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CRITICAL AND SUBCRITICAL FRACTIONAL TRUDINGER-MOSER TYPE INEQUALITIES ON \mathbb{R}^N

FUTOSHI TAKAHASHI

ABSTRACT. In this paper, we are concerned with the critical and subcritical Trudinger-Moser type inequalities for functions in a fractional Sobolev space $H^{1/2,2}$ on the whole real line. We prove the relation between two inequalities and discuss the attainability of the suprema.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a domain with finite volume. Then the Sobolev embedding theorem assures that $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1, +\infty)$, however, a simple example shows that the embedding $W_0^{1,N}(\Omega) \hookrightarrow L^\infty(\Omega)$ does not hold. Instead, functions in $W_0^{1,N}(\Omega)$ enjoy the exponential summability:

$$W_0^{1,N}(\Omega) \hookrightarrow \{u \in L^N(\Omega) : \int_{\Omega} \exp(\alpha|u|^{\frac{N}{N-1}}) dx < \infty \text{ for any } \alpha > 0\},$$

see Yudovich [29], Pohozaev [24], and Trudinger [28]. Later, Moser [18] improved the embedding above as follows, now known as the Trudinger-Moser inequality:

$$TM(\Omega, \alpha) = \sup_{\substack{u \in W_0^{1,N}(\Omega) \\ \|\nabla u\|_{L^N(\Omega)} \leq 1}} \frac{1}{|\Omega|} \int_{\Omega} \exp(\alpha|u|^{\frac{N}{N-1}}) dx \begin{cases} < \infty, & \alpha \leq \alpha_N, \\ = \infty, & \alpha > \alpha_N, \end{cases}$$

here $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and $\omega_{N-1} = |S^{N-1}|$ denotes the area of the unit sphere in \mathbb{R}^N . On the attainability of $TM(\Omega, \alpha)$, Carleson-Chang [4], Flucher [6], and Lin [13] proved that $TM(\Omega, \alpha)$ is attained for any $0 < \alpha \leq \alpha_N$.

On domains with infinite volume, for example on the whole space \mathbb{R}^N , the Trudinger-Moser inequality does not hold as it is. However,

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several variants are known on the whole space. In the following, let

$$\Phi_N(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}$$

denote the truncated exponential function.

First, Ogawa [20], Ogawa-Ozawa [21], Cao [3], Ozawa [23], and Adachi-Tanaka [1] proved that the following inequality holds true, which we call Adachi-Tanaka type Trudinger-Moser inequality:

$$A(N, \alpha) = \sup_{\substack{u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\} \\ \|\nabla u\|_{L^N(\mathbb{R}^N)} \leq 1}} \frac{1}{\|u\|_{L^N(\mathbb{R}^N)}^N} \int_{\mathbb{R}^N} \Phi_N(\alpha |u|^{\frac{N}{N-1}}) dx \begin{cases} < \infty, & \alpha < \alpha_N, \\ = \infty, & \alpha \geq \alpha_N. \end{cases}$$

The inequality enjoys the scale invariance under the scaling $u(x) \mapsto u_\lambda(x) = u(\lambda x)$ for $\lambda > 0$. Note that the critical exponent $\alpha = \alpha_N$ is not allowed for the finiteness of the supremum. Recently, it is proved that $A(N, \alpha)$ is attained for any $\alpha \in (0, \alpha_N)$ by Ishiwata-Nakamura-Wadade [10] and Dong-Lu [5]. In this sense, Adachi-Tanaka type Trudinger-Moser inequality has a subcritical nature of the problem.

On the other hand, Ruf [26] and Li-Ruf [15] proved that the following inequality holds true:

$$B(N, \alpha) = \sup_{\substack{u \in W^{1,N}(\mathbb{R}^N) \\ \|u\|_{W^{1,N}(\mathbb{R}^N)} \leq 1}} \int_{\mathbb{R}^N} \Phi_N(\alpha |u|^{\frac{N}{N-1}}) dx \begin{cases} < \infty, & \alpha \leq \alpha_N, \\ = \infty, & \alpha > \alpha_N. \end{cases}$$

Here $\|u\|_{W^{1,N}(\mathbb{R}^N)} = \left(\|\nabla u\|_{L^N(\mathbb{R}^N)}^N + \|u\|_{L^N(\mathbb{R}^N)}^N \right)^{1/N}$ is the full Sobolev norm. Note that the scale invariance ($u \mapsto u_\lambda$) does not hold for this inequality. Also note that the critical exponent $\alpha = \alpha_N$ is permitted to the finiteness.

Concerning the attainability of $B(N, \alpha)$, the following facts have been proved:

- If $N \geq 3$, then $B(N, \alpha)$ is attained for $0 < \alpha \leq \alpha_N$ [26].
- If $N = 2$, then there exists $\alpha_* > 0$ such that $B(2, \alpha)$ is attained for $\alpha_* < \alpha \leq \alpha_2 (= 4\pi)$ [26], [9].
- If $N = 2$ and $\alpha > 0$ is sufficiently small, then $B(2, \alpha)$ is not attained. [9].

The non-attainability of $B(2, \alpha)$ for α sufficiently small is attributed to the non-compactness of “vanishing” maximizing sequences, as described in [9].

In the following, we focus our attention on the fractional Sobolev spaces.

Let $s \in (0, 1)$, $p \in [1, +\infty)$ and let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. For $s > 0$, let us consider the space

$$L_s(\mathbb{R}^N) = \left\{ u \in L^1_{loc}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|u|}{1 + |x|^{N+s}} dx < \infty \right\}.$$

For $u \in L_s(\mathbb{R}^N)$, we define the fractional Laplacian $(-\Delta)^{s/2}u$ as follows: First, for $\phi \in \mathcal{S}(\mathbb{R}^N)$, the rapidly decreasing functions on \mathbb{R}^N , $(-\Delta)^{s/2}\phi$ is defined via the normalized Fourier transform \mathcal{F} as $(-\Delta)^{s/2}\phi(x) = \mathcal{F}^{-1}(|\xi|^s \mathcal{F}\phi(\xi))(x)$ for $x \in \mathbb{R}^N$. Then for $u \in L_s(\mathbb{R}^N)$, $(-\Delta)^{s/2}u$ is defined as the element of $\mathcal{S}'(\mathbb{R}^N)$, the tempered distributions on \mathbb{R}^N , by the relation

$$\langle \phi, (-\Delta)^{s/2}u \rangle = \langle (-\Delta)^{s/2}\phi, u \rangle = \int_{\mathbb{R}^N} (-\Delta)^{s/2}\phi \cdot u dx, \quad \phi \in \mathcal{S}(\mathbb{R}^N).$$

Note that $L^p(\mathbb{R}^N) \subset L_s(\mathbb{R}^N)$ for any $p \geq 1$. Also note that it could happen $\text{supp}((-\Delta)^{s/2}u) \not\subset \Omega$ even if $\text{supp}(u) \subset \Omega$ for some open set Ω in \mathbb{R}^N .

By using the above notion, we define the *Bessel potential space* $H^{s,p}(\Omega)$ for a (possibly unbounded) set $\Omega \subset \mathbb{R}^N$ as

$$\begin{aligned} H^{s,p}(\mathbb{R}^N) &= \{ u \in L^p(\mathbb{R}^N) : (-\Delta)^{s/2}u \in L^p(\mathbb{R}^N) \}, \\ \tilde{H}^{s,p}(\Omega) &= \{ u \in H^{s,p}(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega \}. \end{aligned}$$

On the other hand, the Sobolev-Slobodeckij space $W^{s,p}(\mathbb{R}^N)$ is defined as

$$\begin{aligned} W^{s,p}(\mathbb{R}^N) &= \{ u \in L^p(\mathbb{R}^N) : [u]_{W^{s,p}(\mathbb{R}^N)} < \infty \}, \\ [u]_{W^{s,p}(\mathbb{R}^N)}^p &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy, \end{aligned}$$

and for a bounded domain $\Omega \subset \mathbb{R}^N$, we define

$$\tilde{W}^{s,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}}$$

where $\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|_{L^p(\mathbb{R}^N)}^p + [u]_{W^{s,p}(\mathbb{R}^N)}^p \right)^{1/p}$. It is known that

$$\tilde{W}^{s,p}(\Omega) = \{ u \in W^{s,p}(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega \}$$

if Ω is a Lipschitz domain and $H^{s,p}(\mathbb{R}^N) = F_{p,2}^s(\mathbb{R}^N)$ (Triebel-Lizorkin space), $W^{s,p}(\mathbb{R}^N) = B_{p,p}^s(\mathbb{R}^N)$ (Besov space). Thus $H^{s,2}(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$, however in general, $H^{s,p}(\mathbb{R}^N) \neq W^{s,p}(\mathbb{R}^N)$ for $p \neq 2$. See [25], [11] and the references therein.

