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メタデータ	言語: English 出版者: OCAMI 公開日: 2019-10-03 キーワード (Ja): キーワード (En): 作成者: 宮地, 秀樹, 能城, 敏博 メールアドレス: 所属: Osaka University, Osaka City University
URL	https://ocu-omu.repo.nii.ac.jp/records/2016771

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Citation	OCAMI Preprint Series
Issue Date	2011
Type	Preprint
Textversion	Author
Relation	<p>The following article has been submitted to Proceedings of the American Mathematical Society.</p> <p>This is not the published version. Please cite only the published version. The article has been published in final form at</p> <p>https://doi.org/10.1090/S0002-9939-2014-12140-X .</p>
Is version of	<p>https://doi.org/10.1090/S0002-9939-2014-12140-X .</p>

From: Osaka City University Advanced Mathematical Institute

<http://www.sci.osaka-cu.ac.jp/OCAMI/publication/preprint/preprint.html>

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HIDEKI MIYACHI AND TOSHIHIRO NOGI

ABSTRACT. Let S be a closed Riemann surface of genus $g(\geq 2)$ and set $\dot{S} = S \setminus \{\hat{z}_0\}$. Then we have the composed map $\varphi \circ r$ of a map $r : T(S) \times U \rightarrow F(S)$ and the Bers isomorphism $\varphi : F(S) \rightarrow T(\dot{S})$, where $F(S)$ is the Bers fiber space of S , $T(X)$ is the Teichmüller space of X and U is the upper half-plane.

The purpose of this paper is to show the map $\varphi \circ r : T(S) \times U \rightarrow T(\dot{S})$ has a continuous extension to some subset of the boundary $T(S) \times \partial U$.

1. INTRODUCTION

Let S be a closed Riemann surface of genus $g(\geq 2)$. Consider any pair (R, f) of a closed Riemann surface R of genus g and a quasiconformal map $f : S \rightarrow R$. Two pairs (R_1, f_1) and (R_2, f_2) are said to be *equivalent* if $f_2 \circ f_1^{-1} : R_1 \rightarrow R_2$ is homotopic to a biholomorphic map $h : R_1 \rightarrow R_2$. Let $[R, f]$ be the equivalence class of such a pair (R, f) . We set

$$T(S) = \{[R, f] \mid f : S \rightarrow R : \text{qc}\}$$

and call $T(S)$ the *Teichmüller space* of S .

It is known that S can be represented as U/G where U is the upper half-plane and G is a torsion free Fuchsian group.

Let $L_\infty(U, G)_1$ be the space of measurable function μ on U satisfying

- (1) $\|\mu\|_\infty = \sup_{z \in U} |\mu(z)| < 1$,
- (2) $(\mu \circ g) \frac{\bar{g}'}{g}$ for all $g \in G$.

For any $\mu \in L_\infty(U, G)_1$, there is a unique quasiconformal map w of U onto U satisfying normalization conditions $w(0) = 0, w(1) = 1$ and $w(\infty) = \infty$. Let $Q(G)$ be the set of all normalized quasiconformal map w such that wGw^{-1} is also Fuchsian. We write $w = w_\mu$. Two maps $w_1, w_2 \in Q(G)$ are said to be *equivalent* if $w_1 = w_2$ on the real axis \mathbb{R} . Let $[w]$ be the equivalence class of $w \in Q(G)$. We set

$$T(G) = \{[w] \mid w \in Q(G)\}$$

and call $T(G)$ the *Teichmüller space* of G .

Then we have a canonical bijection

$$(1.1) \quad T(G) \ni [w_\mu] \mapsto [U/G_\mu, f_\mu] \in T(S)$$

where $G_\mu = w_\mu G w_\mu^{-1}$ and f_μ is the map induced by $w_\mu : U \rightarrow U$. Throughout this paper, we always identify $T(G)$ with $T(S)$ via the bijection (1.1).

2010 *Mathematics Subject Classification*. Primary 30F60, 32G15, 20F67.

This work is partially supported by the JSPS Institutional Program for Young Research Overseas Visits "Promoting international young researchers in mathematics and mathematical sciences led by OCAMI".

For any $\mu \in L_\infty(U, G)_1$, there is a unique quasiconformal map w of $\hat{\mathbb{C}}$ with $w(0) = 0, w(1) = 1, w(\infty) = \infty$, such that w satisfies the Beltrami equation $w_{\bar{z}} = \mu w_z$ on U , and is conformal on the lower half-plane L . We write $w = w^\mu$.

