okounkov bodies, in this section we provide some background and recall some definitions. As in Section 2, we keep discussion very brief since detailed exposition is available in the literature (e.g. [27, 30]).

6.1. Definitions and construction of Newton-Okounkov bodies.

We begin with the definition of a valuation (in our setting). We equip $\mathbb{Z}^n$ with the lexicographic order.

Definition 6.1.
(1) Let V be a $\mathbb{C}$-vector space. A **prevaluation** on V is a function

$$\nu\colon V \setminus \{0\} \to \mathbb{Z}^n$$

satisfying the following:
  (a) $\nu(cf) = \nu(f)$ for all $f \in V \setminus \{0\}$ and $c \in \mathbb{C} \setminus \{0\}$,
  (b) $\nu(f + g) \geqslant \min\{\nu(f), \nu(g)\}$ for all $f, g \in V \setminus \{0\}$ with $f + g \neq 0$.

(2) Let A be a $\mathbb{C}$-algebra. A **valuation** on A is a prevaluation on A, $\nu\colon A \setminus \{0\} \to \mathbb{Z}^n$, which also satisfies the following: $\nu(fg) = \nu(f) + \nu(g)$ for all $f, g \in A \setminus \{0\}$.

(3) The image $\nu(A \setminus \{0\}) \subset \mathbb{Z}^n$ of a valuation ν on a $\mathbb{C}$-algebra A is clearly a semigroup and is called the **value semigroup** of the pair (A, ν).

(4) Moreover, if in addition the valuation has the property that for any pair $f, g \in A \setminus \{0\}$ with same value $\nu(f) = \nu(g)$ there exists a non-zero constant $c \neq 0 \in \mathbb{C}$ such that either $\nu(g - cf) > \nu(g)$ or else $g - cf = 0$ then we say that the valuation has **one-dimensional leaves**.

If ν is a valuation with one-dimensional leaves, then the image of ν is a sublattice of $\mathbb{Z}^n$ of full rank. Hence, by replacing $\mathbb{Z}^n$ with this sublattice if necessary, we will always assume without loss of generality that ν is surjective.

We are interested in valuations which arise naturally in geometric contexts. Let X be a projective variety of dimension d over $\mathbb{C}$. The following is an example of a valuation on $\mathbb{C}(X)$ which is frequently considered in the theory of Newton-Okounkov bodies. Suppose given a flag

$$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{d-1} \supseteq Y_d = \{\mathrm{pt}\}$$

of irreducible subvarieties of X where $\mathrm{codim}_{\mathbb{C}}(Y_\ell) = \ell$ and each Y_ℓ is non-singular at the point $Y_d = \{\mathrm{pt}\}(\ell = 0, 1, \ldots, d)$. Such a flag defines a valuation by an inductive procedure involving restricting to each subvariety and considering the order of vanishing along the next (smaller) subvariety (for details see e.g. [30]). Many computations of Newton-Okounkov bodies that occur in the current literature are for valuations defined in this manner, and the point of Section 7 is to construct a particularly natural and well-behaved such flag of subvarieties for a regular nilpotent Hessenberg variety.

Now let $E := H^0(X, L)$ denote the space of global sections of L; it is a finite dimensional vector space over $\mathbb{C}$. Recall that the line bundle L gives rise to the **Kodaira map** Φ_E of E, from X to the projective space $\mathbb{P}(E^*)$. The assumption that L is very ample implies that the Kodaira map Φ_E is an embedding. Further let E^k denote the image of the k-fold product $E \otimes \cdots \otimes E$ in $H^0(X, L^{\otimes k})$ under the natural map given by taking the product of sections. The homogeneous coordinate ring of X with respect to the embedding $\Phi_E : X \hookrightarrow \mathbb{P}(E^*)$ can be identified with the graded algebra

$$R = R(E) = \bigoplus_{k \geqslant 0} R_k,$$

where $R_k := E^k$. This is a subalgebra of the **ring of sections** $R(L) = \bigoplus_{k \geqslant 0} H^0(X, L^{\otimes k})$.

For a fixed ν we now associate a semigroup $S(R) \subset \mathbb{N} \times \mathbb{Z}^d$ to R. First we identify $E = H^0(X, L)$ with a (finite-dimensional) subspace of $\mathbb{C}(X)$ by choosing a non-zero element $h \in E$ and mapping $f \in E$ to the rational function $f/h \in \mathbb{C}(X)$. Similarly, we can associate the rational function f/h^k to an element $f \in R_k := E^k \subseteq H^0(X, L^{\otimes k})$. We define

$$S = S(R) = S(R, \nu, h) = \bigcup_{k>0} \{(k, \nu(f/h^k)) \mid f \in E^k \setminus \{0\}\}. \tag{6.1}$$

Define $C(R) \subseteq \mathbb{R} \times \mathbb{R}^d$ to be the cone generated by the semigroup $S(R)$, i.e., it is the smallest closed convex cone centered at the origin containing $S(R)$. We can now define the central object of interest.

Definition 6.2. Let $\Delta = \Delta(R) = \Delta(X, R, \nu)$ be the slice of the cone $C(R)$ at level 1, that is, $C(R) \cap (\{1\} \times \mathbb{R}^d)$, projected to $\mathbb{R}^d$ via the projection to the second factor $\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. We have

$$\Delta = \overline{\mathrm{conv}\left(\bigcup_{k>0} \left\{ \frac{x}{k} \;\middle|\; (k, x) \in S(R) \right\} \right)}.$$

The convex body Δ is called the **Newton-Okounkov body of** R with respect to the valuation ν.

The above definition can be naturally extended to cover the case of subrings of the ring $R = R(E)$ of the form $R(W) := \oplus_k W^k \subseteq R(E)$ where W is a choice of subspace $W \subseteq H^0(X, L) = E$ and W^k denotes the image of the k-tensor product $W^{\otimes k} \to E^k$. In this setting we denote the associated Newton-Okounkov body as $\Delta(X, R(W), \nu)$.

6.2. **The degree as an upper bound on the volume of a Newton-Okounkov body.** In this short section we point out a completely elementary fact which follows immediately from the properties of Newton-Okounkov bodies. We do not claim any originality. Nevertheless, because it is an observation which informs much of what we do in Sections 8 and 9 and because we believe it is potentially useful for anyone attempting to compute Newton-Okounkov bodies, we take a moment to record it explicitly here.

Let $H_R(k) := \dim_{\mathbb{C}}(R_k)$ be the Hilbert function of the graded algebra R. It is well-known that the Newton-Okounkov body $\Delta(R)$ defined in Section 6.1 encodes information about the asymptotic behavior of the Hilbert function of R [27, 30, 34, 35].

Theorem 6.3. The Newton-Okounkov body $\Delta(R)$ has real dimension d, and the leading coefficient

$$a_d = \lim_{k \to \infty} \frac{H_R(k)}{k^d},$$

of the Hilbert function $H_R(k)$ of R is equal to $\mathrm{Vol}(\Delta(R))$, the Euclidean volume of $\Delta(R)$ in $\mathbb{R}^d$. In particular, the degree of the projective embedding of X in $\mathbb{P}(E^*)$ is equal to $d!\mathrm{Vol}(\Delta(R))$.

More explicitly, the above theorem implies that

$$\frac{1}{d!} \deg(X \subseteq \mathbb{P}(E^*)) = \mathrm{Vol}(\Delta(R)). \tag{6.2}$$

The discussion above implies the following elementary observation.

Observation 6.4. If we can compute the degree of X in $\mathbb{P}(H^0(X,L)^*) = \mathbb{P}(E^*)$ by some other means, then we have computed the volume of $\Delta(X, R(V), \nu)$, independent of any properties of the semigroup or of the $\nu(R_k)$. In particular, if we are able to obtain, via direct computations, a (finite) set of points in the (projection to the $\mathbb{Z}^d$ factor of the) semigroup $S = S(R)$ whose convex hull has volume equal to $\frac{1}{d!}$ times the degree, then we may immediately conclude that this convex hull is in fact equal to $\Delta(X, R(V), \nu)$.

This is the approach we take in the case of the Peterson variety in Section 9. It also motivates our computation in Section 8, as we explain therein.

7. Flags of subvarieties in regular nilpotent Hessenberg varieties

For a given algebraic variety X, recall from Section 6.1 that the computation of Newton-Okounkov bodies associated to X requires the choice of auxiliary data, one of which is a valuation on the rational functions on X. Natural candidates for such valuations are given by well-behaved flags of subvarieties of X. In general it is natural to choose such flags which are compatible with existing structures on X. For instance, for flag varieties G/B, Kaveh showed in [26] that flags of Schubert varieties give rise to Newon-Okounkov bodies with intimate connections to representation theory. For Hessenberg varieties, which are subvarieties of the flag variety $\mathrm{Flags}(\mathbb{C}^n)$, it seems quite natural to also consider flags of subvarieties obtained by intersecting with Schubert varieties. The point of this section is to show that, in the case of indecomposable regular nilpotent Hessenberg varieties, there is a choice of a sequence of (dual) Schubert varieties which is particularly well-behaved when intersected with $\mathrm{Hess}(N,h)$.

