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Positive solutions of Kirchhoff type elliptic equations involving a critical Sobolev exponent

Daisuke Naimen

Abstract. In this paper we investigate the following Kirchhoff type elliptic boundary value problem involving a critical nonlinearity:

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \mu g(x, u) + u^5, & u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \quad (\text{K1})$$

here $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $a, b \geq 0$ and $a + b > 0$. Under several conditions on $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and $\mu \in \mathbb{R}$, we prove the existence and nonexistence of solutions of (K1). This is some extension of a part of Brezis-Nirenberg's result in 1983.

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1. Introduction

We consider the following Kirchhoff type elliptic equation with a Dirichlet boundary condition:

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \mu g(x, u) + u^5 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{K1})$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. We assume that $a, b \geq 0$ and $a + b > 0$. In addition we emphasize that u^5 in the right hand side of the equation is a critical term, since 6 is a critical exponent in the sense of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. In this paper we prove the existence of solutions of (K1).

(K1) has its origin in the theory of nonlinear vibration. For instance, we give the following equation which describes the free vibration of a stretched

string (cf. [26]):

$$\rho \frac{\partial^2 u}{\partial t^2} = \left(T_0 + \frac{Ea}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2}, \quad (\text{K0})$$

where $\rho > 0$ is the mass per unit length, T_0 is the base tension, E is the Young modulus, a is the area of cross section and L is the initial length of the string. (K0) takes account the change of the tension on the string which is caused by the change of its length during the vibration. The nonlocal equation of this type was first proposed by Kirchhoff in 1876 ([18]). After that, several physicists also consider such equations for their researches in the theory of nonlinear vibrations theoretically or experimentally ([9][10][25][26][30]). Moreover mathematically, the solvability of several Kirchhoff type quasilinear hyperbolic equations have been extensively discussed. See earlier results [6][29] and the work by J. L. Lions ([22]), and recent ones [5][12]. For more details in the physical and mathematical background of Kirchhoff type equations, see the survey [4]. Recently, the Kirchhoff type elliptic equations such like (K1) get so many attentions. Many authors prove the existence of solutions of their problems using variational or topological methods. The main goal of their works is to study the effect of the non-local coefficients: $(a + b \int_{\Omega} |\nabla u|^2 dx)$ (or in general denoted by $m(\int_{\Omega} |\nabla u|^2 dx)$) on the principal term of their equations, on the existence results. See earlier results [1][24] and [2] [3] [13] [14] [19] - [21] [27] [32] [33] and so on.

On the other hand, as is well-known, when a nonlinear elliptic boundary value problem has a critical term such as (K1), a crucial difficulty occurs in proving the existence of solutions of the problem. Such difficulty is caused by the lack of compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$. Because of this difficulty, over these three decades, (K1) with $a = 1$ and $b = 0$ has been extensively studied by many authors. One of the most important results is obtained by Brezis and Nirenberg ([8]). Our main aim in this paper is to extend their results in Section 2.5 in [8] to the case $a, b \geq 0$ and $a + b > 0$. Compare the results below with those in Section 2.5 in [8].

To show our main results, we introduce some conditions on the function $g : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$:

- (g1) g is continuous in $\overline{\Omega} \times \mathbb{R}$, $g(x, u) \geq 0$ if $u \geq 0$ and $g(x, u) = 0$ if $u \leq 0$ for all $x \in \Omega$.
- (g2) $g(x, u) = o(u)$ as $u \rightarrow 0^+$ and $g(x, u) = o(u^5)$ as $u \rightarrow \infty$ uniformly for $x \in \Omega$.
- (g3) There exists a constant $\theta > 0$ such that $4 < \theta < 6$ and $g(x, u)u - \theta G(x, u) \geq 0$ for all $x \in \Omega$ and $u \geq 0$, where $G(x, u) := \int_0^u g(x, t) dt$.
- (g4) There exists a nonempty open set $\omega \subset \Omega$ such that

$$\lim_{u \rightarrow \infty} \frac{g(x, u)}{u^3} = \infty.$$

uniformly for $x \in \omega$.

The following theorem gives our main argument in this paper.

Theorem 1.1. *Let $a, b \geq 0$ and $a + b > 0$. Then if g satisfies (g1)-(g4), (K1) has a solution for all $\mu > 0$.*

In view of the typical power nonlinearities, Theorem 1.1 allows us to ensure the next corollary.

Corollary 1.2. *Let $a, b \geq 0$ and $a + b > 0$. Then if $3 < q < 5$ and $g(x, u) = u^q$, (K1) has a solution for all $\mu > 0$.*

Here we introduce the following assumptions (g5) and (g6) on g which are weaker than (g3) and (g4) respectively.

- (g5) There exists a constant $\theta > 0$ such that $2 < \theta < 6$ and $g(x, u)u - \theta G(x, u) \geq 0$ for all $x \in \Omega$ and $u \geq 0$.
- (g6) There exist a nonempty open set $w \subset \Omega$ and an interval $I \subset (0, \infty)$ such that $g(x, u) > 0$ if $x \in w$ and $u \in I$.

Now we can give the following theorem and corollary. An almost same result for (K1) with a more general nonlocal coefficient has already obtained in [13].

Theorem 1.3. *Let $a > 0$ and $b \geq 0$. Then if g satisfies (g1), (g2), (g5) and (g6), there exists a constant $\mu_* \geq 0$ such that (K1) has a positive solution for all $\mu > \mu_*$.*

Corollary 1.4. *Let $a > 0$ and $b \geq 0$. Then if $1 < q \leq 3$ and $g(x, u) = u^q$, there exists a constant $\mu_* \geq 0$ such that (K1) has a positive solution for all $\mu > \mu_*$.*

As additional results for Corollary 1.2 and Corollary 1.4, we can get the following nonexistence results. The first one is Pohozaev type result ([28]).

Theorem 1.5. *Let $a, b \geq 0$, $a + b > 0$ and further, assume that $\mu \leq 0$ and Ω is strictly star-shaped. Then if $1 < q < 6$ and $g(x, u) = u^q$, (K1) has no solution.*

The next one concludes that the value μ_* in Corollary 1.4 must be strictly positive if Ω is strictly star-shaped.

Theorem 1.6. *In addition to the assumption of Corollary 1.4, we assume $\Omega \subset \mathbb{R}^3$ is strictly star-shaped. Then there exists a constant $\mu_0 > 0$, which is determined by a, q and Ω , such that (K1) has no solution for all $\mu \leq \mu_0$.*

We can obtain these nonexistence results by slightly extension of the argument in [7] or [8]. For reader's convenience, we show the proof in Appendix A.

Recently some results related to (K1) are obtained in [2], [13], [14], and [32]. For (K1) with $\Omega = \mathbb{R}^N$, see [3], [21] and [33]. In [14] and [32], they consider the case $0 < q < 1$ and $g(x, u) = |u|^{q-1}u$. In [14], they show the existence of infinitely many (possibly sign-changing) solutions of (K1) for sufficiently small $\mu > 0$. They use both the minimax theorem which is based on the Clark Theorem ([11]) and the second concentration compactness lemma

by P. L. Lions ([23]). In [32], they prove the existence of a positive minimal solution of (K1) for sufficiently small $\mu > 0$ by applying the method of the Nehari manifold and also using the second concentration compactness lemma. The results in [2] and [13] are closely related to Theorem 1.3. In particular, in [13], they consider the general dimensional case, i.e., $N \geq 3$ and $\Omega \subset \mathbb{R}^N$. Using an appropriate truncation method, they obtain an almost same result with Theorem 1.3 for more general nonlocal problem. In view of these results, Theorem 1.1 is a new result for Kirchhoff type elliptic problems involving a critical Sobolev exponent. Although Theorem 1.3 has already almost proved in [13], inspired by [19], we introduce another truncation method and show the proof of Theorem 1.3 and Corollary 1.4 in section 3. We also note that although our method can be applied to the general dimensional case as in [19], in view of our main purpose of this paper, we only treat 3 dimensional case.

This paper is organized as follows. In section 2, we prove Theorem 1.1 and Corollary 1.2. In section 3, we prove Theorem 1.3 and Corollary 1.4. In section 4, we consider (K1) with $a = 0$, $b = 1$, $1 < q \leq 3$ and $g(x, u) = u^q$ as additional results. In Appendix A, we show the nonexistence result i.e., Theorem 1.5 and Theorem 1.6. In this paper we denote $H_0^1(\Omega)$ as a function space which is defined by the closure of $C_0^\infty(\Omega)$ with the norm $\{\int_\Omega (|\nabla u|^2 + u^2)dx\}^{1/2}$. Thanks to the Poincare inequality, we define the usual $H_0^1(\Omega)$ norm as $\|u\|_{H_0^1(\Omega)} := (\int_\Omega |\nabla u|^2 dx)^{1/2}$. For simplicity we write $\|u\| = \|u\|_{H_0^1(\Omega)}$. Furthermore, we denote $H^{-1}(\Omega)$ as the dual space of $H_0^1(\Omega)$. We define the norm in $H^{-1}(\Omega)$ as $\|f\|_{H^{-1}(\Omega)} := \sup_{\|u\| \leq 1} |\langle f, u \rangle|$.

Before beginning the proof, we define the weak solutions of (K1). We call $u \in H_0^1(\Omega)$ a weak solution of (K1) if and only if u satisfies

$$(a + b\|u\|^2) \int_\Omega \nabla u \cdot \nabla h dx - \mu \int_\Omega g(x, u) h dx - \int_\Omega u_+^5 h = 0$$

for all $h \in H_0^1(\Omega)$. If $u \in H_0^1(\Omega)$ is a nontrivial weak solution for (K1), then we can trivially modify the usual elliptic regularity theorems (see for example, Lemma B.3 in [31] and the regularity theorems in [15]) and ensure the smoothness of u up to $C^2(\bar{\Omega})$ even if $b > 0$. Furthermore if u is nonnegative, by the strong maximum principle, we have $u > 0$ in Ω . Consequently we can conclude that u is a classical solution of (K1).

In the following sections we denote $C > 0$ as some constants. If there occurs no confusion, we use same character C even if the values of constants are different. Furthermore we denote $B(x, r) \subset \mathbb{R}^3$ as an open ball which is concentrated at x with radius r .

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 and Corollary 1.2. Our argument is based on that in [8]. Let $a, b \geq 0$, $a + b > 0$ and fix $\mu > 0$. Assume g satisfies

(g1)-(g4). We define the energy functional associated to (K1) so that

$$I(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \mu \int_{\Omega} G(x, u) dx - \frac{1}{6} \int_{\Omega} u_+^6 dx \quad (u \in H_0^1(\Omega)),$$

where $G(x, u) := \int_0^u g(x, t) dt$ and $u_+ := \max\{u, 0\}$. Thanks to (g1) and (g2), I is well-defined and continuously Fréchet differentiable on $H_0^1(\Omega)$. Furthermore every critical point of I is a weak solution of (K1). Hence we shall find a nontrivial critical point of I . In the following argument, we often use the next fact which says, for all $\delta > 0$ there exists a constant $C_\delta > 0$ such that

$$g(x, u) \leq \delta u^5 + C_\delta u^2 \quad ((x, u) \in \Omega \times \mathbb{R}). \quad (2.1)$$

This is a consequence of our hypotheses (g1) and (g2). We begin with the next lemma.

