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On the number of maximum points of least energy solutions to a two-dimensional Hénon equation with large exponent

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Abstract. In this note, we prove that least energy solutions of the two-dimensional Hénon equation

$$-\Delta u = |x|^{2\alpha} u^p \quad x \in \Omega, \quad u > 0 \quad x \in \Omega, \quad u = 0 \quad x \in \partial\Omega,$$

where Ω is a smooth bounded domain in \mathbb{R}^2 with $0 \in \Omega$, $\alpha \geq 0$ is a constant and $p > 1$, have only one global maximum point when $\alpha > e - 1$ and the nonlinear exponent p is sufficiently large. This answers positively to a recent conjecture by C. Zhao (preprint, 2011).

Keywords: Hénon equation, global maximum point, large exponent.

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1. Introduction.

In this note we consider the problem

$$\begin{cases} -\Delta u = |x|^{2\alpha} u^p & x \in \Omega, \\ u > 0 & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a smooth bounded domain in \mathbb{R}^2 with $0 \in \Omega$, $\alpha \geq 0$ is a constant and $p > 1$. Since the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact for any $p > 1$, we can obtain at least one solution of (1.1) by a standard variational method. In fact, let us consider the constrained minimization problem

$$S_p = \inf \left\{ \int_{\Omega} |\nabla v|^2 dx \mid v \in H_0^1(\Omega), \int_{\Omega} |x|^{2\alpha} |v|^{p+1} dx = 1 \right\}. \quad (1.2)$$

Standard variational method implies that S_p is achieved by a positive function $v_p \in H_0^1(\Omega)$ and then $u_p = S_p^{1/(p-1)}v_p$ solves (1.1). We call u_p a least energy solution to the problem (1.1).

When $\alpha = 0$, several studies on the asymptotic behavior of least energy solutions u_p as $p \rightarrow \infty$ have been done in [4], [5], [3] and [1]. Recently, Chunyi Zhao [9] extended the study to the case when $\alpha > 0$, and obtained some results. First he showed that for any $\alpha > 0$, there exists $\delta > 0$ such that the least energy solution u_p satisfies $1 - \delta \leq \|u_p\|_\infty \leq \sqrt{e} + \delta$ for p large enough. To state his results further, we introduce some notations: Let x_p be a global maximum point of u_p and define $\varepsilon_p > 0$ by the relation

$$\varepsilon_p^2 |x_p|^{2\alpha} p \|u_p\|_\infty^{p-1} = 1.$$

Also define the function $\tilde{u}_p : \Omega_p = \frac{\Omega - x_p}{\varepsilon_p} \rightarrow \mathbb{R}$ such that

$$\tilde{u}_p(y) = \frac{p}{\|u_p\|_\infty} \{u_p(\varepsilon_p y + x_p) - u_p(x_p)\}.$$

By using these symbols, the main result of C. Zhao reads as follows:

Theorem 1 (*Chunyi Zhao [9]*) *Assume $\alpha > e - 1$. Then $\varepsilon_p \rightarrow 0$ and $\Omega_p \rightarrow \mathbb{R}^2$ as $p \rightarrow \infty$. Also for any sequence $p_n \rightarrow \infty$ as $n \rightarrow \infty$, there exists a subsequence (again denoted by the same symbol) such that*

$$\tilde{u}_{p_n}(y) \rightarrow U(y) := -2 \log \left(1 + \frac{|y|^2}{8} \right) \quad \text{in } C_{loc}^2(\mathbb{R}^2), \quad (1.3)$$

$$|x_{p_n}|^{2+2\alpha} p_n \|u_{p_n}\|_\infty^{p_n-1} \rightarrow \infty, \quad \text{dist}(x_{p_n}, \partial\Omega)^2 p_n \|u_{p_n}\|_\infty^{p_n-1} \rightarrow \infty \quad (1.4)$$

as $n \rightarrow \infty$. Moreover, the least energy solution u_p has at most two global maximum points in Ω for large p .

After obtaining these results, Zhao conjectured that u_p has only one global maximum point when p large in Theorem 1.

Main purpose of this note is to answer the conjecture affirmatively.