Recently, Martinazzi [17] (see also [12]) proved a fractional Trudinger-Moser type inequality on $\tilde{H}^{s,p}(\Omega)$ as follows: Let $p \in (1, \infty)$ and

$s = N/p$ for $N \in \mathbb{N}$. Then for any open $\Omega \subset \mathbb{R}^N$ with $|\Omega| < \infty$, it holds

$$\sup_{\substack{u \in \tilde{H}^{s,p}(\Omega) \\ \|(-\Delta)^{s/2}u\|_{L^p(\Omega)} \leq 1}} \frac{1}{|\Omega|} \int_{\Omega} \exp(\alpha|u|^{\frac{p}{p-1}}) dx \begin{cases} < \infty, & \alpha \leq \alpha_{N,p}, \\ = \infty, & \alpha > \alpha_{N,p}. \end{cases}$$

$$\text{Here } \alpha_{N,p} = \frac{N}{\omega_{N-1}} \left(\frac{\Gamma((N-s)/2)}{\Gamma(s/2)2^s\pi^{N/2}} \right)^{-p/(p-1)}.$$

We note that, differently from the classical case, the attainability of the supremum is not known even for $N = 1$ and $p = 2$.

On the Sobolev-Slobodeckij spaces $\tilde{W}^{s,p}(\Omega)$ with $sp = N$, similar fractional Trudinger-Moser inequality is also proved by Parini-Ruf [25] when $N \geq 2$ and Iula [11] when $N = 1$. In this case, the result is weaker and the inequality holds true only for $0 \leq \alpha < \alpha_{N,p}^*$ for some (explicit) value $\alpha_{N,p}^*$. Also, it is not known the inequality holds or not when $\alpha = \alpha_{N,p}^*$.

In the following, we are interested in the simplest one dimensional case, that is, we put $N = 1$, $s = 1/2$ and $p = 2$. In this case, the Bessel potential space $H^{1/2,2}(\mathbb{R})$ coincides with the Sobolev-Slobodeckij space $W^{1/2,2}(\mathbb{R})$ and both seminorms are related as

$$\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi}[u]_{W^{1/2,2}(\mathbb{R})}^2,$$

see Proposition 3.6. in [19]. Then the fractional Trudinger-Moser inequality in [17], [12] can be read as

Proposition 1. *(A fractional Trudinger-Moser inequality on $\tilde{H}^{1/2,2}(I)$)*
Let $I \subset \mathbb{R}$ be an open bounded interval. Then it holds

$$\sup_{\substack{u \in \tilde{H}^{1/2,2}(I) \\ \|(-\Delta)^{1/4}u\|_{L^2(I)} \leq 1}} \frac{1}{|I|} \int_I e^{\alpha|u|^2} dx \begin{cases} < \infty, & \alpha \leq \alpha_{1,2} = \pi, \\ = \infty, & \alpha > \pi \end{cases}$$

For the fractional Adachi-Tanaka type Trudinger-Moser inequality on the whole line, put

$$(1.1) \quad A(\alpha) = \sup_{\substack{u \in H^{1/2,2}(\mathbb{R}) \setminus \{0\} \\ \|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} \leq 1}} \frac{1}{\|u\|_{L^2(\mathbb{R})}^2} \int_{\mathbb{R}} (e^{\alpha u^2} - 1) dx.$$

Then by the precedent results by Ogawa-Ozawa [21] and Ozawa [23], it is known that $A(\alpha) < \infty$ for small exponent α .

On the other hand, it is already known a fractional Li-Ruf type Trudinger-Moser inequality on $H^{1/2,2}(\mathbb{R})$:

Proposition 2. (*Iula-Maalaoui-Martinazzi [12]*)

$$(1.2) \quad B(\alpha) = \sup_{\substack{u \in H^{1/2,2}(\mathbb{R}) \\ \|u\|_{H^{1/2,2}(\mathbb{R})} \leq 1}} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) dx \begin{cases} < \infty, & \alpha \leq \pi, \\ = \infty, & \alpha > \pi. \end{cases}$$

Here

$$\|u\|_{H^{1/2,2}(\mathbb{R})} = \left(\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})}^2 + \|u\|_{L^2(\mathbb{R})}^2 \right)^{1/2}$$

is the full Sobolev norm on $H^{1/2,2}(\mathbb{R})$.

Concerning $A(\alpha)$ in (1.1), a natural question is that to what range of the exponent α the supremum is finite. As pointed out in [8], it remained an open problem for a while. In this paper, first we prove the finiteness of supremum in the full range of values of exponent.

Theorem 1. (*Full range Adachi-Tanaka type on $H^{1/2,2}(\mathbb{R})$*) We have

$$A(\alpha) = \sup_{\substack{u \in H^{1/2,2}(\mathbb{R}) \setminus \{0\} \\ \|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} \leq 1}} \frac{1}{\|u\|_{L^2(\mathbb{R})}^2} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) dx. \begin{cases} < \infty, & \alpha < \pi, \\ = \infty, & \alpha \geq \pi. \end{cases}$$

Ozawa [22] proved that the Adachi-Tanaka type Trudinger-Moser inequality is equivalent to the Gagliardo-Nirenberg type inequality, and he also proved an exact relation between the best constants of both inequalities. As a result, we have the next corollary.

Corollary 1. *Set*

$$\beta_0 = \limsup_{q \rightarrow \infty} \sup_{u \in H^{1/2,2}(\mathbb{R}), u \neq 0} \frac{\|u\|_{L^q(\mathbb{R})}}{q^{1/2} \|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})}^{1-2/q} \|u\|_{L^2(\mathbb{R})}^{2/q}}.$$

Then $\beta_0 = (2\pi e)^{-1/2}$.

Furthermore, we obtain the relation between the suprema of both critical and subcritical Trudinger-Moser type inequalities along the line of Lam-Lu-Zhang [14].

Theorem 2. (*Relation*) We have

$$B(\pi) = \sup_{\alpha \in (0, \pi)} \frac{1 - (\alpha/\pi)}{(\alpha/\pi)} A(\alpha).$$

Also we obtain how Adachi-Tanaka type supremum $A(\alpha)$ behaves when α tends to π .

Theorem 3. (*Asymptotic behavior*) *There exist $C_1, C_2 > 0$ such that for any $\alpha < \pi$ which is close enough to π , it holds*

$$\frac{C_1}{1 - \alpha/\pi} \leq A(\alpha) \leq \frac{C_2}{1 - \alpha/\pi}.$$

Note that the estimate from the above follows from Theorem 2 and Proposition 2. On the other hand, we will see that the estimate from the below follows from a computation using the Moser sequence.

Concerning the existence of maximizers of Adachi-Tanaka type supremum $A(\alpha)$ in (1.1), we see

Theorem 4. (*Attainability of $A(\alpha)$*) *$A(\alpha)$ is attained for any $\alpha \in (0, \pi)$.*

On the other hand, as for $B(\alpha)$ in (1.2), we have

Theorem 5. (*Non-attainability of $B(\alpha)$*) *For $0 < \alpha \ll 1$, $B(\alpha)$ is not attained.*

It is plausible that there exists $\alpha_* > 0$ such that $B(\alpha)$ is attained for $\alpha_* < \alpha \leq \pi$, but we do not have a proof up to now.