Lemma 2.1. *Let g satisfies (g1) and (g2). Then there exist constants $\alpha, \rho > 0$ such that*

$$I(u) \geq \alpha$$

for all $u \in H_0^1(\Omega)$ with $\|u\| = \rho$.

Proof. By (g1) and (g2), there exists a constant $C > 0$ such that

$$g(x, t) \leq \frac{a\lambda_1}{4\mu} t^2 + Ct^5 \quad (2.2)$$

for all $x \in \Omega$ and $t \in \mathbb{R}$, where $\lambda_1 > 0$ is the first eigenvalue of the problem:

$$\begin{cases} -\Delta\phi = \lambda\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Here we recall the well-known inequality:

$$\int_{\Omega} u^2 dx \leq \frac{1}{\lambda_1} \|u\|^2 \quad (u \in H_0^1(\Omega)). \quad (2.3)$$

Take $u \in H_0^1(\Omega)$ with $\|u\| = \rho$. Using (2.2), (2.3) and the Sobolev embedding we get

$$\begin{aligned} I(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{a\lambda_1}{4} \int_{\Omega} u^2 dx - C \int_{\Omega} u^6 dx - \frac{1}{6} \int_{\Omega} u_+^6 dx \\ &\geq \frac{a}{4}\|u\|^2 + \frac{b}{4}\|u\|^4 - C\|u\|^6 \\ &= \frac{a}{4}\rho^2 + \frac{b}{4}\rho^4 - C\rho^6 \end{aligned}$$

for some constant $C > 0$. Since $a, b \geq 0$ and $a+b > 0$, taking $\rho > 0$ sufficiently small, we conclude that there exists a constant $\alpha > 0$ such that

$$I(u) \geq \alpha$$

for all $u \in H_0^1(\Omega)$ with $\|u\| = \rho$. This completes the proof. $\square$

Here we recall (g4):

(g4) There exists a nonempty open set $\omega \subset \Omega$ such that

$$\lim_{u \rightarrow \infty} \frac{g(x, u)}{u^3} = \infty.$$

uniformly for $x \in \omega$.

With no loss of generality, we can assume $0 \in \omega$. For every $\varepsilon > 0$, we define a cut off Talenti function in Ω so that

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{1}{2}} \tau(x)}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}$$

where τ is a smooth function in Ω such that $\tau = 1$ on some neighborhood of 0 and $\text{spt} \tau \subset \omega$. From the estimate in [8], we get

$$\begin{cases} \int_{\Omega} |\nabla u_\varepsilon|^2 dx = K_1 + O(\varepsilon), \\ \int_{\Omega} u_\varepsilon^6 dx = K_2^3 + O(\varepsilon^2), \\ \int_{\Omega} u_\varepsilon^2 dx = O(\varepsilon), \end{cases} \quad (2.4)$$

where $K_1, K_2 > 0$ are some constants with $S = K_1/K_2$. Here $S > 0$ is given by

$$S := \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\left(\int_{\Omega} u^6 dx \right)^{\frac{1}{3}}}.$$

Set

$$v_\varepsilon(x) := \frac{u_\varepsilon(x)}{\left(\int_{\Omega} u_\varepsilon^6 dx \right)^{\frac{1}{6}}}.$$

Then by (2.4), we have

$$\begin{cases} \int_{\Omega} |\nabla v_\varepsilon|^2 dx = S + O(\varepsilon), \\ \int_{\Omega} v_\varepsilon^6 dx = 1, \\ \int_{\Omega} v_\varepsilon^2 dx = O(\varepsilon). \end{cases} \quad (2.5)$$

We prove the following lemma.

Lemma 2.2. *Suppose g satisfies (g1). Then there exists a function $v_0 \in H_0^1(\Omega)$ such that $\|v_0\| > \rho$ and $I(v_0) \leq 0$.*

Proof. We consider tv_ε for all $t > 0$. Note that by (g1), $G(x, u) \geq 0$ for all $x \in \Omega$ and $u \geq 0$. Using this fact and (2.5), we have that there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned} I(tv_\varepsilon) &\leq \frac{at^2}{2} \|v_\varepsilon\|^2 + \frac{bt^4}{4} \|v_\varepsilon\|^4 - \frac{t^6}{6} \\ &\leq \frac{at^2}{2} (S + O(\varepsilon)) + \frac{bt^4}{4} (S^2 + O(\varepsilon)) - \frac{t^6}{6} \\ &\leq aSt^2 + bS^2t^4 - \frac{t^6}{6} \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Fix $\varepsilon \in (0, \varepsilon_0)$. Then we can find $t_0 > 0$ which is independent of our choice of $\varepsilon \in (0, \varepsilon_0)$ such that $I(tu_\varepsilon) \leq 0$ and $\|t_0 v_\varepsilon\| > \rho$. Set $v_0 := t_0 v_\varepsilon$. This concludes the proof. $\square$

Now we define

$$\Gamma := \{\gamma \in C([0, 1], H_0^1(\Omega)) \mid \gamma(0) = 0, \gamma(1) = v_0\}$$

and

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0, 1])} I(u),$$

where v_0 is taken from Lemma 2.2. By Lemma 2.1 and Lemma 2.2, we have $c \geq \alpha$ and ensure the existence of $(PS)_c$ sequences for I . We prove the following lemma which is important to ensure the local compactness of PS sequences for I .

Lemma 2.3. *Let g satisfy (g1), (g2) and (g3) and assume that $\{u_j\}$ is a $(PS)_c$ sequence for I with*

$$c < \frac{a}{2}C_K + \frac{b}{4}C_K^2 - \frac{1}{6S^3}C_K^3,$$

where

$$C_K = \frac{1}{2} \left(bS^3 + \sqrt{(bS^3)^2 + 4aS^3} \right).$$

Then there exists a function $u \in H_0^1(\Omega)$ such that $(u_j)_+ \rightarrow u_+$ in $L^6(\Omega)$ up to subsequences.

Proof. Set

$$c < \frac{a}{2}C_K + \frac{b}{4}C_K^2 - \frac{1}{6S^3}C_K^3,$$

and let $\{u_j\} \subset H_0^1(\Omega)$ be a $(PS)_c$ sequence for I . We first claim $\{u_j\}$ is bounded in $H_0^1(\Omega)$. In fact, since $I(u_j) \rightarrow c$ and $I'(u_j) \rightarrow 0$ in $H^{-1}(\Omega)$, by (g1) and (g3), we have

$$\begin{aligned} c + 1 &\geq I(u_j) - \frac{1}{\theta} \langle I'(u_j), u_j \rangle + \frac{1}{\theta} \langle I'(u_j), u_j \rangle \\ &\geq a \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u_j\|^2 + b \left(\frac{1}{4} - \frac{1}{\theta} \right) \|u_j\|^4 \\ &\quad + \int_{\Omega} \left(\frac{1}{\theta} g(x, u_j) u_j - G(x, u_j) \right) dx + \left(\frac{1}{\theta} - \frac{1}{6} \right) \int_{\Omega} (u_j)_+^6 dx - \|u_j\| \\ &\geq a \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u_j\|^2 + b \left(\frac{1}{4} - \frac{1}{\theta} \right) \|u_j\|^4 - \|u_j\| \end{aligned}$$

for large $j \in \mathbb{N}$. Since $a, b \geq 0$ and $a + b > 0$, we can ensure the claim. Hence by the weak compactness of $H_0^1(\Omega)$ and the compactness of the Sobolev embeddings, there exists a function $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} u_j &\rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ u_j &\rightarrow u \text{ in } L^p(\Omega) \text{ for all } 1 \leq p < 6, \\ u_j &\rightarrow u \text{ a.e. on } \Omega, \end{aligned}$$

up to subsequences but still denoted $\{u_j\}$. Moreover from second concentration compactness lemma by P. L. Lions([23]), there exist an at most countable

set $\mathcal{J}$, points $\{x_k\}_{k \in \mathcal{J}} \subset \bar{\Omega}$ and values $\{\eta_k\}_{k \in \mathcal{J}}, \{\nu_k\}_{k \in \mathcal{J}} \subset \mathbb{R}^+$ such that up to subsequences,

$$\begin{aligned} |\nabla u_j|^2 &\rightharpoonup d\eta \geq |\nabla u|^2 + \sum_{k \in \mathcal{J}} \eta_k \delta_{x_k}, \\ (u_j)_+^6 &\rightharpoonup d\nu = u_+^6 + \sum_{k \in \mathcal{J}} \nu_k \delta_{x_k}, \end{aligned}$$

in the measure sense, here δ_x is the Dirac delta measure concentrated at $x \in \mathbb{R}^3$ with mass 1. In addition we also have the inequality

$$\eta_k \geq S\nu_k^{\frac{1}{3}} \quad (k \in \mathcal{J}). \quad (2.6)$$

Now we claim that $\mathcal{J} = \emptyset$. To ensure this, we suppose $\mathcal{J} \neq \emptyset$ to the contrary. Fix $k \in \mathcal{J}$. Define a smooth function ϕ such that $\phi = 1$ on $B(x_k, \varepsilon)$, $\phi = 0$ on $B(x_k, 2\varepsilon)^c$ and $0 \leq \phi \leq 1$ otherwise. In addition we can assume that $|\nabla \phi| \leq 2/\varepsilon$. As $I'(u_j) \rightarrow 0$ in $H^{-1}(\Omega)$, we have

$$\begin{aligned} 0 &= \lim_{j \rightarrow \infty} \langle I'(u_j), u_j \phi \rangle \\ &= \lim_{j \rightarrow \infty} \left\{ (a + b\|u_j\|^2) \int_{\Omega} \nabla u_j \cdot \nabla (u_j \phi) dx - \mu \int_{\Omega} g(x, u_j) u_j \phi dx \right\} \\ &\quad - \lim_{j \rightarrow \infty} \int_{\Omega} (u_j)_+^6 \phi dx \\ &= \lim_{j \rightarrow \infty} \left\{ (a + b\|u_j\|^2) \int_{\Omega} |\nabla u_j|^2 \phi dx - \int_{\Omega} (u_j)_+^6 \phi dx \right\} + o(1), \end{aligned} \quad (2.7)$$

where $o(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The last inequality comes from the facts that

$$\lim_{j \rightarrow \infty} (a + b\|u_j\|^2) \int_{\Omega} (\nabla u_j \cdot \nabla \phi) u_j dx = o(1) \text{ as } \varepsilon \rightarrow 0, \quad (2.8)$$

and

$$\lim_{j \rightarrow \infty} \int_{\Omega} g(x, u_j) u_j \phi dx = o(1) \text{ as } \varepsilon \rightarrow 0. \quad (2.9)$$