The *Bers fiber space* $F(G)$ over $T(G)$ is defined by

$$F(G) = \{([w_\mu], z) \in T(G) \times \hat{\mathbb{C}} \mid [w_\mu] \in T(G), z \in w^\mu(U)\}.$$

Take a point $z_0 \in U$ and denote the set of all points $g(z_0)$, $g \in G$, by A . Let

$$v : U \rightarrow U - A$$

be a holomorphic universal covering map and define

$$\dot{G} = \{h \in \text{Aut } U \mid v \circ h = g \circ v \text{ for some } g \in G\}.$$

We see that $U/\dot{G} = U/G - \{\pi(z_0)\}$, where $\pi : U \rightarrow S = U/G$ is the natural projection. And set $\dot{S} = U/\dot{G}$. By Lemma 6.3 of Bers[1], every point in $F(G)$ is represented as a point $([w_\mu], w^\mu(z_0))$ for some $\mu \in L_\infty(U, G)_1$. For $\mu \in L_\infty(U, G)_1$, we define $\nu \in L_\infty(U, \dot{G})_1$ by

$$\mu(v(z)) \frac{\overline{v'(z)}}{v'(z)} = \nu(z).$$

Hence we have a map $\varphi : F(G) \rightarrow T(\dot{G})$ by

$$([w_\mu], w^\mu(z_0)) \mapsto [w_\nu].$$

Then the important Bers isomorphism theorem (Theorem 9 of [1]) asserts that φ is a biholomorphic bijection map. Moreover we define a map $r : T(G) \times U \rightarrow F(G)$ by

$$([w_\mu], w_\mu(z_0)) \mapsto ([w_\mu], w^\mu(z_0)).$$

By Lemma 6.4 of [1], this map r is a real analytic bijection.

Via the bijection (1.1), the Bers fiber space $F(S)$ over $T(S)$ is defined by

$$F(S) = \{([R_\mu, f_\mu], z) \in T(S) \times \hat{\mathbb{C}} \mid [R_\mu, f_\mu] \in T(S), z \in w^\mu(U)\}.$$

Similarly, we have the isomorphism $F(S) \rightarrow T(\dot{S})$ and the real analytic bijection $T(\dot{S}) \times U \rightarrow F(S)$, and we denote them by the same symbols φ and r , respectively.

The Teichmüller space $T(S)$ can be regarded canonically as a bounded domain of a complex Banach space $B_2(L, G)$ in the following way: let $B_2(L, G)$ consist of all holomorphic functions ϕ defined on L such that

$$\phi(g(z))g'(z)^2 = \phi(z) \text{ for } g \in G \text{ and } z \in L$$

and

$$\|\phi\|_\infty = \sup_{z \in L} |(\text{Im}z)^2 \phi(z)| < \infty.$$

For any $\mu \in L_\infty(U, G)_1$, we denote by ϕ^μ the Schwarzian derivative of w^μ in L , that is,

$$\phi^\mu = \{w^\mu, z\} = \frac{(w^\mu)'''(z)}{(w^\mu)'(z)} - \frac{3}{2} \left(\frac{(w^\mu)''(z)}{(w^\mu)'(z)} \right)^2.$$

If $\mu \in L_\infty(U, G)_1$, then $\phi^\mu \in B_2(L, G)$ and the *Bers embedding* $T(S) \ni [R_\mu, f_\mu] \mapsto \phi^\mu \in B_2(L, G)$ is a biholomorphic bijection of $T(S)$ onto a holomorphically bounded domain in $B_2(L, G)$. From now on, we will identify $T(S)$ with its canonical image in $B_2(L, G)$.

Similarly, we define the Bers embedding of $T(\dot{S})$ into $B_2(L, \dot{G})$. Since $F(S)$ is a domain of $B_2(L, G) \times \hat{\mathbb{C}}$ and $T(\dot{S})$ is a bounded domain in $B_2(L, \dot{G})$, we define the topological boundaries of them naturally. Let $\overline{F(G)}$ denote the closure of $F(G)$.

Zhang [13] proved the Bers isomorphism φ cannot be continuously extended to $\overline{F(S)}$ if the dimension of $T(S)$ is greater than zero. Then we have the following question: is there a subset of $\overline{F(S)} - F(S)$ to which φ can be continuously extended?

To consider this question, we will use results of Leininger, Mj and Schleimer about the curve complexes of S and of \dot{S} in [7]. To do this, first we compose the isomorphism $\varphi : F(S) \rightarrow T(\dot{S})$ and the map $r : T(S) \times U \rightarrow F(S)$, then we obtain new map $\varphi \circ r : T(S) \times U \rightarrow T(\dot{S})$.