Recall from [19, § 10.6, p.176] that the **dual Schubert variety** $\Omega_w \subseteq \mathrm{Flags}(\mathbb{C}^n)$ for $w \in \mathfrak{S}_n$ is the set of $V_\bullet \in \mathrm{Flags}(\mathbb{C}^n)$ satisfying the condition

$$\dim(V_p \cap \widetilde{F}_{n-q}) \geqslant |\{i \leqslant p \mid w(i) \geqslant q+1\}|$$

for $q, p \in [n]$ where $F_\bullet$ is the **anti-standard flag** given by $\widetilde{F}_j := \mathbb{C}e_{n+1-j} \oplus \mathbb{C}e_{n+2-j} \oplus \cdots \oplus \mathbb{C}e_n$. Recall also from [19, § 10.2, p.159] that $\mathrm{codim}(\Omega_w \subseteq \mathrm{Flags}(\mathbb{C}^n)) = \ell(w)$ the length of $w \in \mathfrak{S}_n$.

For a permutation $w \in \mathfrak{S}_n$, let us define the **rank matrix** $rk(w)$[1] by

$$rk(w)[q,p] := |\{i \leqslant p \mid w(i) \leqslant q\}|.$$

Evidently, $rk(w)[q,p]$ is the rank of the upper left $q \times p$ submatrix of the permutation matrix of w. Recall that the permutation matrix of $w \in \mathfrak{S}_n$ is the matrix which has 1's in the $(w(j), j)$-th entries for $1 \leqslant j \leqslant n$ and 0's elsewhere. For $V_\bullet \in \mathrm{Flags}(\mathbb{C}^n)$, let us consider the composition of the maps

$$V_p \hookrightarrow \mathbb{C}^n \twoheadrightarrow \mathbb{C}^n/\widetilde{F}_{n-q}.$$

Then we have

$$\mathrm{rank}(V_p \to \mathbb{C}^n/\widetilde{F}_{n-q}) = \dim V_p - \dim\ker(V_p \to \mathbb{C}^n/\widetilde{F}_{n-q}) = p - \dim(V_p \cap \widetilde{F}_{n-q})$$

and

$$rk(w)[q,p] = |\{i \leqslant p \mid w(i) \leqslant q\}| = p - |\{i \leqslant p \mid w(i) \geqslant q+1\}|.$$

Hence, we get

$$\Omega_w = \{V_\bullet \in \mathrm{Flags}(\mathbb{C}^n) \mid \mathrm{rank}(V_p \to \mathbb{C}^n/\widetilde{F}_{n-q}) \leqslant rk(w)[q,p] \text{ for } q,p \in [n]\}. \tag{7.1}$$

Now, let us write an element $V_\bullet \in \mathrm{Flags}(\mathbb{C}^n) = \mathrm{GL}_n(\mathbb{C})/B$ in the standard neighbourhood $\mathcal{N}_{w_0}(\subset \mathrm{Flags}(\mathbb{C}^n))$ around w_0B by a matrix of the form

$$V_\bullet = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n-1} & 1 \\ x_{2,1} & x_{2,2} & \cdots & 1 & \\ \vdots & \vdots & \iddots & & \\ x_{n-1,1} & 1 & & & \\ 1 & & & & \end{pmatrix} B. \tag{7.2}$$

[1]This is the notation from [1]. [19] uses $r_w(q,p) = rk(w^{-1})[q,p]$.

Then (7.1) implies that the opposite Schubert variety $\Omega_w \cap \mathcal{N}_{w_0}$ (in this neighbourhood) is described as the set of $V_\bullet \in \mathrm{Flags}(\mathbb{C}^n)$ satisfying the condition:

$$\text{the upper-left } q \times p \text{ matrix in (7.2) has rank at most } rk(w)[q,p] \text{ for all } q,p \in [n].$$

The **diagram** of a permutation $w \in \mathfrak{S}_n$ is obtained from the matrix of w^{-1} by removing all cells in an $n \times n$ array which are weakly to the right and below a 1 in w^{-1}. The remaining cells form the diagram $D(w)$.

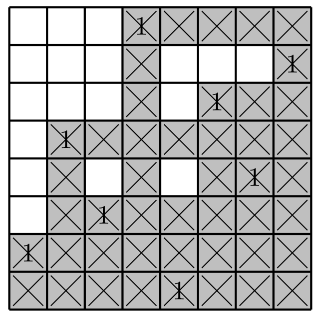


FIGURE 7.1. For $w = 48627315$ in one-line notation, $D(w)$ is the configuration of white boxes in the array above.

It is unfortunate that the diagram is defined in terms of w^{-1}, but that is the most common convention in the literature. In the discussion below, we wish to work with the permutation matrix for w, so we deal with the diagram $D(w^{-1})$ of w^{-1}. Note that the cells of $D(w^{-1})$ are in bijection with the inversions in w^{-1}, and in particular, the Bruhat length $\ell(w) = \ell(w^{-1})$ of w is equal to $|D(w^{-1})|$.

For $w \in \mathfrak{S}_n$, we say that the diagram $D(w^{-1})$ **forms a Young diagram** if all of the boxes in the diagram are connected. From the definitions, the following lemma is immediate.

Lemma 7.1. Let $w \in \mathfrak{S}_n$ and suppose that $D(w^{-1})$ forms a Young diagram. Then we have

$$rk(w)[q,p] = 0 \quad \text{for } (q,p) \in D(w^{-1}).$$

Lemma 7.2. Suppose that $D(w^{-1})$ forms a Young diagram. Then the opposite Schubert variety $\Omega_w \cap \mathcal{N}_{w_0}$ (in the affine chart $\mathcal{N}_{w_0}$) is the set of $V_\bullet \in \mathrm{Flags}(\mathbb{C}^n)$ satisfying the condition

$$x_{q,p} = 0 \quad \text{for } (q,p) \in D(w^{-1})$$

where $x_{i,j}$ are the coordinates for $\mathcal{N}_{w_0}$ given in (7.2).

Proof. Let $Z \subseteq \mathcal{N}_{w_0}$ be the (irreducible) Zariski closed subset of $V_\bullet \in \mathcal{N}_{w_0} (\subset \mathrm{Flags}(\mathbb{C}^n))$ satisfying

$$x_{q,p} = 0 \quad \text{for } (q,p) \in D(w^{-1}).$$

Then, it is clear from Lemma 7.1 that $\Omega_w \cap \mathcal{N}_{w_0} \subseteq Z$. Also, we have

$$\operatorname{codim} \Omega_w \cap \mathcal{N}_{w_0} = \ell(w) = \ell(w^{-1}) = |D(w^{-1})| = \operatorname{codim} Z$$

where the first equality uses the fact that $\Omega_w \cap \mathcal{N}_{w_0} \neq \emptyset$. Hence $\dim \Omega_w \cap \mathcal{N}_{w_0} = \dim Z$, and since Z is irreducible, we obtain $\Omega_w \cap \mathcal{N}_{w_0} = Z$. □

We now build a flag of subvarieties in indecomposable regular nilpotent Hessenberg varieties which looks like a flag of coordinate subspaces near the point w_0. The construction uses a particular sequence of dual Schubert varieties in $\mathrm{Flags}(\mathbb{C}^n)$ which we now describe. First set

$$D := \dim_{\mathbb{C}} \mathrm{Flags}(\mathbb{C}^n) = \frac{1}{2} n(n-1)$$

and let $u_i \in \mathfrak{S}_n$ denote the permutation obtained by multiplying the right-most i simple transpositions of the word

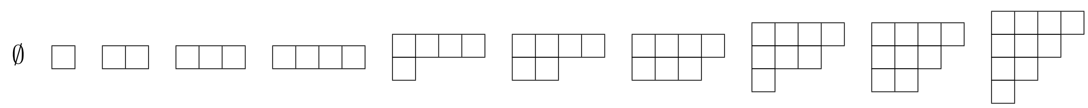
$$(s_1)(s_2 s_1)(s_3 s_2 s_1) \cdots (s_{n-1} s_{n-2} \cdots s_2 s_1),$$

where s_i denotes the simple transposition exchanging i and $i+1$, and we set $u_0 := \text{id}$. Note that $u_D(= w_0)$ is the longest element. It is not hard to check that the diagrams $D(u_i^{-1})$ form Young diagrams, and that the Young diagrams corresponding to the sequence $u_0^{-1}, u_1^{-1}, \ldots, u_{D-1}^{-1}, u_D^{-1} = u_D$ "grow" in sequence by adding boxes from left to right, starting at the top row. We illustrate with an example.