We first verify (2.8). Actually, noting the boundedness and $L^2(\Omega)$ convergence of $\{u_j\}$ and using the Schwartz and the Hölder inequality we have

$$\begin{aligned} \lim_{j \rightarrow \infty} (a + b\|u_j\|^2) \int_{\Omega} (\nabla u_j \cdot \nabla \phi) u_j dx \\ \leq C \lim_{j \rightarrow \infty} \left(\int_{\Omega} |\nabla u_j|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |u_j|^2 |\nabla \phi|^2 dx \right)^{\frac{1}{2}} \\ \leq C \left(\int_{\Omega \cap B(x_k, 2\varepsilon)} u^6 dx \right)^{\frac{1}{6}} \left(\int_{\Omega \cap B(x_k, 2\varepsilon)} |\nabla \phi|^3 dx \right)^{\frac{1}{3}} \\ \leq C \left(\int_{\Omega \cap B(x_k, 2\varepsilon)} u^6 dx \right)^{\frac{1}{6}} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \end{aligned}$$

where for the last inequality we use our assumption $|\nabla\phi| \leq 2/\varepsilon$. This ensures (2.8). We next verify (2.9). By (g1) and (g2), we have the inequality (2.1). Using that, we get

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{\Omega} g(x, u_j) u_j \phi dx &\leq \lim_{j \rightarrow \infty} \left(\delta \int_{\Omega \cap B(x_k, 2\varepsilon)} u_j^6 \phi dx + C_{\delta} \int_{\Omega \cap B(x_k, 2\varepsilon)} u_j^2 \phi dx \right) \\ &\leq C\delta + C_{\delta} \int_{\Omega \cap B(x_k, 2\varepsilon)} u^2 \phi dx, \end{aligned}$$

for some constant $C > 0$, where for the last inequality we use the Sobolev embedding, the boundedness and the $L^2(\Omega)$ convergence of $\{u_j\}$. Hence we have

$$\limsup_{\varepsilon \rightarrow 0} \left\{ \lim_{j \rightarrow \infty} \int_{\Omega} g(x, u_j) u_j \phi dx \right\} \leq C\delta$$

for all $\delta > 0$. This proves (2.9). From (2.7), we have

$$0 \geq \left\{ \left(a + b \int_{\Omega} \phi d\eta \right) \int_{\Omega} \phi d\eta - \int_{\Omega} \phi d\nu \right\} + o(1) \text{ as } \varepsilon \rightarrow 0.$$

Taking $\varepsilon \rightarrow 0$, we have

$$0 \geq (a + b\eta_k)\eta_k - \nu_k.$$

Using the inequality (2.6) we estimate

$$\eta_k \geq \frac{1}{2} \left(bS^3 + \sqrt{(bS^3)^2 + 4aS^3} \right) = C_K. \quad (2.10)$$

Since $I(u_j) \rightarrow c$ and $I'(u_j) \rightarrow 0$ in $H^{-1}(\Omega)$, recalling (g1) and (g3), we get

$$\begin{aligned} c &= \lim_{j \rightarrow \infty} \left\{ I(u_j) - \frac{1}{\theta} \langle I'(u_j), u_j \rangle \right\} \\ &\geq \lim_{j \rightarrow \infty} \left\{ a \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u_j\|^2 + b \left(\frac{1}{4} - \frac{1}{\theta} \right) \|u_j\|^4 \right\} \\ &\quad + \lim_{j \rightarrow \infty} \left\{ \mu \int_{\Omega} \left(\frac{1}{\theta} g(x, u_j) - G(x, u_j) \right) dx + \left(\frac{1}{\theta} - \frac{1}{6} \right) \int_{\Omega} (u_j)_+^6 dx \right\}. \end{aligned}$$

Using (2.6) and (2.10), we have

$$\begin{aligned} c &\geq a \left(\frac{1}{2} - \frac{1}{\theta} \right) \eta_k + b \left(\frac{1}{4} - \frac{1}{\theta} \right) \eta_k^2 + \left(\frac{1}{\theta} - \frac{1}{6} \right) \nu_k \\ &\geq a \left(\frac{1}{2} - \frac{1}{\theta} \right) C_K + b \left(\frac{1}{4} - \frac{1}{\theta} \right) C_K^2 + \frac{1}{S^3} \left(\frac{1}{\theta} - \frac{1}{6} \right) C_K^3 \\ &= \frac{a}{2} C_K + \frac{b}{4} C_K^2 - \frac{1}{6S^3} C_K^3 \end{aligned}$$

where the last equality comes from the fact that $aC_K + bC_K^2 - C_K^3/S^3 = 0$. This is a contradiction. Hence $\mathcal{J} = \emptyset$. Consequently we have

$$\int_{\Omega} (u_j)_+^6 dx \rightarrow \int_{\Omega} u_+^6 \text{ as } j \rightarrow \infty.$$

This leads us to the conclusion. $\square$

Remark 2.4. We can easily check that u is nonnegative. In fact, since $\{u_j\}$ is bounded, there exist a subsequence, still denoted $\{u_j\}$, and a constant $A \geq 0$ such that $\|u_j\| \rightarrow A$. If $A = 0$ the conclusion follows. If $A > 0$, considering the weak convergence, (g1), (g2), and $L^p(\Omega)$ convergence for all $1 \leq p < 6$, we have

$$(a + bA) \int_{\Omega} \nabla u \cdot \nabla h dx = \int_{\Omega} g(x, u) h dx - \int_{\Omega} u_+^5 h \quad (h \in H_0^1(\Omega)).$$

Taking $h = u_-$, where $u_- := -\min\{0, u\}$ and considering (g1), we have

$$\int_{\Omega} |\nabla u_-|^2 dx = 0.$$

The conclusion follows.

We ensure the local PS condition for I .

Lemma 2.5. *Let g satisfy (g1)-(g3) and assume*

$$c < \frac{a}{2} C_K + \frac{b}{4} C_K^2 - \frac{1}{6S^3} C_K^3.$$

Then I satisfies the $(PS)_c$ condition.

Proof. Let $\{u_j\}$ be a $(PS)_c$ sequence for I . Then by (g1) and (g2), we have that $\{u_j\}$ is bounded as in the proof of Lemma 2.3. Furthermore, by Lemma 2.3 and Remark 2.4, there exists a nonnegative function $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} u_j &\rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ u_j &\rightarrow u \text{ in } L^p(\Omega) \text{ for all } 1 \leq p < 6, \\ (u_j)_+ &\rightarrow u \text{ in } L^6(\Omega), \end{aligned}$$

up to subsequences, but still denoted $\{u_j\}$. Since $I'(u_j) \rightarrow 0$ in $H^{-1}(\Omega)$ and $\{u_j\}$ is bounded, we have

$$\langle I'(u_j), (u_j - u) \rangle = o(1) \tag{2.11}$$

where $o(1) \rightarrow 0$ as $j \rightarrow \infty$. Here we claim

$$\int_{\Omega} g(x, u_j)(u_j - u) dx = o(1) \text{ as } j \rightarrow \infty, \tag{2.12}$$

and

$$\int_{\Omega} (u_j)_+^5 (u_j - u) dx = o(1) \text{ as } j \rightarrow \infty. \tag{2.13}$$

We first verify (2.12). Using (2.1) and the Hölder and the Schwartz inequality we have

$$\begin{aligned}
& \limsup_{j \rightarrow \infty} \left| \int_{\Omega} g(x, u_j)(u_j - u) dx \right| \\
& \leq \limsup_{j \rightarrow \infty} \left(\delta \int_{\Omega} |u_j|^5 |u_j - u| dx + C_{\delta} \int_{\Omega} |u_j| |u_j - u| dx \right) \\
& \leq \delta \limsup_{j \rightarrow \infty} \left(\int_{\Omega} u_j^6 dx \right)^{\frac{5}{6}} \left(\int_{\Omega} |u_j - u|^6 dx \right)^{\frac{1}{6}} \\
& \quad + C_{\delta} \limsup_{j \rightarrow \infty} \left(\int_{\Omega} u_j^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |u_j - u|^2 dx \right)^{\frac{1}{2}} \\
& \leq C_{\delta}
\end{aligned}$$

for all $\delta > 0$, where for the last inequality we use the Sobolev embedding and the boundedness and $L^2(\Omega)$ convergence of $\{u_j\}$. This proves (2.12). Next we show (2.13). Actually, by the Hölder inequality, the boundedness of $\{u_j\}$ and the $L^6(\Omega)$ convergence of $(u_j)_+$ we get

$$\begin{aligned}
\limsup_{j \rightarrow \infty} \left| \int_{\Omega} (u_j)_+^5 (u_j - u) dx \right| & \leq \limsup_{j \rightarrow \infty} \left(\int_{\Omega} (u_j)_+^6 dx \right)^{\frac{5}{6}} \left(\int_{\Omega} |(u_j)_+ - u|^6 dx \right)^{\frac{1}{6}} \\
& = 0,
\end{aligned}$$

here we use the relation $(u_j)_+^5 u_j = (u_j)_+^6$. This shows (2.13). Consequently from (2.11)-(2.13), we conclude that

$$(a + b\|u_j\|^2) \int_{\Omega} \nabla u_j \cdot \nabla (u_j - u) dx = o(1) \text{ as } j \rightarrow \infty.$$

By the weak convergence, we have $\|u_j\| \rightarrow \|u\|$ as $j \rightarrow \infty$. This completes the proof. $\square$

The following lemma gives a sufficient condition on g (which is weaker than (g4)) to ensure the compactness of $(PS)_c$ sequences for I . The argument is strictly based on Lemma 2.1 in [8].

Lemma 2.6. *Let g satisfy (g1) and (g2). We assume that there exist a nonempty open set $\omega \subset \Omega$ and a measurable function $g(u)$ such that $g(x, u) \geq g(u) \geq 0$ for all $x \in \omega$ and $u \geq 0$ and further,*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \int_0^{\varepsilon^{-1}} G\left(\frac{\varepsilon^{-\frac{1}{2}}}{(1+s^2)^{\frac{1}{2}}}\right) s^2 ds = \infty,$$

then there exists a constant $\varepsilon_1 > 0$ such that

$$\max_{t \geq 0} I(tv_{\varepsilon}) < \frac{a}{2} C_K + \frac{b}{4} C_K^2 - \frac{1}{6S^3} C_K^3$$

for all $\varepsilon \in (0, \varepsilon_1)$.

