# EHRHART POLYNOMIALS OF 3-DIMENSIONAL SIMPLE INTEGRAL CONVEX POLYTOPES

YUSUKE SUYAMA

# EHRHART POLYNOMIALS OF 3-DIMENSIONAL SIMPLE INTEGRAL CONVEX POLYTOPES

YUSUKE SUYAMA

Abstract. We give an explicit formula on the Ehrhart polynomial of a 3-dimensional simple integral convex polytope by using toric geometry.

## 1. Introduction

Let $P \subset \mathbb{R}^d$ be an integral convex polytope of dimension $d$, that is, a convex polytope whose vertices have integer coordinates. For a non-negative integer $l$, we write $lP = \{lx \mid x \in P\}$. Ehrhart [2] proved that the number of lattice points in $lP$ can be expressed by a polynomial in $l$ of degree $d$:

$$|(lP) \cap \mathbb{Z}^d| = c_d l^d + c_{d-1} l^{d-1} + \cdots + c_0.$$

This polynomial is called the *Ehrhart polynomial* of $P$. It is known that:

(1) $c_0 = 1$.
(2) $c_{d-1}$ is half of the sum of relative volumes of facets of $P$ ([1, Theorem 5.6]).
(3) $c_d$ is the volume of $P$ ([1, Corollary 3.20]).

However, we have no formula on other coefficients of Ehrhart polynomials. In particular, we do not know a formula on $c_1$ for a general 3-dimensional integral convex polytope. In this paper, we find an explicit formula on $c_1$ of the Ehrhart polynomial of a 3-dimensional *simple* integral convex polytope, see Theorem 5.

Pommersheim [4] gave a method for computing the $(d-2)$-nd coefficient of the Ehrhart polynomial of a $d$-dimensional simple integral convex polytope $P$ by using toric geometry. He obtained an explicit description of the Ehrhart polynomial of a tetrahedron by using this method. Our formula is obtained by using this method for a general 3-dimensional simple integral convex polytope.

The structure of the paper is as follows. In Section 2, we state the main theorem and give a few examples. In Section 3, we give a proof of the main theorem.

Acknowledgment. This work was supported by Grant-in-Aid for JSPS Fellows 15J01000. The author wishes to thank his supervisor, Professor Mikiya Masuda, for his continuing support.

## 2. The main theorem

Let $P \subset \mathbb{R}^3$ be a 3-dimensional simple integral convex polytope, and let $F_1, \dots, F_n$ be the facets of $P$. For $k = 1, \dots, n$, we denote by $v_k \in \mathbb{Z}^3$ the inward-pointing primitive normal vector of $F_k$. For an edge $E$ of $P$, we denote by $\mathrm{Vol}(E)$ the relative

*Date*: May 16, 2016.
2010 *Mathematics Subject Classification.* Primary 52B20; Secondary 52B10, 14M25.
*Key words and phrases.* integral convex polytopes, Ehrhart polynomials, toric geometry.

volume of $E$, that is, the length of $E$ measured with respect to the lattice of rank one in the line containing $E$.

Definition 1. For each edge $E = F_{k_1} \cap F_{k_2}$ of $P$, we define an integer $m(E)$ and a rational number $s(E)$ as follows:

(1) We define $m(E) = |((\mathbb{R}v_{k_1} + \mathbb{R}v_{k_2}) \cap \mathbb{Z}^3)/(\mathbb{Z}v_{k_1} + \mathbb{Z}v_{k_2})|$.

(2) There exists a basis $e_1, e_2$ for $(\mathbb{R}v_{k_1} + \mathbb{R}v_{k_2}) \cap \mathbb{Z}^3$ such that $v_{k_1} = e_1$ and $v_{k_2} = pe_1 + qe_2$ for some $q > p \geq 0$. Then we define $s(E) = s(p, q)$, where $s(p, q)$ is the Dedekind sum, which is defined by

$$s(p,q) = \sum_{i=1}^{q} \left(\left(\frac{i}{q}\right)\right)\left(\left(\frac{pi}{q}\right)\right), \quad ((x)) = \begin{cases} x - [x] - \frac{1}{2} & (x \notin \mathbb{Z}), \\ 0 & (x \in \mathbb{Z}). \end{cases}$$

Remark 2. We have $q = m(E)$. Although $p$ is not uniquely determined, $s(p, q)$ does not depend on the choice of $e_1, e_2$. Thus $s(E)$ is well-defined.