Example 7.3. Suppose $n = 5$. Then

$$\begin{aligned}
u_0 &= & \text{id},\\
u_1 &= & s_1,\\
u_2 &= & s_2s_1,\\
u_3 &= & s_3s_2s_1,\\
u_4 &= & s_4s_3s_2s_1,\\
u_5 &= & s_1\ s_4s_3s_2s_1,\\
u_6 &= & s_2s_1\ s_4s_3s_2s_1,\\
u_7 &= & s_3s_2s_1\ s_4s_3s_2s_1,\\
u_8 &= & s_1\ s_3s_2s_1\ s_4s_3s_2s_1,\\
u_9 &= & s_2s_1\ s_3s_2s_1\ s_4s_3s_2s_1,\\
u_{10} &= & s_1\ s_2s_1\ s_3s_2s_1\ s_4s_3s_2s_1.
\end{aligned}$$

The Young diagrams of $u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}$ are



We can now define a sequence of subvarieties of $\text{Hess}(N, h)$ by intersecting with a sequence of dual Schubert varieties, as follows:

$$\text{Hess}(N,h) = \Omega_{u_0} \cap \text{Hess}(N,h) \supseteq \Omega_{u_1} \cap \text{Hess}(N,h) \supseteq \cdots \supseteq \Omega_{u_D} \cap \text{Hess}(N,h) = \{w_0B\}. \tag{7.3}$$

This sequence is not proper in the sense that it may happen that $\Omega_{u_i} \cap \text{Hess}(N,h) = \Omega_{u_{i+1}} \cap \text{Hess}(N,h)$ for some i. Nevertheless, by omitting redundancies of the above form, we obtain a flag of subvarieties of $\text{Hess}(N,h)$ with well-behaved geometric properties within the open dense subset $\mathcal{N}_{w_0}$. This is the content of the next theorem and is the main result of this section. Recall from Section 3 that the defining equations of $\mathcal{N}_{w_0,h} = \text{Hess}(N,h) \cap \mathcal{N}_{w_0}$ in $\mathcal{N}_{w_0}$ have the property that some of the coordinates $x_{i,j}$ are free and others are non-free variables (cf. remarks after proof of Lemma 3.8).

Theorem 7.4. Let $h\colon [n] \to [n]$ be an indecomposable Hessenberg function. Let $\{u_\ell\}_{\ell=0}^D$ be the sequence in $\mathfrak{S}_n$ defined above, where $D = n(n-1)/2$. Let $\mathcal{N}_{w_0,h} = \text{Hess}(N,h) \cap \mathcal{N}_{w_0}$ be the open affine chart of $\text{Hess}(N,h)$ around w_0B. Then the subvarieties

$$\mathcal{N}_{w_0,h} = \Omega_{u_0} \cap \mathcal{N}_{w_0,h} \supseteq \Omega_{u_1} \cap \mathcal{N}_{w_0,h} \supseteq \cdots \supseteq \Omega_{u_D} \cap \mathcal{N}_{w_0,h} = \{w_0B\} \tag{7.4}$$

satisfy the following:

(1) if the lowest lower-right corner of the Young diagram formed by $D(u_\ell^{-1})$ is located at the position of a free variable, then $\Omega_{u_{\ell-1}} \cap \mathcal{N}_{w_0,h} \neq \Omega_{u_\ell} \cap \mathcal{N}_{w_0,h}$ and
$$\dim \Omega_{u_\ell} \cap \mathcal{N}_{w_0,h} = \dim \Omega_{u_{\ell-1}} \cap \mathcal{N}_{w_0,h} - 1;$$
otherwise, $\Omega_{u_\ell - 1} \cap \mathcal{N}_{w_0,h} = \Omega_{u_\ell} \cap \mathcal{N}_{w_0,h}$;

(2) each $\Omega_{u_\ell} \cap \mathcal{N}_{w_0,h}$ is isomorphic to an affine space, and in particular is non-singular and irreducible in $\mathcal{N}_{w_0,h}$.

Proof. Throughout this argument we use the explicit list of $D = n(n-1)/2$ coordinates on $\mathcal{N}_{w_0} \cong \mathbb{A}^D = \mathbb{A}^{n(n-1)/2}$ given in (7.2), totally ordered by reading the variables from left to right and top to bottom, i.e.

$$x_{1,1}, x_{1,2}, \cdots, x_{1,n-1}, x_{2,1}, x_{2,2}, \cdots, x_{2,n-2}, \cdots, x_{n-1,1}. \tag{7.5}$$

Note also that there are exactly as many variables in the list above as there are elements in the sequence

$$u_1, u_2, \cdots, u_D.$$

As already observed above, from the construction of the sequence u_ℓ it follows that the associated diagrams $D(u_\ell^{-1})$ form Young diagrams, and for a given ℓ, $1 \leqslant \ell \leqslant D$, the Young diagram of $D(u_\ell^{-1})$ contains the boxes corresponding to the first ℓ variables in the list (7.5). We already saw in Lemma 7.2 that $\Omega_{u_\ell} \cap \mathcal{N}_{w_0}$ is equal to the coordinate subspace given by $\{x_{q,p} = 0 \mid (q,p) \in D(u_\ell^{-1})\}$, so it follows that the sequence of intersections $\Omega_{u_\ell} \cap \mathcal{N}_{w_0}$ can be described explicitly in coordinates by setting the first ℓ variables in (7.5) equal to 0, i.e. we have

$$\mathcal{N}_{w_0} \supset \{x_{1,1} = 0\} \supset \{x_{1,1} = x_{1,2} = 0\} \supset \cdots \supset \{x_{1,1} = x_{1,2} = \cdots = x_{n-1,1} = 0\} = \{w_0 B\}. \tag{7.6}$$

In order to prove the statements in the theorem, we must now also analyze the intersection of these $\Omega_{u_\ell} \cap \mathcal{N}_{w_0}$ with $\mathrm{Hess}(N,h)$. We proceed by induction on ℓ.

For $\ell = 1$, we have $\mathbb{C}[\Omega_{u_1} \cap \mathcal{N}_{w_0,h}] \cong \mathbb{C}[\mathcal{N}_{w_0,h}]/\langle x_{1,1}\rangle$. As shown in Lemma 3.8, $\mathbb{C}[\mathcal{N}_{w_0,h}] \cong \mathbb{C}[\mathbf{x}_{w_0}]/J_{w_0,h}$ is isomorphic to a polynomial ring. Moreover, $D(u_1^{-1})$ is a single box located at the position of $x_{1,1}$, which is always a free variable. Therefore $\mathbb{C}[\Omega_{u_1} \cap \mathcal{N}_{w_0,h}]$ is isomorphic to a polynomial ring of dimension one less than $\mathbb{C}[\Omega_{u_0} \cap \mathcal{N}_{w_0,h}] \cong \mathbb{C}[\mathcal{N}_{w_0,h}]$, and $\Omega_{u_1} \cap \mathcal{N}_{w_0,h}$ satisfies properties (1) and (2).

For $\ell > 1$, let $x_{i,j}$ denote the ℓ-th variable in the ordered list (7.5), so that $\Omega_{u_\ell} \cap \mathcal{N}_{w_0,h}$ is obtained from $\Omega_{u_{\ell-1}} \cap \mathcal{N}_{w_0,h}$ by setting $x_{i,j}$ equal to 0. (Visually, the position (i,j) is the lowest lower-right corner of the Young diagram corresponding to $D(u_\ell^{-1})$.) First we consider the case when $x_{i,j}$ is a free variable. Then it is clear that $x_{i,j} = 0$ places a new linear condition on $\Omega_{u_{\ell-1}} \cap \mathcal{N}_{w_0,h}$. Moreover, $\mathbb{C}[\Omega_{u_{\ell-1}} \cap \mathcal{N}_{w_0,h}]$ is irreducible by inductive hypothesis. Therefore the new condition $x_{i,j} = 0$ forces $\Omega_{u_\ell} \cap \mathcal{N}_{w_0,h} \neq \Omega_{u_{\ell-1}} \cap \mathcal{N}_{w_0,h}$ and $\dim \Omega_{u_\ell} \cap \mathcal{N}_{w_0,h} = \dim \Omega_{u_{\ell-1}} \cap \mathcal{N}_{w_0,h} - 1$. Next suppose that $x_{i,j}$ is a non-free variable. As we saw in Lemma 3.8, the defining equations of $\mathrm{Hess}(N,h)$ within the affine coordinate chart $\mathcal{N}_{w_0}$ take the form

$$x_{i,j} = g$$

where $x_{i,j}$ is a non-free variable and where g is a polynomial in the free variables which is contained in the ideal generated by $x_{i-1,t}$ for $t > j$. Since the sequence (7.6) sets variables equal to 0 in order from left to right and top to bottom, we know that at this ℓ-th step, all variables $x_{i-1,t}$ for $t > j$, which are contained in the row directly above that of $x_{i,j}$, have already been set equal to 0, and hence $x_{i,j}$ is already equal to 0 in $\Omega_{u_{\ell-1}} \cap \mathcal{N}_{w_0,h}$. Thus the placement of the additional condition $x_{i,j} = 0$ does not affect the intersection and we conclude that in this case $\Omega_{u_\ell} \cap \mathcal{N}_{w_0,h} = \Omega_{u_{\ell-1}} \cap \mathcal{N}_{w_0,h}$, as was to be shown.

