# COHOMOGENEITY ONE SPECIAL LAGRANGIAN SUBMANIFOLDS IN THE COTANGENT BUNDLE OF THE SPHERE

# COHOMOGENEITY ONE SPECIAL LAGRANGIAN SUBMANIFOLDS IN THE COTANGENT BUNDLE OF THE SPHERE

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Abstract. We construct cohomogeneity one special Lagrangian submanifolds in the cotangent bundle of the sphere $S^n$ invariant under $SO(p) \times SO(q)$ $(p + q = n + 1)$. We describe the asymptotic behavior of these special Lagrangian submanifolds.

## 1. Introduction

In 1993, Stenzel [12] constructed cohomogeneity one Ricci-flat Kähler metrics on the cotangent bundles of rank one symmetric spaces of compact type. Anciaux [1] constructed special Lagrangian submanifolds in the cotangent bundle of the sphere, applying the symmetry of the Stenzel metric. Ionel and Min-Oo [7] studied special Lagrangian submanifolds in the deformed and resolved conifolds of dimension 3, using the moment map technique. In this paper, generalizing there results, we study cohomogeneity one special Lagrangian submanifolds in the cotangent bundle of the sphere $S^n$ invariant under $SO(p) \times SO(q)$ $(p + q = n + 1)$. First we construct Lagrangian submanifolds using the moment map technique. Since these Lagrangian submanifolds are cohomogeneity one, the condition for them to be special Lagrangian is reduced to an ODE. We analyse the solution of this ODE, and observe the asymptotic behavior of those special Lagrangian submanifolds. We note that special Lagrangian submanifolds with this kind of symmetry were also studied by Kanemitsu [10] independently.

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## 2. Preliminaries

### 2.1. Calabi-Yau manifolds and special Lagrangian submanifolds.

We shall review some definitions and basic notions of Calabi-Yau manifolds and special Lagrangian submanifolds. See [9] for details.

There are several different definitions of Calabi-Yau manifolds. In this paper, we use the following definition.

**Definition 2.1.** Let $n \geq 2$. An *almost Calabi-Yau* $n$-fold is a quadruple $(M, J, \omega, \Omega)$ such that $(M, J, \omega)$ is a Kähler manifold of complex dimension $n$ with a complex structure $J$ and a Kähler form $\omega$, and $\Omega$ is a nonvanishing holomorphic $(n, 0)$-form on $M$. In addition, if $\omega$ and $\Omega$ satisfy

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \bar{\Omega}, \tag{2.1}$$

then we call $(M, J, \omega, \Omega)$ a *Calabi-Yau* $n$-fold.

If $\omega$ and $\Omega$ satisfy (2.1), then the Kähler metric $g$ of $(M, J, \omega)$ is Ricci-flat and its holonomy group $\mathrm{Hol}(g)$ is a subgroup of $SU(n)$, that is another definition of a Calabi-Yau manifold.

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A closed $p$-form $\varphi$ on a Riemannian manifold $(M, g)$ is called a *calibration* if $\varphi|_V \leq \mathrm{vol}_V$ for any oriented $p$-plane $V \subset T_xM$ for all $x \in M$. A $p$-dimensional submanifold $N$ of $M$ is said to be *calibrated* by a calibration $\varphi$ if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

**Remark 2.2.** The constant factor in (2.1) is chosen so that $\mathrm{Re}(e^{\sqrt{-1}\theta}\Omega)$ is a calibration for any $\theta \in \mathbb{R}$.

**Definition 2.3.** Let $(M, J, \omega, \Omega)$ be a Calabi-Yau $n$-fold and $L$ be a real $n$-dimensional submanifold of $M$. Then, for $\theta \in \mathbb{R}$, $L$ is called a *special Lagrangian submanifold* of phase $\theta$ if it is calibrated by the calibration $\mathrm{Re}(e^{\sqrt{-1}\theta}\Omega)$.

Harvey and Lawson gave the following alternative characterization of special Lagrangian submanifolds.

**Proposition 2.4** ([4])**.** *Let $(M, J, \omega, \Omega)$ be a Calabi-Yau $n$-fold and $L$ be a real $n$-dimensional submanifold of $M$. Then $L$ is a special Lagrangian submanifold of phase $\theta$ if and only if $\omega|_L \equiv 0$ and $\mathrm{Im}(e^{\sqrt{-1}\theta}\Omega)|_L \equiv 0$.*

2.2. **Stenzel metric on the cotangent bundle of the sphere.** In [12], Stenzel constructed complete Ricci-flat Kähler metrics on the cotangent bundles of rank one symmetric spaces of compact type. For our use, here we shall recall the Stenzel metric on the cotangent bundle of the sphere. We denote the cotangent bundle of the $n$-sphere $S^n \cong SO(n+1)/SO(n)$ by

$$T^*S^n = \left\{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1, \langle x, \xi\rangle = 0\right\}.$$

We identify the tangent bundle and cotangent bundle of $S^n$ by the Riemannian metric on $S^n$. Since any unit cotangent vector of $S^n$ can be translated to another one, the Lie group $SO(n+1)$ acts on $T^*S^n$ with cohomogeneity one by $g \cdot (x, \xi) = (gx, g\xi)$ for $g \in SO(n+1)$. Let $Q^n$ be a complex quadric in $\mathbb{C}^{n+1}$ defined by

$$Q^n = \left\{ z = (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} z_i^2 = 1 \right\}.$$

The group $SO(n+1, \mathbb{C})$ acts on $Q^n$ transitively, hence $Q^n \cong SO(n+1, \mathbb{C})/SO(n, \mathbb{C})$. According to Szöke [13], we can identify $T^*S^n$ with $Q^n$ by the following diffeomorphism:

$$\begin{aligned} \Phi : T^*S^n &\longrightarrow Q^n \\ (x, \xi) &\longmapsto x\cosh(\|\xi\|) + \sqrt{-1}\frac{\xi}{\|\xi\|}\sinh(\|\xi\|). \end{aligned}$$

The diffeomorphism $\Phi$ is equivariant under the action of $SO(n+1)$. Thus we frequently identify $T^*S^n$ with $Q^n$. We give the complex structure $J_{Stz}$ on $T^*S^n$ by pulling back the complex structure of $Q^n$ via the map $\Phi$. With respect to this complex structure, Stenzel [12] constructed a complete Ricci-flat Kähler metric on $Q^n$, whose Kähler form is given by

$$\omega_{Stz} = \sqrt{-1}\partial\bar{\partial}u(r^2) = \sqrt{-1}\sum_{i,j=1}^{n+1} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} u(r^2) dz_i \wedge d\bar{z}_j,$$

where $r^2 = \|z\|^2 = \sum_{i=1}^{n+1} z_i\bar{z}_i$ and $u$ is a smooth real function satisfying the following differential equation:

$$\frac{d}{dt}(U'(t))^n = cn(\sinh t)^{n-1} \qquad (c > 0), \tag{2.2}$$

where $U(t) = u(\cosh t)$. In the case of $n = 2$, the Stenzel metric coincides with the hyperkähler metric discovered by Eguchi and Hanson [3].

The Kähler form $\omega_{Stz}$ is exact and $\omega_{Stz} = d\alpha_{Stz}$, where the 1-form $\alpha_{Stz} = -\mathrm{Im}\left(\bar{\partial}u(r^2)\right)$. We give the Liouville form $\alpha_0$ on $\mathbb{C}^{n+1}$ by $\alpha_0(v) = \langle Jz, v\rangle$, where $\langle\ ,\ \rangle$ and $J$ are the standard real inner product and complex structure on $\mathbb{C}^{n+1}$, respectively. Then one can show that $\alpha_{Stz} = u'(r^2)\alpha_0$. Hence $\alpha_{Stz}$ has the following expression:

$$\alpha_{Stz}(v) = u'(r^2)\alpha_0(v) = u'(r^2)\langle Jz, v\rangle \qquad (v \in T_zQ^n, z \in Q^n).$$

From this, $\omega_{Stz}$ can be evaluated as

$$\begin{aligned}\omega_{Stz}(v,w) &= d\alpha_{Stz}(v,w)\\ &= v(\alpha_{Stz}(w)) - w(\alpha_{Stz}(v)) - \alpha_{Stz}([v,w])\\ &= 2u'(r^2)\langle Jv, w\rangle + 2u''(r^2)\Big(\langle z,v\rangle\langle Jz,w\rangle - \langle z,w\rangle\langle Jz,v\rangle\Big)\end{aligned} \tag{2.3}$$

for $v, w \in T_zQ^n$ and $z \in Q^n$.

The holomorphic $(n,0)$-form $\Omega_{Stz}$ on $Q^n$ is given by

$$\frac{1}{2}d(z_1^2 + z_2^2 + \cdots + z_{n+1}^2 - 1) \wedge \Omega_{Stz} = \Omega_0,$$

where $\Omega_0 = dz_1 \wedge \ldots \wedge dz_{n+1}$ is the standard holomorphic $(n+1,0)$-form on $\mathbb{C}^{n+1}$. We can express $\Omega_{Stz}$ as

$$\Omega_{Stz}(v_1,\ldots,v_n) = \Omega_0(z, v_1, \ldots, v_n)$$

or

$$\Omega_{Stz}(v_1,\ldots,v_n) = \frac{1}{\|z\|^2}\Omega_0(\bar{z}, v_1, \ldots, v_n),$$

where $v_1,\ldots,v_n \in T_zQ^n, z \in Q^n$ and $z = z_1\frac{\partial}{\partial z_1} + \cdots + z_{n+1}\frac{\partial}{\partial z_{n+1}}, \bar{z} = \bar{z}_1\frac{\partial}{\partial z_1} + \cdots + \bar{z}_{n+1}\frac{\partial}{\partial z_{n+1}}$.