Proof. We define v_ε as in the paragraph above Lemma 2.2. Take $t \geq 0$. By (2.5), we have

$$I(tv_\varepsilon) = \frac{at^2}{2}\|v_\varepsilon\|^2 + \frac{bt^4}{4}\|v_\varepsilon\|^4 - \mu \int_{\Omega} G(x, tv_\varepsilon) dx - \frac{t^6}{6} =: f(t). \quad (2.14)$$

We put t_ε so that $f(t_\varepsilon) = \max_{t \geq 0} f(t)$. If $t_\varepsilon = 0$, the conclusion follows. We assume $t_\varepsilon > 0$ and put $A_\varepsilon := \|v_\varepsilon\|^2$. Then noting (g1) and (g2), we get

$$aA_\varepsilon + bt_\varepsilon^2 A_\varepsilon^2 - \mu \int_{\Omega} \frac{g(x, t_\varepsilon v_\varepsilon) v_\varepsilon}{t_\varepsilon} dx - t_\varepsilon^4 = 0.$$

Since $g(x, u) \geq 0$ for all $x \in \Omega$ and $u \geq 0$ by (g1), we have

$$t_\varepsilon \leq \left\{ \frac{1}{2} \left(bA_\varepsilon^2 + \sqrt{(bA_\varepsilon^2)^2 + 4aA_\varepsilon} \right) \right\}^{\frac{1}{2}} =: T_\varepsilon. \quad (2.15)$$

Since the map

$$t \mapsto \frac{at^2}{2} A_\varepsilon + \frac{bt^4}{4} A_\varepsilon^2 - \frac{t^6}{6}$$

is increasing on the interval $[0, T_\varepsilon]$, we have

$$\begin{aligned} I(tv_\varepsilon) &\leq I(t_\varepsilon v_\varepsilon) \\ &\leq \frac{aT_\varepsilon^2}{2} A_\varepsilon + \frac{bT_\varepsilon^4}{4} A_\varepsilon^2 - \frac{T_\varepsilon^6}{6} - \mu \int_{\Omega} G(x, t_\varepsilon v_\varepsilon) dx \\ &\leq \frac{a}{2} C_K + \frac{b}{4} C_K^2 - \frac{1}{6S^3} C_K^3 + O(\varepsilon) - \mu \int_{\Omega} G(x, t_\varepsilon v_\varepsilon) dx \end{aligned}$$

after some calculation. Hence if

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{\Omega} G(x, t_\varepsilon v_\varepsilon) dx = \infty, \quad (2.16)$$

we can take $\varepsilon_1 > 0$ so small that

$$\sup_{t \geq 0} I(tv_\varepsilon) < \frac{a}{2} C_K + \frac{b}{4} C_K^4 - \frac{1}{6S^3} C_K^3$$

for all $\varepsilon \in (0, \varepsilon_1)$. This finishes the proof. Now we show (2.16). To this end, first notice that

$$t_\varepsilon \rightarrow \left\{ \frac{1}{2} \left(bS^4 + \sqrt{(bS^4)^2 + 4aS^2} \right) \right\}^{\frac{1}{2}}. \quad (2.17)$$

(2.17) comes from the fact that

$$\int_{\Omega} \frac{g(x, t_\varepsilon v_\varepsilon) v_\varepsilon}{t_\varepsilon} dx \rightarrow 0 \quad (2.18)$$

as $\varepsilon \rightarrow 0$. In fact, by (2.1), we get

$$\int_{\Omega} \frac{g(x, t_\varepsilon v_\varepsilon) v_\varepsilon}{t_\varepsilon} dx \leq \delta t_\varepsilon^4 + C_\delta t_\varepsilon \int_{\Omega} v_\varepsilon^2 dx$$

for all $\delta > 0$ and some constant $C_\delta > 0$. On the other hand, since T_ε is bounded, t_ε is also bounded by (2.15). Using this fact and (2.5), we have

$$\limsup_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{g(x, t_\varepsilon v_\varepsilon) v_\varepsilon}{t_\varepsilon} dx \leq C\delta$$

for some constant $C > 0$ and all $\delta > 0$. This ensures (2.18). After this, the argument is completely same as that in the proof of Lemma 2.1 in [8]. But we shall proceed with the rest for reader's convenience. By the assumption on g , recalling (2.17) and (2.5), we have

$$\begin{aligned} \varepsilon^{-1} \int_{\Omega} G(x, t_\varepsilon v_\varepsilon) dx &\geq \varepsilon^{-1} \int_{\omega} G(t_\varepsilon v_\varepsilon) dx \\ &\geq \varepsilon^{-1} \int_{\omega} G\left(C_\varepsilon \frac{\varepsilon^{\frac{1}{2}} \tau(x)}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}\right) dx \\ &\geq C\varepsilon^{-1} \int_0^r G\left(C_\varepsilon \frac{\varepsilon^{\frac{1}{2}}}{(\varepsilon^2 + s^2)^{\frac{1}{2}}}\right) s^2 ds \\ &\geq C\varepsilon^2 \int_0^{\frac{r}{\varepsilon}} G\left(C_\varepsilon \frac{\varepsilon^{-\frac{1}{2}}}{(1 + s^2)^{\frac{1}{2}}}\right) s^2 ds \\ &\geq C\varepsilon^2 \int_0^{\frac{C}{\varepsilon}} G\left(\frac{\varepsilon^{-\frac{1}{2}}}{(1 + s^2)^{\frac{1}{2}}}\right) s^2 ds. \end{aligned}$$

for some constant $C > 0$ where C_ε denotes some constant which converges some positive values as $\varepsilon \rightarrow 0$ and for the last inequality we perform an appropriate rescaling for ε . If $C \geq 1$, (2.18) immediately follows. If $C < 1$, we have

$$\begin{aligned} \varepsilon^2 \int_0^{\frac{C}{\varepsilon}} G\left(\frac{\varepsilon^{-\frac{1}{2}}}{(1 + s^2)^{\frac{1}{2}}}\right) s^2 dx &= \varepsilon^2 \int_0^{\frac{1}{\varepsilon}} G\left(\frac{\varepsilon^{-\frac{1}{2}}}{(1 + s^2)^{\frac{1}{2}}}\right) s^2 dx \\ &\quad - \varepsilon^2 \int_{\frac{1}{\varepsilon}}^{\frac{C}{\varepsilon}} G\left(\frac{\varepsilon^{-\frac{1}{2}}}{(1 + s^2)^{\frac{1}{2}}}\right) s^2 dx. \end{aligned}$$

Put

$$B_\varepsilon := \varepsilon^2 \int_{\frac{1}{\varepsilon}}^{\frac{C}{\varepsilon}} G\left(\frac{\varepsilon^{-\frac{1}{2}}}{(1 + s^2)^{\frac{1}{2}}}\right) s^2 dx.$$

Using (2.1), we get for some $\delta > 0$,

$$\begin{aligned} B_\varepsilon &\leq \varepsilon^2 \left(\delta \int_{\frac{1}{\varepsilon}}^{\frac{C}{\varepsilon}} \left| \frac{\varepsilon^{-\frac{1}{2}}}{(1 + s^2)^{\frac{1}{2}}} \right|^6 s^2 ds + C_\delta \int_{\frac{1}{\varepsilon}}^{\frac{C}{\varepsilon}} \left| \frac{\varepsilon^{-\frac{1}{2}}}{(1 + s^2)^{\frac{1}{2}}} \right|^2 s^2 ds \right) \\ &\leq C \end{aligned}$$

for some constant $C > 0$. This concludes (2.18). $\square$

We can also conclude the following lemma as in the proof of Corollary 2.3 in [8].

Lemma 2.7. *Suppose g satisfies (g1), (g2) and (g4). Then g satisfies the hypothesis of Lemma 2.6.*

Proof. Set $g(u) := \inf_{x \in \omega} g(x, u)$. By (g4), for all $M > 0$, there exists a constant $R > 0$ such that $G(u) \geq Mu^4$ for all $u \geq R$. We note that there exists a constant $C > 0$ which is independent of $\varepsilon > 0$ such that

$$\frac{\varepsilon^{-\frac{1}{2}}}{(1+s^2)^{\frac{1}{2}}} \geq R$$

for all $s \leq C\varepsilon^{-\frac{1}{2}}$ if $\varepsilon > 0$ is sufficiently small. Using this fact, we get

$$\begin{aligned} \varepsilon^2 \int_0^{\varepsilon^{-1}} G\left(\frac{\varepsilon^{-\frac{1}{2}}}{(1+s^2)^{\frac{1}{2}}}\right) s^2 ds &\geq \varepsilon^2 \int_0^{C\varepsilon^{-\frac{1}{2}}} G\left(\frac{\varepsilon^{-\frac{1}{2}}}{(1+s^2)^{\frac{1}{2}}}\right) s^2 ds \\ &\geq M \int_0^{C\varepsilon^{-\frac{1}{2}}} \frac{s^2}{(1+s^2)^2} \end{aligned}$$

for all $M > 0$ and small $\varepsilon > 0$. Hence we have

$$\liminf_{\varepsilon \rightarrow 0} \left\{ \varepsilon^2 \int_0^{\varepsilon^{-1}} G\left(\frac{\varepsilon^{-\frac{1}{2}}}{(1+s^2)^{\frac{1}{2}}}\right) s^2 ds \right\} \geq CM$$

for all $M > 0$ and some constant $C > 0$ which is independent of $M > 0$. This concludes the proof. $\square$

Finally we prove Theorem 1.1.

Proof of Theorem 1.1. Let g satisfy (g1)-(g4). By Lemma 2.7, g satisfy the assumption of Lemma 2.6. Take $\varepsilon \in (0, \min\{\varepsilon_0, \varepsilon_1\})$ where $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ are took from the proof of Lemma 2.1 and Lemma 2.6 respectively. Define $v_0 := t_0 v_\varepsilon$ as in the proof of Lemma 2.2. Now by Lemma 2.1 and Lemma 2.2, I satisfies the mountain pass geometry. Moreover from Lemma 2.6 and Lemma 2.5, I satisfies the $(PS)_c$ condition. Hence by the mountain pass theorem, we conclude that I has a nontrivial critical point $u \in H_0^1(\Omega)$ with its critical value $c \geq \alpha$. Furthermore, similarly to Remark 2.4, we can show that $u \in H_0^1(\Omega)$ is nonnegative. The smoothness and positivity of u is shown as we note in the end of section 1. This finishes the proof. $\square$

Proof of Corollary 1.2. We assume $4 < q < 6$ and $g(x, u) := u^q$. For the proof, we consider instead $g(x, u) := (u_+)^q$. Then clearly g satisfies (g1)-(g4). By Theorem 1.1, we conclude the proof. $\square$

3. Proof of Theorem 1.3

In section 3, we prove Theorem 1.3 and Corollary 1.4. Assume $a > 0$ and $b \geq 0$. In addition, let g satisfy (g1), (g2), (g5) and (g6). But if there exists a constant $4 < \theta < 6$ such that $g(x, u)u - \theta G(u) \geq 0$ for all $x \in \Omega$ and $t \geq 0$, the proof is simpler. We can refer to the argument in [2] for that situation. Hence we only consider the following condition instead of (g5):

(g5)' There exists a constant $\theta > 0$ such that $2 < \theta \leq 4$ and $g(x, u)u - \theta G(x, u) \geq 0$ for all $x \in \Omega$ and $u \geq 0$.