Finally, we improve the subcritical Adachi-Tanaka type inequality along the line of Dong-Lu [5]:

Theorem 6. *For $\alpha > 0$, set*

$$(1.3) \quad E(\alpha) = \sup_{\substack{u \in H^{1/2,2}(\mathbb{R}) \setminus \{0\} \\ \|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} \leq 1}} \frac{1}{\|u\|_{L^2(\mathbb{R})}^2} \int_{\mathbb{R}} u^2 e^{\alpha u^2} dx.$$

Then we have

$$E(\alpha) \begin{cases} < \infty, & \alpha < \pi, \\ = \infty, & \alpha \geq \pi. \end{cases}$$

Furthermore, $E(\alpha)$ is attained for all $\alpha \in (0, \pi)$.

Since $e^{\alpha t^2} - 1 \leq \alpha t^2 e^{\alpha t^2}$ for $t \in \mathbb{R}$, Theorem 6 extends Theorem 1. In the classical case, Dong-Lu used a rearrangement technique to reduce the problem to one-dimension and obtained the similar inequality by estimating a one-dimensional integral. The method is similar to [4]. In the fractional setting $H^{1/2,2}$, we cannot follow this argument and we need a new idea.

The organization of the paper is as follows: In section 2, we prove Theorem 1, 2, and 3. In section 3, we prove Theorem 4 and 5. In section 4, we prove Theorem 6.

2. PROOF OF THEOREM 1, 2, AND 3

For the proofs of Theorem 1, 2, and 3, we prepare several lemmas.

Lemma 1. *Set*

$$(2.1) \quad \tilde{A}(\alpha) = \sup_{\substack{u \in H^{1/2,2}(\mathbb{R}) \setminus \{0\} \\ \|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} \leq 1 \\ \|u\|_{L^2(\mathbb{R})} = 1}} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) dx.$$

Then $\tilde{A}(\alpha) = A(\alpha)$ for any $\alpha > 0$.

Proof. For any $u \in H^{1/2,2}(\mathbb{R}) \setminus \{0\}$ and $\lambda > 0$, we put $u_\lambda(x) = u(\lambda x)$ for $x \in \mathbb{R}$. Then we have

$$(2.2) \quad \begin{cases} \|(-\Delta)^{1/4}u_\lambda\|_{L^2(\mathbb{R})} = \|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})}, \\ \|u_\lambda\|_{L^2(\mathbb{R})}^2 = \lambda^{-1}\|u\|_{L^2(\mathbb{R})}^2, \end{cases}$$

since

$$\begin{aligned} 2\pi \|(-\Delta)^{1/4}u_\lambda\|_{L^2(\mathbb{R})}^2 &= [u_\lambda]_{W^{1/2,2}(\mathbb{R})}^2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(\lambda x) - u(\lambda y)|^2}{|x - y|^2} dx dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(\lambda x) - u(\lambda y)|^2}{|\lambda x - \lambda y|^2} d(\lambda x) d(\lambda y) \\ &= [u]_{W^{1/2,2}(\mathbb{R})}^2 = 2\pi \|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Thus for any $u \in H^{1/2,2}(\mathbb{R}) \setminus \{0\}$ with $\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} \leq 1$, if we choose $\lambda = \|u\|_{L^2(\mathbb{R})}^2$, then $u_\lambda \in H^{1/2,2}(\mathbb{R})$ satisfies

$$\|(-\Delta)^{1/4}u_\lambda\|_{L^2(\mathbb{R})} \leq 1 \quad \text{and} \quad \|u_\lambda\|_{L^2(\mathbb{R})}^2 = 1.$$

Thus

$$\frac{1}{\|u\|_{L^2(\mathbb{R})}^2} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) dx = \int_{\mathbb{R}} \left(e^{\alpha u_\lambda^2} - 1 \right) dx \leq \tilde{A}(\alpha),$$

which implies $A(\alpha) \leq \tilde{A}(\alpha)$. The opposite inequality is trivial. \square

Lemma 2. *For any $0 < \alpha < \pi$, it holds*

$$A(\alpha) \leq \frac{(\alpha/\pi)}{1 - (\alpha/\pi)} B(\pi).$$

Proof. Choose any $u \in H^{1/2,2}(\mathbb{R})$ with $\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} \leq 1$ and $\|u\|_{L^2(\mathbb{R})} = 1$. Put $v(x) = Cu(\lambda x)$ where $C^2 = \alpha/\pi \in (0, 1)$ and $\lambda = \frac{C^2}{1-C^2}$. Then

by scaling rules (2.2), we see

$$\begin{aligned} \|v\|_{H^{1/2,2}(\mathbb{R})}^2 &= \|(-\Delta)^{1/4}v\|_{L^2(\mathbb{R})}^2 + \|v\|_{L^2(\mathbb{R})}^2 \\ &= C^2 \|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})}^2 + \lambda^{-1}C^2 \|u\|_{L^2(\mathbb{R})}^2 \\ &\leq C^2 + \lambda^{-1}C^2 = 1. \end{aligned}$$

Also we have

$$\begin{aligned} \int_{\mathbb{R}} \left(e^{\pi v^2} - 1 \right) dx &= \int_{\mathbb{R}} \left(e^{\pi C^2 u^2(\lambda x)} - 1 \right) dx \\ &= \lambda^{-1} \int_{\mathbb{R}} \left(e^{\pi C^2 u^2(y)} - 1 \right) dy \\ &= \frac{1 - C^2}{C^2} \int_{\mathbb{R}} \left(e^{\alpha u^2(y)} - 1 \right) dy \\ &= \frac{1 - (\alpha/\pi)}{(\alpha/\pi)} \int_{\mathbb{R}} \left(e^{\alpha u^2(y)} - 1 \right) dy. \end{aligned}$$

Thus testing $B(\pi)$ by v , we see

$$B(\pi) \geq \int_{\mathbb{R}} \left(e^{\pi v^2} - 1 \right) dx \geq \frac{1 - (\alpha/\pi)}{(\alpha/\pi)} \int_{\mathbb{R}} \left(e^{\alpha u^2(y)} - 1 \right) dy.$$

By taking the supremum for $u \in H^{1/2,2}(\mathbb{R})$ with $\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} \leq 1$ and $\|u\|_{L^2(\mathbb{R})} = 1$, we have

$$B(\pi) \geq \frac{1 - (\alpha/\pi)}{(\alpha/\pi)} \tilde{A}(\alpha).$$

Finally, Lemma 1 implies the result. \square

Proof of Theorem 1: The assertion that $A(\alpha) < \infty$ for $\alpha < \pi$ follows from Lemma 2 and the fact $B(\pi) < \infty$ by Proposition 2.

For the proof of $A(\pi) = \infty$, we use the Moser sequence

$$(2.3) \quad u_\varepsilon = \begin{cases} (\log(1/\varepsilon))^{1/2}, & \text{if } |x| < \varepsilon, \\ \frac{\log(1/|x|)}{(\log(1/\varepsilon))^{1/2}}, & \text{if } \varepsilon < |x| < 1, \\ 0, & \text{if } 1 \leq |x|, \end{cases}$$

and its estimates

$$(2.4) \quad \|(-\Delta)^{1/4}u_\varepsilon\|_{L^2(\mathbb{R})}^2 = \pi + o(1),$$

$$(2.5) \quad \|(-\Delta)^{1/4}u_\varepsilon\|_{L^2(\mathbb{R})}^2 \leq \pi \left(1 + (C \log(1/\varepsilon))^{-1} \right),$$

$$(2.6) \quad \|u_\varepsilon\|_{L^2(\mathbb{R})}^2 = O \left((\log(1/\varepsilon))^{-1} \right)$$

as $\varepsilon \rightarrow 0$ for some $C > 0$. Note $u_\varepsilon \in \tilde{W}^{1/2,2}((-1,1)) \subset W^{1/2,2}(\mathbb{R}) = H^{1/2,2}(\mathbb{R})$. For the estimate (2.4), we refer to Iula [11] Proposition 2.2. For the estimate (2.5), we refer to [11] equation (35). Actually, after a careful look of the proof of Proposition 2.2 in [11], we confirm that