On the other hand, Leininger, Mj and Schleimer defined a map $\Phi : \mathcal{C}(S) \times U \rightarrow \mathcal{C}(\dot{S})$, where $\mathcal{C}(S)$ and $\mathcal{C}(\dot{S})$ are the curve complexes of S and of \dot{S} , respectively. (For definitions and more details, see §3). Let \mathbb{A} be a subset of ∂U consisting of all points filling S . Then they proved that the map $\Phi(v, \cdot)$ can be continuously extended to $\{v\} \times \mathbb{A}$ for any $v \in \mathcal{C}(S)$.

To use their results, we define a map $\mathcal{E} : T(S) \rightarrow \mathcal{C}(S)$ by sending p to a simple closed curve on S of the minimal extremal length Ext_p (similarly, define $\dot{\mathcal{E}} : T(\dot{S}) \rightarrow \mathcal{C}(\dot{S})$) then we consider the following diagram

$$\begin{array}{ccc} T(S) \times U & \xrightarrow{\varphi \circ r} & T(\dot{S}) \\ \mathcal{E} \times id \downarrow & & \dot{\mathcal{E}} \downarrow \\ \mathcal{C}(S) \times U & \xrightarrow{\Phi} & \mathcal{C}(\dot{S}) \end{array}$$

Our main theorem is as follows:

Theorem 4.1 *The map $\varphi \circ r : T(S) \times U \rightarrow T(\dot{S})$ has a limit in $\{p_0\} \times \mathbb{A}$ for any point $p_0 \in T(S)$.*

2. GROMOV-HYPERBOLIC SPACES

In this section, we shall give the boundary at infinity of hyperbolic space. For details, see Klarreich [6].

Let (Δ, d) be a metric space. If Δ is equipped with a basepoint 0, we define the *Gromov product* $\langle x|y \rangle$ of points x and y in Δ by

$$\langle x|y \rangle = \langle x|y \rangle_0 = \frac{1}{2} \{d(x, 0) + d(y, 0) - d(x, y)\}.$$

For $\delta \geq 0$, the metric space Δ is said to be δ -hyperbolic if

$$\langle x|y \rangle \geq \min\{\langle x|z \rangle, \langle y|z \rangle\} - \delta$$

holds for every $x, y, z \in \Delta$ and for every choice of basepoint. We say that Δ is *hyperbolic in the sense of Gromov* if Δ is δ -hyperbolic for some $\delta \geq 0$.

If Δ is a hyperbolic space, we can define a boundary of Δ in the following way: We say that a sequence $\{x_n\}_{n=1}^{\infty}$ of points in Δ *converges at infinity* if it satisfies $\lim_{m, n \rightarrow \infty} \langle x_m|x_n \rangle = \infty$. Given two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ that converge at infinity, they are called to be *equivalent* if $\lim_{m, n \rightarrow \infty} \langle x_m|y_n \rangle = \infty$. Since Δ is a hyperbolic, we see that this is an equivalence relation (\sim). We set

$$\partial_{\infty} \Delta = \{\{x_n\}_{n=1}^{\infty} \mid \{x_n\}_{n=1}^{\infty} \text{ converges at infinity}\} / \sim$$

and call $\partial_\infty\Delta$ the *boundary at infinity* of Δ . If $\xi \in \partial_\infty\Delta$, then we say that a sequence of points in Δ *converges to* ξ if the sequence belongs to the equivalence class ξ . We set

$$\bar{\Delta} = \Delta \cup \partial_\infty\Delta.$$

3. LEININGER, MJ AND SCHLEIMER'S WORK

3.1. Curve Complex. Let $S = U/G$ be a closed Riemann surface of genus $g(\geq 2)$ and $\pi : U \rightarrow S$ be the natural projection. We take a point z_0 in U and set $\hat{z}_0 = \pi(z_0)$. Put $\dot{S} = S \setminus \{\hat{z}_0\}$.

We begin to define the curve complex $\mathcal{C}(S)$ of S in the following way: the vertices of $\mathcal{C}(S)$ are homotopy classes of non-peripheral simple closed curves on S . Two curves are connected by an edge if they can be realized disjointly on S , and in general a collection of curves spans a simplex if the curves can be realized disjointly on S . Similarly, we may define $\mathcal{C}(\dot{S})$.

We turn $\mathcal{C}(S)$ (resp $\mathcal{C}(\dot{S})$) into a metric space by specifying that each edge has length 1, and define the distance $d_{\mathcal{C}(S)}$ (resp $d_{\mathcal{C}(\dot{S})}$) by taking shortest paths.