Definition 3. For each facet $F$ of $P$, we define a rational number $C(F)$ as follows. We name vertices and facets around $F$ as in Figure 1. We denote by $v \in \mathbb{Z}^3$ the inward-pointing primitive normal vector of $F$.

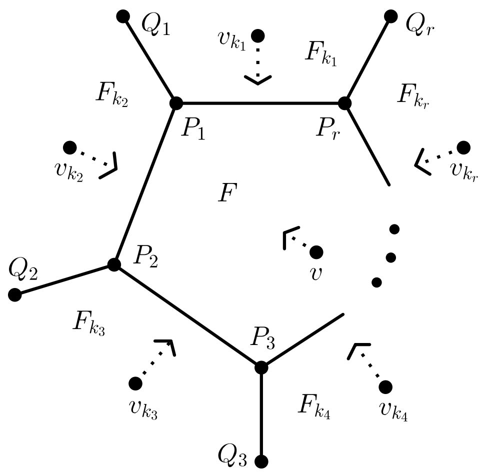


Figure 1. vertices and facets around $F$.

For $i = 1, \dots, r$, we define

$$\varepsilon_i = \det(v, v_{k_{i+1}}, v_{k_i}) > 0, \quad a_i = \frac{\langle \overrightarrow{P_{i-1}Q_{i-1}}, v_{k_{i+1}} \rangle}{\varepsilon_i \langle \overrightarrow{P_{i-1}Q_{i-1}}, v \rangle}, \quad b_i = \frac{\langle \overrightarrow{P_iP_{i+1}}, v_{k_{i-1}} \rangle}{\varepsilon_{i-1} \langle \overrightarrow{P_iP_{i+1}}, v_{k_i} \rangle},$$

where $v_{k_0} = v_{k_r}, v_{k_{r+1}} = v_{k_1}, \varepsilon_0 = \varepsilon_r, P_0 = P_r, P_{r+1} = P_1, Q_0 = Q_r$ and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^3$. Then we define

$$C(F) = -\sum_{2 \le i < j \le r} a_i \begin{vmatrix} b_{i+1} & \varepsilon_{i+1}^{-1} & 0 & \cdots & 0 \\ \varepsilon_{i+1}^{-1} & b_{i+2} & \varepsilon_{i+2}^{-1} & \ddots & \vdots \\ 0 & \varepsilon_{i+2}^{-1} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_{j-2} & \varepsilon_{j-2}^{-1} \\ 0 & \cdots & 0 & \varepsilon_{j-2}^{-1} & b_{j-1} \end{vmatrix} \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-1} \frac{\mathrm{Vol}(P_{j-1}P_j)}{m(P_{j-1}P_j)},$$

where $P_{j-1}P_j$ is the edge whose endpoints are $P_{j-1}$ and $P_j$, and the determinants above are understood to be one when $j = i + 1$.

Remark 4. The proof of Theorem 5 below shows that $C(F)$ does not depend on the choice of $F_{k_1}$.

The following is our main theorem:

**Theorem 5.** *Let $P \subset \mathbb{R}^3$ be a 3-dimensional simple integral convex polytope, and let $E_1, \ldots, E_m$ and $F_1, \ldots, F_n$ be the edges and the facets of $P$, respectively. Then the coefficient $c_1$ of the Ehrhart polynomial $|(lP) \cap \mathbb{Z}^3| = c_3 l^3 + c_2 l^2 + c_1 l + c_0$ is given by*

$$\sum_{j=1}^{m} \left( s(E_j) + \frac{1}{4} \right) \mathrm{Vol}(E_j) + \frac{1}{12} \sum_{k=1}^{n} C(F_k).$$