Theorem 2 *Under the assumption of Theorem 1, the number of global maximum points of least energy solution u_p is exactly 1 for p large enough.*

For the proof, we will use the Morse index characterization of least energy solutions and an argument of [6]. Relations between the number of blowing-up points and the Morse indices of blowing-up solutions to a two-dimensional Liouville equation have been studied in [7], [8].

2. Proof of Theorem 2.

As in §1, let v_p denote a solution of (1.2), which may be chosen positive. Then v_p solves the equation

$$-\Delta v_p = S_p |x|^{2\alpha} v_p^p \quad x \in \Omega, \quad v_p > 0 \quad x \in \Omega, \quad v_p = 0 \quad x \in \partial\Omega.$$

Let $u_p = S_p^{1/(p-1)} v_p$ be a least energy solution to (1.1). First, we recall the well-known fact, which says that the Morse index of u_p is less than or equal to 1 for any $p > 1$.

Lemma 3 *Let $L_p = -\Delta_x - p|x|^{2\alpha} u_p^{p-1}(x) : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ denote the linearized operator around u_p . Then the second eigenvalue of L_p , denoted by $\lambda_2(L_p, \Omega)$, is nonnegative.*

Proof. When $\alpha = 0$, a proof of this lemma is shown, for example, in [2]. Proof in the case of $\alpha > 0$ is similar. Here we recall it for the sake of completeness.

Let $\bar{L}_p = -\Delta_x - pS_p |x|^{2\alpha} v_p^{p-1}(x) : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ denote the linearized operator around v_p . Since $S_p v_p^{p-1} = u_p^{p-1}$, it is enough to show that the second eigenvalue of \bar{L}_p , denoted by $\lambda_2(\bar{L}_p, \Omega)$, is nonnegative. For this purpose, let us define

$$f(t) = \frac{\int_{\Omega} |\nabla(v_p + t\varphi)|^2 dx}{\left(\int_{\Omega} |x|^{2\alpha} |v_p + t\varphi|^{p+1} dx\right)^{\frac{2}{p+1}}}$$

for any $\varphi \in H_0^1(\Omega)$. By the minimality of v_p , we have $f'(0) = 0$ and $f''(0) \geq 0$. Calculation using $\int_{\Omega} |x|^{2\alpha} v_p^{p+1} dx = 1$ and $\int_{\Omega} |\nabla v_p|^2 dx = S_p$ shows that

$$\begin{aligned} f''(0) &= 2 \left\{ \int_{\Omega} |\nabla \varphi|^2 dx - pS_p \int_{\Omega} |x|^{2\alpha} v_p^{p-1} \varphi^2 dx + (p-1)S_p \left(\int_{\Omega} |x|^{2\alpha} v_p^p \varphi dx \right)^2 \right\} \\ &= 2(\bar{L}_p \varphi, \varphi)_{L^2(\Omega)} + 2(p-1)S_p \left(\int_{\Omega} |x|^{2\alpha} v_p^p \varphi dx \right)^2. \end{aligned}$$

Combining this to a variational characterization of $\lambda_2(\bar{L}_p, \Omega)$, we have

$$\begin{aligned} \lambda_2(\bar{L}_p, \Omega) &= \sup_{L \subset H_0^1(\Omega), \text{codim} L=1} \inf_{\varphi \in L} \frac{(\bar{L}_p \varphi, \varphi)_{L^2(\Omega)}}{\|\varphi\|_{L^2(\Omega)}^2} \\ &\geq \inf_{\varphi \in H_0^1(\Omega), \varphi \perp |x|^{2\alpha} v_p^p} \frac{(\bar{L}_p \varphi, \varphi)_{L^2(\Omega)}}{\|\varphi\|_{L^2(\Omega)}^2} = \inf_{\varphi \in H_0^1(\Omega), \varphi \perp |x|^{2\alpha} v_p^p} \frac{1}{2} \frac{f''(0)}{\|\varphi\|_{L^2(\Omega)}^2} \geq 0. \end{aligned}$$

□

By using this fact, we prove Theorem 2 by a contradiction argument.