Theorem 3.1 (Masur and Minsky [9], Theorem 1.1). *The spaces $\mathcal{C}(S)$ and $\mathcal{C}(\dot{S})$ are δ -hyperbolic for some $\delta > 0$.*

We put $\bar{\mathcal{C}}(S) = \mathcal{C}(S) \cup \partial_\infty\mathcal{C}(S)$ and $\bar{\mathcal{C}}(\dot{S}) = \mathcal{C}(\dot{S}) \cup \partial_\infty\mathcal{C}(\dot{S})$, respectively.

3.2. Definition of Φ . Denote by $\text{Diff}^+(S)$ the group of all orientation preserving diffeomorphisms of S onto itself. Let $\text{Diff}_0(S)$ be a group which consists of all elements in $\text{Diff}^+(S)$ isotopic to the identity map *id*.

We define the evaluation map

$$\text{ev} : \text{Diff}^+(S) \rightarrow S$$

by $\text{ev}(f) = f(\hat{z}_0)$. A theorem of Earle and Eells asserts that $\text{Diff}_0(S)$ is contractible. Hence, for the map $\text{ev}|_{\text{Diff}_0(S)}$, there is a unique lift

$$\tilde{\text{ev}} : \text{Diff}_0(S) \rightarrow U$$

under the condition that $\tilde{\text{ev}}(\text{id}) = z_0$.

Next, we will define a map $\tilde{\Phi} : \mathcal{C}(S) \times \text{Diff}_0(S) \rightarrow \mathcal{C}(\dot{S})$. To give an idea of the definition of $\tilde{\Phi}$, we consider the case of $\mathcal{C}^0(S) \times \text{Diff}_0(S)$. Take a point $(v, f) \in \mathcal{C}^0(S) \times \text{Diff}_0(S)$. Then there is an isotopy f_t , $t \in [0, 1]$, between $f_0 = \text{id}$ and $f_1 = f$. Setting $C(t) = f_t(\hat{z}_0)$ for every $t \in [0, 1]$, we have a path C from \hat{z}_0 to $f(\hat{z}_0)$ on S . Move a point in S from $f(\hat{z}_0)$ to \hat{z}_0 along C and drag v back along the moving point. Then we obtain new simple closed curve on \dot{S} and denote the curve by $f^{-1}(v)$. Thus we define $\tilde{\Phi}(v, f) = f^{-1}(v)$.

However, when $f(\hat{z}_0) \in v$, we can not define $\tilde{\Phi}(v, f)$ as above. We solve this problem in the following way: Now choose $\{\epsilon(v)\}_{v \in \mathcal{C}^0(S)} \subset \mathbb{R}_{>0}$ so that the $\epsilon(v)$ -neighborhood $N(v) = N_{\epsilon(v)}$ of v has the following properties:

- (i) $N(v)$ is homeomorphic to $S^1 \times [0, 1]$
- (ii) $N(v_1) \cap N(v_2) = \emptyset$ if $v_1 \cap v_2 = \emptyset$.

Let $N^\circ(v)$ be the interior of $N(v)$ and v^\pm the boundary components of $N(v)$. For instance, we may take $\epsilon(v)$ as the half of the width of the collar neighborhood of

the geodesic representative of v . Notice that $\epsilon(v)$ is depending only on the length of the geodesic representative of v (cf. [4]).

If $v \subset \mathcal{C}(S)$ is a simplex with vertices $\{v_0, v_1, \dots, v_k\}$, then we consider the barycentric coordinates for points in v :

$$\left\{ \sum_{j=0}^k s_j v_j \mid \sum_{j=0}^k s_j = 1 \text{ and } s_j \geq 0, \text{ for } j = 0, 1, \dots, k \right\}$$

For a point (v, f) with v a vertex of $\mathcal{C}(S)$, we can define $\tilde{\Phi}$ in the following way: If $f(\hat{z}_0) \notin N^\circ(v)$, then we define

$$\tilde{\Phi}(v, f) = f^{-1}(v)$$

as above.

If $f(\hat{z}_0) \in N^\circ(v)$, then $f^{-1}(v^+)$ and $f^{-1}(v^-)$ are not isotopic in \dot{S} . We set

$$t = \frac{d(v^+, f(\hat{z}_0))}{2\epsilon(v)},$$

where $d(v^+, f(\hat{z}_0))$ is the distance inside $N(v)$ from $f(\hat{z}_0)$ to v^+ . Then we define

$$\tilde{\Phi}(v, f) = t f^{-1}(v^+) + (1-t) f^{-1}(v^-)$$

in barycentric coordinates on the edge $[f^{-1}(v^+), f^{-1}(v^-)]$.