Clearly the action of $SO(n+1)$ on $Q^n$ preserves $J_{Stz}$, $\omega_{Stz}$ and $\Omega_{Stz}$. Moreover one can show that there exists a constant $\lambda \in \mathbb{R}$ such that

$$\frac{\omega_{Stz}^n}{n!} = (-1)^{\frac{n(n-1)}{2}}\left(\frac{\sqrt{-1}}{2}\right)^n \lambda^2 \Omega_{Stz} \wedge \bar{\Omega}_{Stz}.$$

Hence $(T^*S^n \cong Q^n, J_{Stz}, \omega_{Stz}, \lambda\Omega_{Stz})$ is a cohomogeneity one Calabi-Yau manifold.

2.3. **Moment maps and Lagrangian submanifolds.** Let $(M,\omega)$ be a symplectic manifold, and $G$ be a Lie group acting on $M$. We denote the Lie algebra of $G$ by $\mathfrak{g}$. Let $X^*$ denote the fundamental vector field of $X \in \mathfrak{g}$ on $M$, i.e.,

$$X_x^* = \frac{d}{dt}\Big|_{t=0}\exp(tX)x \qquad (x \in M).$$

Now we suppose that the action of $G$ on $M$ is Hamiltonian with the moment map $\mu : M \to \mathfrak{g}^*$. We define the center of $\mathfrak{g}^*$ to be $Z(\mathfrak{g}^*) = \{X \in \mathfrak{g}^* \mid \mathrm{Ad}^*(g)X = X\ (\forall g \in G)\}$. It is easy to see that the inverse image $\mu^{-1}(c)$ of the moment map $\mu$ for $c \in \mathfrak{g}^*$ is $G$-invariant if and only if $c \in Z(\mathfrak{g}^*)$.

**Proposition 2.5.** *Let $L$ be a connected isotropic submanifold, i.e., $\omega|_L \equiv 0$, of $M$ invariant under the action of $G$. Then $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.*

*Proof.* For $X \in \mathfrak{g}$, we define a function $\mu_X$ on $M$ by $\mu_X(x) = (\mu(x))(X)$. Then, from the definition of the moment map, $\mu_X$ is the Hamiltonian function of $X^*$. Since $L$ is an isotropic submanifold of $M$, we have

$$\mathcal{L}_Y(\mu_X) = d\mu_X(Y) = \omega(X_x^*, Y) = 0$$

for all $X \in \mathfrak{g}, Y \in T_xL$ and $x \in L$. This implies that $\mu_X$ is constant on $L$ for all $X \in \mathfrak{g}$, hence $\mu : M \to \mathfrak{g}^*$ is also constant on $L$. Thus $L \subset \mu^{-1}(c)$ for some $c \in \mathfrak{g}^*$. Moreover, since $L$ is $G$-invariant, we have $c \in Z(\mathfrak{g}^*)$. □

**Proposition 2.6.** *Let $L$ be a connected submanifold of $M$ invariant under the action of $G$. Suppose that the action of $G$ on $L$ is cohomogeneity one (possibly transitive). Then $L$ is an isotropic submanifold, i.e., $\omega|_L \equiv 0$, if and only if $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.*

*Proof.* By Proposition 2.5, we know that $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$ if $L$ is isotropic. So it suffices to prove the converse.

Suppose that $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$. This means that $\mu$ is constant on $L$, so $\mu_X$ is also constant on $L$ for all $X \in \mathfrak{g}$. Therefore

$$\omega(X^*_x, Y) = \mathcal{L}_Y(\mu_X) = 0$$

for all $X \in \mathfrak{g}, Y \in T_xL$ and $x \in L$. Let $x \in L$ be a regular point of the action of $G$ on $L$. It is known that the set of regular points is open dense in $L$. Since the action of $G$ on $L$ is cohomogeneity one, if we take $Y_1 \in T_xL$ which is transverse to the orbit of $G$ at $x$, then $T_xL = \mathrm{span}\{X^*_x, Y_1 \mid X \in \mathfrak{g}\}$. Therefore $\omega|_{T_xL} \equiv 0$. Since $\omega$ vanishes on an open dense subset of $L$, it vanishes on $L$ entirely. Thus $L$ is isotropic. □

## 3. Construction of cohomogeneity one special Lagrangian submanifolds in $T^*S^n$

In this section we shall construct cohomogeneity one special Lagrangian submanifolds in $T^*S^n$ with respect to the Stenzel metric, using the moment map technique. Since the zero-section $S^n$ of $T^*S^n$ is a homogeneous special Lagrangian submanifold in $T^*S^n$, a homogeneous hypersurface in $S^n$ is a $(n-1)$-dimensional isotropic submanifold in $T^*S^n$. Extending it to an $n$-dimensional submanifold in $T^*S^n$, we can construct a cohomogeneity one Lagrangian submanifold. For such a Lagrangian submanifold, the condition to be special Lagrangian can be described by an ordinary differential equation.

Let $G$ be a compact Lie subgroup of $SO(n+1)$ and $\mathfrak{g}$ be its Lie algebra. Then the action of $G$ on $Q^n$ is Hamiltonian, and its moment map $\mu : Q^n \to \mathfrak{g}^*$ is given by

$$(\mu(z))(X) = \mu_X(z) = \alpha_{Stz}(X^*_z) = \alpha_{Stz}(Xz) = u'(r^2)\langle Jz, Xz\rangle \qquad (z \in Q^n, X \in \mathfrak{g}). \tag{3.1}$$

In this paper we shall study special Lagrangian submanifolds especially invariant under

$$G = \left(\begin{array}{c|c} SO(p) & O \\ \hline O & SO(q) \end{array}\right) \cong SO(p) \times SO(q) \quad (p+q = n+1,\ 1 \leq p \leq q \leq n).$$

In this case, $G$-action on $S^n$ is cohomogeneity one, and its principal orbits are diffeomorphic to $S^{p-1} \times S^{q-1}$. Let us take

$$X_{ij} = E_{ji} - E_{ij} \in \mathfrak{so}(n+1),$$

where $E_{ij}$ denotes the $(n+1) \times (n+1)$-matrix whose $(i,j)$-component is 1 and all of others are 0. Then

$$\{X_{ij} \mid 1 \leq i < j \leq p\} \cup \{X_{ij} \mid p+1 \leq i < j \leq n+1\}$$

forms a basis of the Lie algebra $\mathfrak{g} = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$ of $G$. We denote by $\{\theta_{ij}\}$ the dual basis of $\{X_{ij}\}$. Then the moment map $\mu : Q^n \to \mathfrak{g}^*$ of $G$-action on $Q^n$ can be expressed as

$$\mu(z) = \sum_{i,j} \mu_{ij}(z)\theta_{ij},$$

where $\mu_{ij}$ is defined by $\mu_{ij}(z) = \mu_{X_{ij}}(z) = (\mu(z))(X_{ij})$. From (3.1) we have

$$\mu_{ij}(z) = u'(r^2)\langle Jz, X_{ij}z\rangle = 2u'(r^2)\mathrm{Im}(z_i\bar{z}_j).$$

Thus, using the basis $\{\theta_{ij}\}$ of $\mathfrak{g}^*$, the moment map $\mu : Q^n \to \mathfrak{g}^*$ of $G$-action on $Q^n$ can be evaluated as

$$\mu(z) = 2u'(r^2)\Big(\mathrm{Im}(z_i\bar{z}_j)_{1\leq i<j\leq p},\ \mathrm{Im}(z_i\bar{z}_j)_{p+1\leq i<j\leq n+1}\Big).$$

From Proposition 2.6, a Lagrangian submanifold of $Q^n$ invariant under $G$ should be contained in $\mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$. In the case of $p = 2$ or $q = 2$, since $SO(2)$ is abelian, $\mathfrak{g}^*$ has the non-trivial center. In the case of $p = 1$, the orbit space of $G$ action on the zero-section $S^n$ is different from the case of $p \geq 2$. Therefore we shall discuss the following five cases individually.

(1) $3 \leq p \leq q$ (2) $p = 1, q \geq 3$ (3) $p = 2, q \geq 3$ (4) $p = q = 2$ (5) $p = 1, q = 2$

In the case of $p = 1$, we have special Lagrangian submanifolds invariant under $SO(n)$, which were first studied by Anciaux [1]. Ionel and Min-Oo [7] investigated special Lagrangian submanifolds in $Q^3$ invariant under $SO(2) \times SO(2)$ or $SO(3)$. So this paper is a generalization of these two preceding studies. Special Lagrangian submanifolds with this kind of symmetry were also constructed by Kanemitsu [10] independently.