Example 6. Let $a, b, c$ be positive integers with $\gcd(a, b, c) = 1$ and let $P \subset \mathbb{R}^3$ be the tetrahedron with vertices

$$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}.$$

We put $A = \gcd(b, c), B = \gcd(a, c), C = \gcd(a, b)$ and $d = ABC$. Then we have the following table:

| edge $E$ | $OP_1$ | $OP_2$ | $OP_3$ | $P_1P_2$ | $P_1P_3$ | $P_2P_3$ |
|---|---|---|---|---|---|---|
| $\mathrm{Vol}(E)$ | $a$ | $b$ | $c$ | $C$ | $B$ | $A$ |
| $m(E)$ | 1 | 1 | 1 | $cC/d$ | $bB/d$ | $aA/d$ |
| $s(E)$ | 0 | 0 | 0 | $-s\left(\frac{ab}{d}, \frac{cC}{d}\right)$ | $-s\left(\frac{ac}{d}, \frac{bB}{d}\right)$ | $-s\left(\frac{bc}{d}, \frac{aA}{d}\right)$ |

| facet $F$ | $OP_1P_2$ | $OP_1P_3$ | $OP_2P_3$ | $P_1P_2P_3$ |
|---|---|---|---|---|
| inward-pointing primitive normal vector of $F$ | $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} -bc/d \\ -ac/d \\ -ab/d \end{pmatrix}$ |
| $C(F)$ | $ab/c$ | $ac/b$ | $bc/a$ | $d^2/(abc)$ |

Table 1. the values of $\mathrm{Vol}(E), s(E)$ and $C(F)$.

Thus we have

$$\begin{aligned}
&\sum_{E:\text{edge}} \left( s(E) + \frac{1}{4} \right) \mathrm{Vol}(E) + \frac{1}{12} \sum_{F:\text{facet}} C(F) \\
&= \frac{a}{4} + \frac{b}{4} + \frac{c}{4} + \left( -s\left( \frac{ab}{d}, \frac{cC}{d} \right) + \frac{1}{4} \right) C + \left( -s\left( \frac{ac}{d}, \frac{bB}{d} \right) + \frac{1}{4} \right) B \\
&+ \left( -s\left( \frac{bc}{d}, \frac{aA}{d} \right) + \frac{1}{4} \right) A + \frac{1}{12} \left( \frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a} + \frac{d^2}{abc} \right),
\end{aligned}$$

which coincides with the formula in [4, Theorem 5].

EXAMPLE 7. Let $a$ and $c$ be positive integers and $b$ be a non-negative integer. Consider the convex hull $P \subset \mathbb{R}^3$ of the six points

$$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix},$$
$$O' = \begin{pmatrix} b \\ 0 \\ c \end{pmatrix}, \quad A' = \begin{pmatrix} a+b \\ 0 \\ c \end{pmatrix}, \quad B' = \begin{pmatrix} b \\ a \\ c \end{pmatrix}.$$

$P$ is a 3-dimensional simple polytope. We put $g = \gcd(b, c)$. Then we have the following table:

| edge $E$ | $OA$ | $OB$ | $AB$ | $OO'$ | $AA'$ | $BB'$ | $O'A'$ | $O'B'$ | $A'B'$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{Vol}(E)$ | $a$ | $a$ | $a$ | $g$ | $g$ | $g$ | $a$ | $a$ | $a$ |
| $m(E)$ | 1 | $c/g$ | $c/g$ | 1 | 1 | $c/g$ | 1 | $c/g$ | $c/g$ |
| $s(E)$ | 0 | $-s\left(\frac{b}{g}, \frac{c}{g}\right)$ | $s\left(\frac{b}{g}, \frac{c}{g}\right)$ | 0 | 0 | $-s\left(1, \frac{c}{g}\right)$ | 0 | $s\left(\frac{b}{g}, \frac{c}{g}\right)$ | $-s\left(\frac{b}{g}, \frac{c}{g}\right)$ |

| facet $F$ | $OAB$ | $OAA'O'$ | $OBB'O'$ | $ABB'A'$ | $O'A'B'$ |
|---|---|---|---|---|---|
| inward-pointing primitive normal vector of $F$ | $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} c/g \\ 0 \\ -b/g \end{pmatrix}$ | $\begin{pmatrix} -c/g \\ -c/g \\ b/g \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ |
| $C(F)$ | 0 | $c$ | $g^2/c$ | $g^2/c$ | 0 |

TABLE 2. the values of $\mathrm{Vol}(E), s(E)$ and $C(F)$.