In general, for a point $(x, f) \in \mathcal{C}(S) \times \text{Diff}_0(S)$ with $x = \sum_{j=0}^k s_j v_j$, we define $\tilde{\Phi}(x, f)$ as follows: If $f(\hat{z}_0) \notin \bigcup_{j=0}^k N^\circ(v_j)$, then we define

$$\tilde{\Phi}(x, f) = \sum_j s_j f^{-1}(v_j).$$

If $f(\hat{z}_0) \in N^\circ(v_i)$ for exactly one i , we set

$$t = \frac{d(v^+, f(\hat{z}_0))}{2\epsilon(v_i)},$$

and define

$$\tilde{\Phi}(x, f) = s_i (t f^{-1}(v_i^+) + (1-t) f^{-1}(v_i^-)) + \sum_{j \neq i} s_j f^{-1}(v_j).$$

Finally, by Proposition 2.2 in [7], if $\tilde{e}\tilde{v}(f_1) = \tilde{e}\tilde{v}(f_2)$ in U , then we see that $\tilde{\Phi}(x, f_1) = \tilde{\Phi}(x, f_2)$. From this, we have a map $\Phi : \mathcal{C}(S) \times U \rightarrow \mathcal{C}(\dot{S})$ satisfying $\tilde{\Phi} = \Phi \circ (id \times \tilde{e}\tilde{v})$.

3.3. Properties of Φ . A subsurface of S is said to be an *essential* if it is either a component of the complement of a geodesic multicurve in S , the annular neighborhood $N(v)$ of some geodesic $v \in \mathcal{C}^0(S)$, or else S .

If a point $x \in \partial U$ has the following properties,

- (i) for every geodesic ray $r \subset U$ ending at x and for every $v \in \mathcal{C}^0(S)$ which nontrivially intersects an essential subsurface Y , we have $\pi(r) \cap v \neq \emptyset$ and
- (ii) there is a geodesic ray $r \subset U$ ending at x such that $\pi(r) \subset Y$,

we call such a point x a *filling point* for Y (or simply, x *fills* Y). We set

$$\mathbb{A} = \{x \in \partial U \mid x \text{ fills } S\}.$$

Next, we take a geodesic ℓ in U whose projection $\pi(\ell)$ is a non-simple closed geodesic. Let $\{\ell_n\}_{n=1}^{\infty}$ be a set of all pairwise distinct $\pi_1(S)$ -translates of ℓ such that

$$H(\ell_1) \supset H(\ell_2) \supset \cdots,$$

where $H(\ell_k)$ is the half space bounded by ℓ_k . We denote the closure of $H(\ell_k)$ in $U \cup \partial U$ by $\overline{H(\ell_k)}$. Since ℓ are all distinct and $\pi_1(S)$ acts properly discontinuously on U , we see that

$$\bigcap_{n=1}^{\infty} \overline{H(\ell_n)} = \{x\}$$

for some $x \in \partial U$.

We have the following results.

Proposition 3.1 ([7], Proposition 3.4). *If $\{\ell_n\}_{n=1}^{\infty}$ is a sequence nesting down to a point $x \in \mathbb{A}$, then for any choice of basepoint $u_0 \in \mathcal{C}(\dot{S})$,*

$$d_{\mathcal{C}(\dot{S})}(\Phi(\mathcal{C}(S) \times H(\ell_n)), u_0) \rightarrow \infty$$

as $n \rightarrow \infty$.

Theorem 3.2 ([7], Theorem 3.5). *For any $v \in \mathcal{C}(S)$, the map*

$$\Phi(v, \cdot) : U \rightarrow \mathcal{C}(\dot{S})$$

can be continuously extended to

$$\overline{\Phi}(v, \cdot) : U \cup \mathbb{A} \rightarrow \overline{\mathcal{C}(\dot{S})}.$$

4. MAIN THEOREM

Let α be a nontrivial simple closed curve on a Riemann surface R . Denote by $\text{Mod}(A)$ the modulus of an annulus in R whose core curve is homotopic in R to α . We define the extremal length $\text{Ext}(\alpha)$ of α on R by

$$\text{Ext}_R(\alpha) = \inf_A 1/\text{Mod}(A),$$

where the infimum is over all annuli $A \subset R$ whose core curve is homotopic in R to α .

Given any point $p = (R, f) \in T(S)$ and a nontrivial simple closed curve γ on S , we define the extremal length $\text{Ext}_p(\gamma)$ by

$$\text{Ext}_p(\gamma) = \text{Ext}_R(f(\gamma)).$$

Then there is a natural map $\mathcal{E} : T(S) \rightarrow \mathcal{C}(S)$ which sends any $p \in T(S)$ to an element of $\mathcal{C}^0(S)$ of minimal Ext_p . Similarly, we define a map $\dot{\mathcal{E}} : T(\dot{S}) \rightarrow \mathcal{C}(\dot{S})$.