3.1. **Case of $3 \leq p \leq q$.** We give a parametrization of the orbit space of the action of $G = SO(p) \times SO(q)$ on $T^*S^n = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1,\ \langle x, \xi \rangle = 0\}$. First, $x \in S^n$ can be moved to

$$x = (\underbrace{\cos t}_{1}, 0, \ldots, 0, \underbrace{\sin t}_{p+1}, 0, \ldots, 0) \qquad (t \in \mathbb{R})$$

by the action of $G$. Furthermore $\xi \in T^*_x S^n$ can be moved to

$$\xi = (\underbrace{-\xi_1 \sin t}_{1}, \underbrace{\xi_2}_{2}, 0, \ldots, 0, \underbrace{\xi_1 \cos t}_{p+1}, \underbrace{\xi_3}_{p+2}, 0, \ldots, 0) \qquad (\xi_1, \xi_2, \xi_3 \in \mathbb{R})$$

by the isotropy subgroup $G_x$ of $G$-action on $S^n$ at $x$. Therefore we define a subset $\Sigma$ of $T^*S^n$ by

$$\Sigma = \left\{ (x, \xi) \;\middle|\; \begin{array}{l} x = (\cos t, 0, \ldots, 0, \sin t, 0, \ldots, 0) \\ \xi = (-\xi_1 \sin t, \xi_2, 0, \ldots, 0, \xi_1 \cos t, \xi_3, 0, \ldots, 0) \end{array} \right\}.$$

Then each $G$-orbit in $T^*S^n$ meets $\Sigma$, i.e., $G \cdot \Sigma = T^*S^n$.

In this case, the center of $\mathfrak{g}^*$ is $Z(\mathfrak{g}^*) = \{0\}$. We determine the subset $\mu^{-1}(0) \cap \Phi(\Sigma)$ of $Q^n$. Now $z = \Phi(x, \xi) \in \Phi(\Sigma)$ can be expressed as

$$\begin{aligned} z = \Bigg( &\cos t \cosh \rho - \sqrt{-1}\frac{\xi_1 \sin t}{\rho} \sinh \rho, \sqrt{-1}\frac{\xi_2}{\rho} \sinh \rho, 0, \ldots, 0, \\ &\sin t \cosh \rho + \sqrt{-1}\frac{\xi_1 \cos t}{\rho} \sinh \rho, \sqrt{-1}\frac{\xi_3}{\rho} \sinh \rho, 0, \ldots, 0 \Bigg), \end{aligned}$$

where $\rho = \|\xi\| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$. Then $\mu(z) = 0$ if and only if

$$\begin{aligned} 0 &= \operatorname{Im}(z_1 \bar{z}_2) = -\frac{\xi_2}{\rho} \cos t \sinh \rho \cosh \rho, \\ 0 &= \operatorname{Im}(z_{p+1} \bar{z}_{p+2}) = -\frac{\xi_3}{\rho} \sin t \sinh \rho \cosh \rho. \end{aligned}$$

So we have $\xi_2 = \xi_3 = 0$, hence

$$z = \left(\cos(t + \sqrt{-1}\xi_1), 0, \ldots, 0, \sin(t + \sqrt{-1}\xi_1), 0, \ldots, 0\right).$$

Consequently we obtain

$$\mu^{-1}(0) \cap \Phi(\Sigma) = \left\{(\cos \tau, 0, \ldots, 0, \sin \tau, 0, \ldots, 0) \mid \tau = t + \sqrt{-1}\xi_1 \ (t, \xi_1 \in \mathbb{R})\right\}.$$

Since $\mu^{-1}(0)$ is $G$-invariant, we have

$$\mu^{-1}(0) = G \cdot (\mu^{-1}(0) \cap \Phi(\Sigma)).$$

Thus the orbit space $\mu^{-1}(0)/G$ of $G$-action on $\mu^{-1}(0)$ is parametrized by $t$ and $\xi_1$.

**Remark 3.1.** The $(t, \xi_1)$-plane can be regarded as the covering space of the orbit space $\mu^{-1}(0)/G$. In fact, we can take $t \in [0, \pi/2]$, and $\mu^{-1}(0)/G \cong \mathbb{C}/(\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2)$, where the action of $\mathbb{Z}$ on $\mathbb{C}$ is the parallel translation of period $2\pi$ and the actions of $\mathbb{Z}_2$ are reflections across the points $(t, \xi_1) = (0, 0)$ and $(\pi/2, 0)$ respectively. Principal orbits of $G$-action on $\mu^{-1}(0)$ are diffeomorphic to $S^{p-1} \times S^{q-1}$. There are two singular orbits $S^{p-1}$ and $S^{q-1}$ at $(t, \xi_1) = (0, 0)$ and $(\pi/2, 0)$, respectively. This implies that the orbit space $\mu^{-1}(0)/G$ is an orbifold with two singular points.

**Theorem 3.2.** *Let $\tau$ be a regular curve in the complex plane $\mathbb{C}$. We define a curve $\sigma$ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by*

$$\sigma(s) = (\cos\tau(s), 0, \ldots, 0, \sin\tau(s), 0, \ldots, 0).$$

*Then the $G$-orbit $L = G \cdot \sigma$ through $\sigma$ is a Lagrangian submanifold in $Q^n$. Moreover, $L$ is a special Lagrangian submanifold of phase $\theta$ if and only if $\tau$ satisfies*

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\tau'(\cos\tau)^{p-1}(\sin\tau)^{q-1}\right) = 0. \tag{3.2}$$

*Proof.* Since $L = G\cdot\sigma$ is a cohomogeneity one (possibly homogeneous) submanifold of dimension $n$ contained in $\mu^{-1}(0)$, from Proposition 2.6, $L$ is a Lagrangian submanifold in $Q^n$. We shall look for $\sigma$ so that $L$ is a special Lagrangian submanifold in $Q^n$. We take a basis of the tangent space $T_{\sigma(s)}L$ of $L$ at $\sigma(s)$ as follows:

$$\begin{aligned}
X^*_{1,2} &= X_{1,2}\sigma(s) = (0, \underbrace{\cos\tau(s)}_{2}, 0, \ldots, 0),\\
&\vdots\\
X^*_{1,p} &= X_{1,p}\sigma(s) = (0, \ldots, 0, \underbrace{\cos\tau(s)}_{p}, 0, \ldots, 0),\\
X^*_{p+1,p+2} &= X_{p+1,p+2}\sigma(s) = (0, \ldots, 0, \underbrace{\sin\tau(s)}_{p+2}, 0, \ldots, 0),\\
&\vdots\\
X^*_{p+1,n+1} &= X_{p+1,n+1}\sigma(s) = (0, \ldots, 0, \underbrace{\sin\tau(s)}_{n+1}),\\
\sigma'(s) &= (\underbrace{-\tau'(s)\sin\tau(s)}_{1}, 0, \ldots, 0, \underbrace{\tau'(s)\cos\tau(s)}_{p+1}, 0, \ldots, 0).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\Omega_{Stz}(X^*_{1,2}, \ldots, X^*_{1,p}, \sigma'(s), X^*_{p+1,p+2}, \ldots, X^*_{p+1,n+1})\\
&= (dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n+1})(\sigma(s), X^*_{1,2}, \ldots, X^*_{1,p}, \sigma'(s), X^*_{p+1,p+2}, \ldots, X^*_{p+1,n+1})\\
&= \begin{vmatrix}
\cos\tau(s) & 0 & \cdots & 0 & -\tau'(s)\sin\tau(s) & 0 & \cdots & 0\\
0 & \cos\tau(s) & & \vdots & 0 & \vdots & & \vdots\\
\vdots & 0 & \ddots & 0 & \vdots & \vdots & & \vdots\\
0 & \vdots & & \cos\tau(s) & 0 & \vdots & & \vdots\\
\sin\tau(s) & \vdots & & 0 & \tau'(s)\cos\tau(s) & 0 & & \vdots\\
0 & \vdots & & \vdots & 0 & \sin\tau(s) & & \vdots\\
\vdots & \vdots & & \vdots & \vdots & 0 & \ddots & 0\\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \sin\tau(s)
\end{vmatrix}\\
&= \tau'(s)(\cos\tau(s))^{p-1}(\sin\tau(s))^{q-1}.
\end{aligned}$$

Thus $L$ is a special Lagrangian submanifold of phase $\theta$ if and only if $\tau$ satisfies (3.2). □

For a curve $\tau$ in the complex plane $\mathbb{C}$, $L$ coincides with the image of the following map:

$$\begin{aligned}
\Psi : I \times S^{p-1} \times S^{q-1} &\longrightarrow Q^n\\
(s, x, y) &\longmapsto (\cos\tau(s)x_1, \ldots, \cos\tau(s)x_p, \sin\tau(s)y_1, \ldots, \sin\tau(s)y_q).
\end{aligned}$$

Here $I$ is an open interval in $\mathbb{R}$. When $\tau$ passes through $m\pi/2$ ($m \in \mathbb{Z}$), the map $\Psi$ degenerates at that point. If $\tau$ does not pass through $m\pi/2$ ($m \in \mathbb{Z}$), then $L$ is diffeomorphic to $I \times S^{p-1} \times S^{q-1}$ and immersed in $Q^n$ by the map $\Psi$.