$$\lim_{\varepsilon \rightarrow 0} (\log(1/\varepsilon)) \left(\|(-\Delta)^{1/4} u_\varepsilon\|_{L^2(\mathbb{R})}^2 - \pi \right) \leq C$$

for a positive $C > 0$, which implies (2.5). For (2.6), we compute

$$\begin{aligned} \|u_\varepsilon\|_{L^2(\mathbb{R})}^2 &= \int_{|x| \leq \varepsilon} (\log(1/\varepsilon)) dx + \int_{\varepsilon < |x| \leq 1} \left(\frac{\log(1/|x|)}{(\log(1/\varepsilon))^{1/2}} \right)^2 dx \\ &= 2\varepsilon \log(1/\varepsilon) + \frac{2}{\log(1/\varepsilon)} \int_{\log(1/\varepsilon)}^0 t^2 (-e^t) dx \\ &= 2\varepsilon \log(1/\varepsilon) + \frac{2}{\log(1/\varepsilon)} (\Gamma(3) + o(1)) \end{aligned}$$

as $\varepsilon \rightarrow 0$. Thus we obtain (2.6).

By testing $A(\pi)$ by $v_\varepsilon = u_\varepsilon / \|(-\Delta)^{1/4} u_\varepsilon\|_{L^2(\mathbb{R})}$, we have

$$\begin{aligned} A(\pi) &\geq \frac{1}{\|v_\varepsilon\|_{L^2(\mathbb{R})}^2} \int_{\mathbb{R}} \left(e^{\pi v_\varepsilon^2} - 1 \right) dx \\ &\geq \frac{\|(-\Delta)^{1/4} u_\varepsilon\|_{L^2(\mathbb{R})}^2}{\|u_\varepsilon\|_{L^2(\mathbb{R})}^2} \int_{|x| \leq \varepsilon} \left(e^{\pi v_\varepsilon^2} - 1 \right) dx \\ &\geq \frac{\|(-\Delta)^{1/4} u_\varepsilon\|_{L^2(\mathbb{R})}^2}{\|u_\varepsilon\|_{L^2(\mathbb{R})}^2} \varepsilon \exp \left(\pi \frac{\log(1/\varepsilon)}{\|(-\Delta)^{1/4} u_\varepsilon\|_{L^2(\mathbb{R})}^2} \right) \\ &\geq \frac{\|(-\Delta)^{1/4} u_\varepsilon\|_{L^2(\mathbb{R})}^2}{\|u_\varepsilon\|_{L^2(\mathbb{R})}^2} \varepsilon \exp \left(\frac{\log(1/\varepsilon)}{1 + (C \log(1/\varepsilon))^{-1}} \right) \end{aligned}$$

since $e^t - 1 \geq (1/2)e^t$ for t large and (2.5). Also since

$$\frac{t}{1 + \frac{1}{Ct}} - t = \frac{-1/C}{1 + \frac{1}{Ct}} \rightarrow -\frac{1}{C} \quad \text{as } t \rightarrow \infty,$$

we see $\frac{t}{1 + \frac{1}{Ct}} = t - 1/C + o(1)$ as $t \rightarrow \infty$. Put $t = \log(1/\varepsilon)$, we see

$$\exp \left(\frac{\log(1/\varepsilon)}{1 + (C \log(1/\varepsilon))^{-1}} \right) = \exp(\log(1/\varepsilon) - 1/C + o(1)) = (1/\varepsilon) e^{-1/C + o(1)},$$

which leads to

$$\varepsilon \exp \left(\frac{\log(1/\varepsilon)}{1 + (C \log(1/\varepsilon))^{-1}} \right) \geq e^{-1/C + o(1)} \geq \delta > 0$$

for some $\delta > 0$ independent of $\varepsilon \rightarrow 0$. Therefore, by (2.4), (2.5), (2.6), we have for $\delta' > 0$

$$A(\pi) \geq \frac{\pi + o(1)}{(C \log(1/\varepsilon))^{-1}} \delta \geq \delta' (\log(1/\varepsilon)) \rightarrow \infty$$

as $\varepsilon \rightarrow 0$. This proves $A(\pi) = \infty$. \square

Proof of Theorem 2: By Lemma 2, we have

$$B(\pi) \geq \sup_{\alpha \in (0, \pi)} \frac{1 - (\alpha/\pi)}{(\alpha/\pi)} A(\alpha).$$

Let us prove the opposite inequality. Let $\{u_n\} \subset H^{1/2,2}(\mathbb{R})$, $u_n \neq 0$, $\|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})}^2 + \|u_n\|_{L^2(\mathbb{R})}^2 \leq 1$, be a maximizing sequence of $B(\pi)$.

We may assume $\|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})}^2 < 1$ for any $n \in \mathbb{N}$. Put

$$\begin{cases} v_n(x) = \frac{u_n(\lambda_n x)}{\|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})}}, & (x \in \mathbb{R}) \\ \lambda_n = \frac{1 - \|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})}^2}{\|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})}^2} > 0. \end{cases}$$

Thus by (2.2), we see

$$\|(-\Delta)^{1/4} v_n\|_{L^2(\mathbb{R})}^2 = 1,$$

$$\|v_n\|_{L^2(\mathbb{R})}^2 = \frac{\lambda_n^{-1}}{\|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})}^2} \|u_n\|_{L^2(\mathbb{R})}^2 = \frac{\|u_n\|_{L^2(\mathbb{R})}^2}{1 - \|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})}^2} \leq 1,$$

since $\|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})}^2 + \|u_n\|_{L^2(\mathbb{R})}^2 \leq 1$. Thus, setting $\alpha_n = \pi \|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})}^2 < \pi$ for any $n \in \mathbb{N}$, we may test $A(\alpha_n)$ by $\{v_n\}$, which results in

$$\begin{aligned} B(\pi) + o(1) &= \int_{\mathbb{R}} \left(e^{\pi u_n^2(y)} - 1 \right) dy \\ &= \lambda_n \int_{\mathbb{R}} \left(e^{\pi \|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})}^2 v_n^2(x)} - 1 \right) dx \\ &\leq \lambda_n \frac{1}{\|v_n\|_{L^2(\mathbb{R})}^2} \int_{\mathbb{R}} \left(e^{\alpha_n v_n^2(x)} - 1 \right) dx \\ &\leq \lambda_n A(\alpha_n) = \frac{1 - (\alpha_n/\pi)}{(\alpha_n/\pi)} A(\alpha_n) \\ &\leq \sup_{\alpha \in (0, \pi)} \frac{1 - (\alpha/\pi)}{(\alpha/\pi)} A(\alpha). \end{aligned}$$

Here we have used a change of variables $y = \lambda_n x$ for the second equality, and $\|v_n\|_{L^2(\mathbb{R})}^2 \leq 1$ for the first inequality. Letting $n \rightarrow \infty$, we have the desired result. \square

Proof of Theorem 3:

We need to prove that there exists $C_1 > 0$ such that for any $\alpha < \pi$ which is sufficiently close to π , it holds that

$$A(\alpha) \geq \frac{C_1}{1 - \alpha/\pi}.$$

Again we use the Moser sequence (2.3) and we test $A(\alpha)$ by $v_\varepsilon = u_\varepsilon / \|(-\Delta)^{1/4} u_\varepsilon\|_{L^2(\mathbb{R})}$. As in the similar calculations in the proof of Theorem 1, we have

$$\begin{aligned} A(\alpha) &\geq \frac{1}{\|v_\varepsilon\|_{L^2(\mathbb{R})}^2} \int_{\mathbb{R}} \left(e^{\alpha v_\varepsilon^2} - 1 \right) dx \\ &\geq \frac{(1/2)}{\|v_\varepsilon\|_{L^2(\mathbb{R})}^2} \int_{|x| \leq \varepsilon} e^{\alpha v_\varepsilon^2} dx \\ &\geq C\varepsilon (\log(1/\varepsilon)) \exp\left(\frac{\alpha}{\pi} \frac{\log(1/\varepsilon)}{1 + (C \log(1/\varepsilon))^{-1}}\right) \\ &= C\varepsilon (\log(1/\varepsilon)) \exp(\delta_\varepsilon \log(1/\varepsilon)) \end{aligned}$$

where we put $\delta_\varepsilon = \left(\frac{\alpha}{\pi}\right) \frac{1}{1 + (C \log(1/\varepsilon))^{-1}} \in (0, 1)$.