By virtue of Bers' theorem and Maskit's comparizon theorem, there is a constant E_0 depending only on the topology of S such that

$$(4.1) \quad \text{Ext}_{p_0}(\mathcal{E}(p_0)) \leq E_0$$

([2] and [8]). Henceforth, we fix such E_0 and we may suppose that such E_0 is available for simple closed curves on both S and \dot{S} .

Theorem 4.1. *The map $\varphi \circ r : T(S) \times U \rightarrow T(\dot{S})$ has a limit in $\{p_0\} \times \mathbb{A}$ for any point $p_0 \in T(S)$.*

Proof.

We may assume that p_0 is the base point (S, id) of $T(S)$. Let $\{(p_m, z_m)\}_{m=1}^\infty$ be any sequence in $T(S) \times U$ converging to $(p_0, z_\infty) \in T(S) \times \mathbb{A}$. We set $(\xi_m, z_m) = (\mathcal{E} \times id)(p_m, z_m)$ and $q_m = \varphi \circ r(p_m, z_m)$. Moreover, put

$$\delta_m = \Phi(\xi_m, z_m)$$

and $\gamma_m = \dot{\mathcal{E}}(q_m)$.

By filling at the puncture \widehat{z}_0 of \dot{S} , for each m there is an element $\gamma_{0,m} \in \mathcal{C}(S)$ such that

$$\gamma_m = \Phi(\gamma_{0,m}, z_m).$$

We first check the following lemma.

Lemma 4.1. $\lim_{m \rightarrow \infty} \delta_m = \lim_{n \rightarrow \infty} \gamma_n$ in $\partial_\infty \mathcal{C}(\dot{S})$, that is,

$$(4.2) \quad \lim_{m, n \rightarrow \infty} \langle \delta_m | \gamma_n \rangle_0 = \infty.$$

Proof. To show this, we begin with the following two claims.

Claim 1. $d_{\mathcal{C}(\dot{S})}(\delta_m, 0) \rightarrow \infty$ and $d_{\mathcal{C}(\dot{S})}(\gamma_m, 0) \rightarrow \infty$ as $m \rightarrow \infty$.

Proof of Claim 1. Let $\{\ell_n\}_{n=1}^\infty$ be a sequence nesting down to the point $z_\infty \in \mathbb{A}$. Then there is a sequence of half spaces $\{H(\ell_n)\}_{n=1}^\infty$ having following properties

$$H(\ell_1) \supset H(\ell_2) \supset \dots$$

and

$$\bigcap_{n=1}^\infty \overline{H(\ell_n)} = \{z_\infty\}.$$

For a sufficiently large number N_0 , there is a number n_0 such that z_m ($m = n_0, n_0 + 1, n_0 + 2, \dots$) are all contained in $H(\ell_{N_0})$. For each m , there is a number N_m such that z_m is contained in $H(\ell_{N_m})$ but not in $H(\ell_{N_m+1})$. From $\delta_m = \Phi(\xi_m, z_m)$ and $\gamma_m = \Phi(\gamma_{0,m}, z_m)$, we see

$$\begin{aligned} \delta_m &\in \Phi(\mathcal{C}(S) \times H(\ell_{N_m})), \\ \gamma_m &\in \Phi(\mathcal{C}(S) \times H(\ell_{N_m})). \end{aligned}$$

Since Theorem 3.1 shows that

$$d_{\mathcal{C}(\dot{S})}(\Phi(\mathcal{C}(S) \times H(\ell_m)), 0) \rightarrow \infty \quad (m \rightarrow \infty),$$

we have $d_{\mathcal{C}(\dot{S})}(\delta_m, 0) \rightarrow \infty$ and $d_{\mathcal{C}(\dot{S})}(\gamma_m, 0) \rightarrow \infty$ as $m \rightarrow \infty$, as desired.

Claim 2. $d_{\mathcal{C}(\dot{S})}(\delta_m, \gamma_m) = O(1)$ as $m \rightarrow \infty$.

Proof of Claim 2. To clarify the argument, we first assume that $p_m = p_0$ for all m .

Take $f_m \in \text{Diff}_0(S)$ with $(id \times \tilde{v})(\xi, f_m) = (\xi, z_m)$. Let $N(\xi)$ as §3.2. Since $\xi = \mathcal{E}(p_0)$ and (4.1), we have

$$(4.3) \quad \text{Mod}(N(\xi)) \geq 1/E_1$$

where $E_1 > 0$ is a constant depending only on the topology of S .