It follows from the above that each $\Omega_{u_\ell} \cap \mathcal{N}_{w_0,h}$ is isomorphic to an affine space with codimension equal to the number of free variables contained within the first ℓ variables in the sequence (7.5). In particular, it is non-singular and irreducible. This completes the proof. □

The practical consequence of the above discussion is the following. By omitting the redundancies in the sequence (7.4) caused by the non-free variables, we obtain a flag of subvarieties in $\mathrm{Hess}(N,h)$ (defined in a geometrically natural fashion by intersecting with dual Schubert varieties) such that, near $w_0 B$, the flag is simply a sequence of affine coordinate subspaces. It would be interesting to compute Newton-Okounkov bodies of regular nilpotent Hessenberg varieties associated to this natural flag. Indeed, the computation of the special case of the Peterson variety in Section 9 uses the flag described above.

8. Degrees of regular nilpotent Hessenberg varieties

Let $\mathrm{Hess}(X,h)$ be a Hessenberg variety in $\mathrm{Flags}(\mathbb{C}^n)$ and consider a Plücker embedding of $\mathrm{Flags}(\mathbb{C}^n) \hookrightarrow \mathbb{P}(V_\lambda^*)$, where λ is a strict partition and V_λ is the irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ associated with λ. It is then natural to consider the induced embedding $\mathrm{Hess}(X,h) \hookrightarrow \mathbb{P}(V_\lambda^*)$, and to ask for its degree. Aside from its intrinstic interest as one of the most basic invariants of an embedding, we also observed in Section 6.2 that a computation of this degree would yield an upper bound on the volume of a Newton-Okounkov body computed with respect to this embedding and is therefore important in the study of Newton-Okounkov bodies of Hessenberg varieties. In this section, we apply results of Section 5 to give an efficient computation of the degree of $\mathrm{Hess}(N,h) \hookrightarrow \mathbb{P}(V_\lambda^*)$ for all indecomposable regular nilpotent Hessenberg varieties. Throughout this section, we let $S\colon \mathbb{C}^n \to \mathbb{C}^n$ be a semisimple operator with pairwise distinct eigenvalues, and we consider the associated regular semisimple Hessenberg variety $\mathrm{Hess}(S,h)$.

In Theorem 5.1 we showed that a certain family $\mathfrak{X}_h \to \mathbb{A}^1$ of Hessenberg varieties – whose generic fibres are regular semisimple Hessenberg varieties and the special fibre is a regular nilpotent Hessenberg variety – is both flat and has reduced fibres. Since Hilbert polynomials are constant along fibres of a flat family [23, Theorem 9.9] and because the special fibre is reduced, Theorem 5.1 therefore immediately implies the following.

Corollary 8.1. Let λ be a dominant weight and let $\mathrm{Flags}(\mathbb{C}^n) \hookrightarrow \mathbb{P}(V_\lambda^*)$ be the corresponding Plücker embedding. By composing with the natural inclusion maps, we obtain embeddings $\mathrm{Hess}(N,h) \hookrightarrow \mathbb{P}(V_\lambda^*)$ and $\mathrm{Hess}(S,h) \hookrightarrow \mathbb{P}(V_\lambda^*)$. If h is indecomposable, then the degrees of these two embeddings are equal, i.e.,

$$\deg(\mathrm{Hess}(N,h) \hookrightarrow \mathbb{P}(V_\lambda^*)) = \deg(\mathrm{Hess}(S,h) \hookrightarrow \mathbb{P}(V_\lambda^*)).$$

Example 8.2. Let $h\colon [n] \to [n]$ satisfy $h(j) = j+1$ for $1 \leqslant j < n$. Then $\mathrm{Hess}(N,h)$ is the Peterson variety Pet_n. The regular semisimple Hessenberg variety $\mathrm{Hess}(S,h)$ is isomorphic to the toric variety $X_{A_{n-1}}$ associated to the root system of type A_{n-1} [14]. We now have

$$\deg(\mathrm{Pet}_n \hookrightarrow \mathbb{P}(V_\lambda^*)) = \deg(X_{A_{n-1}} \hookrightarrow \mathbb{P}(V_\lambda^*)).$$

It is known that regular semisimple Hessenberg varieties are smooth and are equipped with an action of the maximal torus T of $\mathrm{GL}_n(\mathbb{C})$ [14]. In what follows, we use these facts to maximum effect by using both the recent work of Abe, Horiguchi, Masuda, Murai, and Sato [4] as well as the classical Atiyah-Bott-Berline-Vergne formula to obtain a computationally efficient formula for the degree of the embedding $\mathrm{Hess}(S,h) \hookrightarrow \mathbb{P}(V_\lambda^*)$, expressed as a polynomial in the components of $\lambda = (\lambda_1 > \lambda_2 > \ldots, \lambda_{n-1} > \lambda_n)$. By Corollary 8.1, the formula also computes the degree of $\mathrm{Hess}(N,h)$.

We now turn to the details. For simplicity, throughout this discussion we restrict to the special case of $\mathrm{GL}_n(\mathbb{C})$. Let $\lambda = (\lambda_1 > \lambda_2 > \ldots, \lambda_{n-1} > \lambda_n) \in \mathbb{Z}^n$ be a strict partition. It is well-known that there is a unique irreducible representation V_λ of $\mathrm{GL}_n(\mathbb{C})$ associated with λ, and a corresponding Plücker embedding

$$\mathrm{Flags}(\mathbb{C}^n) \cong \mathrm{GL}_n(\mathbb{C})/B \hookrightarrow \mathbb{P}(V_\lambda^*)$$

given by mapping $\mathrm{Flags}(\mathbb{C}^n)$ to the $\mathrm{GL}_n(\mathbb{C})$-orbit of the highest weight vector in V_λ^*. Composing with the canonical inclusion map $\mathrm{Hess}(N,h) \hookrightarrow \mathrm{Flags}(\mathbb{C}^n)$, this gives us a closed embedding

$$\mathrm{Hess}(N,h) \hookrightarrow \mathrm{Flags}(\mathbb{C}^n) \hookrightarrow \mathbb{P}(V_\lambda^*).$$

Define the **volume** of this embedding (or of the corresponding line bundle) by

$$\mathrm{Vol}(\mathrm{Hess}(N,h) \hookrightarrow \mathbb{P}(V_\lambda^*)) := \frac{1}{d!} \deg(\mathrm{Hess}(N,h) \hookrightarrow \mathbb{P}(V_\lambda^*)) \tag{8.1}$$

where $d := \dim_{\mathbb{C}} \mathrm{Hess}(N,h) = \sum_{j=1}^n (h(j) - j)$. As already observed in (6.2), the right-hand side of (8.1) is precisely the volume of any Newton-Okounkov body of $\mathrm{Hess}(N,h)$ computed with respect to this embedding; this justifies the choice of terminology.

Using the result from [3] that the cohomology ring $H^*(\mathrm{Hess}(N,h);\mathbb{Q})$ is a Poincaré duality algebra generated by degree 2 elements, the recent work of [4] relates the cohomology ring of $\mathrm{Hess}(N,h)$ to other combinatorial and algebraic invariants; in particular, in [4, § 11] they define, purely algebraically, a certain polynomial (denoted P_I in [4, § 11]) associated to $H^*(\mathrm{Hess}(N,h);\mathbb{Q})$. The main result of this section is that this polynomial computes the volume $\mathrm{Vol}(\mathrm{Hess}(N,h) \hookrightarrow \mathbb{P}(V_\lambda^*))$ defined in geometric terms in (8.1). To state the result precisely, we first concretely define the polynomial (up to a scalar multiple) given in [4] for our special case of Lie type A_{n-1}. Let $\mathbb{Q}[x_1,\ldots,x_n]$ be a polynomial ring in n variables and for any $i \in [n]$ let ∂_{x_i} denote the usual derivative with respect to the variable x_i. Also for any $i,j \in [n]$ we define $\partial_{i,j} := \partial_{x_j} - \partial_{x_i}$. With this notation in place we may now define, following [4],

$$P_h(x_1,\ldots,x_n) := \left(\prod_{h(j)<i} \partial_{i,j} \right) \prod_{1 \leqslant k < \ell \leqslant n} \frac{x_k - x_\ell}{\ell - k} \in \mathbb{Q}[x_1,\ldots,x_n]. \tag{8.2}$$

The theorem below is the main result of this section.