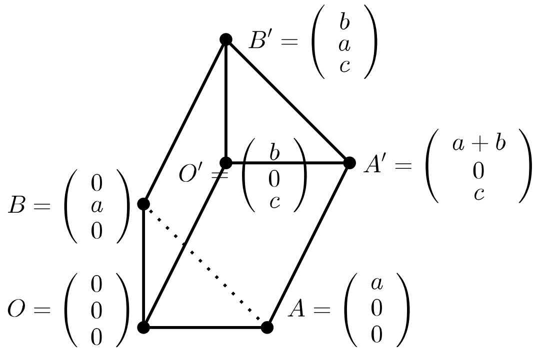


FIGURE 2. the simple polytope $P$.

Thus we have

$$\begin{aligned}
&\sum_{E:\text{edge}} \left(s(E)+\frac{1}{4}\right)\mathrm{Vol}(E)+\frac{1}{12}\sum_{F:\text{facet}} C(F)\\
&= -s\left(1,\frac{c}{g}\right)g+\frac{3a}{2}+\frac{3g}{4}+\frac{1}{12}\left(c+\frac{2g^2}{c}\right)\\
&= -g\sum_{i=1}^{c/g-1}\left(\frac{i}{\frac{c}{g}}-\frac{1}{2}\right)^2+\frac{3a}{2}+\frac{3g}{4}+\frac{c}{12}+\frac{g^2}{6c}\\
&= -g\sum_{i=1}^{c/g-1}\left(\frac{g^2}{c^2}i^2-\frac{g}{c}i+\frac{1}{4}\right)+\frac{3a}{2}+\frac{3g}{4}+\frac{c}{12}+\frac{g^2}{6c}\\
&= -\frac{g^3}{c^2}\frac{\left(\frac{c}{g}-1\right)\frac{c}{g}\left(\frac{2c}{g}-1\right)}{6}+\frac{g^2}{c}\frac{\left(\frac{c}{g}-1\right)\frac{c}{g}}{2}-g\frac{\frac{c}{g}-1}{4}+\frac{3a}{2}+\frac{3g}{4}+\frac{c}{12}+\frac{g^2}{6c}\\
&= \frac{3a}{2}+g.
\end{aligned}$$

On the other hand, since

$$\#\{(x,y)\in\mathbb{Z}^2 \mid (x,y,z)\in lP\} = \begin{cases} \dfrac{(al+1)(al+2)}{2} & ((c/g)|z), \\ \dfrac{al(al+1)}{2} & ((c/g)\nmid z) \end{cases}$$

for $z=0,1,\dots,cl$, we have

$$\begin{aligned}
|(lP)\cap\mathbb{Z}^3| &= \frac{(al+1)(al+2)}{2}(gl+1)+\frac{al(al+1)}{2}((cl+1)-(gl+1))\\
&= \frac{a^2c}{2}l^3+\frac{1}{2}\left(a^2+ac+2ag\right)l^2+\left(\frac{3a}{2}+g\right)l+1.
\end{aligned}$$

The coefficient of $l$ is also $3a/2+g$.

## 3. Proof of Theorem 5

First we recall some facts about toric geometry, see [3] for details. Let $P\subset\mathbb{R}^d$ be a $d$-dimensional integral convex polytope. We define a cone

$$\sigma_F=\{v\in\mathbb{R}^d \mid \langle u'-u,v\rangle\geq 0\ \forall u'\in P, \forall u\in F\}$$

for each face $F$ of $P$. Then the set

$$\Delta_P=\{\sigma_F \mid F \text{ is a face of } P\}$$

of such cones forms a fan in $\mathbb{R}^d$, which is called the *normal fan* of $P$. Let $X(\Delta_P)$ be the associated projective toric variety. We denote by $V(\sigma)$ the subvariety of $X(\Delta_P)$ corresponding to $\sigma\in\Delta_P$. Let $\mathrm{Td}_i(X(\Delta_P))\in A_i(X(\Delta_P))_{\mathbb{Q}}$ be the $i$-th Todd class in the Chow group of $i$-cycles with rational coefficients.