Now, for $\alpha < \pi$ which is sufficiently close to π , we fix $\varepsilon > 0$ small such that

$$(2.7) \quad \frac{1}{1 - \alpha/\pi} \leq \log(1/\varepsilon) \leq \frac{2}{1 - \alpha/\pi},$$

which implies

$$\exp\left(-\frac{2}{1 - \alpha/\pi}\right) \leq \varepsilon \leq \exp\left(-\frac{1}{1 - \alpha/\pi}\right).$$

With this choice of $\varepsilon > 0$, we have

$$(2.8) \quad \begin{aligned} A(\alpha) &\geq C\varepsilon (\log(1/\varepsilon)) \exp(\delta_\varepsilon \log(1/\varepsilon)) \\ &= C\varepsilon (\log(1/\varepsilon)) (1/\varepsilon)^{\delta_\varepsilon} = C\varepsilon^{1-\delta_\varepsilon} (\log(1/\varepsilon)). \end{aligned}$$

Now, we estimate that

$$\begin{aligned}
\varepsilon^{1-\delta_\varepsilon} &\geq \left(\exp \left(-\frac{2}{1-\alpha/\pi} \right) \right)^{1-\delta_\varepsilon} = \exp \left(-\frac{2}{1-\alpha/\pi} (1-\delta_\varepsilon) \right) \\
&= \exp \left(-\left(\frac{2}{1-\alpha/\pi} \right) \left\{ (1-\alpha/\pi) + (\alpha/\pi) \left(1 - \frac{1}{1+(C \log 1/\varepsilon)^{-1}} \right) \right\} \right) \\
&= \exp \left(-2 - \left(\frac{2(\alpha/\pi)}{1-\alpha/\pi} \right) \left(\frac{1}{1+C \log 1/\varepsilon} \right) \right) \\
&\geq \exp \left(-2 - \left(\frac{2(\alpha/\pi)}{1-\alpha/\pi} \right) \left(\frac{1}{1+\frac{C}{1-\alpha/\pi}} \right) \right) \\
&= e^{-2} \cdot e^{-\frac{2(\alpha/\pi)}{C+1-\alpha/\pi}} = e^{-2} \cdot e^{-f(\alpha/\pi)}
\end{aligned}$$

where $f(t) = \frac{2t}{C+1-t}$ for $t \in [0, 1]$ and we have used (2.7) in the last inequality. We easily see that $f(0) = 0$, $f'(t) = \frac{2(C+1)}{(C+1-t)^2} > 0$ for $t > 0$, thus $f(t)$ is strictly increasing in t and $\max_{t \in [0, 1]} f(t) = f(1) = 2/C$. Thus we have

$$\varepsilon^{1-\delta_\varepsilon} \geq e^{-2} \cdot e^{-2/C} =: C_0$$

which is independent of α . Backing to (2.8) with (2.7), we observe that

$$A(\alpha) \geq C\varepsilon^{1-\delta_\varepsilon} (\log(1/\varepsilon)) \geq CC_0 (\log(1/\varepsilon)) \geq \frac{CC_0}{1-\alpha/\pi}$$

which proves the result. \square

3. PROOF OF THEOREM 4 AND 5

For $u \in H^{1/2,2}(\mathbb{R})$, u^* will denote its symmetric decreasing rearrangement defined as follows: For a measurable set $A \subset \mathbb{R}$, let A^* denote an open interval $A^* = (-|A|/2, |A|/2)$. We define u^* by

$$u^*(x) = \int_0^\infty \chi_{\{y \in \mathbb{R}: |u(y)| > t\}^*}(x) dt$$

where χ_A denote the indicator function of a measurable set $A \subset \mathbb{R}$. Note that u^* is nonnegative, even, and decreasing on the positive line $\mathbb{R}_+ = [0, +\infty)$. It is known that

$$(3.1) \quad \int_{\mathbb{R}} F(u^*) dx = \int_{\mathbb{R}} F(|u|) dx$$

for any nonnegative measurable function $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is the difference of two monotone increasing functions F_1, F_2 with $F_1(0) =$

$F_2(0) = 0$ such that either $F_1 \circ |u|$ or $F_2 \circ |u|$ is integrable. Also the inequality of Pólya-Szegö type

$$\int_{\mathbb{R}} |(-\Delta u^*)^{1/4}|^2 dx \leq \int_{\mathbb{R}} |(-\Delta u)^{1/4}|^2 dx$$

holds true for $u \in H^{1/2,2}(\mathbb{R})$, see for example, [2] and [16].

Remark 1. Note that Radial Compactness Lemma by Strauss [27] is violated on \mathbb{R} . More precisely, let

$$H_{rad}^{1/2,2}(\mathbb{R}) = \{u \in H^{1/2,2}(\mathbb{R}) : u(x) = u(-x), x \geq 0\},$$

then $H_{rad}^{1/2,2}(\mathbb{R})$ cannot be embedded compactly in $L^q(\mathbb{R})$ for any $q > 0$. To see this, let $\psi \neq 0$ be an even function in $C_c^\infty(\mathbb{R})$ with $\text{supp}(\psi) \subset (-1, 1)$ and put $u_n(x) = \psi(x - n) + \psi(x + n)$. Then we see u_n is even, compactly supported smooth function, and $u_n \rightharpoonup 0$ weakly in $H^{1/2,2}(\mathbb{R})$ as $n \rightarrow \infty$. But $\{u_n\}$ does not have any strong convergent subsequence in $L^q(\mathbb{R})$, because $\|u_n\|_{L^q(\mathbb{R})}^q = 2\|\psi\|_{L^q(\mathbb{R})}^q > 0$ for any n sufficient large.

However, for a sequence $\{u_n\}_{n \in \mathbb{N}} \subset H^{1/2,2}(\mathbb{R})$ with u_n even, nonnegative and decreasing on \mathbb{R}_+ , we have the following compactness result.

Proposition 3. *Assume $\{u_n\} \subset H^{1/2,2}(\mathbb{R})$ be a sequence such that u_n is even, nonnegative and decreasing on \mathbb{R}_+ . Let $u_n \rightharpoonup u$ weakly in $H^{1/2,2}(\mathbb{R})$. Then $u_n \rightarrow u$ strongly in $L^q(\mathbb{R})$ for any $q \in (2, +\infty)$ for a subsequence.*

Proof. Since $\{u_n\} \subset H^{1/2,2}(\mathbb{R})$ is a weakly convergent sequence, we have $\sup_{n \in \mathbb{N}} \|u_n\|_{H^{1/2,2}(\mathbb{R})} \leq C$ for some $C > 0$. We also have $u_n(x) \rightarrow u(x)$ a.e $x \in \mathbb{R}$ for a subsequence, thus u is even, nonnegative and decreasing on \mathbb{R}_+ . Now, we use the estimate below, which is referred to a Simple Radial Lemma: If $u \in L^2(\mathbb{R})$ is even, nonnegative and decreasing on \mathbb{R}_+ , then it holds

$$(3.2) \quad u^2(x) \leq \frac{1}{2|x|} \int_{-|x|}^{|x|} u^2(y) dy \leq \frac{1}{2|x|} \|u\|_{L^2(\mathbb{R})}^2 \quad (x \neq 0).$$

Thus $u_n^2(x) \leq \frac{C}{2|x|}$ for $x \neq 0$ by $\sup_{n \in \mathbb{N}} \|u_n\|_{H^{1/2,2}(\mathbb{R})} \leq C$ and $u^2(x) \leq \frac{C}{2|x|}$ by the pointwise convergence. Now, set $v_n = |u_n - u|^q$ for $q > 2$.