3.2. **Case of $p = 1, q \geq 3$.** The orbit space of the action of

$$G = \left(\begin{array}{c|c} 1 & O \\ \hline O & SO(n) \end{array}\right) \cong SO(n)$$

on $T^*S^n$ is parametrized as

$$\Sigma = \left\{ (x,\xi) \;\middle|\; \begin{array}{l} x = (\cos t, \sin t, 0, \ldots, 0) \\ \xi = (-\xi_1 \sin t, \xi_1 \cos t, \xi_2, 0, \ldots, 0) \end{array} \right\}.$$

Then each $G$-orbit in $T^*S^n$ meets $\Sigma$, i.e., $G \cdot \Sigma = T^*S^n$.

In this case, the center of $\mathfrak{g}^*$ is $Z(\mathfrak{g}^*) = \{0\}$. We determine the subset $\mu^{-1}(0) \cap \Phi(\Sigma)$ of $Q^n$. For $z = \Phi(x,\xi) \in \Phi(\Sigma)$, $\mu(z) = 0$ is satisfied if and only if

$$0 = \mathrm{Im}(z_2 \bar{z}_3) = -\frac{\xi_2}{\rho} \sin t \sinh\rho \cosh\rho,$$

where $\rho = \|\xi\| = \sqrt{\xi_1^2 + \xi_2^2}$. Thus $\xi_2 = 0$ and we obtain

$$\mu^{-1}(0) \cap \Phi(\Sigma) = \left\{ (\cos\tau, \sin\tau, 0, \ldots, 0) \mid \tau = t + \sqrt{-1}\xi_1 \ (t, \xi_1 \in \mathbb{R}) \right\}.$$

Since $\mu^{-1}(0)$ is $G$-invariant, we have

$$\mu^{-1}(0) = G \cdot (\mu^{-1}(0) \cap \Phi(\Sigma)).$$

Thus the orbit space $\mu^{-1}(0)/G$ of $G$-action on $\mu^{-1}(0)$ is parametrized by $t$ and $\xi_1$.

**Remark 3.3.** *In this case, we can take $t \in [0, \pi]$, and $\mu^{-1}(0)/G \cong \mathbb{C}/(\mathbb{Z} \times \mathbb{Z}_2)$. Principal orbits of $G$-action on $\mu^{-1}(0)$ are diffeomorphic to $S^{n-1}$. There are two singular orbits at $(t, \xi_1) = (0, 0)$ and $(\pi, 0)$, that is, fixed orbits at the north pole and the south pole of the zero-section $S^n$. Thus the orbit space $\mu^{-1}(0)/G$ is an orbifold with two singular points.*

**Theorem 3.4.** *Let $\tau$ be a regular curve in the complex plane $\mathbb{C}$. We define a curve $\sigma$ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by*

$$\sigma(s) = (\cos\tau(s), \sin\tau(s), 0, \ldots, 0).$$

*Then the $G$-orbit $L = G \cdot \sigma$ through $\sigma$ is a Lagrangian submanifold in $Q^n$. Moreover, $L$ is a special Lagrangian submanifold of phase $\theta$ if and only if $\tau$ satisfies*

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\tau'(\sin\tau)^{n-1}\right) = 0. \tag{3.3}$$

*Proof.* Same with the proof of Theorem 3.2. □

For a curve $\tau$ in the complex plane $\mathbb{C}$, $L$ coincides with the image of the following map:

$$\begin{array}{rccl} \Psi : I \times S^{n-1} & \longrightarrow & Q^n \\ (s, y) & \longmapsto & (\cos\tau(s), \sin\tau(s) y_1, \ldots, \sin\tau(s) y_n). \end{array}$$

When $\tau$ passes through $m\pi$ $(m \in \mathbb{Z})$, the map $\Psi$ degenerates at that point. If $\tau$ does not pass through $m\pi$ $(m \in \mathbb{Z})$, then $L$ is diffeomorphic to $I \times S^{n-1}$ immersed in $Q^n$ by the map $\Psi$.

3.3. **Case of $p = 2, q \geq 3$.** The orbit space of the action of

$$G = \left(\begin{array}{c|c} SO(2) & O \\ \hline O & SO(n-1) \end{array}\right) \cong SO(2) \times SO(n-1)$$

on $T^*S^n$ is parametrized as

$$\Sigma = \left\{ (x,\xi) \;\middle|\; \begin{array}{l} x = (\cos t, 0, \sin t, 0, \ldots, 0) \\ \xi = (-\xi_1 \sin t, \xi_2, \xi_1 \cos t, \xi_3, 0, \ldots, 0) \end{array} \right\}.$$

Then each $G$-orbit in $T^*S^n$ meets $\Sigma$, i.e., $G \cdot \Sigma = T^*S^n$.

In this case, the center of $\mathfrak{g}^*$ is $Z(\mathfrak{g}^*) = \mathbb{R}\theta_{12}$. For $c_1 \in \mathbb{R}$, we determine the subset $\mu^{-1}(c_1\theta_{12}) \cap \Phi(\Sigma)$ of $Q^n$. For $z = \Phi(x, \xi) \in \Phi(\Sigma)$, $\mu(z) = c_1\theta_{12}$ is satisfied if and only if

$$c_1 = 2u'(r^2)\mathrm{Im}(z_1\bar{z}_2) = -2u'(\cosh(2\rho))\frac{\xi_2}{\rho}\cos t \sinh\rho\cosh\rho,$$
$$0 = \mathrm{Im}(z_3\bar{z}_4) = -\frac{\xi_3}{\rho}\sin t \sinh\rho\cosh\rho,$$

where $\rho = \|\xi\| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$. Thus $\xi_3 = 0$ and we obtain

$$\Phi^{-1}(\mu^{-1}(c_1\theta_{12})) \cap \Sigma = \left\{ (x, \xi) \;\middle|\; \begin{array}{l} x = (\cos t, 0, \sin t, 0, \dots, 0) \\ \xi = (-\xi_1 \sin t, \xi_2, \xi_1 \cos t, 0, \dots, 0) \\ c_1 = -u'(\cosh(2\rho))\frac{\xi_2}{\rho}\cos t \sinh(2\rho) \end{array} \right\}.$$

Since $\mu^{-1}(c_1\theta_{12})$ is $G$-invariant, we have

$$\mu^{-1}(c_1\theta_{12}) = G \cdot (\mu^{-1}(c_1\theta_{12}) \cap \Phi(\Sigma)).$$

**Theorem 3.5.** *Let $\sigma$ be a regular curve in $\mu^{-1}(c_1\theta_{12}) \cap \Phi(\Sigma)$. We express $\sigma$ as*

$$\sigma(s) = (z_1(s), z_2(s), z_3(s), 0, \dots, 0).$$

*Then the $G$-orbit $L = G \cdot \sigma$ through $\sigma$ is a Lagrangian submanifold in $Q^n$. Moreover, $L$ is a special Lagrangian submanifold of phase $\theta$ if and only if $\sigma$ satisfies*

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta} z_3^{n-1}\right) = c_2 \tag{3.4}$$

*for some $c_2 \in \mathbb{R}$.*

*Proof.* Since $L = G \cdot \sigma$ is a cohomogeneity one submanifold contained in $\mu^{-1}(c_1\theta_{12})$, from Proposition 2.6, $L$ is a Lagrangian submanifold in $Q^n$. We shall look for $\sigma$ so that $L$ is a special Lagrangian submanifold in $Q^n$. We take a basis of the tangent space $T_{\sigma(s)}L$ of $L$ at $\sigma(s)$ as follows:

$$\begin{aligned} X_{12}^* &= X_{12}\sigma(s) = (-z_2(s), z_1(s), 0, \dots, 0), \\ X_{34}^* &= X_{34}\sigma(s) = (0, 0, 0, z_3(s), 0, \dots, 0), \\ &\vdots \\ X_{3,n+1}^* &= X_{3,n+1}\sigma(s) = (0, \dots, 0, z_3(s)), \\ \sigma'(s) &= (z_1'(s), z_2'(s), z_3'(s), 0, \dots, 0). \end{aligned}$$

Since $z$ is in $Q^n$, we note that

$$z_1(s)^2 + z_2(s)^2 + z_3(s)^2 = 1,$$
$$z_1(s)z_1'(s) + z_2(s)z_2'(s) + z_3(s)z_3'(s) = 0.$$

Using these equalities, we have

$$\begin{aligned} &\Omega_{Stz}(X_{12}^*, \sigma'(s), X_{34}^*, \dots, X_{3,n+1}^*) \\ &= (dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n+1})(\sigma(s), X_{12}^*, \sigma'(s), X_{34}^*, \dots, X_{3,n+1}^*) \\ &= \begin{vmatrix} z_1(s) & -z_2(s) & z_1'(s) & 0 & \cdots & 0 \\ z_2(s) & z_1(s) & z_2'(s) & 0 & \cdots & 0 \\ z_3(s) & 0 & z_3'(s) & 0 & \cdots & 0 \\ 0 & \vdots & 0 & z_3(s) & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & z_3(s) \end{vmatrix} \\ &= z_3(s)^{n-2} z_3'(s). \end{aligned}$$

Therefore $L$ is a special Lagrangian submanifold of phase $\theta$ in $Q^n$ if and only if $\sigma$ satisfies

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta} z_3^{n-2} z_3'\right) = 0.$$

This condition is equivalent to (3.4) for some $c_2 \in \mathbb{R}$. □

For a curve $\sigma$ in $\mu^{-1}(c_1\theta_{12}) \cap \Phi(\Sigma)$, $L$ coincides with the image of the following map:

$$\begin{array}{rcl} \Psi : I \times S^1 \times S^{n-2} & \longrightarrow & Q^n \\ (s, x, y) & \longmapsto & (z_1(s)x_1 - z_2(s)x_2, z_1(s)x_2 + z_2(s)x_1, z_3(s)y_1, \ldots, z_3(s)y_{n-1}). \end{array}$$

When $\sigma$ passes through $z = (\pm\cosh(\xi_2), \sqrt{-1}\sinh(\xi_2), 0, \ldots, 0)$ or $z = (0, 0, \pm 1, 0, \ldots, 0)$, the map $\Psi$ degenerates at that point. If $\sigma$ does not pass through the points of singular orbits, $L$ is diffeomorphic to $I \times S^1 \times S^{n-2}$ and immersed in $Q^n$ by the map $\Psi$.