**Theorem 8.** *Let $P\subset\mathbb{R}^d$ be a $d$-dimensional integral convex polytope and $|(lP)\cap\mathbb{Z}^d|=c_dl^d+c_{d-1}l^{d-1}+\cdots+c_0$ be its Ehrhart polynomial. If $\mathrm{Td}_i(X(\Delta_P))$ has an expression of the form $\sum_F r_F[V(\sigma_F)]$ with $r_F\in\mathbb{Q}$, then we have $c_i=\sum_F r_F\mathrm{Vol}(F)$, where $[V(\sigma_F)]$ is the class of $V(\sigma_F)$ in the Chow group and $\mathrm{Vol}(F)$ is the relative volume of $F$.*

Now we assume that $d = 3$ and $P$ is simple. Then the associated toric variety $X(\Delta_P)$ is $\mathbb{Q}$-factorial and we know the ring structure of the Chow ring $A^*(X(\Delta_P))_{\mathbb{Q}}$ with rational coefficients. Let $E_1, \dots, E_m$ and $F_1, \dots, F_n$ be the edges and the facets of $P$, respectively. We have

$$\sum_{k=1}^{n} \langle u, v_k \rangle [V(\sigma_{F_k})] = 0 \quad \forall u \in (\mathbb{Q}^3)^*. \tag{3.1}$$

If $F_{k_1}$ and $F_{k_2}$ are distinct, then

$$[V(\sigma_{F_{k_1}})][V(\sigma_{F_{k_2}})] = \begin{cases} \frac{1}{m(E_j)}[V(\sigma_{E_j})] & (1 \le \exists j \le m : F_{k_1} \cap F_{k_2} = E_j), \\ 0 & (F_{k_1} \cap F_{k_2} = \emptyset) \end{cases} \tag{3.2}$$

in $A^*(X(\Delta_P))_{\mathbb{Q}}$.

Pommersheim gave an expression of $\mathrm{Td}_{d-2}(X(\Delta_P))$ for a $d$-dimensional simple integral convex polytope $P \subset \mathbb{R}^d$. In the case where $d = 3$, we have the following:

**Theorem 9** (Pommersheim [4])**.** *If $P \subset \mathbb{R}^3$ is a 3-dimensional simple integral convex polytope, then*

$$\mathrm{Td}_1(X(\Delta_P)) = \sum_{j=1}^{m} \left( s(E_j) + \frac{1}{4} \right) [V(\sigma_{E_j})] + \frac{1}{12} \sum_{k=1}^{n} [V(\sigma_{F_k})]^2.$$

We use the notation in Definition 3. It suffices to show

$$[V(\sigma_F)]^2 = - \sum_{2 \le i < j \le r} a_i \begin{vmatrix} b_{i+1} & \varepsilon_{i+1}^{-1} & 0 & \cdots & 0 \\ \varepsilon_{i+1}^{-1} & b_{i+2} & \varepsilon_{i+2}^{-1} & \ddots & \vdots \\ 0 & \varepsilon_{i+2}^{-1} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_{j-2} & \varepsilon_{j-2}^{-1} \\ 0 & \cdots & 0 & \varepsilon_{j-2}^{-1} & b_{j-1} \end{vmatrix} \frac{\varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-1}}{m(P_{j-1}P_j)} [V(\sigma_{P_{j-1}P_j})]$$

for each facet $F$ of $P$.

We put

$$D(s,t) = \begin{vmatrix} b_s & \varepsilon_s^{-1} & 0 & \cdots & 0 \\ \varepsilon_s^{-1} & b_{s+1} & \varepsilon_{s+1}^{-1} & \ddots & \vdots \\ 0 & \varepsilon_{s+1}^{-1} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_{t-1} & \varepsilon_{t-1}^{-1} \\ 0 & \cdots & 0 & \varepsilon_{t-1}^{-1} & b_t \end{vmatrix}$$

for $2 < s \le t < r$ and $D(s,t) = 1$ for $s > t$. Define $u \in (\mathbb{Q}^3)^*$ by $\langle u, v \rangle = 1, \langle u, v_{k_1} \rangle = 0, \langle u, v_{k_2} \rangle = 0$. By (3.1) and (3.2), we have

$$[V(\sigma_F)]^2 = -[V(\sigma_F)] \sum_{j=1}^{r} \langle u, v_{k_j} \rangle [V(\sigma_{F_{k_j}})] = - \sum_{j=3}^{r} \frac{\langle u, v_{k_j} \rangle}{m(P_{j-1}P_j)} [V(\sigma_{P_{j-1}P_j})].$$