Then we see $v_n(x) \rightarrow 0$ a.e. $x \in \mathbb{R}$. Moreover,

$$\begin{aligned} \int_{|x| \geq R} |u_n - u|^q dx &= 2 \int_R^\infty |u_n - u|^q dx \\ &\leq 2^q \left(\int_R^\infty |u_n|^q dx + \int_R^\infty |u|^q dx \right) \\ &\leq C \int_R^\infty \frac{dx}{|x|^{q/2}} = \frac{CR^{1-q/2}}{(q/2) - 1} \rightarrow 0 \end{aligned}$$

as $R \rightarrow \infty$ since $q > 2$. Thus $\{v_n\}_{n \in \mathbb{N}}$ is uniformly integrable. Also by [19] Theorem 6.9, we know that

$$H^{1/2,2}(\mathbb{R}) \subset L^{q_0}(\mathbb{R}) \quad \text{for any } q_0 \geq 2 \text{ and } \|u\|_{L^{q_0}(\mathbb{R})} \leq C \|u\|_{H^{1/2,2}(\mathbb{R})}.$$

For any $q > 2$, take q_0 such that $2 < q < q_0 < \infty$. Since $u_n - u$ is uniformly bounded in $H^{1/2,2}(\mathbb{R})$, we have $\|u_n - u\|_{L^{q_0}(\mathbb{R})} \leq C$, and

$$\int_I v_n dx = \int_I |u_n - u|^q dx \leq \left(\int_I |u_n - u|^{q_0} dx \right)^{q/q_0} |I|^{1-q/q_0}$$

for any bounded measurable set $I \subset \mathbb{R}$. Therefore $\int_I v_n dx \rightarrow 0$ if $|I| \rightarrow 0$, which implies $\{v_n\}$ is uniformly absolutely continuous. Thus by Vitali's Convergence Theorem (see for example, [7] p.187), we obtain $v_n = |u_n - u|^q \rightarrow 0$ strongly in $L^1(\mathbb{R})$, which is the desired conclusion. \square

Proposition 4. *Assume $\{u_n\} \subset H^{1/2,2}(\mathbb{R})$ be a sequence with $\|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})} \leq 1$. Let $u_n \rightharpoonup u$ weakly in $H^{1/2,2}(\mathbb{R})$ for some u and assume u_n is even, nonnegative and decreasing on \mathbb{R}_+ . Then we have*

$$\int_{\mathbb{R}} \left(e^{\alpha u_n^2} - 1 - \alpha u_n^2 \right) dx \rightarrow \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 - \alpha u^2 \right) dx$$

for any $\alpha \in (0, \pi)$.

Proof. The similar proposition above is already appeared, see [10] Lemma 3.1, and [5] Lemma 5.5. We prove it here for the reader's convenience.

Put $\Phi_\alpha(t) = e^{\alpha t^2} - 1$ and $\Psi_\alpha(t) = e^{\alpha t^2} - 1 - \alpha t^2$. Note that $\Phi_\alpha(t)$ is nonnegative, strictly convex and $\Psi'_\alpha(t) = 2\alpha t \Phi_\alpha(t)$. Thus by the mean value theorem, we have

$$\begin{aligned} |\Psi_\alpha(u_n) - \Psi_\alpha(u)| &\leq \Psi'_\alpha(\theta u_n + (1-\theta)u) |u_n - u| \\ &\leq 2\alpha |\theta u_n + (1-\theta)u| \Phi_\alpha(\theta u_n + (1-\theta)u) |u_n - u| \\ &\leq 2\alpha (|u_n| + |u|) (\theta \Phi_\alpha(u_n) + (1-\theta)\Phi_\alpha(u)) |u_n - u| \\ &\leq 2\alpha (|u_n| + |u|) (\Phi_\alpha(u_n) + \Phi_\alpha(u)) |u_n - u|. \end{aligned}$$

Thus we have

$$\begin{aligned}
\int_{\mathbb{R}} |\Psi_{\alpha}(u_n) - \Psi_{\alpha}(u)| dx &\leq 2\alpha \int_{\mathbb{R}} (|u_n| + |u|) (\Phi_{\alpha}(u_n) + \Phi_{\alpha}(u)) |u_n - u| dx \\
(3.3) \qquad &\leq 2\alpha \| |u_n| + |u| \|_{L^a(\mathbb{R})} \| \Phi_{\alpha}(u_n) + \Phi_{\alpha}(u) \|_{L^b(\mathbb{R})} \| u_n - u \|_{L^c(\mathbb{R})}
\end{aligned}$$

by Hölder's inequality, where $a, b, c > 1$ and $1/a + 1/b + 1/c = 1$ are chosen later.

First, direct calculation shows that

$$(3.4) \qquad (\Phi_{\alpha}(t))^b < e^{b\alpha t^2} - 1 \quad (t \in \mathbb{R})$$

for all $b > 1$. Thus if we fix $1 < b < \pi/\alpha$ so that $b\alpha < \pi$ is realized, then we have

$$\begin{aligned}
\| \Phi_{\alpha}(u_n) + \Phi_{\alpha}(u) \|_{L^b(\mathbb{R})}^b &\leq \left(\| \Phi_{\alpha}(u_n) \|_{L^b(\mathbb{R})} + \| \Phi_{\alpha}(u) \|_{L^b(\mathbb{R})} \right)^b \\
&\leq 2^{b-1} \left(\int_{\mathbb{R}} (\Phi_{\alpha}(u_n))^b dx + \int_{\mathbb{R}} (\Phi_{\alpha}(u))^b dx \right) \\
&\leq 2^{b-1} \left(\int_{\mathbb{R}} (e^{b\alpha u_n^2} - 1) dx + \int_{\mathbb{R}} (e^{b\alpha u^2} - 1) dx \right) \\
&\leq 2^{b-1} A(b\alpha) \left(\| u_n \|_{L^2(\mathbb{R})}^2 + \| u \|_{L^2(\mathbb{R})}^2 \right),
\end{aligned}$$

here we used (3.4) for the third inequality and Theorem 1 for the last inequality, the use of which is valid since $\| (-\Delta)^{1/4} u_n \|_{L^2(\mathbb{R})} \leq 1$ and $\| (-\Delta)^{1/4} u \|_{L^2(\mathbb{R})} \leq 1$ by the weak lower semicontinuity. Note that $\{u_n\}$ satisfies $\sup_{n \in \mathbb{N}} \| u_n \|_{H^{1/2,2}(\mathbb{R})} \leq C$ for some $C > 0$. Thus we have obtained $\| \Phi_{\alpha}(u_n) + \Phi_{\alpha}(u) \|_{L^b(\mathbb{R})} = O(1)$ independent of n .

Next, we estimate the term $\| |u_n| + |u| \|_{L^a(\mathbb{R})}$. Since $\{u_n\}$ is a bounded sequence in $H^{1/2,2}(\mathbb{R})$, we have by [19] Theorem 6.9 that $\| u \|_{L^q(\mathbb{R})} \leq C \| u_n \|_{H^{1/2,2}(\mathbb{R})}$ for any $q \geq 2$. Thus we see $\| |u_n| + |u| \|_{L^a(\mathbb{R})} \leq C$ for some $C > 0$ independent of n for $a \geq 2$. Now, note that if we choose $1 < b < \pi/\alpha$ and $a > 2$ sufficiently large, then we can find $c > 2$ such that $1/a + 1/b + 1/c = 1$.