3.4. **Case of $p = q = 2$.** The orbit space of the action of

$$G = \left(\begin{array}{c|c} SO(2) & O \\ \hline O & SO(2) \end{array}\right) \cong SO(2) \times SO(2)$$

on $T^*S^3$ is parametrized as

$$\Sigma = \left\{ (x, \xi) \;\middle|\; \begin{array}{l} x = (\cos t, 0, \sin t, 0) \\ \xi = (-\xi_1 \sin t, \xi_2, \xi_1 \cos t, \xi_3) \end{array} \right\}.$$

Then each $G$-orbit in $T^*S^3$ meets $\Sigma$, i.e., $G \cdot \Sigma = T^*S^3$.

In this case, the center of $\mathfrak{g}^*$ is $Z(\mathfrak{g}^*) = \mathbb{R}\theta_{12} + \mathbb{R}\theta_{34} = \mathfrak{g}^*$. For $c_1, c_2 \in \mathbb{R}$, we determine the subset $\mu^{-1}(c_1\theta_{12} + c_2\theta_{34}) \cap \Phi(\Sigma)$ of $Q^3$. For $z = \Phi(x, \xi) \in \Phi(\Sigma)$, $\mu(z) = c_1\theta_{12} + c_2\theta_{34}$ is satisfied if and only if

$$\begin{aligned} c_1 &= 2u'(r^2)\mathrm{Im}(z_1\bar{z}_2) = -2u'(\cosh(2\rho))\frac{\xi_2}{\rho}\cos t \sinh\rho \cosh\rho, \\ c_2 &= 2u'(r^2)\mathrm{Im}(z_3\bar{z}_4) = -2u'(\cosh(2\rho))\frac{\xi_3}{\rho}\sin t \sinh\rho \cosh\rho, \end{aligned}$$

where $\rho = \|\xi\| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$. Therefore we obtain

$$\Phi^{-1}(\mu^{-1}(c_1\theta_{12} + c_2\theta_{34})) \cap \Sigma = \left\{ (x, \xi) \;\middle|\; \begin{array}{l} x = (\cos t, 0, \sin t, 0) \\ \xi = (-\xi_1 \sin t, \xi_2, \xi_1 \cos t, \xi_3) \\ c_1 = -u'(\cosh(2\rho))\frac{\xi_2}{\rho}\cos t \sinh(2\rho) \\ c_2 = -u'(\cosh(2\rho))\frac{\xi_3}{\rho}\sin t \sinh(2\rho) \end{array} \right\}.$$

Since $\mu^{-1}(c_1\theta_{12} + c_2\theta_{34})$ is $G$-invariant, we have

$$\mu^{-1}(c_1\theta_{12} + c_2\theta_{34}) = G \cdot (\mu^{-1}(c_1\theta_{12} + c_2\theta_{34}) \cap \Phi(\Sigma)).$$

**Theorem 3.6.** *Let $\sigma$ be a regular curve in $\mu^{-1}(c_1\theta_{12} + c_2\theta_{34}) \cap \Phi(\Sigma)$. We express $\sigma$ as*

$$\sigma(s) = (z_1(s), z_2(s), z_3(s), z_4(s)).$$

*Then the $G$-orbit $L = G \cdot \sigma$ through $\sigma$ is a Lagrangian submanifold in $Q^3$. Moreover, $L$ is a special Lagrangian submanifold of phase $\theta$ if and only if $\sigma$ satisfies*

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\left(z_1^2 + z_2^2\right)\right) = c_3 \tag{3.5}$$

*for some $c_3 \in \mathbb{R}$.*

*Proof.* The proof is similar with the previous theorem. We take a basis of the tangent space $T_{\sigma(s)}L$ of $L$ at $\sigma(s)$ as follows:

$$\begin{aligned} X_{12}^* &= X_{12}\sigma(s) = (-z_2(s), z_1(s), 0, 0), \\ X_{34}^* &= X_{34}\sigma(s) = (0, 0, -z_4(s), z_3(s)), \\ \sigma'(s) &= (z_1'(s), z_2'(s), z_3'(s), z_4'(s)). \end{aligned}$$

Then we have

$$\begin{aligned} \Omega_{Stz}(X_{12}^*, X_{34}^*, \sigma'(s)) &= (dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4)(\sigma(s), X_{12}^*, X_{34}^*, \sigma'(s)) \\ &= \begin{vmatrix} z_1(s) & -z_2(s) & 0 & z_1'(s) \\ z_2(s) & z_1(s) & 0 & z_2'(s) \\ z_3(s) & 0 & -z_4(s) & z_3'(s) \\ z_4(s) & 0 & z_3(s) & z_4'(s) \end{vmatrix} \\ &= z_1(s)z_1'(s) + z_2(s)z_2'(s). \end{aligned}$$

Therefore $L$ is a special Lagrangian submanifold of phase $\theta$ in $Q^3$ if and only if

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\left(z_1z_1' + z_2z_2'\right)\right) = 0.$$

This condition is equivalent to (3.5) for some $c_3 \in \mathbb{R}$. □

**Remark 3.7.** Since $G = SO(2) \times SO(2)$ is abelian and $Z(\mathfrak{g}^*) = \mathfrak{g}^*$, arbitrary $z \in Q^3$ lies in $\mu^{-1}(c_1\theta_{12} + c_2\theta_{34})$ for some $c_1, c_2 \in \mathbb{R}$. Furthermore $z \in Q^3$ satisfies (3.5) for some $c_3 \in \mathbb{R}$. This yields that, for a fixed $\theta$, the family of special Lagrangian submanifolds, which is constructed in Theorem 3.6, foliates $T^*S^3 \cong Q^3$.

For a curve $\sigma$ in $\mu^{-1}(c_1\theta_{12} + c_2\theta_{34}) \cap \Phi(\Sigma)$, $L$ coincides with the image of the following map:

$$\begin{aligned} \Psi : I \times S^1 \times S^1 &\longrightarrow Q^3 \\ (s, x, y) &\longmapsto (z_1(s)x_1 - z_2(s)x_2, z_1(s)x_2 + z_2(s)x_1, \\ &\qquad\qquad z_3(s)y_1 - z_4(s)y_2, z_3(s)y_2 + z_4(s)y_1). \end{aligned}$$

When $\sigma$ passes through $z = (\pm\cosh(\xi_2), \sqrt{-1}\sinh(\xi_2), 0, 0)$ or $(0, 0, \pm\cosh(\xi_3), \sqrt{-1}\sinh(\xi_3))$, the map $\Psi$ degenerates at that point. If $\sigma$ does not pass through the points of singular orbits, then $L$ is diffeomorphic to $I \times S^1 \times S^1$ and immersed in $Q^3$ by the map $\Psi$.

3.5. **Case of $p = 1, q = 2$.** The orbit space of the action of

$$G = \left(\begin{array}{c|c} 1 & O \\ \hline O & SO(2) \end{array}\right) \cong SO(2)$$

on $T^*S^2$ is parametrized as

$$\Sigma = \left\{ (x, \xi) \;\middle|\; \begin{array}{l} x = (\cos t, \sin t, 0) \\ \xi = (-\xi_1 \sin t, \xi_1 \cos t, \xi_2) \end{array} \right\}$$

Then each $G$-orbit in $T^*S^2$ meets $\Sigma$, i.e., $G \cdot \Sigma = T^*S^2$.