Suppose first that $\widehat{z}_m = f_m(\widehat{z}_0) \notin N^\circ(\xi)$. Then, by definition, δ_m is homotopic to $f_m^{-1}(\xi)$ on \dot{S} . By the assumption, the interior of the annulus $N(\xi)$ is embedded in $S - \{z_m\}$. Therefore, by (4.3), we have

$$\text{Ext}_{q_m}(\delta_m) \leq 1/\text{Mod}(N(\xi)) \leq E_1.$$

Meanwhile, $\text{Ext}_{q_m}(\gamma_m) \leq E_0$ because $\gamma_m = \dot{\mathcal{E}}(q_m)$. Thus by Minsky and Masur's lemma [9] and Minsky's lemma [10], we get

$$d_{\mathcal{C}(\dot{S})}(\gamma_m, \delta_m) \leq 2i(\gamma_m, \delta_m) + 1 \leq 2(E_1 E_0)^{1/2} + 1,$$

which is what we desired.

Suppose $\hat{z}_m \in N^\circ(\xi)$. Let ξ^* be the core geodesic of $N(\xi)$. Take a conformal (not isometric) coordinates

$$h_m : \xi^* \times [-\epsilon(\xi), \epsilon(\xi)] \rightarrow N(\xi)$$

such that $\xi^* \times \{0\}$ maps to the core geodesic of $N(\xi)$ and for each t , $\xi^* \times \{t\}$ is sent to the equidistant circle to the core geodesic. Let $t_m \in [-\epsilon(\xi), \epsilon(\xi)]$ such that $\hat{z}_m \in h_m(\xi^* \times \{t_m\})$. Then, by definition,

$$\delta_m = \left(1 + \frac{t_m}{2\epsilon(\xi)}\right) f_m^{-1}(\xi^+) + \left(1 - \frac{t_m}{2\epsilon(\xi)}\right) f_m^{-1}(\xi^-)$$

where ξ^\pm is the components of $\partial N(\xi)$. Henceforth, we suppose $t_m > 0$. The case $t_m \geq 0$ can be dealt with the same manner.

Let A_m be the component of $N(\xi) \setminus h_m(\xi^* \times \{t_m\})$ which containing ξ^* . Since h_m is conformal,

$$\text{Mod}(A_m) \geq (\text{Mod}N(\xi))/2.$$

and the core of A_m is homotopic to ξ^- in $S - \{\hat{z}_m\}$. Therefore,

$$\text{Ext}_{q_m}(\xi^-) \leq 2E_1,$$

where we recognize ξ^- as a simple closed curve on $S - \{\hat{z}_m\}$. Therefore, we have

$$\begin{aligned} d_{\mathcal{C}(\dot{S})}(f_m^{-1}(\xi^-), \gamma_m) &\leq 2i(f_m^{-1}(\xi^-), \gamma_m) + 1 \\ &\leq 2\text{Ext}_{q_m}(\xi^-)^{1/2}\text{Ext}_{q_m}(\gamma_m)^{1/2} + 1 \\ &\leq 2\sqrt{2}(E_1 E_0)^{1/2} + 1. \end{aligned}$$

Thus we deduce

$$\begin{aligned} d_{\mathcal{C}(\dot{S})}(\gamma_m, \delta_m) &\leq d_{\mathcal{C}(\dot{S})}(\gamma_m, f_m^{-1}(\xi^-)) + d_{\mathcal{C}(\dot{S})}(f_m^{-1}(\xi^-), \delta_m) \\ &\leq 2\sqrt{2}(E_1 E_0)^{1/2} + 2, \end{aligned}$$

which implies Claim 2 holds when $p_m = p_0$ for all m .

We next deal with the general case. Let S_m be the underlying Riemann surface for p_m . Let $w_m \in Q_{norm}$ be a quasiconformal deformation from p_0 to p_m , and $G_m = w_m G w_m^{-1}$. We let $\hat{z}'_m \in S_m$ be the projection of z_m via the covering projection $\mathbb{H} \rightarrow \mathbb{H}/G_m = S_m$. Let $N_m(\xi_m) \subset S_m$ be the collar neighborhood of the geodesic representative of ξ_m on S_m . Since $\xi_m = \mathcal{E}(p_m)$, the modulus of $N_m(\xi_m)$ is bounded by a constant independent of m . By the same argument as above, we can find an essential subannulus B_m in $N_m(\xi_m) \setminus \{\hat{z}'_m\}$ such that the core of B_m is homotopic to ξ_m on S_m and the modulus of B_m is uniformly bounded above and below.