Hence it suffices to show

$$\langle u, v_{k_j} \rangle = \sum_{i=2}^{j-1} a_i D(i+1, j-1) \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-1} \tag{3.3}$$

for any $j = 3, \dots, r$.

First we claim that

$$\varepsilon_{j-1}^{-1} v_{k_{j-1}} + \varepsilon_j^{-1} v_{k_{j+1}} = a_j v + b_j v_{k_j} \tag{3.4}$$

for any $j = 2, \ldots, r-1$. By Cramer's rule, we have

$$\begin{aligned} v_{k_{j+1}} &= \frac{\det(v_{k_{j+1}}, v_{k_j}, v_{k_{j-1}})}{\det(v, v_{k_j}, v_{k_{j-1}})} v + \frac{\det(v, v_{k_{j+1}}, v_{k_{j-1}})}{\det(v, v_{k_j}, v_{k_{j-1}})} v_{k_j} + \frac{\det(v, v_{k_j}, v_{k_{j+1}})}{\det(v, v_{k_j}, v_{k_{j-1}})} v_{k_{j-1}} \\ &= \frac{\det(v_{k_{j+1}}, v_{k_j}, v_{k_{j-1}})}{\varepsilon_{j-1}} v + \frac{\det(v, v_{k_{j+1}}, v_{k_{j-1}})}{\varepsilon_{j-1}} v_{k_j} - \frac{\varepsilon_j}{\varepsilon_{j-1}} v_{k_{j-1}}. \end{aligned}$$

So we have

$$\begin{aligned} &\varepsilon_{j-1}^{-1} v_{k_{j-1}} + \varepsilon_j^{-1} v_{k_{j+1}} \\ &= \varepsilon_{j-1}^{-1}\varepsilon_j^{-1} \det(v_{k_{j+1}}, v_{k_j}, v_{k_{j-1}}) v + \varepsilon_{j-1}^{-1}\varepsilon_j^{-1} \det(v, v_{k_{j+1}}, v_{k_{j-1}}) v_{k_j}. \end{aligned} \tag{3.5}$$

Taking the inner product of both sides of (3.5) with $\overrightarrow{P_{j-1}Q_{j-1}}$ gives

$$\varepsilon_j^{-1} \langle \overrightarrow{P_{j-1}Q_{j-1}}, v_{k_{j+1}} \rangle = \varepsilon_{j-1}^{-1}\varepsilon_j^{-1} \det(v_{k_{j+1}}, v_{k_j}, v_{k_{j-1}}) \langle \overrightarrow{P_{j-1}Q_{j-1}}, v \rangle,$$

which means $a_j = \varepsilon_{j-1}^{-1}\varepsilon_j^{-1} \det(v_{k_{j+1}}, v_{k_j}, v_{k_{j-1}})$. Taking the inner product of both sides of (3.5) with $\overrightarrow{P_j P_{j+1}}$ gives

$$\varepsilon_{j-1}^{-1} \langle \overrightarrow{P_j P_{j+1}}, v_{k_{j-1}} \rangle = \varepsilon_{j-1}^{-1}\varepsilon_j^{-1} \det(v, v_{k_{j+1}}, v_{k_{j-1}}) \langle \overrightarrow{P_j P_{j+1}}, v_{k_j} \rangle,$$

which means $b_j = \varepsilon_{j-1}^{-1}\varepsilon_j^{-1} \det(v, v_{k_{j+1}}, v_{k_{j-1}})$. Thus (3.4) follows.