As in section 2, we define the energy functional associated to (K1) so that

$$I_\mu(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \mu \int_\Omega G(x, u)dx - \frac{1}{6} \int_\Omega u_+^6,$$

where $G(x, u) := \int_0^u g(x, t)dt$ and $u_+ := \max\{u, 0\}$. Inspired by [19], we introduce a truncation method. That is also seen in earlier researches [16] and [17]. Let ψ be a smooth function on $[0, \infty)$ such that $\psi = 1$ on $[0, 1]$, $\psi = 0$ on $[2, \infty)$ and $0 \leq \psi \leq 1$ otherwise. Furthermore, we assume $-2 \leq \psi' \leq 0$ on $[0, \infty)$. For $T > 0$, define a C^1 cut off functional $\Phi_T(u)$ on $H_0^1(\Omega)$ so that

$$\Phi_T(u) := \psi\left(\frac{\|u\|^2}{T^2}\right).$$

We consider a truncated functional on $H_0^1(\Omega)$ such that

$$J_\mu^T(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4\Phi_T(u) - \mu \int_\Omega G(x, u)dx - \frac{1}{6} \int_\Omega u_+^6 dx.$$

Then by (g1) and (g2), we can easily verify that J_λ^T is well defined and continuously Fréchet differentiable on $H_0^1(\Omega)$. Its first Fréchet derivative is given by

$$\begin{aligned} \langle J_\mu^{T'}(u), h \rangle &= \left\{ (a + b\|u\|^2\Phi_T(u) + \frac{b}{2T^2}\|u\|^4\psi'\left(\frac{\|u\|^2}{T^2}\right)) \int_\Omega \nabla u \cdot \nabla h dx \right. \\ &\quad \left. - \int_\Omega g(x, u)h dx - \int_\Omega u_+^5 h dx \quad (h \in H_0^1(\Omega)). \right. \end{aligned}$$

Choose $T > 0$ as

$$T = \min \left\{ \left(\frac{a}{8b} \right)^{\frac{1}{2}}, \left\{ \frac{a(\theta - 2)}{4b(4 - \theta)} \right\}^{\frac{1}{2}} \right\}. \quad (3.1)$$

Note that by (3.1) and the facts that $|\|u\|^4\psi'(\|u\|^2/T^2)| \leq 8T^4$ for all $u \in H_0^1(\Omega)$, the following relations hold.

$$a + \frac{b}{2T^2}\|u\|^4\psi'\left(\frac{\|u\|^2}{T^2}\right) \geq \frac{a}{2}, \quad (3.2)$$

$$a\left(\frac{1}{2} - \frac{1}{\theta}\right) - 2b\left(\frac{1}{\theta} - \frac{1}{4}\right)T^2 \geq 0, \quad (3.3)$$

and

$$8b\left(\frac{1}{\theta} - \frac{1}{4}\right)T^4 \leq a\left(\frac{1}{2} - \frac{1}{\theta}\right)T^2. \quad (3.4)$$

Note also that if $u \in H_0^1(\Omega)$ and $\|u\| < T$, $J_\mu^T(u) = I_\mu(u)$ on some neighborhood of u . Thus if u is a critical point of J_μ^T with $\|u\| < T$, u is also a critical point of I_μ . From now on, we show the existence of a nontrivial critical point u of J_μ^T with $\|u\| < T$. Since we fix $T > 0$ as in (3.1), we denote $J_\mu^T(u)$ and $\Phi_T(u)$ as $J_\mu(u)$ and $\Phi(u)$ respectively for simplicity.

We begin with following two lemmas.

Lemma 3.1. *We assume g satisfies (g1) and (g2). Then there exist constants $\alpha, \rho > 0$ such that $J_\mu(u) \geq \alpha$ for all $u \in H_0^1(\Omega)$ with $\|u\| = \rho$.*

Proof. For $\rho > 0$, take $u \in H_0^1(\Omega)$ with $\|u\| = \rho$. By (g1) and (g2), as in the proof of Lemma 2.1, we reach to the inequality

$$J_\mu(u) \geq \frac{a}{4}\|u\|^2 - C\|u\|^6 = \frac{a}{4}\rho^2 - C\rho^6$$

for some constant $C > 0$. Hence by taking ρ small enough, we conclude that there exists a constant $\alpha > 0$ such that

$$J_\mu(u) \geq \alpha.$$

This finishes the proof. $\square$

Lemma 3.2. *Suppose g satisfies (g1), (g2) and (g6). Then there exists a function $v_0 \in H_0^1(\Omega)$ such that $\|v_0\| \geq \rho$ and $J_\mu(v_0) \leq 0$.*

Proof. As in the proof of corollary 2.4 in [8], we assume $0 \in \omega$ and take a function $\xi \in C_0^\infty(\omega)$ with $\xi(0) = 1$ and a constant $0 < k < 1/2$. Set $v := \xi|x|^{-k}$. Then $v \in H_0^1(\Omega)$ by a null extension. Take $t > 0$. Since by (g1), $G(x, t) \geq 0$ for all $x \in \Omega$ and $t \in \mathbb{R}$, we get

$$J_\mu(tv) \leq \frac{at^2}{2}\|v\|^2 + \frac{bt^4}{4}\|v\|^4 - \frac{t^6}{6} \int_\Omega v^6 dx.$$

Thus we have $J_\mu(tv) \rightarrow -\infty$ as $t \rightarrow \infty$. Hence there exists a constant $t_0 > 0$ such that if $v_0 := t_0 v$, $\|v_0\| \geq \rho$ and $J_\mu(v_0) \leq 0$. This concludes the proof. $\square$

As in section 2 we define

$$\Gamma := \{\gamma \in C([0, 1], H_0^1(\Omega)) \mid \gamma(0) = 0, \gamma(1) = v_0\}$$

and

$$c_\mu := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0, 1])} J_\mu(u).$$

We prove the next lemma.

Lemma 3.3. *Let g satisfy (g1), (g2) and (g6). Then $c_\mu \rightarrow 0$ as $\mu \rightarrow \infty$.*

Proof. Let v be as defined in the proof of Lemma 3.2. By the definition of v and (g1), we have

$$J_\mu(tv) \leq \frac{at^2}{2}\|v\|^2 + \frac{bt^4}{4}\|v\|^4 - \mu \int_\omega G(x, tv) dx - \frac{t^6}{6} \int_\Omega v^6 dx =: f(t).$$

We put $t_\mu > 0$ so that $f(t_\mu) = \max_{t \geq 0} f(t)$. Then, using (g2) we have

$$a\|v\|^2 + bt_\mu^2\|v\|^4 - \frac{\mu}{t_\mu} \int_\omega g(x, t_\mu v) v dx - t_\mu^4 \int_\Omega v^6 = 0. \quad (3.5)$$

By (g1), we get

$$a\|v\|^2 + bt_\mu^2\|v\|^4 \geq t_\mu^4 \int_\Omega v^6.$$

Hence there exists a constant $C > 0$ such that $t_\mu \leq C$ for all $\mu > 0$. Furthermore $t_\mu \rightarrow 0$ as $\mu \rightarrow \infty$. If not, there exists a sequence $\{\mu_n\}$ and a constant

$\beta > 0$ such that $\mu_n \rightarrow \infty$ and $t_{\mu_n} \rightarrow \beta$ as $n \rightarrow \infty$. Then by (g1) and (g2), we have

$$\frac{1}{t_{\mu_n}} \int_{\omega} g(x, t_{\mu_n} v) v dx \rightarrow \frac{1}{\beta} \int_{\omega} g(x, \beta v) v dx > 0.$$

The positivity of the right hand side is ensured by (g6) and the definition of v . But in view of (3.5), $\int_{\omega} g(x, \beta v) v dx$ must be 0. This is a contradiction. Therefore we obtain

$$0 < c_{\mu} \leq \max_{t \geq 0} J_{\mu}(tv) \leq \max_{t \geq 0} f(t) < \frac{at_{\mu}^2}{2} \|v\|^2 + \frac{bt_{\mu}^4}{4} \|v\|^4 \rightarrow 0.$$

This completes the proof. $\square$

Next we prove an important lemma which ensures the local compactness of the PS sequences for J_{μ} .

Lemma 3.4. *We assume g satisfies (g1), (g2) and (g5)' and further, $\{u_j\}$ is a $(PS)_c$ sequence for J_{μ} with*

$$c < \left(\frac{1}{\theta} - \frac{1}{6} \right) \left(\frac{aS}{2} \right)^{\frac{3}{2}}.$$

Then $\{(u_j)_+\}$ has a subsequence which strongly converges in $L^6(\Omega)$.

Proof. Let $\{u_j\} \subset H_0^1(\Omega)$ be a $(PS)_c$ sequence for J_{μ} with

$$c < \left(\frac{1}{\theta} - \frac{1}{6} \right) \left(\frac{aS}{2} \right)^{\frac{3}{2}}.$$