In this case, the center of $\mathfrak{g}^*$ is $Z(\mathfrak{g}^*) = \mathbb{R}\theta_{23} = \mathfrak{g}^*$. For $c_1 \in \mathbb{R}$, we determine the subset $\mu^{-1}(c_1\theta_{23}) \cap \Phi(\Sigma)$ of $Q^2$. For $z = \Phi(x, \xi) \in \Phi(\Sigma)$, $\mu(z) = c_1\theta_{23}$ is satisfied if and only if

$$c_1 = 2u'(r^2)\mathrm{Im}(z_2\bar{z}_3) = -2u'(\cosh(2\rho))\frac{\xi_2}{\rho}\sin t \sinh\rho \cosh\rho,$$

where $\rho = \|\xi\| = \sqrt{\xi_1^2 + \xi_2^2}$. Therefore we obtain

$$\Phi^{-1}(\mu^{-1}(c_1\theta_{23})) \cap \Sigma = \left\{ (x, \xi) \;\middle|\; \begin{array}{l} x = (\cos t, \sin t, 0) \\ \xi = (-\xi_1 \sin t, \xi_1 \cos t, \xi_2) \\ c_1 = -u'(\cosh(2\rho))\frac{\xi_2}{\rho}\sin t \sinh(2\rho) \end{array} \right\}.$$

Since $\mu^{-1}(c_1\theta_{23})$ is $G$-invariant, we have

$$\mu^{-1}(c_1\theta_{23}) = G \cdot (\mu^{-1}(c_1\theta_{23}) \cap \Phi(\Sigma)).$$

**Theorem 3.8.** *Let $\sigma$ be a regular curve in $\mu^{-1}(c_1\theta_{23}) \cap \Phi(\Sigma)$. We express $\sigma$ as*

$$\sigma(s) = (z_1(s), z_2(s), z_3(s)).$$

*Then the $G$-orbit $L = G \cdot \sigma$ through $\sigma$ is a Lagrangian submanifold in $Q^2$. Moreover, $L$ is a special Lagrangian submanifold of phase $\theta$ if and only if $\sigma$ satisfies*

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta} z_1\right) = c_2 \tag{3.6}$$

*for some $c_2 \in \mathbb{R}$.*

*Proof.* Similar with the previous theorems. □

**Remark 3.9.** Since $G = SO(2)$ is abelian and $Z(\mathfrak{g}^*) = \mathfrak{g}^*$, arbitrary $z \in Q^2$ lies in $\mu^{-1}(c_1\theta_{23})$ for some $c_1 \in \mathbb{R}$. Furthermore $z \in Q^2$ satisfies (3.6) for some $c_2 \in \mathbb{R}$. This yields that, for a fixed $\theta$, the family of special Lagrangian submanifolds, which is constructed in Theorem 3.8, foliates $T^*S^2 \cong Q^2$.

For a curve $\sigma$ in $\mu^{-1}(c_1\theta_{23}) \cap \Phi(\Sigma)$, $L$ coincides with the image of the following map:

$$\begin{array}{rccc} \Psi : I \times S^1 & \longrightarrow & Q^2 \\ (s, y) & \longmapsto & (z_1(s), z_2(s)y_1 - z_3(s)y_2, z_2(s)y_2 + z_3(s)y_1) \end{array}$$

When $\sigma$ passes through $z = (\pm 1, 0, 0)$, the map $\Psi$ degenerates at that point. If $\sigma$ does not pass through $z = (\pm 1, 0, 0)$, then $L$ is diffeomorphic to $I \times S^1$ and immersed in $Q^2$ by the map $\Psi$.

3.6. **Conormal bundle special Lagrangian submanifolds.** Harvey and Lawson [4] introduced the notion of austere submanifolds in order to construct special Lagrangian submanifolds in $T^*\mathbb{R}^n \cong \mathbb{C}^n$ as the conormal bundles of submanifolds in $\mathbb{R}^n$. A submanifold $M$ of a Riemannian manifold $\tilde{M}$ is said to be *austere* if the set of eigenvalues of the shape operator of $M$ is invariant under the multiplication of $-1$ concerning the multiplicities. As a generalization of the construction of conormal bundle special Lagrangian submanifolds due to Harvey and Lawson, Karigiannis and Min-Oo proved the following theorem.

**Theorem 3.10** ([11])**.** *Let $M$ be a submanifold of $S^n$. Then the conormal bundle $N^*M$ of $M$ is a Lagrangian submanifold of $T^*S^n$ with respect to the Stenzel metric. Moreover, $N^*M$ is a special Lagrangian submanifold of $T^*S^n$ if and only if $M$ is an austere submanifold of $S^n$.*

In [6], we determined all austere orbits of the isotropy representations of irreducible symmetric spaces of compact type. All austere orbits of the action of $SO(p) \times SO(q)$ $(p + q = n + 1)$ on $S^n$ are the following.

(1) When $p = q$, a minimal principal orbit of the action of $SO(p) \times SO(p)$ on $S^n$, that is called a minimal Clifford hypersurface $S^{p-1}(1/\sqrt{2}) \times S^{p-1}(1/\sqrt{2}) \subset S^n(1)$.
(2) When $p = 1$, a minimal principal orbit of the action of $SO(1) \times SO(n)$ on $S^n$, that is a totally geodesic hypersphere $S^{n-1}(1) \subset S^n(1)$.
(3) Singular orbits of the action of $SO(p) \times SO(q)$ on $S^n$, that are totally geodesic spheres $S^{p-1}(1) \subset S^n(1)$ and $S^{q-1}(1) \subset S^n(1)$.

From Theorem 3.10, the conormal bundles of the above austere orbits are special Lagrangian submanifolds in $T^*S^n$. In fact, we can catch these special Lagrangian submanifolds by the construction we gave in this section.

(1) Let $\tau(s) = \pi/4 + \sqrt{-1}s$ and define a curve $\sigma$ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by

$$\sigma(s) = (\overbrace{\cos\tau(s)}^{1}, 0, \ldots, 0, \overbrace{\sin\tau(s)}^{p+1}, 0, \ldots, 0).$$

Then the orbit $L = G \cdot \sigma$ of the action of $G = SO(p) \times SO(p)$ through $\sigma$ is the conormal bundle of a minimal Clifford hypersurface in $Q^n \cong T^*S^n$. In fact, $\tau$ satisfies the condition (3.2) for $\theta = \pi/2$, hence $L$ is a special Lagrangian submanifold of phase $\pi/2$ in $Q^n$.

(2) Let $\tau(s) = \pi/2 + \sqrt{-1}s$ and define a curve $\sigma$ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by

$$\sigma(s) = (\cos\tau(s), \sin\tau(s), 0, \dots, 0).$$

Then the orbit $L = G \cdot \sigma$ of the action of $SO(1) \times SO(n)$ through $\sigma$ is the conormal bundle of a totally geodesic hypersphere in $Q^n \cong T^*S^n$. In fact, $\tau$ satisfies the condition (3.3) for $\theta = \pi/2$, hence $L$ is a special Lagrangian submanifold of phase $\pi/2$ in $Q^n$.

(3) Let $\tau(s) = 0 + \sqrt{-1}s$ and define a curve $\sigma$ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by

$$\sigma(s) = (\overset{1}{\smile}\!\!\!\!\cos\tau(s), 0, \dots, 0, \overset{p+1}{\smile}\!\!\!\!\sin\tau(s), 0, \dots, 0).$$

Then the orbit $L = G \cdot \sigma$ of the action of $SO(p) \times SO(q)$ through $\sigma$ is the conormal bundle of a totally geodesic sphere in $Q^n \cong T^*S^n$. In fact, when $q$ is even, $L$ is a special Lagrangian submanifold of phase 0 in $Q^n$. When $q$ is odd, $L$ is a special Lagrangian submanifold of phase $\pi/2$ in $Q^n$.

## 4. Ricci flat Kähler metric and special Lagrangian submanifolds in the complex cone

We define the complex cone $Q_0^n$ in $\mathbb{C}^{n+1}$ by

$$Q_0^n = \left\{ z = (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} z_i^2 = 0 \right\}.$$

$Q_0^n$ has a (unique) singularity at the origin of $\mathbb{C}^{n+1}$. As $r = \|z\|$ tends to $\infty$, $Q^n$ is asymptotic to $Q_0^n$ in $\mathbb{C}^{n+1}$. In this section, we give a (singular) Ricci-flat Kähler metric on $Q_0^n$ as the limit of the Stenzel metric on $Q^n$.

The holomorphic $(n,0)$-form $\Omega_{cone}$ on $Q_0^n$ is given by

$$\frac{1}{2} d(z_1^2 + \cdots + z_{n+1}^2) \wedge \Omega_{cone} = \Omega_0.$$

We can express $\Omega_{cone}$ as

$$\Omega_{cone}(v_1, \dots, v_n) = \frac{1}{\|z\|^2} (dz_1 \wedge \cdots \wedge dz_{n+1})(\bar{z}, v_1, \dots, v_n),$$

where $v_1, \dots, v_n \in T_z Q^n$ and $z \in Q^n$.

As $t \to \infty$, the differential equation (2.2) is asymptotic to

$$\frac{d}{dt}(F'(t))^n = \left(\frac{1}{2}\right)^{n-1} n\, c\, e^{t(n-1)} \qquad (c > 0).$$

Then

$$F(t) = \left(\frac{1}{2}\right)^{\frac{n-1}{n}} \left(\frac{n}{n-1}\right)^{\frac{n+1}{n}} c^{\frac{1}{n}} e^{\frac{n-1}{n}t}$$

is a solution of this differential equation. Since $\cosh t \to (1/2)e^t$ as $t \to \infty$, we define a function $f$ as $F(t) = f((1/2)e^t)$. Then we have

$$f(t) = \left(\frac{n}{n-1}\right)^{\frac{n+1}{n}} c^{\frac{1}{n}} t^{\frac{n-1}{n}}.$$

**Proposition 4.1.** *Let* $f_{cone}(t) = c\, r^{\frac{n-1}{n}}$ $(c>0)$ *and define a Kähler form* $\omega_{cone}$ *on* $Q_0^n$ *by*

$$\omega_{cone} = \sqrt{-1}\partial\overline{\partial} f_{cone}(r^2) = \sqrt{-1}\sum_{i,j=1}^{n+1} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} f_{cone}(r^2) dz_i \wedge d\bar{z}_j.$$

*Then* $\omega_{cone}$ *gives a Ricci-flat Kähler metric on* $Q_0^n$.