We show (3.3) by induction on $j$. If $j = 3$, then both sides are $a_2\varepsilon_2$. If $j = 4$, then both sides are $a_2 b_3 \varepsilon_2 \varepsilon_3 + a_3 \varepsilon_3$. Suppose $4 \le j \le r-1$. By (3.4) and the hypothesis of induction, we have

$$\begin{aligned} \langle u, v_{k_{j+1}} \rangle &= \langle u, a_j \varepsilon_j v + b_j \varepsilon_j v_{k_j} - \varepsilon_{j-1}^{-1} \varepsilon_j v_{k_{j-1}} \rangle \\ &= a_j \varepsilon_j + b_j \varepsilon_j \langle u, v_{k_j} \rangle - \varepsilon_{j-1}^{-1} \varepsilon_j \langle u, v_{k_{j-1}} \rangle \\ &= a_j \varepsilon_j + b_j \varepsilon_j \sum_{i=2}^{j-1} a_i D(i+1, j-1) \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-1} \\ &\quad - \varepsilon_{j-1}^{-1} \varepsilon_j \sum_{i=2}^{j-2} a_i D(i+1, j-2) \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\sum_{i=2}^{j} a_i D(i+1, j) \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_j \\ &= a_j \varepsilon_j + a_{j-1} b_j \varepsilon_{j-1} \varepsilon_j + \sum_{i=2}^{j-2} a_i D(i+1, j) \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_j. \end{aligned}$$

Since

$$\begin{aligned}&\sum_{i=2}^{j-2} a_i D(i+1,j)\varepsilon_i\varepsilon_{i+1}\cdots\varepsilon_j\\&=\sum_{i=2}^{j-2} a_i(b_j D(i+1,j-1)-\varepsilon_{j-1}^{-2}D(i+1,j-2))\varepsilon_i\varepsilon_{i+1}\cdots\varepsilon_j\\&=b_j\varepsilon_j\sum_{i=2}^{j-2} a_i D(i+1,j-1)\varepsilon_i\varepsilon_{i+1}\cdots\varepsilon_{j-1}\\&-\varepsilon_{j-1}^{-1}\varepsilon_j\sum_{i=2}^{j-2} a_i D(i+1,j-2)\varepsilon_i\varepsilon_{i+1}\cdots\varepsilon_{j-2},\end{aligned}$$

we have

$$\begin{aligned}&\sum_{i=2}^{j} a_i D(i+1,j)\varepsilon_i\varepsilon_{i+1}\cdots\varepsilon_j\\&=a_j\varepsilon_j+a_{j-1}b_j\varepsilon_{j-1}\varepsilon_j+b_j\varepsilon_j\sum_{i=2}^{j-2} a_i D(i+1,j-1)\varepsilon_i\varepsilon_{i+1}\cdots\varepsilon_{j-1}\\&-\varepsilon_{j-1}^{-1}\varepsilon_j\sum_{i=2}^{j-2} a_i D(i+1,j-2)\varepsilon_i\varepsilon_{i+1}\cdots\varepsilon_{j-2}\\&=a_j\varepsilon_j+b_j\varepsilon_j\sum_{i=2}^{j-1} a_i D(i+1,j-1)\varepsilon_i\varepsilon_{i+1}\cdots\varepsilon_{j-1}\\&-\varepsilon_{j-1}^{-1}\varepsilon_j\sum_{i=2}^{j-2} a_i D(i+1,j-2)\varepsilon_i\varepsilon_{i+1}\cdots\varepsilon_{j-2}\\&=\langle u, v_{k_{j+1}}\rangle.\end{aligned}$$

Thus (3.3) holds for $j+1$. This completes the proof of Theorem 5.

## References

[1] M. Beck and S. Robins: Computing the Continuous Discretely, Undergraduate Texts in Mathematics, Springer, 2007.

[2] E. Ehrhart: Polynômes Arithmétiques et Méthode des Polyèdres en Combinatoire, Birkhäuser, Boston-Basel-Stuttgart, 1977.

[3] W. Fulton: Introduction to Toric Varieties, Annals of Mathematics Studies **131**, Princeton Univ. Press, Princeton, NJ, 1993.

[4] J. E. Pommersheim: *Toric varieties, lattice points and Dedekind sums*, Math. Ann. **295** (1993), 1–24.

Department of Mathematics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585 JAPAN

*E-mail address*: d15san0w03@st.osaka-cu.ac.jp