By these choices and Proposition 3, we conclude that $\| u_n - u \|_{L^c(\mathbb{R})} \rightarrow 0$ as $n \rightarrow \infty$. Backing to (3.3) with all together, we conclude that

$$\int_{\mathbb{R}} \Psi_{\alpha}(u_n) dx \rightarrow \int_{\mathbb{R}} \Psi_{\alpha}(u) dx \quad (n \rightarrow \infty),$$

which is the desired conclusion. \square

Now, we prove Theorem 4. We will show that $A(\alpha)$ in (1.1) is attained for any $0 < \alpha < \pi$. Since $A(\alpha) = \tilde{A}(\alpha)$ by Lemma 1, we choose

a maximizing sequence for $\tilde{A}(\alpha)$:

$$\int_{\mathbb{R}} \left(e^{\alpha u_n^2} - 1 \right) dx = A(\alpha) + o(1) \quad (n \rightarrow \infty).$$

Here $\{u_n\}_{n \in \mathbb{N}} \subset H^{1/2,2}(\mathbb{R})$ satisfies $\|(-\Delta)^{1/4} u_n\|_{L^2(\mathbb{R})} \leq 1$ and $\|u_n\|_{L^2(\mathbb{R})} = 1$. By appealing to the use of rearrangement, we may furthermore assume that u_n is nonnegative, even, and decreasing on \mathbb{R}_+ . Since $\{u_n\}_{n \in \mathbb{N}} \subset H^{1/2,2}(\mathbb{R})$ is a bounded sequence, we have $u \in H^{1/2,2}(\mathbb{R})$ such that $u_n \rightharpoonup u$ in $H^{1/2,2}(\mathbb{R})$. By Proposition 4, we see

$$\int_{\mathbb{R}} \left(e^{\alpha u_n^2} - 1 - \alpha u_n^2 \right) dx = \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 - \alpha u^2 \right) dx$$

as $n \rightarrow \infty$. Therefore, since $\|u_n\|_{L^2(\mathbb{R})}^2 = 1$, we have, letting $n \rightarrow \infty$,

$$(3.5) \quad A(\alpha) = \alpha + \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 - \alpha u^2 \right) dx.$$

Next, we claim that $A(\alpha) > \alpha$ for any $0 < \alpha < \pi$. Indeed, take any $u_0 \in H^{1/2,2}(\mathbb{R})$ such that $u_0 \not\equiv 0$, $\|(-\Delta)^{1/4} u_0\|_{L^2(\mathbb{R})} \leq 1$ and $\|u_0\|_{L^2(\mathbb{R})} = 1$. Then we have

$$A(\alpha) = \tilde{A}(\alpha) \geq \int_{\mathbb{R}} \left(e^{\alpha u_0^2} - 1 \right) dx = \alpha + \int_{\mathbb{R}} \left(e^{\alpha u_0^2} - 1 - \alpha u_0^2 \right) dx.$$

Now, since $e^{\alpha t^2} - 1 - \alpha t^2 > 0$ for any $t > 0$, we have

$$\int_{\mathbb{R}} \left(e^{\alpha u_0^2} - 1 - \alpha u_0^2 \right) dx > 0$$

for $u_0 \not\equiv 0$, which results in $A(\alpha) > \alpha$, the claim.

By the claim and (3.5), we conclude that the weak limit u satisfies $u \not\equiv 0$. By the weak lower semi continuity, we have $u \not\equiv 0$ satisfies $\|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R})} \leq 1$ and $\|u\|_{L^2(\mathbb{R})} \leq 1$. Thus by (3.5) again, we see

$$\begin{aligned} A(\alpha) &= \alpha + \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 - \alpha u^2 \right) dx \\ &\leq \alpha + \frac{1}{\|u\|_{L^2(\mathbb{R})}^2} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 - \alpha u^2 \right) dx \\ &= \alpha + \frac{1}{\|u\|_{L^2(\mathbb{R})}^2} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) dx - \alpha \frac{\|u\|_{L^2(\mathbb{R})}^2}{\|u\|_{L^2(\mathbb{R})}^2} \\ &= \frac{1}{\|u\|_{L^2(\mathbb{R})}^2} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) dx. \end{aligned}$$

Thus we have shown that $u \in H^{1/2,2}(\mathbb{R})$ maximizes $A(\alpha)$. \square

Next, we prove Theorem 5. We follow Ishiwata's argument in [9].
Let

$$M = \{u \in H^{1/2,2}(\mathbb{R}) : \|u\|_{H^{1/2,2}(\mathbb{R})} = 1\},$$

$$J_\alpha : M \rightarrow \mathbb{R}, \quad J_\alpha(u) = \int_{\mathbb{R}} (e^{\alpha u^2} - 1) dx.$$

Actually, we will show a stronger claim that J_α has no critical point on M for sufficiently small $\alpha > 0$. Assume the contrary that there exists a critical point $v \in M$ of J_α for small $\alpha > 0$. Then we define an orbit on M through v as

$$v_\tau(x) = \sqrt{\tau}v(\tau x) \quad \tau \in (0, \infty), \quad w_\tau = \frac{v_\tau}{\|v_\tau\|_{H^{1/2}}} \in M.$$

Note that $w_1 = v$ thus it must be $\left. \frac{d}{d\tau} \right|_{\tau=1} J_\alpha(w_\tau) = 0$. By scaling rules (2.2), we see for any $p \geq 2$,

$$\|v_\tau\|_{L^p(\mathbb{R})}^p = \tau^{p/2-1} \|v\|_{L^p(\mathbb{R})}^p \quad \text{and} \quad \|(-\Delta)^{1/4}v_\tau\|_{L^2(\mathbb{R})} = \tau \|(-\Delta)^{1/4}v\|_{L^2(\mathbb{R})}.$$

Now, we see

$$\begin{aligned} J_\alpha(w_\tau) &= \int_{\mathbb{R}} (e^{\alpha w_\tau^2} - 1) dx = \int_{\mathbb{R}} \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \frac{v_\tau^{2j}(x)}{(\|v_\tau\|_2^2 + \|(-\Delta)^{1/4}v_\tau\|_2^2)^j} \\ &= \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \frac{\|v_\tau\|_{2j}^{2j}}{(\|v_\tau\|_2^2 + \|(-\Delta)^{1/4}v_\tau\|_2^2)^j} = \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \frac{\tau^{j-1} \|v\|_{2j}^{2j}}{(\|v\|_2^2 + \tau \|(-\Delta)^{1/4}v\|_2^2)^j} \\ &= \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} f_j(\tau) \end{aligned}$$

where $f_j(\tau) = \frac{\tau^{j-1}c}{(b+\tau a)^j}$ with $a = \|(-\Delta)^{1/4}v\|_2^2$, $b = \|v\|_2^2$ and $c = \|v\|_{2j}^{2j}$.
Since

$$f'_j(\tau) = \frac{\tau^{j-2}c}{(b+\tau a)^{j+1}} \{-\tau a + (j-1)b\}$$

and $\|(-\Delta)^{1/4}v\|_2^2 + \|v\|_2^2 = 1$, we calculate

$$\begin{aligned}
& \left. \frac{d}{d\tau} J_\alpha(w_\tau) \right|_{\tau=1} \\
&= \sum_{j=1}^{\infty} \left[\frac{\alpha^j}{j!} \frac{\tau^{j-2} \|v\|_{2j}^{2j}}{(\|v\|_2^2 + \tau \|(-\Delta)^{1/4}v\|_2^2)^{j+1}} \left\{ -\tau \|(-\Delta)^{1/4}v\|_2^2 + (j-1) \|v\|_2^2 \right\} \right]_{\tau=1} \\
&\leq -\alpha \|(-\Delta)^{1/4}v\|_2^2 \|v\|_2^2 + \sum_{j=2}^{\infty} \frac{\alpha^j}{(j-1)!} \|v\|_{2j}^{2j} \\
&= \alpha \|(-\Delta)^{1/4}v\|_2^2 \|v\|_2^2 \left\{ -1 + \sum_{j=2}^{\infty} \frac{\alpha^{j-1}}{(j-1)!} \frac{\|v\|_{2j}^{2j}}{\|(-\Delta)^{1/4}v\|_2^2 \|v\|_2^2} \right\}.
\end{aligned}$$

Here, we need the following lemma:

Lemma 3. (Ogawa-Ozawa [21]) *There exists $C > 0$ such that for any $u \in H^{1/2,2}(\mathbb{R})$ and $p \geq 2$, it holds*

$$\|u\|_{L^p(\mathbb{R})}^p \leq Cp^{p/2} \|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})}^{p-2} \|u\|_{L^2(\mathbb{R})}^2.$$