**Remark 4.2.** When $n=3$ and $c=3/2$, then $f_{cone}(r^2) = (3/2) r^{\frac{4}{3}}$ coincides with the potential of the Ricci-flat Kähler metric on $Q_0^3$ due to Candelas and de la Ossa [2].

**Proof of Proposition 4.1.** Henceforth we write $f$ as $f_{cone}$ shortly. In a similar way with (2.3), we can evaluate

$$\omega_{cone}(v,w) = 2f'(r^2)\langle Jv, w\rangle + 2f''(r^2)\big(\langle v,z\rangle\langle Jz,w\rangle - \langle w,z\rangle\langle Jz,v\rangle\big)$$

for $v, w \in T_z Q_0^n, z \in Q_0^n$. From this, we have

$$\omega_{cone}(v,\bar{w}) = 2\sqrt{-1}\big(f'(r^2)(v,w) + 2f''(r^2)(v,z)(z,w)\big),$$

where $(\ ,\ )$ is the standard Hermitian inner product on $\mathbb{C}^{n+1}$.

Now we show that there exists a constant $\lambda \in \mathbb{R}$ such that

$$\frac{\omega_{cone}^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n \lambda\, \Omega_{cone} \wedge \overline{\Omega}_{cone}. \tag{4.1}$$

Let $v_1, \dots, v_n$ be a basis of $T_z Q_0^n$ which satisfies $(v_i, v_j) = \delta_{ij}$, and $\theta_1, \dots, \theta_n$ be its dual basis. Using this basis, we can express $\omega_{cone}$ as

$$\omega_{cone} = \sum_{i,j=1}^{n} \omega_{ij} \theta_i \wedge \bar{\theta}_j,$$

where

$$\omega_{ij} = \omega_{cone}(v_i, \bar{v}_j) = 2\sqrt{-1}\big(f'(r^2)\delta_{ij} + 2f''(r^2)(v_i,z)(z,v_j)\big).$$

Then the left-hand side of (4.1) is

$$\frac{\omega_{cone}^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \det(\omega_{ij}) \theta_1 \wedge \cdots \wedge \theta_n \wedge \bar{\theta}_1 \wedge \cdots \wedge \bar{\theta}_n.$$

Here we can compute

$$\begin{aligned}
\det(\omega_{ij}) &= (2\sqrt{-1})^n (f'(r^2))^n \left(1 + 2\frac{f''(r^2)}{f'(r^2)}\left(|(v_1,z)|^2 + \cdots + |(v_n,z)|^2\right)\right) \\
&= \big(2\sqrt{-1}\big)^n \left(\frac{c(n-1)}{n}\right)^n \left(\frac{n-2}{n}\right) \frac{1}{r^2}. \qquad (4.2)
\end{aligned}$$

On the other hand, $\Omega_{cone}$ can be computed as follows:

$$\begin{aligned}
\Omega_{cone}(v_1, \dots, v_n) &= \frac{1}{\|z\|^2}(dz_1 \wedge \cdots \wedge dz_{n+1})(\bar{z}, v_1, \dots, v_n) \\
&= \frac{1}{\|z\|} \det\left(\frac{\bar{z}}{\|z\|}, v_1, \dots, v_n\right).
\end{aligned}$$

Hence we have

$$\Omega_{cone} \wedge \overline{\Omega}_{cone}(v_1, \dots, v_n, \bar{v}_1, \dots, \bar{v}_n) = \frac{1}{\|z\|^2} = \frac{1}{r^2}. \tag{4.3}$$

From (4.2) and (4.3), consequently we obtain

$$\frac{\omega_{cone}^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n \left(\frac{4c(n-1)}{n}\right)^n \left(\frac{n-2}{n}\right) \Omega_{cone} \wedge \overline{\Omega}_{cone}.$$

Thus we proved the proposition. □

We shall construct cohomogeneity one special Lagrangian submanifolds in $Q_0^n$ in a similar way with the previous section, using the moment map technique.

Let $T^\circ S^n$ denote the subset of $T^*S^n$ excluding the zero-section. Then we can identify $T^\circ S^n$ and $Q_0^n \backslash \{0\}$ by the following diffeomorphism:

$$\begin{array}{rcl} \Pi : T^\circ S^n & \longrightarrow & Q_0^n \setminus \{0\} \\ (x,\xi) & \longmapsto & \|\xi\| x + \sqrt{-1}\xi. \end{array}$$

The diffeomorphism $\Pi$ is equivariant under the action of $SO(n+1)$.

Here we consider

$$G = \left( \begin{array}{c|c} SO(p) & O \\ \hline O & SO(q) \end{array} \right) \cong SO(p) \times SO(q) \quad (p+q = n+1,\ 1 \leq p \leq q \leq n)$$

as a Lie subgroup of $SO(n+1)$. The action of $G$ on $Q_0^n$ is Hamiltonian, and its moment map $\mu : Q_0^n \to \mathfrak{g}^*$ can be expressed as

$$\mu(z) = 2f'(r^2)\Big(\mathrm{Im}(z_i\bar{z}_j)_{1 \leq i<j \leq p},\ \mathrm{Im}(z_i\bar{z}_j)_{p+1 \leq i<j \leq n+1}\Big)$$

using the basis $\{\theta_{ij}\}$ of $\mathfrak{g}^*$.

From Proposition 2.6, a special Lagrangian submanifold of $Q_0^n$ invariant under $G$ should be contained in the inverse image $\mu^{-1}(c)$ of some $c \in Z(\mathfrak{g}^*)$. Although we should consider each type of the center $Z(\mathfrak{g}^*)$ individually, here we shall work on the generic case, $3 \leq p \leq q$. For other cases, we can study similarly as in the previous section.

4.1. **Case of** $3 \leq p \leq q$**.** The orbit space of the action of $G = SO(p) \times SO(q)$ on $T^\circ S^n$ is parametrized as

$$\Sigma = \left\{ (x,\xi) \;\middle|\; \begin{array}{l} x = (\cos t, 0, \ldots, 0, \sin t, 0, \ldots, 0) \\ \xi = (-\xi_1 \sin t, \xi_2, 0, \ldots, 0, \xi_1 \cos t, \xi_3, 0, \ldots, 0) \\ (\xi_1,\xi_2,\xi_3) \neq (0,0,0) \end{array} \right\}.$$

Then each $G$-orbit in $T^*S^n$ meets $\Sigma$, i.e., $G \cdot \Sigma = T^\circ S^n$.

In this case, the center of $\mathfrak{g}^*$ is $Z(\mathfrak{g}^*) = \{0\}$. We determine the subset $\mu^{-1}(0) \cap \Pi(\Sigma)$ of $Q_0^n$. Now $z \in \Pi(x,\xi) \in \Pi(\Sigma)$ can be expressed as

$$z = (\rho\cos t - \sqrt{-1}\xi_1 \sin t, \sqrt{-1}\xi_2, 0, \ldots, 0, \rho \sin t + \sqrt{-1}\xi_1 \cos t, \sqrt{-1}\xi_3, 0, \ldots, 0),$$

where $\rho = \|\xi\| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$. Then $\mu(z) = 0$ if and only if

$$\begin{aligned} 0 &= \mathrm{Im}(z_1\bar{z}_2) = -\xi_2 \rho \cos t, \\ 0 &= \mathrm{Im}(z_{p+1}\bar{z}_{p+2}) = -\xi_3 \rho \sin t. \end{aligned}$$

Thus $\xi_2 = \xi_3 = 0$ and we obtain

$$\mu^{-1}(0) \cap \Pi(\Sigma) = \left\{ \Big(|\xi_1|\cos t - \sqrt{-1}\xi_1 \sin t, 0, \ldots, 0, |\xi_1| \sin t + \sqrt{-1}\xi_1 \cos t, 0, \ldots, 0\Big) \;\middle|\; \xi_1 \neq 0 \right\}.$$

Since $\mu^{-1}(0)$ is $G$-invariant, we have

$$\mu^{-1}(0) = G \cdot (\mu^{-1}(0) \cap \Pi(\Sigma)).$$

Thus the orbit space $\mu^{-1}(0)/G$ of $G$-action on $\mu^{-1}(0)$ is parametrized by $t$ and $\xi_1$.