Theorem 8.3. Let $h\colon [n] \to [n]$ be an indecomposable Hessenberg function and let $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_n) \in \mathbb{Z}^n$ be a strict partition. Then

$$\mathrm{Vol}(\mathrm{Hess}(N,h) \hookrightarrow \mathbb{P}(V_\lambda^*)) = P_h(\lambda_1,\ldots,\lambda_n).$$

Proof. Consider the regular semisimple Hessenberg variety $\mathrm{Hess}(S,h)$ corresponding to the same Hessenberg function h and define the volume $\mathrm{Vol}(\mathrm{Hess}(S,h))$ by the same formula (8.1) (replacing N by S). From the right-hand side of (8.1) and by Corollary 8.1 it follows that it suffices to prove that the volume of the regular semisimple Hessenberg variety is computed by P_h, i.e. it is enough to show

$$\mathrm{Vol}(\mathrm{Hess}(S,h) \hookrightarrow \mathbb{P}(V_\lambda^*)) = P_h(\lambda_1,\ldots,\lambda_n).$$

Since $\mathrm{Hess}(S,h)$ is non-singular [14], the degree of a projective embedding is equal to its symplectic volume [22, § 1.3, pg. 171]:

$$\mathrm{Vol}(\mathrm{Hess}(S,h) \hookrightarrow \mathbb{P}(V_\lambda^*)) = \frac{1}{d!}\deg(\mathrm{Hess}(S,h) \hookrightarrow \mathbb{P}(V_\lambda^*)) = \frac{1}{d!}\int_{\mathrm{Hess}(S,h)} c_1(L_\lambda)^d \tag{8.3}$$

where $c_1(L_\lambda)$ is the first Chern class of the pullback line bundle L_λ on $\mathrm{Hess}(S,h)$ with respect to the Plücker embedding and $d = \dim_{\mathbb{C}} \mathrm{Hess}(S,h) = \sum_{j=1}^n (h(j)-j)$. Since the maximal torus T of $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathrm{Hess}(S,h)$ [14], the Atiyah-Bott-Berline-Vergne localization formula [6, 8] computes this integral using the local data around the torus fixed points:

$$\frac{1}{d!}\int_{\mathrm{Hess}(S,h)} c_1(L_\lambda)^d = \frac{1}{d!}\sum_{w\in\mathfrak{S}_n}\frac{\lambda_w}{e_w}, \tag{8.4}$$

where λ_w denotes the weight of the T-action on the fiber of L_λ at the fixed point wB and e_w denotes the T-equivariant Euler class of the normal bundle to the fixed point wB, i.e., the product of the weights of the T-representation on the tangent space $T_w \mathrm{Hess}(S,h)$.

To proceed further we need a more explicit description of the line bundle L_λ. Let L_i denote the i-th tautological line bundle over $\mathrm{Flags}(\mathbb{C}^n)$, i.e., the fiber of L_i at a flag $V_\bullet \in \mathrm{Flags}(\mathbb{C}^n)$ is V_i/V_{i-1}. Then it is well-known [19, § 9.3] that

$$L_\lambda \cong (L_1^*)^{\lambda_1} \otimes (L_2^*)^{\lambda_2} \otimes \cdots \otimes (L_n^*)^{\lambda_n} \tag{8.5}$$

is the pullback to $\mathrm{Flags}(\mathbb{C}^n)$ of $\mathcal{O}(1) \to \mathbb{P}(V_\lambda^*)$. By slight abuse of notation we also denote by L_λ this line bundle restricted on $\mathrm{Hess}(S,h)$.

We can now compute the right-hand side of (8.4). Recall that the torus T in question is the diagonal torus $T = \{\mathrm{diag}(t_1,t_2,\ldots,t_n) \mid t_i \in \mathbb{C}^\times\}$ of $\mathrm{GL}_n(\mathbb{C})$. In this context, T-weights are elements of $\mathbb{Z}[t_1,\ldots,t_n]$ where each t_i denotes the weight $T \to \mathbb{C}^\times$ defined by $\mathrm{diag}(t_1,t_2,\ldots,t_n) \mapsto t_i$. The weight of the i-th tautological line bundle L_i at the fixed point $w \in \mathfrak{S}_n$ is given by $t_{w(i)}$ since the fiber is $\mathrm{span}_{\mathbb{C}}\, e_{w(i)} \subset \mathbb{C}^n$ by definition of L_i where $e_1,\ldots,e_n$ are the standard basis of $\mathbb{C}^n$. Thus the weight λ_w is

$$\lambda_w = -\sum_{i=1}^n \lambda_i t_{w(i)}. \tag{8.6}$$

It is also known [14] that the weight e_w is given by

$$e_w = \prod_{j<i\leqslant h(j)} (t_{w(i)} - t_{w(j)}) = (-1)^d \prod_{j<i\leqslant h(j)} (t_{w(j)} - t_{w(i)}). \tag{8.7}$$

Putting together (8.3), (8.4), (8.6) and (8.7) we therefore obtain

$$\mathrm{Vol}(\mathrm{Hess}(S,h) \hookrightarrow \mathbb{P}(V_\lambda^*)) = \frac{1}{d!}\sum_{w\in\mathfrak{S}_n}\frac{\left(\sum_{i=1}^n \lambda_i t_{w(i)}\right)^d}{\prod_{j<i\leqslant h(j)}(t_{w(j)} - t_{w(i)})}. \tag{8.8}$$

The essential idea of what follows, due to [4], is to now think of the right-hand side of (8.8) as a polynomial in the variables λ_i. More precisely, let us define

$$Q_{\mathrm{Hess}(S,h)}(x_1,\ldots,x_n) := \frac{1}{d!}\sum_{w\in\mathfrak{S}_n}\frac{\left(\sum_{i=1}^n x_i t_{w(i)}\right)^d}{\prod_{j<i\leqslant h(j)}(t_{w(j)} - t_{w(i)})}.$$

This is in fact a polynomial in $\mathbb{R}[x_1,\ldots,x_n]$ since after taking the summation over $\mathfrak{S}_n$ the right-hand side does not depend on $t_1,\ldots,t_n$ [6, 8]. From the definition it follows that for any strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_n)$ we have

$$\mathrm{Vol}(\mathrm{Hess}(S,h) \hookrightarrow \mathbb{P}(V_\lambda^*)) = Q_{\mathrm{Hess}(S,h)}(\lambda_1,\ldots,\lambda_n). \tag{8.9}$$

Now a straightforward computation shows that

$$\partial_{i,j}\left(\sum_{i=1}^{n} x_i t_{w(i)}\right) = t_{w(j)} - t_{w(i)}.$$

From this, it follows from an easy induction argument that

$$Q_{\mathrm{Hess}(S,h)}(x_1,\ldots,x_n) = \left(\prod_{h(j)<i} \partial_{i,j}\right) Q_{\mathrm{Flags}(\mathbb{C}^n)}(x_1,\ldots,x_n) \tag{8.10}$$

where we think of $\mathrm{Flags}(\mathbb{C}^n)$ as the regular semisimple Hessenberg variety with $h = (n,\ldots,n)$. For a strict partition λ, the volume of $\mathrm{Flags}(\mathbb{C}^n)$ with respect to the Plücker embedding into $\mathbb{P}(V_\lambda^*)$ is well-known to be the volume of the Gelfand-Cetlin polytope associated to λ, for which a formula is known (e.g. [32] and [36]), and we conclude

$$\mathrm{Vol}(\mathrm{Flags}(\mathbb{C}^n) \hookrightarrow \mathbb{P}(V_\lambda^*)) = Q_{\mathrm{Flags}(\mathbb{C}^n)}(x_1,\ldots,x_n) = \prod_{1\leqslant k<\ell\leqslant n} \frac{x_k - x_\ell}{\ell - k}. \tag{8.11}$$

From (8.10) and (8.11) we therefore deduce that

$$Q_{\mathrm{Hess}(S,h)}(x_1,\ldots,x_n) = P_h(x_1,\ldots,x_n). \tag{8.12}$$

Thus, from (8.9) and (8.12), we conclude that for a strict partition λ

$$\mathrm{Vol}(\mathrm{Hess}(S,h) \hookrightarrow \mathbb{P}(V_\lambda^*)) = P_h(\lambda_1,\ldots,\lambda_n)$$

as was to be shown. □

Remark 8.4. Since the line bundle $L_1 \otimes \cdots \otimes L_n$ is trivial, we have $L_n \cong L_1^* \otimes \cdots \otimes L_{n-1}^*$. So we can always assume that $\lambda_n = 0$.