Firstly we claim that $\{u_j\}$ is bounded in $H_0^1(\Omega)$. Actually, since $J_{\mu}(u_j) \rightarrow c$ and $J'_{\mu}(u_j) \rightarrow 0$ in $H^{-1}(\Omega)$, we have by (g1) and (g5)',

$$\begin{aligned} c + 1 &\geq J_{\mu}(u_j) - \frac{1}{\theta} \langle J'_{\mu}(u_j), u_j \rangle + \frac{1}{\theta} \langle J'_{\mu}(u_j), u_j \rangle \\ &\geq a \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u_j\|^2 - b \left(\frac{1}{\theta} - \frac{1}{4} \right) \|u_j\|^4 \Phi(u) - \frac{b}{2\theta T^2} \|u_j\|^6 \psi' \left(\frac{\|u_j\|^2}{T^2} \right) \\ &\quad + \int_{\Omega} \left(\frac{1}{\theta} g(x, u_j) - G(x, u_j) \right) dx + \left(\frac{1}{\theta} - \frac{1}{6} \right) \int_{\Omega} (u_j)_+^6 dx - \|u_j\| \\ &\geq a \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u_j\|^2 - 4b \left(\frac{1}{\theta} - \frac{1}{4} \right) T^4 - \|u_j\|, \end{aligned}$$

for large $j \in \mathbb{N}$, where for the last inequality we use the facts that $\|u\|^4 \Phi(u) \leq 4T^4$ for all $u \in H_0^1(\Omega)$ and $\psi' \leq 0$ on $[0, \infty)$. This inequality proves the claim. Consequently as in the previous section, we conclude that there exists a function $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} u_j &\rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ u_j &\rightarrow u \text{ in } L^p(\Omega) \text{ for all } 1 \leq p < 6, \\ u_j &\rightarrow u \text{ a.e. on } \Omega, \end{aligned}$$

up to subsequences but still denoted $\{u_j\}$. Furthermore by the second concentration compactness lemma, there exist an at most countable set $\mathcal{J}$, points $(x_k)_{k \in \mathcal{J}} \subset \bar{\Omega}$ and positive values $(\eta_k)_{k \in \mathcal{J}}, (\nu_k)_{k \in \mathcal{J}}$ with

$$\eta_k \geq S\nu_k^{\frac{1}{3}} \quad (k \in \mathcal{J}),$$

such that

$$\begin{aligned} |\nabla u_j|^2 &\rightharpoonup d\eta \geq |\nabla u|^2 + \sum_{k \in \mathcal{J}} \eta_k \delta_{x_k}, \\ (u_j)_+^6 &\rightharpoonup d\nu = u_+^6 + \sum_{k \in \mathcal{J}} \nu_k \delta_{x_k}, \end{aligned}$$

in the measure sense. To complete the proof, we claim $\mathcal{J} = \emptyset$. If not, choose $k \in \mathcal{J}$. We define a test function $\phi \in C_0^\infty(\mathbb{R}^3)$ such that $\phi = 1$ on $B(x_k, \varepsilon)$, $\phi = 0$ on $B(x_k, 2\varepsilon)^c$, and $0 \leq \phi \leq 1$ otherwise. In addition we assume $|\nabla \phi| \leq 2/\varepsilon$. Then we have

$$\begin{aligned} 0 &= \lim_{j \rightarrow \infty} \langle J'_\mu(u_j), u_j \phi \rangle \\ &= \lim_{j \rightarrow \infty} \left\{ a + b\|u_j\|^2 \Phi(u_j) + \frac{b}{2T^2} \|u_j\|^4 \psi' \left(\frac{\|u_j\|^2}{T^2} \right) \right\} \int_{\Omega} \nabla u_j \cdot \nabla (u_j \phi) dx \\ &\quad - \lim_{j \rightarrow \infty} \int_{\Omega} g(x, u_j) u_j \phi dx - \lim_{j \rightarrow \infty} \int_{\Omega} (u_j)_+^6 \phi dx. \end{aligned} \tag{3.6}$$

Now we estimate the first and second terms in the right hand side of (3.6). We compute the first term so that

$$\begin{aligned} &\lim_{j \rightarrow \infty} \left\{ a + b\|u_j\|^2 \Phi(u_j) + \frac{b}{2T^2} \|u_j\|^4 \psi' \left(\frac{\|u_j\|^2}{T^2} \right) \right\} \int_{\Omega} \nabla u_j \cdot \nabla (u_j \phi) dx \\ &= \lim_{j \rightarrow \infty} \left\{ a + b\|u_j\|^2 \Phi(u_j) + \frac{b}{2T^2} \|u_j\|^4 \psi' \left(\frac{\|u_j\|^2}{T^2} \right) \right\} \int_{\Omega} |\nabla u_j|^2 \phi dx \\ &\quad + \lim_{j \rightarrow \infty} \left\{ a + b\|u_j\|^2 \Phi(u_j) + \frac{b}{2T^2} \|u_j\|^4 \psi' \left(\frac{\|u_j\|^2}{T^2} \right) \right\} \int_{\Omega} (\nabla u_j \cdot \nabla \phi) u_j dx \\ &\geq \lim_{j \rightarrow \infty} \frac{a}{2} \int_{\Omega} |\nabla u_j|^2 \phi dx + o(1), \end{aligned} \tag{3.7}$$

where $o(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The last inequality comes from (3.2) and the fact that

$$\begin{aligned} &\lim_{j \rightarrow \infty} \left| \left\{ a + b\|u_j\|^2 \Phi(u_j) + \frac{b}{2T^2} \|u_j\|^4 \psi' \left(\frac{\|u_j\|^2}{T^2} \right) \right\} \int_{\Omega} (\nabla u_j \cdot \nabla \phi) u_j dx \right| \\ &\quad = o(1) \text{ as } \varepsilon \rightarrow 0. \end{aligned} \tag{3.8}$$

We verify (3.8). Using the facts $\|u\|^2\Phi(u) \leq 2T^2$ and $|||u_j||^4\psi'(\|u_j\|^2/T^2)| \leq 8T^4$ for all $u \in H_0^1(\Omega)$ we estimate

$$\begin{aligned}
& \lim_{j \rightarrow \infty} \left| \left(a + b\|u_j\|^2\Phi(u_j) + \frac{b}{2T^2}\|u_j\|^4\psi' \left(\frac{\|u_j\|^2}{T^2} \right) \right) \int_{\Omega} \nabla u_j \cdot \nabla \phi u_j dx \right| \\
& \leq (a + 6bT^2) \lim_{j \rightarrow \infty} \int_{\Omega \cap B(x_k, 2\varepsilon)} |\nabla u_j| |u_j \nabla \phi| dx \\
& \leq (a + 6bT^2) \lim_{j \rightarrow \infty} \left\{ \left(\int_{\Omega} |\nabla u_j|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega \cap B(x_k, 2\varepsilon)} |u_j \nabla \phi|^2 dx \right)^{\frac{1}{2}} \right\} \\
& \leq C \left(\int_{\Omega \cap B(x_k, 2\varepsilon)} u^6 dx \right)^{\frac{1}{6}} \left(\int_{\Omega \cap B(x_k, 2\varepsilon)} |\nabla \phi|^3 dx \right)^{\frac{1}{3}} \\
& \leq C \left(\int_{\Omega \cap B(x_k, 2\varepsilon)} u^6 dx \right)^{\frac{1}{6}} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,
\end{aligned}$$

where for the second inequality we use the Schwartz inequality, and for the third inequality we use the boundedness and $L^2(\Omega)$ convergence of $\{u_j\}$ and the Hölder inequality, and for the fourth inequality we use our assumption that $|\nabla \phi| \leq 2/\varepsilon$. This verifies (3.8). Next using (2.1), we compute the second term in the right hand side of (3.6) similarly to (2.9) in the previous section so that

$$\lim_{j \rightarrow \infty} \int_{\Omega} g(x, u_j) u_j \phi dx = o(1) \text{ as } \varepsilon \rightarrow 0. \quad (3.9)$$

Using the computations (3.7) and (3.9) for (3.6), we get

$$\begin{aligned}
0 & \geq \lim_{j \rightarrow \infty} \left(\frac{a}{2} \int_{\Omega} |\nabla u_j|^2 \phi dx - \int_{\Omega} (u_j)_+^6 \phi dx \right) + o(1) \\
& \geq \frac{a}{2} \int_{\Omega} \phi d\eta - \int_{\Omega} \phi d\nu + o(1) \text{ as } \varepsilon \rightarrow 0.
\end{aligned}$$

Taking $\varepsilon \rightarrow 0$, we obtain

$$\nu_k \geq \frac{a}{2} \eta_k.$$

Recalling that $\eta_k \geq S\nu_k^{\frac{1}{3}}$, we estimate

$$\nu_k \geq \left(\frac{aS}{2} \right)^{\frac{3}{2}}. \quad (3.10)$$

Since $J_\mu(u_j) \rightarrow c$ and $J'_\mu(u_j) \rightarrow 0$ in $H^{-1}(\Omega)$, we have by (3.3), (g1), (g5)', and the fact that $\psi' \leq 0$,

$$\begin{aligned} c &= \lim_{j \rightarrow \infty} \left\{ J_\mu(u_j) - \frac{1}{\theta} \langle J'_\mu(u_j), u_j \rangle \right\} \\ &= \lim_{j \rightarrow \infty} \left[\left\{ a \left(\frac{1}{2} - \frac{1}{\theta} \right) - b \left(\frac{1}{\theta} - \frac{1}{4} \right) \|u_j\|^2 \Phi(u_j) \right\} \int_\Omega |\nabla u_j|^2 dx \right] \\ &\quad - \lim_{j \rightarrow \infty} \frac{b}{2\theta T^2} \|u_j\|^6 \psi' \left(\frac{\|u_j\|^2}{T^2} \right) + \lim_{j \rightarrow \infty} \int_\Omega \frac{1}{\theta} \{g(u_j)u_j - G(u_j)\} dx \\ &\quad + \lim_{j \rightarrow \infty} \left(\frac{1}{\theta} - \frac{1}{6} \right) \int_\Omega (u_j)_+^6 dx \\ &\geq \left(\frac{1}{\theta} - \frac{1}{6} \right) \left(\frac{aS}{2} \right)^{\frac{3}{2}}. \end{aligned}$$

This is a contradiction. Hence $\mathcal{J} = \emptyset$. Consequently we have

$$\int_\Omega (u_j)_+^6 \rightarrow \int_\Omega u_+^6 dx \text{ as } j \rightarrow \infty,$$

up to subsequences but still denoted $\{u_j\}$. This completes the proof. $\square$

Remark 3.5. As in Remark 2.4, we can show that u is nonnegative.

Using Lemma 3.1-Lemma 3.4 we prove the existence of a nontrivial critical point of J_μ .

Lemma 3.6. *Let g satisfies (g1), (g2), (g5)' and (g6). Then there exists a constant $\mu^* \geq 0$ such that J_μ has a nontrivial critical point for all $\mu > \mu^*$.*

Proof. From Lemma 3.3, there exists a constant $\mu^* \geq 0$ such that

$$c_\mu < \left(\frac{1}{\theta} - \frac{1}{6} \right) \left(\frac{aS}{2} \right)^{\frac{3}{2}}$$

for all $\mu > \mu^*$. For each $\mu > \mu^*$, let $\{u_j\}$ be a $(PS)_{c_\mu}$ sequence for J_μ . Then by Lemma 3.4 and Remark 3.5, there exists a nonnegative function $u \in H_0^1(\Omega)$ such that

$$(u_j)_+ \rightarrow u \text{ in } L^6(\Omega) \quad (3.11)$$

up to subsequences. From now on we prove that J_μ satisfies the $(PS)_{c_\mu}$ condition. Since $(u_j - u)$ is bounded in $H_0^1(\Omega)$, by the definition, we have

$$\langle J'_\mu(u_j), u_j - u \rangle = o(1) \quad (3.12)$$

where $o(1) \rightarrow 0$ as $j \rightarrow \infty$. Furthermore, noting u is nonnegative, similarly to (2.12) and (2.13) in section 2, we have

$$\int_\Omega g(u_j)(u_j - u) dx = o(1) \text{ as } j \rightarrow \infty, \quad (3.13)$$

and

$$\int_\Omega (u_j)_+^5 (u_j - u) dx = o(1) \text{ as } j \rightarrow \infty. \quad (3.14)$$

Combining (3.12)-(3.14), we get

$$\left\{ a + b\|u_j\|^2\Phi(u_j) + \frac{b}{2T^2}\|u_j\|^4\psi'\left(\frac{\|u_j\|^2}{T^2}\right) \right\} \int_{\Omega} \nabla u_j \cdot \nabla(u_j - u) dx = o(1) \text{ as } j \rightarrow \infty.$$

Noting (3.2) and the weak convergence of $\{u_j\}$, we have

$$\|u_j\| \rightarrow \|u\| \text{ as } j \rightarrow \infty.$$

This proves the $(PS)_{c_\mu}$ condition for J_μ . Now noting Lemma 3.1, Lemma 3.2, and the $(PS)_{c_\mu}$ conditions for I_μ , the mountain pass theorem concludes the proof. $\square$

Finally we prove Theorem 1.3.