For $p = 2j$, Lemma 3 implies

$$\frac{\|v\|_{2j}^{2j}}{\|(-\Delta)^{1/4}v\|_2^2 \|v\|_2^2} \leq C(2j)^j \underbrace{\|(-\Delta)^{1/4}v\|_2^{2j-4}}_{\leq 1 (j \geq 2)} \leq C(2j)^j.$$

Thus for $0 < \alpha \ll 1$ sufficiently small (it would be enough that $\alpha < 1/(2e)$), Stirling's formula $j! \sim j^j e^{-j} \sqrt{2\pi j}$ implies that

$$\sum_{j=2}^{\infty} \frac{\alpha^{j-1}}{(j-1)!} \frac{\|v\|_{2j}^{2j}}{\|(-\Delta)^{1/4}v\|_2^2 \|v\|_2^2} \leq \sum_{j=2}^{\infty} \frac{\alpha^{j-1}}{(j-1)!} (2j)^j \leq \alpha C$$

for some $C > 0$ independent of α . Therefore we have $\left. \frac{d}{d\tau} J_\alpha(w_\tau) \right|_{\tau=1} < 0$ for small α , which is a desired contradiction. \square

4. PROOF OF THEOREM 6.

In order to prove Theorem 6, first we set

$$(4.1) \quad F(\beta) = \sup_{\substack{u \in H^{1/2,2}(\mathbb{R}) \\ \|u\|_{H^{1/2,2}(\mathbb{R})} \leq 1}} \int_{\mathbb{R}} u^2 e^{\beta u^2} dx$$

for $\beta > 0$. Then we have

Proposition 5. *We have $F(\beta) < \infty$ for $\beta < \pi$*

Proof. We follow the proof of Theorem 1.5 in [12]. Take any $u \in H^{1/2,2}(\mathbb{R})$ with $\|u\|_{H^{1/2,2}(\mathbb{R})} \leq 1$ in the admissible sets for $F(\beta)$ in (4.1). By appealing to the rearrangement, we may assume that u is even, nonnegative and decreasing on \mathbb{R}_+ . We divide the integral

$$\int_{\mathbb{R}} u^2 e^{\beta u^2} dx = \int_{\mathbb{R} \setminus I} u^2 e^{\beta u^2} dx + \int_I u^2 e^{\beta u^2} dx = (I) + (II),$$

where $I = (-1/2, 1/2)$.

First, we estimate (I). By the Radial Lemma (3.2), we see for any $k \in \mathbb{N}$, $k \geq 2$,

$$u^{2k}(x) \leq \left(\frac{\|u\|_{L^2(\mathbb{R})}^2}{2|x|} \right)^k = \frac{\|u\|_{L^2(\mathbb{R})}^{2k}}{2^k} \frac{1}{|x|^k} \quad \text{for } x \neq 0.$$

Thus

$$\begin{aligned} \int_{\mathbb{R} \setminus I} u^{2k}(x) dx &\leq \frac{\|u\|_{L^2(\mathbb{R})}^{2k}}{2^k} \int_{\mathbb{R} \setminus I} \frac{dx}{|x|^k} \\ &= \frac{\|u\|_{L^2(\mathbb{R})}^{2k}}{2^{k-1}} \int_{1/2}^{\infty} \frac{dx}{x^k} = \frac{\|u\|_{L^2(\mathbb{R})}^{2k}}{k-1}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} (I) &= \int_{\mathbb{R} \setminus I} u^2 e^{\beta u^2} dx = \int_{\mathbb{R} \setminus I} u^2 \left(1 + \sum_{k=1}^{\infty} \frac{\beta^k u^{2k}}{k!} \right) dx \\ &= \int_{\mathbb{R} \setminus I} u^2 dx + \sum_{k=2}^{\infty} \frac{\beta^{k-1}}{(k-1)!} \int_{\mathbb{R} \setminus I} u^{2k} dx \\ &\leq \|u\|_{L^2(\mathbb{R})}^2 + \sum_{k=2}^{\infty} \frac{\beta^{k-1}}{(k-1)!} \frac{\|u\|_{L^2(\mathbb{R})}^{2k}}{k-1} \\ &= \|u\|_{L^2(\mathbb{R})}^2 \left(1 + \sum_{k=2}^{\infty} \frac{\beta^{k-1}}{(k-1)(k-1)!} \|u\|_{L^2(\mathbb{R})}^{2(k-1)} \right). \end{aligned}$$

Now by the constraint $\|u\|_{H^{1/2,2}(\mathbb{R})} \leq 1$, we have $\|u\|_{L^2(\mathbb{R})} \leq 1$. Also if we put $a_k = \frac{\beta^{k-1}}{(k-1)(k-1)!}$, then $\sum_{k=2}^{\infty} a_k$ converges since $a_{k+1}/a_k = \beta \frac{k-1}{k^2} \rightarrow 0$ as $k \rightarrow \infty$. Thus we obtain

$$(I) \leq 1 + \sum_{k=2}^{\infty} \frac{\beta^{k-1}}{(k-1)(k-1)!} \leq C$$

where $C > 0$ is independent of $u \in H^{1/2,2}(\mathbb{R})$ with $\|u\|_{H^{1/2,2}(\mathbb{R})} \leq 1$.

Next, we estimate (II). Set

$$v(x) = \begin{cases} u(x) - u(1/2), & |x| \leq 1/2, \\ 0, & |x| > 1/2. \end{cases}$$

Then by the argument of [12], we know that

$$\begin{aligned} \|(-\Delta)^{1/4}v\|_{L^2(\mathbb{R})}^2 &\leq \|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})}^2, \\ u^2(x) &\leq v^2(x) \left(1 + \|u\|_{L^2(\mathbb{R})}^2\right) + 2 \end{aligned}$$

for $x \in I$. Put $w = v\sqrt{1 + \|u\|_{L^2(\mathbb{R})}^2}$. Then we have $w \in \tilde{H}^{1/2,2}(I)$ since $v \equiv 0$ on $\mathbb{R} \setminus I$, and

$$\begin{aligned} \|(-\Delta)^{1/4}w\|_{L^2(\mathbb{R})}^2 &= \left(1 + \|u\|_{L^2(\mathbb{R})}^2\right) \|(-\Delta)^{1/4}v\|_{L^2(\mathbb{R})}^2 \\ &\leq \left(1 + \|u\|_{L^2(\mathbb{R})}^2\right) \left(1 - \|u\|_{L^2(\mathbb{R})}^2\right) \leq 1. \end{aligned}$$

Thus we may use the fractional Trudinger-Moser inequality (Proposition 1) to w to obtain

$$\int_I e^{\pi w^2} dx \leq C$$

for some $C > 0$ independent of u . By $u^2 \leq w^2 + 2$ on I , we conclude that

$$\int_I e^{\pi u^2} dx \leq \int_I e^{\pi(w^2+2)} dx = e^{2\pi} \int_I e^{\pi w^2} dx \leq C'.$$

Now, since $\beta < \pi$, there is an absolute constant C_0 such that $t^2 e^{\beta t^2} \leq C_0 e^{\pi t^2}$ for any $t \in \mathbb{R}$. Finally, we obtain

$$(II) = \int_I u^2 e^{\beta u^2} dx \leq C_0 \int_I e^{\pi u^2} dx \leq C_0 C'.$$

Proposition 5 follows from the estimates (I) and (II). \square

By using Proposition 5 and arguing as in the proof of Theorem 1 (after establishing the similar claims as in Lemma 1 and Lemma 2), it is easy to obtain the following Proposition:

Proposition 6. *For any $0 < \alpha < \beta < \pi$, we have*

$$E(\alpha) \leq \left(\frac{1}{1 - \alpha/\beta}\right) F(\beta).$$

Since $F(\beta) < \infty$ for any $\beta < \pi$, this proves the first part of Theorem 6. For the attainability of $E(\alpha)$ for $\alpha \in (0, \pi)$, it is enough to argue as in the proof of Theorem 4. We omit the details. \square

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