We now use Theorem 8.3 to compute the volume of a special case of a regular nilpotent Hessenberg variety which is studied in Section 9.

Example 8.5. Let $n = 3$ and $h = (2,3,3)$, and consider the corresponding regular nilpotent Hessenberg variety $\mathrm{Pet}_3 := \mathrm{Hess}(N,h) \subset \mathrm{Flags}(\mathbb{C}^3)$. Then

$$\begin{aligned} P_h(x_1,x_2,x_3) &= (\partial_{x_1} - \partial_{x_3})\left(\frac{(x_1-x_2)(x_1-x_3)(x_2-x_3)}{2}\right) \\ &= \frac{1}{2}(x_1-x_2)^2 + 2(x_1-x_2)(x_2-x_3) + \frac{1}{2}(x_2-x_3)^2. \end{aligned}$$

So we obtain

$$\mathrm{Vol}(\mathrm{Pet}_3 \hookrightarrow \mathbb{P}(V_\lambda^*)) = \frac{1}{2}(\lambda_1-\lambda_2)^2 + 2(\lambda_1-\lambda_2)(\lambda_2-\lambda_3) + \frac{1}{2}(\lambda_2-\lambda_3)^2$$

for any strict partition $\lambda = (\lambda_1 > \lambda_2 > \lambda_3)$. Let us introduce the notation $a_1 := \lambda_2 - \lambda_3$ and $a_2 := \lambda_1 - \lambda_2$ and set $\lambda_3 = 0$ following Remark 8.4. Then we have

$$\mathrm{Vol}(\mathrm{Pet}_3 \hookrightarrow \mathbb{P}(V_\lambda^*)) = \frac{1}{2}a_1^2 + 2a_1a_2 + \frac{1}{2}a_2^2.$$

9. Newton-Okounkov bodies of Peterson varieties

With the results of Section 7 and 8 in hand, we are now in a position to give a concrete computation of many examples of Newton-Okounkov bodies associated to the Peterson variety Pet_3 as defined in Definition 2.5. Specifically, we compute the Newton-Okounkov bodies $\Delta(\mathrm{Pet}_3, R(W_\lambda), \nu)$, where here W_λ is the image of $H^0(\mathrm{Flags}(\mathbb{C}^3), L_\lambda)$ in $H^0(\mathrm{Pet}_3, L_\lambda|_{\mathrm{Pet}_3})$ and L_λ is the Plücker line bundle over $\mathrm{Flags}(\mathbb{C}^3)$ corresponding to λ (see [19, § 9.3] or (8.5)).

We need some notation. Let $\lambda = (\lambda_1 > \lambda_2 > \lambda_3) \in \mathbb{Z}^3$ be a dominant weight where we may assume without loss of generality that $\lambda_3 = 0$. In fact it will be convenient to set the notation $a_1 := \lambda_2$ and $a_2 := \lambda_1 - \lambda_2$ so that $\lambda = (a_1 + a_2, a_1, 0)$. Let L_λ denote the Plücker line bundle obtained from the Plücker embedding $\varphi_\lambda\colon \mathrm{Flags}(\mathbb{C}^3) \to \mathbb{P}(V_\lambda^*)$ where V_λ denotes the irreducible $\mathrm{GL}_3(\mathbb{C})$-representation associated with λ. Let W_λ denote the image of $H^0(\mathrm{Flags}(\mathbb{C}^3), L_\lambda)$ in $H^0(\mathrm{Pet}_3, L_\lambda|_{\mathrm{Pet}_3})$ and let $R(W_\lambda)$ denote the corresponding graded ring. We use a geometric valuation on Pet_3 coming from the flag of subvarieties constructed in Section 7. More specifically, on the affine open chart $\mathcal{N}_{w_0}$ near the longest permutation $w_0 = (321) \in \mathfrak{S}_3$, it follows from the analysis in Section 3 of the defining equations of regular nilpotent Hessenberg varieties that $\mathrm{Pet}_3^\circ := \mathcal{N}_{w_0} \cap \mathrm{Pet}_3$ can be identified with matrices of the form

$$\begin{pmatrix} y & x & 1 \\ x & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{9.1}$$

for arbitrary $x, y \in \mathbb{C}$, and applying Theorem 7.4 in this case, we obtain the flag (restricted to Pet_3°)

$$\mathrm{Pet}_3^\circ \supset \{x = 0\} \supset \{x = y = 0\} = \{\mathrm{pt}\}.$$

Letting ν denote the valuation corresponding to the above flag, Theorems 9.6 and 9.10 of this section compute $\Delta(\mathrm{Pet}_3, R(W_{(a_1+a_2,a_1,0)}), \nu)$ for all values of $a_1, a_2 \in \mathbb{Z}_{>0}$ (we argue separately the cases $a_2 \geqslant a_1$ and $a_1 \geqslant a_2$). It is not hard to see that for the usual lexicographic order on $\mathbb{Z}^2$ with $x > y$, the valuation ν is the lowest-term valuation.

Before launching into the computations, we briefly recall a well-known basis for $H^0(\mathrm{Flags}(\mathbb{C}^3), L_\lambda)$ and compute its restriction to Pet_3^0 in terms of the variables x and y above. The following discussion is valid for more general flags and partitions but we restrict to our case for simplicity; see [19] for details. Let $\lambda = (a_1 + a_2, a_1, 0)$ as above, which we now interpret as a Young diagram. For each semistandard Young tableau T with shape λ there is an associated section σ_T of $H^0(\mathrm{Flags}(\mathbb{C}^3), L_\lambda)$ obtained by taking the product of the Plücker coordinates corresponding to each column of T. We illustrate with an example. When restricted to the affine open chart $\mathcal{N}_{w_0}$, we obtain expressions which are polynomials in the matrix entries (suitably interpreted as Plücker coordinates).

Example 9.1. Let

$$A = \begin{pmatrix} y & x & 1 \\ x & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

represent a flag and suppose $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$. Then the left column corresponds to the determinant

$$\det \begin{pmatrix} y & x \\ x & 1 \end{pmatrix}$$

of the first and second rows of the left 3×2 submatrix of A, while the second column corresponds to the determinant $\det(1) = 1$ of the third row of the left 3×1 submatrix. Thus $\sigma_T = y - x^2$.

It is well-known that sections obtained from semistandard Young tableaux form a basis of $H^0(\mathrm{Flags}(\mathbb{C}^3), L_\lambda)$ [19, § 8 and 9].

Theorem 9.2. *The set $\{\sigma_T\}$ of all sections corresponding to semistandard Young tableaux of shape λ, as described above, form a basis for $H^0(\mathrm{Flags}(\mathbb{C}^3), L_\lambda)$.*

Motivated by the above theorem, we now analyze the set $\mathcal{S}_\lambda$ of all semistandard Young tableau of shape $\lambda = (a_1 + a_2, a_1, 0)$ with entries in $\{1, 2, 3\}$. First observe that, from the definition of λ, our Young tableau contains columns of length at most 2. Moreover, since columns must be strictly increasing, the only possible length-2 columns which can appear in $T \in \mathcal{S}_\lambda$ are $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}$, $\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}$, and $\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}$. The only possible length-1 columns are $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$. Moreover, because rows must be weakly increasing (from left to right), a column $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}$ must appear to the left of a $\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}$ or a $\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}$, and a $\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}$ can only appear to the left of a $\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}$, and so on. Thus it is not hard to see that we can uniquely represent a semistandard Young tableau of shape $\lambda = (a_1 + a_2, a_1, 0)$ by recording the number of times each type of column appears. More formally, let

$$k_{12}(T) := \text{ the number of times the column } \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} \text{ appears in } T$$

and

$$k_1(T) := \text{ the number of times the column } \boxed{1} \text{ appears in } T$$

and similarly for $k_{13}(T), k_{23}(T), k_2(T)$ and $k_3(T)$. The following lemma is straightforward.

Lemma 9.3. Let $T \in \mathcal{S}_\lambda$. Then:

(1) T is completely determined by the 6 integers $k_{12}(T), k_{13}(T), k_{23}(T), k_1(T), k_2(T)$ and $k_3(T)$;
(2) we must have $k_{12}(T) + k_{13}(T) + k_{23}(T) = a_1, \quad k_1(T) + k_2(T) + k_3(T) = a_2$, and if $k_{23}(T) \neq 0$ then $k_1(T) = 0$.