Proof of Theorem 1.3. We first choose $\mu_* \geq 0$ so that

$$c_\mu < \min \left\{ \left(\frac{1}{\theta} - \frac{1}{6} \right) \left(\frac{aS}{2} \right)^{\frac{3}{2}}, 4b \left(\frac{1}{\theta} - \frac{1}{4} \right) T^4 \right\}$$

for all $\mu > \mu_*$. Then by Lemma 3.6, for every $\mu > \mu_*$, there exists a nontrivial critical point $u \in H_0^1(\Omega)$ of J_μ with critical value c_μ . As we see in the first part of this section, it is enough to show that $\|u\| < T$. In fact, since $J_\mu(u) = c_\mu$, we have

$$\begin{aligned} \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4\Phi(u) &= \mu \int_{\Omega} G(x, u) dx + \frac{1}{6} \int_{\Omega} u_+^6 dx + c_\mu \\ &\leq \frac{\mu}{\theta} \int_{\Omega} g(x, u) u dx + \frac{1}{\theta} \int_{\Omega} u_+^6 dx + c_\mu \end{aligned} \quad (3.15)$$

by (g5)'. On the other hand, since u is a critical point of J_μ , we obtain

$$\begin{aligned} \frac{1}{\theta} \left\{ a + b\|u\|^2\Phi(u) + \frac{b}{2T^2}\|u\|^4\psi'\left(\frac{\|u\|^2}{T^2}\right) \right\} \|u\|^2 \\ = \frac{\mu}{\theta} \int_{\Omega} g(x, u) u dx + \frac{1}{\theta} \int_{\Omega} u_+^6 dx. \end{aligned} \quad (3.16)$$

Substituting (3.16) into (3.15), we compute

$$\begin{aligned} a \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u\|^2 &\leq b \left(\frac{1}{\theta} - \frac{1}{4} \right) \|u\|^4\Phi(u) + \frac{b}{2\theta T^2} \|u\|^6\psi'\left(\frac{\|u\|^2}{T^2}\right) + c_\mu \\ &\leq 4b \left(\frac{1}{\theta} - \frac{1}{4} \right) T^4 + c_\mu, \end{aligned} \quad (3.17)$$

here we use the fact that $\|u\|^4\Phi(u) \leq 4T^4$ and $\psi' \leq 0$ on $[0, \infty)$. Now we assume $\|u\| \geq T$ to the contrary. Then we have

$$a \left(\frac{1}{2} - \frac{1}{\theta} \right) T^2 \leq 4b \left(\frac{1}{\theta} - \frac{1}{4} \right) T^4 + c_\mu.$$

But from our choice of $T = T(a, b, \theta) > 0$ and $\mu > \mu_*$, recalling (3.4), we have

$$4b \left(\frac{1}{\theta} - \frac{1}{4} \right) T^4 \leq c_\mu < 4b \left(\frac{1}{\theta} - \frac{1}{4} \right) T^4.$$

This is a contradiction. Hence $\|u\| < T$ and thus, u is a critical point of I_μ . As in the proof of Theorem 1.1, we can ensure the smoothness and the positivity of u . This finishes the proof. $\square$

Proof of Corollary 1.4. Considering $1 < q \leq 3$ and $g(x, u) := (u_+)^q$, we can prove Corollary 1.4 by Theorem 1.3. $\square$

4. Additional results

In Theorem 1.3 and Corollary 1.4, we do not consider the case $a = 0$, $b > 0$. They also do not consider such a case in [13]. Thus we naturally ask about the existence of solutions for such situation. As we shall discuss below, the situation seems to be different from the case $a > 0$ and $b \geq 0$. We have partially but positive result for the existence. For our additional results, we consider the following problem. Let $1 < q \leq 3$ and consider

$$\begin{cases} -\|u\|^2 \Delta u = \mu u^q + u^5 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (\text{K2})$$

Firstly we can prove the existence of minimal solution of (K2) when $1 < q < 3$ (and also get the same conclusion when $0 < q \leq 1$). We give the following theorem.

Theorem 4.1. *Let $1 < q < 3$. Then there exists a constant $m_0 > 0$ such that (K2) has a positive (minimal) solution for all $0 < \mu < m_0$.*

We can prove Theorem 4.1 by a few modification of the argument in [32]. Thus we leave the proof for readers. How about the case $q = 4$? For this case, we define

$$\mu_1 = \inf \left\{ \|u\|^4 \mid \int_{\Omega} u^4 dx = 1 \right\}.$$

We can easily check that μ_1 is strictly positive and achieved by some function $\phi_1 > 0$ in Ω . Then it is natural to consider the Kirchhoff type nonlinear eigenvalue problem:

$$\begin{cases} -\|\phi\|^2 \Delta \phi = \mu \phi^3 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \quad (\text{E})$$

In [27], they prove the existence of unbounded sequences of eigenvalues of (E). Some additional results for the problem (E) is obtained in [19]. Using the principal eigenvalue of (E) i.e. $\mu_1 = \mu_1(\Omega) > 0$, we state the following theorem.

Theorem 4.2. *We assume $0 \in \Omega$ and there exists a constant $R > 0$ such that $B(0, R) \subset \Omega$. Then if $3\pi^3/4R < \mu < \mu_1(\Omega)$, there exists a solution of (K2).*

Remark 4.3. Since we do not know the precise lower bound for the value $\mu_1 = \mu_1(\Omega) > 0$ (but we can easily confirm that at least $\mu_1(\Omega) \geq \{(3\pi)^7/4^5\}^{1/3}/R$ if $\Omega = B(0, R)$), the set $\{\mu \in \mathbb{R} \mid 3\pi^3/4R < \mu < \mu_1(\Omega)\}$ may be empty. Furthermore the necessity of the condition on $\mu > 0$ seems to be difficult to prove. These are left for our future works.

In this section, we prove Theorem 4.2. As a matter of the fact, for this case we can employ the minimizing argument as in [8]. Define

$$K := \inf \left\{ \|u\|^4 \mid u \in H_0^1(\Omega), \int_B u^6 dx = 1 \right\}$$

and

$$K_\mu := \inf \left\{ \|u\|^4 - \mu \int_B u^4 dx \mid u \in H_0^1(\Omega), \int_B u^6 dx = 1 \right\}.$$

Notice that $K = S^2$ where S is defined as in the previous sections. Following the argument in [8], we shall show the existence of a minimizer of K_μ .

Lemma 4.4. *If $K_\mu < S^2$, K_μ has a nontrivial and nonnegative minimizer.*

Proof. We assume $K_\mu < S^2$. Let $\{u_j\}$ be a minimizing sequence of K_μ . Then there holds $\int_B u_j^6 dx = 1$ and

$$\|u_j\|^4 - \mu \int_\Omega u_j^4 = K_\mu + o(1) \quad (4.1)$$

where $o(1) \rightarrow 0$ as $j \rightarrow \infty$. Then from the embedding $L^6(\Omega) \hookrightarrow L^4(\Omega)$, we have that $\{u_j\}$ is bounded in $H_0^1(\Omega)$. Hence by the weak compactness of $H_0^1(\Omega)$ and the compactness of the Sobolev embeddings, we get

$$\begin{aligned} u_j &\rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ u_j &\rightarrow u \text{ in } L^p(\Omega) \text{ for all } 1 \leq p < 6, \\ u_j &\rightarrow u \text{ a.e. on } \Omega, \end{aligned}$$

up to subsequences. Notice that $\|u_j\|^4 \geq S^2$ by the definition of S . Then from (4.1), $L^4(\Omega)$ convergence of $\{u_j\}$ and our assumption, we have

$$\int_\Omega u^4 dx \geq \frac{1}{\mu} (S^2 - K_\mu) > 0.$$

Hence u is nontrivial. Now we put $v_j := u_j - u$. Then by the weak convergence, we have

$$\|u_j\|^4 = \|v_j\|^4 + \|u\|^4 + 2\|v_j\|^2\|u\|^2 + o(1) \text{ as } j \rightarrow \infty. \quad (4.2)$$

In addition by the Vitali convergence theorem,

$$1 = \int_\Omega u_j^6 dx = \int_\Omega v_j^6 dx + \int_\Omega u^6 dx + o(1) \text{ as } j \rightarrow \infty.$$

Thus we have by the definition of S ,

$$\begin{aligned}
1 &= \int_{\Omega} v_j^6 dx + \int_{\Omega} u^6 dx + o(1) \\
&\leq \left(\int_{\Omega} v_j^6 dx \right)^{\frac{2}{3}} + \left(\int_{\Omega} u^6 dx \right)^{\frac{2}{3}} + o(1) \\
&\leq S^{-2} \|v_j\|^4 + \left(\int_{\Omega} u^6 dx \right)^{\frac{2}{3}} + o(1),
\end{aligned} \tag{4.3}$$

as $j \rightarrow \infty$. Now we claim

$$\|u\|^4 - \mu \int_{\Omega} u^4 dx \leq K_{\mu} \left(\int_{\Omega} u^6 dx \right)^{\frac{2}{3}}. \tag{4.4}$$

In fact, let us first assume $K_{\mu} > 0$. Then we have by (4.3) that

$$K_{\mu} \leq \frac{K_{\mu}}{S^2} \|v_j\|^4 + K_{\mu} \left(\int_{\Omega} u^6 dx \right)^{\frac{2}{3}} + o(1).$$

Combining this inequality and (4.1), and using the relation (4.2), we have (4.4). Next if $K_{\mu} \leq 0$, we have

$$K_{\mu} \leq K_{\mu} \left(\int_{\Omega} u^6 dx \right)^{\frac{2}{3}}.$$

Again combining this inequality and (4.1), and using (4.2), we obtain (4.4). In addition, from above arguments we can also ensure that v_j must converge to 0 in $H_0^1(\Omega)$. Thus we have $\int_{\Omega} u^6 dx = 1$. From this fact and (4.4), we conclude that u is a nontrivial minimizer of K_{μ} . Furthermore since we can trivially replace u by $|u|$, we get the conclusion of Lemma (4.4). $\square$