Proof of Theorem 2. Assume the contrary that there exist two global maximum points $x_{p_n}^1, x_{p_n}^2$ of u_{p_n} , $x_{p_n}^i \in \Omega$, $\|u_{p_n}\|_\infty = u_{p_n}(x_{p_n}^i)$, $(i = 1, 2)$ for some sequence $p_n \rightarrow \infty$. Define $\varepsilon_{p_n}^i > 0$ by the relation

$$(\varepsilon_{p_n}^i)^2 |x_{p_n}^i|^{2\alpha} p_n \|u_{p_n}\|_\infty^{p_n-1} = 1, \quad (2.1)$$

and the scaled functions

$$\begin{aligned} \tilde{u}_{p_n}^i(y) &= \frac{p_n}{\|u_{p_n}\|_\infty} \{u_{p_n}(\varepsilon_{p_n}^i y + x_{p_n}^i) - u_{p_n}(x_{p_n}^i)\}, \\ y \in \Omega_{p_n}^i &:= \frac{\Omega - x_{p_n}^i}{\varepsilon_{p_n}^i} \end{aligned} \quad (2.2)$$

for $i = 1, 2$. Now we assume $\alpha > e - 1$, so all results of Theorem 1 hold true for $\tilde{u}_{p_n}^i$, $(i = 1, 2)$. In particular, there exists a subsequence (denoted by the same symbol again) such that (1.3) holds for both $\tilde{u}_{p_n}^i$.

Next, we define elliptic operators

$$\tilde{L}_{p_n}^i = -\Delta_y - \left| \frac{\varepsilon_{p_n}^i}{|x_{p_n}^i|} y + \frac{x_{p_n}^i}{|x_{p_n}^i|} \right|^{2\alpha} \left(1 + \frac{\tilde{u}_{p_n}^i}{p_n}(y) \right)^{p_n-1}, \quad (i = 1, 2) \quad (2.3)$$

acting on $H_0^1(\Omega_{p_n}^i)$. These operators are related to the operator L_{p_n} by the formula

$$(\varepsilon_{p_n}^i)^2 L_{p_n} \Big|_{x=\varepsilon_{p_n}^i y + x_{p_n}^i} = \tilde{L}_{p_n}^i, \quad y \in \Omega_{p_n}^i,$$

for $i = 1, 2$. Also, eigenvalues are related with each other by the formula

$$(\varepsilon_{p_n}^i)^2 \lambda_j(L_{p_n}, D) = \lambda_j(\tilde{L}_{p_n}^i, D_{p_n}^i), \quad D_{p_n}^i = \frac{D - x_{p_n}^i}{\varepsilon_{p_n}^i}, \quad (2.4)$$

where $\lambda_j(L_{p_n}, D)$ will denote a j -th eigenvalue of the operator L_{p_n} acting on $H_0^1(D)$ for a domain D , etc.

Let $B(a, R) = B_R(a)$ denote an open ball of center $a \in \mathbb{R}^2$ with radius R . We prove the following:

Lemma 4 *There exist disjoint balls B^i ($i = 1, 2$), each ball is of the form $B(x_{p_n}^i, \varepsilon_{p_n}^i R)$ for some $R > 0$, such that $\lambda_1(L_{p_n}, B^i) < 0$ for $i = 1, 2$ when n sufficiently large.*

Proof. For $R > 0$, we define

$$w_R(y) = 2 \log \frac{8 + R^2}{8 + |y|^2}.$$

Since $w_R = 0$ on $\partial B_R(0)$, we see $w_R \in H_0^1(B_R(0))$.

We will prove that $(\tilde{L}_{p_n}^i w_R, w_R)_{L^2(B_R(0))} < 0$ for $R > 0$ sufficiently large and $B_R(0) \subset \Omega_{p_n}^i$. Indeed,

$$\begin{aligned} (\tilde{L}_{p_n}^i w_R, w_R)_{L^2(B_R(0))} &= \int_{B_R(0)} |\nabla w_R|^2 dy \\ &\quad - \int_{B_R(0)} \left| \frac{\varepsilon_{p_n}^i}{|x_{p_n}^i|} y + \frac{x_{p_n}^i}{|x_{p_n}^i|} \right|^{2\alpha} \left(1 + \frac{\tilde{u}_{p_n}^i}{p_n}(y) \right)^{p_n-1} w_R^2(y) dy \\ &=: I_1 - I_2. \end{aligned}$$