Thus the set $\mathcal{S}_\lambda$ is in bijective correspondence with the set

$$\left\{(k_{12}, k_{13}, k_{23}, k_1, k_2, k_3) \in \mathbb{Z}^6_{\geqslant 0} \mid k_{12} + k_{13} + k_{23} = a_1, \ \ k_1 + k_2 + k_3 = a_2, \text{ and } k_{23} \neq 0 \Rightarrow k_1 = 0\right\}. \tag{9.2}$$

Proof. By definition, a Young tableau of shape $\lambda = (a_1 + a_2, a_1, 0)$, reading left to right, has a_1 columns of size 2 and a_2 columns of size 1. A semistandard Young tableau $T \in \mathcal{S}_\lambda$ must have weakly increasing rows. Hence the only possible arrangement of the length-2 columns is to place (starting from the left) all the $\begin{smallmatrix}1\\2\end{smallmatrix}$'s, then the $\begin{smallmatrix}1\\3\end{smallmatrix}$'s, and then the $\begin{smallmatrix}2\\3\end{smallmatrix}$'s. Since the diagram has a_1 many columns of length 2, it is immediate that $k_{12}(T) + k_{13}(T) + k_{23}(T) = a_1$. It also follows that the left a_1 columns are determined by these 3 integers. Next consider the length-1 columns. Again, since rows must be weakly increasing, all $\boxed{1}$'s must be placed first, followed by $\boxed{2}$'s, followed by the $\boxed{3}$'s. Finally, if $k_{23}(T) \neq 0$, this means that there is already a 2 in the top row before reaching the length-1 columns, so there cannot be any $\boxed{1}$'s among the length-1 columns, i.e. $k_1(T) = 0$ as claimed. Again it follows that these are completely determined by $k_1(T), k_2(T)$ and $k_3(T)$ and that $k_1(T) + k_2(T) + k_3(T) = a_2$. Moreover, it is clear that any 6 positive integers satisfying the conditions of (9.2) correspond to some $T \in \mathcal{S}_\lambda$. ☐

Based on the above lemma, henceforth we specify a semistandard Young tableau T by a tuple of integers $(k_{12}, k_{13}, k_{23}, k_1, k_2, k_3)$ satisfying the conditions (9.2), and we also use the notation

$$(12)^{k_{12}}(13)^{k_{13}}(23)^{k_{23}}(1)^{k_1}(2)^{k_2}(3)^{k_3}. \tag{9.3}$$

Example 9.4. Suppose $\lambda = (5, 2, 0)$ so that $a_2 = 3$ and $a_1 = 2$. The tableau $\begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&3\\\hline 3&3\\\cline{1-2}\end{array}$ corresponds to $(0, 2, 0, 0, 2, 1)$ and we also write it as $(13)^2(2)^2(3)$.

We need the following computation.

Lemma 9.5. Let T be a semistandard Young tableau

$$T := (12)^{k_{12}}(13)^{k_{13}}(23)^{k_{23}}(1)^{k_1}(2)^{k_2}(3)^{k_3}$$

as above. Then the section σ_T, restricted to Pet_3° and expressed in terms of the variables x and y in (9.1), takes the form

$$(y - x^2)^{k_{12}}(-x)^{k_{13}}(-1)^{k_{23}}y^{k_1}x^{k_2}1^{k_3}.$$

Proof. Let A denote a 3×3 matrix as in (9.1). By its construction, the section σ_T evaluated at A takes the form [19]

$$(P_{12})^{k_{12}}(P_{13})^{k_{13}}(P_{23})^{k_{23}}(P_1)^{k_1}(P_2)^{k_2}(P_3)^{k_3}$$

where

$$P_{12} = \begin{vmatrix} y & x \\ x & 1 \end{vmatrix} = y - x^2, \quad P_{13} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x, \quad P_{23} = \begin{vmatrix} x & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad P_1 = y, \quad P_2 = x, \quad P_3 = 1.$$

The result follows. ☐

We can now compute the Newton-Okounkov bodies $\Delta(\mathrm{Pet}_3, R(W_{(a_1+a_2,a_1,0)}), \nu)$. Recall from Section 8 (especially Observation 6.4) that if we can find vertices contained in $\Delta(\mathrm{Pet}_3, R(W_{(a_1+a_2,a_1,0)}), \nu)$ whose convex hull Δ has volume equal to the degree of $\mathrm{Pet}_3 \hookrightarrow \mathbb{P}(V_\lambda^*)$, then $\Delta = \Delta(\mathrm{Pet}_3, R(W_{(a_1+a_2,a_1,0)}), \nu)$. Since we know the degree of $\mathrm{Pet}_3 \hookrightarrow \mathbb{P}(V_\lambda^*)$ by Theorem 8.3 and Example 8.5, we take this approach in our arguments below. We need to consider the cases $a_2 \geqslant a_1$ and $a_1 \geqslant a_2$ separately. We begin with the slightly simpler case, $a_2 \geqslant a_1$.

Theorem 9.6. Let $\lambda = (a_1 + a_2, a_1, 0)$ and suppose $a_2 \geqslant a_1$. Then the corresponding Newton-Okounkov body $\Delta(\mathrm{Pet}_3, R(W_{(a_1+a_2,a_1,0)}), \nu)$ is the convex hull of the vertices

$$\{(0, 0), (2a_1 + a_2, 0), (0, a_1 + a_2), (3a_1, a_2 - a_1)\}.$$

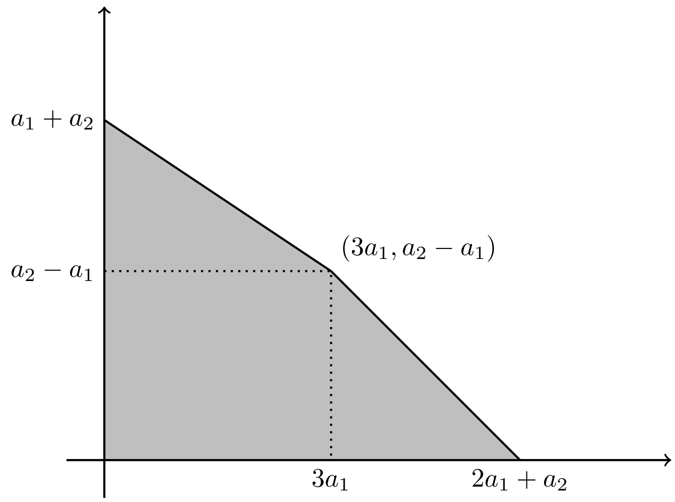


FIGURE 9.1. Newton-Okounkov body $\Delta(\mathrm{Pet}_3, R(W_{(a_1+a_2,a_1,0)}), \nu)$ for $a_2 \geqslant a_1$.

Proof. First, notice that the area of the convex hull described in the statement of the theorem is

$$3a_1(a_2 - a_1) + \frac{1}{2}(3a_1)(2a_1) + \frac{1}{2}(a_2 - a_1)^2 = \frac{1}{2}a_1^2 + 2a_1a_2 + \frac{1}{2}a_2^2.$$

Therefore, by Observation 6.4, it suffices to show that the four stated vertices all lie in $\nu(W_{(a_1+a_2,a_1,0)})$. We deal with the four cases separately.

We begin with $(0,0)$. The semistandard Young tableau $(23)^{a_1}(3)^{a_2}$ corresponds to the polynomial 1 (by Lemma 9.5), and $\nu(1) = (0,0)$. Hence, $(0,0)$ is in the image $\nu(W_{(a_1+a_2,a_1,0)})$.

Next we consider $(0, a_1+a_2)$. The semistandard Young tableau $(12)^{a_1}(1)^{a_2}$ corresponds to the polynomial $(y - x^2)^{a_1}y^{a_2}$, and $\nu((y - x^2)^{a_1}y^{a_2}) = (0, a_1 + a_2)$.

Now we consider $(2a_1+a_2, 0)$, for which we look at the set of tableaux $(12)^k(13)^{a_1-k}(1)^{a_1-k}(2)^{a_2-a_1+k}$ for $0 \leqslant k \leqslant a_1$. Notice that these are valid tableaux because $a_2 \geqslant a_1$. By Lemma 9.5 these have corresponding polynomials (up to sign)

$$\begin{aligned} g_k := (y - x^2)^k x^{a_2} y^{a_1-k} &= \left[\sum_{j=0}^{k} (-1)^j \binom{k}{j} y^{k-j} x^{2j}\right] x^{a_2} y^{a_1-k} \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{a_2+2j} y^{a_1-j}. \end{aligned}$$

Note that the set of $a_1 + 1$ monomials $x^\alpha y^\beta$ that appear in the $a_1 + 1$ polynomials $\{g_0, \ldots, g_{a_1}\}$ is precisely:

$$\left\{x^{a_2}y^{a_1}, x^{a_2+2}y^{a_1-1}, x^{a_2+4}y^{a_1-2}, \ldots, x^{a_2+2a_1}\right\} \tag{9.4}$$

and also that, with respect to this ordered basis, the $(a_1 + 1) \times (a_1 + 1)$ matrix of coefficients of the g_k is triangular and invertible. Thus $x^{a_2+2a_1}$ is equal to an appropriate linear combination of the g_k's and in particular is in $W_{(a_1+a_2,a_1,0)}$. Since $\nu(x^{a_2+2a_1}) = (a_2 + 2a_1, 0)$ we see that this vertex lies in the image of ν.