Let us ensure the hypothesis of Lemma 4.4. We assume $0 \in \Omega$ and there exists a constant $R > 0$ such that $B(0, R) \subset \Omega$. We define a cut off Talenti function u_{ε} in Ω similarly to the one in Section 2 so that

$$u_{\varepsilon}(x) := \frac{\varepsilon^{\frac{1}{2}} \sigma(|x|)}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}},$$

here $\sigma(r)$ is an appropriate cut off function such that $\sigma(0) = 1$, $\sigma'(0) = \sigma(R) = 0$ and $\sigma(r) = 0$ if $r > R$. Then we have $u_{\varepsilon} \in H_0^1(\Omega)$. We estimate as in [8],

$$\begin{cases} \int_{\Omega} |\nabla u_{\varepsilon}|^2 dx = K_1 + 4\pi\varepsilon \int_0^R |\sigma'(r)|^2 dr + O(\varepsilon^2), \\ \int_{\Omega} u_{\varepsilon}^6 dx = K_2^3 + O(\varepsilon^2), \\ \int_{\Omega} u_{\varepsilon}^4 dx = \pi^2 + O(\varepsilon^2), \end{cases} \tag{4.5}$$

where $K_1 := (3\pi^2)/4$ and $K_2 := (\pi^2/4)^{1/3}$ with $K_1/K_2 = S$. As in section 2, we define

$$v_{\varepsilon}(x) = \frac{u_{\varepsilon}(x)}{(\int_{\Omega} u_{\varepsilon}^6 dx)^{\frac{1}{6}}}.$$

Then we get

$$\begin{cases} \int_{\Omega} |\nabla v_{\varepsilon}|^2 dx = S + \frac{4\pi\varepsilon}{K_2} \int_0^R |\sigma'(r)|^2 dr + O(\varepsilon^2), \\ \int_{\Omega} v_{\varepsilon}^6 dx = 1, \\ \int_{\Omega} v_{\varepsilon}^4 dx = \frac{\pi^2}{K_2^2} \varepsilon + O(\varepsilon^2). \end{cases} \quad (4.6)$$

We prove the next lemma.

Lemma 4.5. *If $\mu > 3\pi^3/4R$, $K_{\mu} < S^2$.*

Proof. Using (4.6), we estimate

$$\begin{aligned} K_{\mu} &\leq \|v_{\varepsilon}\|^4 - \mu \int_{\Omega} v_{\varepsilon}^4 dx \\ &\leq S^2 - \frac{1}{K_2^2} \left(\pi^2 \mu - 8\pi K_1 \int_0^R |\sigma'(r)|^2 dr \right) \varepsilon + O(\varepsilon^2). \end{aligned}$$

We take $\sigma(r) = \cos(\pi r/2R)$. Then $\int_0^1 |\sigma'(r)|^2 dr = \pi^2/8R$. Hence taking $\varepsilon > 0$ sufficiently small, we conclude that if $\mu > 3\pi^3/4R$, $K_{\mu} < S^2$. This gives the conclusion. $\square$

Finally we prove Theorem 4.2.

Proof of Theorem 4.2. First choose $3\pi^3/4R < \mu < \mu_1$. Then $K_{\mu} < S^2$ by Lemma 4.5. Hence by Lemma 4.4, we have a nontrivial minimizer $u \in H_0^1(\Omega)$ of K_{μ} . Thus there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $-\|u\|^2 \Delta u - \mu u^3 = \lambda u^5$ in Ω . We have further, $\lambda = K_{\mu}$. Since $\mu < \mu_1$, $K_{\mu} > 0$ from the definition of $\mu_1 > 0$. Consequently, after an appropriate rescaling, we have the existence of a nontrivial and nonnegative solution of (K2). We can ensure the smoothness and positivity of the solution similarly to the argument in previous sections. This completes the proof. $\square$

Appendix A. Nonexistence results

In Appendix A, we prove the nonexistence results for (K1), that is, we prove Theorem 1.5 and Theorem 1.6. To do that, we introduce a Kirchhoff type Pohozaev identity.

Lemma A.1. *We assume $a, b \geq 0$ and $a + b > 0$ and suppose g satisfies (g1) and (g2). Then if u is a solution of (K1), the following identity holds.*

$$\begin{aligned} \frac{(a + b\|u\|^2)}{2} \|u\|^2 - 3\mu \int_{\Omega} G(x, u) dx - \frac{1}{2} \int_{\Omega} u^6 dx \\ + \frac{a + b\|u\|^2}{2} \int_{\partial\Omega} (x \cdot \nu) \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma = 0, \end{aligned} \quad (A.1)$$

where ν and $\partial/\partial \nu$ denote the outer normal vector and the outer normal derivative on $\partial\Omega$ respectively, and further σ is the 2 dimensional surface measure on $\partial\Omega$.

Proof. Fix a solution u of (K1). We define

$$m(\|u\|^2) := (a + b\|u\|^2)^{-1}$$

and

$$f(x, u) := \mu g(x, u) + u^5.$$

Then u is a solution of

$$\begin{cases} -\Delta u = m(\|u\|^2)f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The Pohozaev type identity for the solutions of the above equation is shown by an usual procedure. We get

$$\frac{1}{2}\|u\|^2 - 3m(\|u\|^2) \int_{\Omega} F(x, u) dx + \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma = 0,$$

where $F(x, u) := \int_0^u f(x, t) dt$. This concludes the proof. $\square$

Next we prove Theorem 1.5.

Proof of Theorem 1.5. Let $a, b \geq 0$, $a + b > 0$, $1 < q < 5$ and $g(x, t) = t^q$. In addition we suppose that $\mu \leq 0$ and Ω is strictly star-shaped. To conclude the proof, we assume that there exists a solution u of (K1) to the contrary. Then we have

$$(a + b\|u\|^2)\|u\|^2 - \mu \int_{\Omega} u^{q+1} dx - \int_{\Omega} u^6 = 0.$$

Combining this relation and (A.1), we get

$$\left(\frac{6}{q+1} - 1 \right) \mu \int_{\Omega} u^{q+1} dx = (a + b\|u\|^2) \int_{\partial\Omega} (x \cdot \nu) \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma.$$

If $\mu < 0$, the right hand side of the above equality is strictly less than 0. This is a contradiction since $x \cdot \nu > 0$ on $\partial\Omega$. Now we assume $\mu = 0$. Then we have from the above equality,

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Therefore we have

$$0 = - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\sigma = - \int_{\Omega} \Delta u dx = \frac{1}{a + b\|u\|^2} \int_{\Omega} (u^q + u^5) dx.$$

But the right hand side of the above equality must be strictly positive. This is a contradiction. Hence there exists no solution for (K1) with $\mu \leq 0$. This is the desired conclusion. $\square$

Lastly we prove Theorem 1.6. We refer to the argument in [7].

Proof of Theorem 1.6. Assume $a > 0$ and $b \geq 0$ and let $1 < q \leq 3$ and $g(x, t) = t^q$. If u is a solution of (K1), there holds

$$(a + b\|u\|^2)\|u\|^2 - \mu \int_{\Omega} u^{q+1} dx - \int_{\Omega} u^6 dx = 0.$$

By this relation and (A.1), we have

$$\mu \int_{\Omega} u^{q+1} dx = C(a + b\|u\|^2) \int_{\partial\Omega} (x \cdot \nu) \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma,$$

where $C > 0$ is some constant depending only on q . Then by the assumption that Ω is strictly star shaped and the embedding $L^2(\partial\Omega) \hookrightarrow L^1(\partial\Omega)$, we have

$$\begin{aligned} \mu \int_{\Omega} u^{q+1} dx &\geq C(a + b\|u\|^2) \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma \\ &\geq C(a + b\|u\|^2) \left(\int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\sigma \right)^2 \\ &= C(a + b\|u\|^2) \left(\int_{\Omega} |\Delta u| dx \right)^2 \\ &\geq C(a + b\|u\|^2) [u]_{3,w}^2, \end{aligned} \tag{A.2}$$

for some constant $C > 0$ which depends only on q and Ω , where $[u]_{p,w} := \sup_{\lambda > 0} [\lambda |\{u > \lambda\}|^{\frac{1}{p}}]$ denotes the L^p weak norm of u and $|A|$ denotes the Lebesgue measure of the set $A \subset \mathbb{R}^3$. Further here, for the third inequality of the above inequality we use the Green representation formula, the facts that $|x|^{-1}$ belongs to weak L^3 space and the Young's inequality for weak L^3 norm of convolution $|x|^{-1} * |\Delta u|$. On the other hand, since $(a + b\|u\|^2)|\Delta u| \geq u^5$, we have

$$\mu \int_{\Omega} u^{q+1} dx \geq \frac{1}{a + b\|u\|^2} \left(\int_{\Omega} u^5 dx \right)^2. \tag{A.3}$$

Now define

$$r := \frac{6(q+1)}{q+3}.$$

Then clearly $q+1 \leq r$ and $3 < r < 5$. Put $0 < \gamma < 1$ so that

$$\gamma := \frac{3(5-r)}{2r}.$$

Using $r, a, \gamma > 0$, (A.2) and (A.3) we get

$$\begin{aligned} &\left(\int_{\Omega} u^{q+1} dx \right)^{\frac{1}{q+1}} \\ &\leq C \left(\int_{\Omega} u^r dx \right)^{\frac{1}{r}} \\ &\leq C [u]_{3,w}^{\gamma} \left(\int_{\Omega} u^5 dx \right)^{\frac{1-\gamma}{5}} \\ &\leq C \left\{ (a + b\|u\|^2)^{-1} \mu \int_{\Omega} u^{q+1} \right\}^{\frac{\gamma}{2}} \left\{ (a + b\|u\|^2) \mu \int_{\Omega} u^{q+1} \right\}^{\frac{1-\gamma}{10}}, \end{aligned}$$

for some constant $C > 0$ depending only on q and Ω . Since

$$\frac{\gamma}{2} + \frac{1-\gamma}{10} = \frac{1}{q+1},$$

we have

$$(a + b\|u\|^2)^{\frac{\gamma}{2} - \frac{1-\gamma}{10}} \leq C\mu.$$

for some constant $C > 0$ depending only on a, q and Ω . Since

$$\frac{\gamma}{2} - \frac{1-\gamma}{10} > 0,$$

we finally have

$$\mu \geq \mu_0$$

for some constant $\mu_0 > 0$ which depends only on a, q and Ω . This proves Theorem 1.6. $\square$

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