We see

$$I_1 = \int_{B_R(0)} \frac{16|y|^2}{(8 + |y|^2)^2} dy = 2\pi \int_0^R \frac{16r^2}{(8 + r^2)^2} r dr = 32\pi [\log R + o_R(1)],$$

where $o_R(1) \rightarrow 0$ as $R \rightarrow \infty$. As for I_2 , (1.4) implies $\frac{\varepsilon_{p_n}^i}{|x_{p_n}^i|} \rightarrow 0$ as $n \rightarrow \infty$ even if $x_{p_n}^i \rightarrow 0$. Also $\frac{x_{p_n}^i}{|x_{p_n}^i|} \rightarrow \exists y_0, |y_0| = 1$ for a subsequence. Therefore by choosing a subsequence, we have

$$\left| \frac{\varepsilon_{p_n}^i}{|x_{p_n}^i|} y + \frac{x_{p_n}^i}{|x_{p_n}^i|} \right|^{2\alpha} \left(1 + \frac{\tilde{u}_{p_n}^i}{p_n}(y) \right)^{p_n-1} \rightarrow e^{U(y)}$$

in $C_{loc}^2(\mathbb{R}^2)$ by (1.3). Thus,

$$\begin{aligned}
I_2 &= \int_{B_R(0)} \left| \frac{\varepsilon_{p_n}^i}{|x_{p_n}^i|} y + \frac{x_{p_n}^i}{|x_{p_n}^i|} \right|^{2\alpha} \left(1 + \frac{\tilde{u}_{p_n}^i(y)}{p_n} \right)^{p_n-1} w_R^2(y) dy \\
&= \int_{B_R(0)} \frac{1}{\left(1 + \frac{|y|^2}{8}\right)^2} \left\{ \log \frac{8 + R^2}{8 + |y|^2} \right\}^2 dy + o_n(1) \\
&= 2\pi \int_0^R \frac{r}{\left(1 + \frac{r^2}{8}\right)^2} \left\{ \log(8 + R^2) - \log(8 + r^2) \right\}^2 dr + o_n(1) \\
&= 2\pi \cdot 8^2 \left\{ \log(8 + R^2) \right\}^2 \left[\frac{1}{16} + o_R(1) \right] + o_n(1) \\
&= 32\pi (\log R)^2 [1 + o_R(1)] + o_n(1),
\end{aligned}$$

where we have used $\int_0^\infty \frac{r}{(8+r^2)^2} dr = \frac{1}{16}$. Hence we obtain

$$(\tilde{L}_{p_n}^i w_R, w_R)_{L^2(B_R(0))} = I_1 - I_2 = -32\pi (\log R)^2 [1 + o_R(1)] < 0$$

by taking n sufficiently large first, and then $R > 0$ large such that $B_R(0) \subset \Omega_{p_n}^i$. This implies that the first eigenvalue of the operator $\tilde{L}_{p_n}^i$ on B_R is negative: $\lambda_1(\tilde{L}_{p_n}^i, B_R) < 0$. By this and the scaling formula (2.4) proves the first half part of the Lemma.

Recall that, under the assumption considered here, the following estimate is proved in [9] Lemma 5.1:

$$\frac{|x_p^1 - x_p^2|}{\max\{\varepsilon_p^1, \varepsilon_p^2\}} \rightarrow \infty$$

as $p \rightarrow \infty$. This implies that these two balls $B(x_{p_n}^i, \varepsilon_{p_n}^i R)$ are disjoint for n sufficiently large. \square

By Lemma 4, we have

$$\lambda_1(L_{p_n}, B^i) < 0 \quad i = 1, 2 \tag{2.5}$$

for n sufficiently large. On the other hand, a well known estimate of $\lambda_2(L_{p_n}, \Omega)$ claims

$$\lambda_2(L_{p_n}, \Omega) \leq \sum_{i=1}^2 \lambda_1(L_{p_n}, B^i). \tag{2.6}$$

See, for example, [7] Appendix. From (2.5) and (2.6), we have $\lambda_2(L_{p_n}, \Omega) < 0$ for n large, which contradicts to Lemma 3. \square

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