Finally, for the case of the vertex $(3a_1, a_2 - a_1)$ we consider the tableaux $(12)^k(13)^{a_1-k}(1)^{a_2-k}(2)^k$ for $0 \leqslant k \leqslant a_1$. Notice that these are valid tableaux because $a_2 \geqslant a_1$. Again by Lemma 9.5, we can compute the corresponding polynomials to be

$$h_k := (y - x^2)^k x^{a_1} y^{a_2-k} = \left[\sum_{j=0}^{k} (-1)^j \binom{k}{j} y^{k-j} x^{2j} \right] x^{a_1} y^{a_2-k}$$
$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{a_1+2j} y^{a_2-j}.$$

By an argument similar to that above, we can see that there is an appropriate linear combination of the h_k which equals $x^{3a_1} y^{a_2-a_1}$, and since $\nu(x^{3a_1} y^{a_2-a_1}) = (3a_1, a_2 - a_1)$ we conclude that it is in the image, as desired. This concludes the proof. $\square$

In order to prove the $a_1 \geqslant a_2$ case, we will need the following terminology.

Definition 9.7. An **upper-triangular Pascal matrix** T is an infinite matrix with (i, j)-th entry for $i, j \in \mathbb{Z}_{\geqslant 0}$ equal to the binomial coefficient $\binom{j-1}{i-1}$, i.e.

$$T := \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \cdots \\ 0 & \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots \\ 0 & 0 & \binom{2}{2} & \binom{3}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix},$$

where we take the convention that $\binom{j-1}{i-1} := 0$ if $i - 1 > j - 1$.

Definition 9.8. A **truncated Pascal matrix** is a matrix obtained from an upper-triangular Pascal matrix T by selecting some arbitrary finite subsets of the rows and columns of T of equal size, i.e.

$$T(r, s) := \begin{pmatrix} \binom{s_0}{r_0} & \binom{s_1}{r_0} & \cdots & \binom{s_d}{r_0} \\ \binom{s_0}{r_1} & \binom{s_1}{r_1} & \cdots & \binom{s_d}{r_1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{s_0}{r_d} & \binom{s_1}{r_d} & \cdots & \binom{s_d}{r_d} \end{pmatrix},$$

for some sets $r = \{r_0 < r_1 < \cdots < r_d\}$ and $s = \{s_0 < s_1 < \cdots < s_d\}$, for $s_i, r_i \in \mathbb{N}$.

We will need the following result [28].

Theorem 9.9. Following the notation above, a truncated Pascal matrix is invertible if and only if $r_i \leqslant s_i$ for all i.

We now compute the Newton-Okounkov body for the case $a_1 \geqslant a_2$.

Theorem 9.10. Let $\lambda = (a_1 + a_2, a_1, 0)$ and suppose $a_1 \geqslant a_2$. Then the corresponding Newton-Okounkov body $\Delta(\mathrm{Pet}_3, R(W_{(a_1+a_2,a_1,0)}), \nu)$ is the convex hull of the vertices

$$(0, 0), (0, a_1 + a_2), (2a_2 + a_1, 0), (3a_2, a_1 - a_2).$$

Proof. By the same reasoning as in the proof of Theorem 9.6, it suffices to show that the four vertices given in the statement of the theorem lie in $\nu(W_{(a_1+a_2,a_1,0)})$. Many of the arguments are similar to those for Theorem 9.6 so we will be brief.

For $(0, 0)$ and $(0, a_1 + a_2)$ it suffices to consider the tableaux $(23)^{a_1}(3)^{a_2}$ and $(12)^{a_1}(1)^{a_2}$ respectively. For $(2a_2 + a_1, 0)$, the collection of tableaux $(12)^k(13)^{a_1-k}(1)^{a_2-k}(2)^k$ for varying k as in the proof of Theorem 9.6 does the job.

The last case of $(3a_2, a_1 - a_2)$ follows the same basic strategy but now also uses truncated Pascal matrices. Consider the tableaux $(12)^{a_1-a_2+k}(13)^{a_2-k}(1)^{a_2-k}(2)^k$ where $0 \leqslant k \leqslant a_2$. As before we can compute the corresponding polynomials h_k to be

$$h_k = \sum_{j=0}^{a_1-a_2+k} (-1)^j \binom{a_1 - a_2 + k}{j} x^{a_2+2j} y^{a_1-j}, \text{ for } 0 \leqslant k \leqslant a_2.$$

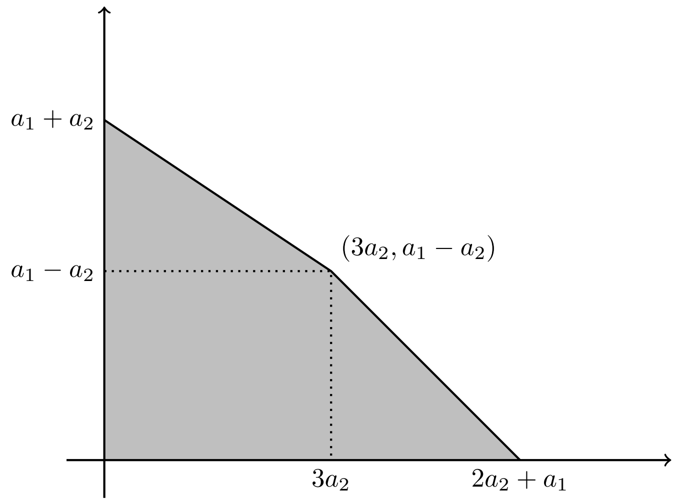


FIGURE 9.2. Newton-Okounkov body $\Delta(\mathrm{Pet}_3, R(W_{(a_1+a_2,a_1,0)}), \nu)$ for $a_1 \geqslant a_2$.

There are $a_1 + 1$ many monomials $x^\alpha y^\beta$ appearing in these $a_2 + 1$ polynomials; listed in increasing lex order, they are

$$\{x^{a_2}y^{a_1}, x^{a_2+2}y^{a_1-1}, x^{a_2+4}y^{a_1-2}, \ldots, x^{3a_2}y^{a_1-a_2}, \ldots, x^{a_2+2a_1-2}y, x^{a_2+2a_1}\}. \tag{9.5}$$

The $(a_1 + 1) \times (a_2 + 1)$ matrix of coefficients of the h_k with respect to the ordered basis (9.5) has (j, k)-th entry equal to $(-1)^j \binom{a_1-a_2+k}{j}$.

We wish to find a suitable linear combination of the h_k so that its lowest term is a multiple of $x^{3a_2}y^{a_1-a_2}$. Some elementary linear algebra shows that it suffices to prove that the upper-left $(a_2+1)\times(a_2+1)$ submatrix A of the matrix of coefficients above, with entries equal to $(-1)^j \binom{a_1-a_2+k}{j}$ for $0 \leqslant j, k \leqslant a_2$, is invertible. For this it suffices in turn to show that $\det A \neq 0$. Let A' denote the matrix obtained from A by multiplying every other row by (-1); then $\det A' = \pm \det A$ so it suffices to show $\det A' \neq 0$. Finally observe that A' is (up to sign) a truncated Pascal matrix $T(r, s)$ for $r = \{0 < 1 < 2 < \cdots < a_2\}$ and $s = \{a_1 - a_2 < a_1 - a_2 + 1 < \cdots < a_1\}$. By our assumption that $a_1 \geqslant a_2$ we have that $r_i \leqslant s_i$ for all i, so by Theorem 9.9 we conclude that $\det A' \neq 0$ as desired. This completes the proof. □

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Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S4K1, Canada / Osaka City University Advanced Mathematical Institute, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, JAPAN

E-mail address: hirakuabe@globe.ocn.ne.jp

Department of Mathematics, Vincent Hall 9, University of Minnesota, 206 Church St. SE, Minneapolis, Minnesota 55455, U.S.A.

E-mail address: dedieu@umn.edu

URL: http://www.math.umn.edu/~dedieu/

Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S4K1, Canada

E-mail address: galettof@math.mcmaster.ca

URL: http://math.galetto.org

Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S4K1, Canada

E-mail address: Megumi.Harada@math.mcmaster.ca

URL: http://www.math.mcmaster.ca/Megumi.Harada/