Let $\eta_m \in \mathcal{C}(\dot{S})$ be the element corresponding to the core of B_m . Since $\gamma_m = \dot{\mathcal{E}}(q_m)$ and the argument above, the extremal lengths of γ_m and η_m on q_m is uniformly bounded above. Therefore, by Minsky's inequality, we have

$$d_{\mathcal{C}(\dot{S})}(\eta_m, \gamma_m) = O(1)$$

for all m . On the other hand, Since η_m is the core of an essential subannulus B_m of $N_m(\xi_m)$, η_m is homotopic to one of the components of $\partial N_m(\xi_m)$ in $S_m - \{z'_m\}$. Hence, by the definition of δ_m , we get

$$d_{\mathcal{C}(\dot{S})}(\eta_m, \delta_m) = O(1).$$

Therefore, we conclude that

$$d_{\mathcal{C}(\dot{S})}(\delta_m, \gamma_m) \leq d_{\mathcal{C}(\dot{S})}(\delta, \eta_m) + d_{\mathcal{C}(\dot{S})}(\eta_m, \delta_m) = O(1),$$

which is what we desired.

We now check that the equation (4.2) holds. From two claims above, we get

$$\lim_{m \rightarrow \infty} \langle \delta_m | \gamma_m \rangle = \infty.$$

Since $\mathcal{C}(\dot{S})$ is δ -hyperbolic,

$$\langle \delta_m | \gamma_n \rangle \geq \min\{\langle \delta_m | \gamma_m \rangle, \langle \gamma_m | \gamma_n \rangle\} - \delta$$

holds. Therefore we conclude $\lim_{m, n \rightarrow \infty} \langle \delta_m | \gamma_n \rangle = \infty$. Namely,

$$(4.4) \quad \lim_{m \rightarrow \infty} \Phi \circ (\mathcal{E} \times id)(p_m, z_m) = \lim_{n \rightarrow \infty} \dot{\mathcal{E}} \circ (\varphi \circ r)(p_n, z_n),$$

holds, which implies Lemma 4.1. ■

We now return to the proof of Theorem 4.1. Since (4.4) holds for any sequence $\{(p_m, z_m)\}_{m=1}^{\infty}$ in $T(S) \times U$ converging to $(p_0, z_\infty) \in T(S) \times \mathbb{A}$, from now we may consider the case of $p_m = p_0$ for every $m \geq 1$. For a sequence $\{(p_0, z_m)\}_{m=1}^{\infty}$ converging to $(p_0, z_\infty) \in \{p_0\} \times \mathbb{A}$, we assume $\{\varphi \circ r(p_0, z_m)\}$ converges to q_∞ . Then by using (4.4), we obtain

$$\begin{aligned} \lim_{m \rightarrow \infty} \dot{\mathcal{E}} \circ (\varphi \circ r)(p_0, z_m) &= \lim_{m \rightarrow \infty} \Phi \circ (\mathcal{E} \times id)(p_0, z_m) \\ &= \lim_{m \rightarrow \infty} \Phi(\xi, z_m), \end{aligned}$$

where $\xi = \mathcal{E}(p_0) \in \mathcal{C}(S)$. Theorem 3.2 shows that there is a γ_∞ in $\partial_\infty \mathcal{C}(\dot{S})$ such that

$$\lim_{m \rightarrow \infty} \Phi(\xi, z_m) = \gamma_\infty.$$

By Klarreich's work of [6], we can identify $\partial_\infty \mathcal{C}(\dot{S})$ with the space of ending lamination $\mathcal{EL}(\dot{S})$. Thus γ_∞ is an ending lamination.

Put $q_m = \varphi \circ r(p_0, z_m)$. We regard $\{q_m\}_{m=1}^{\infty}$ as the sequence in a Bers slice $T(\dot{S}) \times \{q_0\}$. For each pair (q_m, q_0) , there is a unique quasifuchsian group Γ_m up to conjugation such that $\Omega(\Gamma_m)/\Gamma_m = \dot{S}_{q_m} \cup \dot{S}_{q_0}$, where $\Omega(\Gamma_m)$ is the region of discontinuity of Γ_m and the symbol \dot{S}_q means the Riemann surface corresponding to $q \in T(\dot{S})$. Since $\{q_m\}_{m=1}^{\infty}$ converges to q_∞ , by using Ending lamination theorem for surface groups of [3], there is a unique Kleinian group Γ_∞ up to conjugation such that $\{\Gamma_m\}_{m=1}^{\infty}$ converges to Γ_∞ algebraically. This implies that the sequence $\{q_m\}_{m=1}^{\infty}$ converges to q_∞ without depending on the choice of a convergent sequence to (p_0, z_∞) . This shows $\varphi \circ r$ has a limit in $\{p_0\} \times \mathbb{A}$. ■

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