**Proposition 4.3.** *Let $\sigma$ be a curve in $\mu^{-1}(0) \cap \Pi(\Sigma)$. We express $\sigma$ as*

$$\sigma(s) = (z_1(s), 0, \ldots, 0, z_{p+1}(s), 0, \ldots, 0).$$

*Then the $G$-orbit $L = G \cdot \sigma$ through $\sigma$ is a Lagrangian submanifold in $Q_0^n$. Moreover, $L$ is a special Lagrangian submanifold of phase $\theta$ if and only if*

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}(-1)^{\frac{q}{2}} z_1(s)^{n-1}\right) = c \tag{4.4}$$

*for some $c \in \mathbb{R}$.*

*Proof.* Since $L = G \cdot \sigma$ is a cohomogeneity one submanifold of dimension $n$ contained in $\mu^{-1}(0)$, from Proposition 2.6, $L$ is a Lagrangian submanifold in $Q_0^n$. We shall look for $\sigma$ so that $L$ is a special Lagrangian submanifold in $Q_0^n$. Since $\sigma(s) \in Q_0^n$, we note that

$$z_1^2(s) + z_{p+1}^2(s) = 0,$$
$$z_1(s)z_1'(s) + z_{p+1}(s)z_{p+1}'(s) = 0.$$

We take a basis of the tangent space $T_{\sigma(s)}L$ of $L$ at $\sigma(s)$ as follows:

$$\begin{aligned}
X_{1,2}^* &= X_{1,2}\sigma(s) = (0, \underbrace{z_1(s)}_{2}, 0, \dots, 0),\\
&\vdots\\
X_{1,p}^* &= X_{1,p}\sigma(s) = (0, \dots, 0, \underbrace{z_1(s)}_{p}, 0, \dots, 0),\\
X_{p+1,p+2}^* &= X_{p+1,p+2}\sigma(s) = (0, \dots, 0, \underbrace{z_{p+1}(s)}_{p+2}, 0, \dots, 0),\\
&\vdots\\
X_{p+1,n+1}^* &= X_{p+1,n+1}\sigma(s) = (0, \dots, 0, \underbrace{z_{p+1}(s)}_{n+1}),\\
\sigma'(s) &= (\underbrace{z_1'(s)}_{1}, 0, \dots, 0, \underbrace{z_{p+1}'(s)}_{p+1}, 0, \dots, 0).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\Omega_{cone}(X_{1,2}^*, \dots, X_{1,p}^*, \sigma'(s), X_{p+1,p+2}^*, \dots, X_{p+1,n+1}^*)\\
&= \frac{1}{\|z\|^2}(dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n+1})(\overline{\sigma(s)}, X_{1,2}^*, \dots, X_{1,p}^*, \sigma'(s), X_{p+1,p+2}^*, \dots, X_{p+1,n+1}^*)\\
&= \frac{1}{\|z\|^2}\begin{vmatrix}
\bar{z}_1(s) & 0 & \cdots & 0 & z_1'(s) & 0 & \cdots & 0\\
0 & z_1(s) & & \vdots & 0 & \vdots & & \vdots\\
\vdots & 0 & \ddots & 0 & \vdots & \vdots & & \vdots\\
0 & \vdots & & z_1(s) & 0 & \vdots & & \vdots\\
\bar{z}_{p+1}(s) & \vdots & & 0 & z_{p+1}'(s) & 0 & & \vdots\\
0 & \vdots & & \vdots & 0 & z_{p+1}(s) & & \vdots\\
\vdots & \vdots & & \vdots & \vdots & 0 & \ddots & 0\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & z_{p+1}(s)
\end{vmatrix}\\
&= (-1)^{\frac{q}{2}} z_1^{n-2}(s) z_1'(s).
\end{aligned}$$

Thus $L$ is a special Lagrangian submanifold of phase $\theta$ if and only of $\sigma$ satisfies

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}(-1)^{\frac{q}{2}} z_1^{n-2} z_1'\right) = 0.$$

This condition is equivalent to (4.4) for some $c \in \mathbb{R}$. □

We express $z_1 = |\xi_1|\cos t - \sqrt{-1}\xi_1 \sin t$. When $\xi_1 > 0$, the condition (4.4) becomes

$$\operatorname{Im}\left((-1)^{\frac{q}{2}} e^{\sqrt{-1}(\theta-(n-1)t)}\right) = c \tag{4.5}$$

for some $c \in \mathbb{R}$. In particular, when $c = 0$ we have

$$\theta - (n-1)t = \begin{cases} 0 \pmod{\pi} & (q : \text{even}) \\ \frac{\pi}{2} \pmod{\pi} & (q : \text{odd}) \end{cases}$$

When $c \neq 0$, solution curves of (4.5) are asymptotic to the following lines:

$$\left\{\tau = t + \sqrt{-1}\xi_1 \;\middle|\; t = \frac{\theta - k\pi}{n-1},\ \xi_1 \in \mathbb{R}\right\} \quad (k \in \mathbb{Z}) \qquad (q : \text{even}),$$

$$\left\{\tau = t + \sqrt{-1}\xi_1 \;\middle|\; t = \frac{2\theta - (2k+1)\pi}{2(n-1)},\ \xi_1 \in \mathbb{R}\right\} \quad (k \in \mathbb{Z}) \qquad (q : \text{odd}).$$

Therefore, when $c = 0$, the cones over the orbits of the action of $SO(p) \times SO(q)$ through

$$\frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}\frac{k\pi-\theta}{n-1}}, 0, \dots, 0, \sqrt{-1}e^{\sqrt{-1}\frac{k\pi-\theta}{n-1}}, 0, \dots, 0\right) \qquad (q : \text{even})$$

$$\frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}\frac{(2k+1)\pi-2\theta}{2(n-1)}}, 0, \dots, 0, \sqrt{-1}e^{\sqrt{-1}\frac{(2k+1)\pi-2\theta}{2(n-1)}}, 0, \dots, 0\right) \qquad (q : \text{odd})$$

are special Lagrangian cones of phase $\theta$ in $Q_0^n$. When $c \neq 0$, special Lagrangian submanifolds are diffeomorphic to $\mathbb{R} \times S^{p-1} \times S^{q-1}$, and their ends are asymptotic to the above special Lagrangian cones.

## 5. Asymptotic behavior of cohomogeneity one special Lagrangian submanifolds in $T^*S^n$

Cohomogeneity one special Lagrangian submanifolds in $Q^n$ which we constructed in Section 3 are diffeomorphic to $\mathbb{R} \times S^{p-1} \times S^{q-1}$ generically. In this section, we shall study the asymptotic behavior of their ends and the singular sets.

5.1. **Case of $3 \leq p \leq q$.** We shall analyse solution curves of the differential equation (3.2). In the phase space $\mathbb{C}$, the orbit space of $G$-action on $\mu^{-1}(0)$ can be reduced to

$$\{\tau = t + \sqrt{-1}\xi_1 \mid 0 \leq t \leq \frac{\pi}{2},\ \xi_1 \in \mathbb{R}\}.$$

In this area, (3.2) has singularities at 0 and $\pi/2$. When $\theta = 0$, the real segment $[0, \pi/2]$ is a trivial solution, and its corresponding special Lagrangian submanifold is the zero-section $S^n$ of $T^*S^n$.

As $\xi_1$ tends to $\infty$, $\cos\tau$ and $\sin\tau$ are asymptotic to

$$\cos\tau \longrightarrow \frac{1}{2}e^{-\sqrt{-1}\tau}, \qquad \sin\tau \longrightarrow \frac{\sqrt{-1}}{2}e^{-\sqrt{-1}\tau}.$$

Then (3.2) is asymptotic to

$$\mathrm{Im}\left(\sqrt{-1}^{\,q-1}\tau' e^{\sqrt{-1}(\theta-(n-1)\tau)}\right) = 0.$$

This condition becomes

$$\mathrm{Im}\left(\sqrt{-1}\tau' e^{\sqrt{-1}(\theta-(n-1)\tau)}\right) = 0 \qquad (q : \text{even}),$$

$$\mathrm{Im}\left(\tau' e^{\sqrt{-1}(\theta-(n-1)\tau)}\right) = 0 \qquad (q : \text{odd}),$$

and it is equivalent to the equation

$$\mathrm{Im}\left(e^{\sqrt{-1}(\theta-(n-1)\tau)}\right) = c \qquad (q : \text{even}), \tag{5.1}$$

$$\mathrm{Re}\left(e^{\sqrt{-1}(\theta-(n-1)\tau)}\right) = c \qquad (q : \text{odd})$$

for some $c \in \mathbb{R}$. In particular, when $c = 0$ we have

$$\theta - (n-1)t = 0 \pmod{\pi} \qquad (q : \text{even}),$$

$$\theta - (n-1)t = \frac{\pi}{2} \pmod{\pi} \qquad (q : \text{odd}).$$

When $c \neq 0$, solution curves of (5.1) are asymptotic to these lines. Therefore, as $\xi_1 \to \infty$, solution curves of (3.2) are asymptotic to the following lines:

$$\left\{\tau = t + \sqrt{-1}\xi_1 \;\middle|\; t = \frac{\theta - k\pi}{n-1},\ \xi_1 \in \mathbb{R}\right\} \quad (k \in \mathbb{Z}) \qquad (q : \text{even}),$$
$$\left\{\tau = t + \sqrt{-1}\xi_1 \;\middle|\; t = \frac{2\theta - (2k+1)\pi}{2(n-1)},\ \xi_1 \in \mathbb{R}\right\} \quad (k \in \mathbb{Z}) \qquad (q : \text{odd}).$$

A special Lagrangian submanifold $L$ in $Q^n$ is given as the orbit through a curve

$$\sigma(s) = (\underbrace{\cos\tau(s)}_{1}, 0, \ldots, 0, \underbrace{\sin\tau(s)}_{p+1}, 0, \ldots, 0)$$

in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by the action of $SO(p) \times SO(q)$. The unit vector is

$$\frac{\sigma}{\|\sigma\|} \to \frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}\frac{k\pi-\theta}{n-1}}, 0, \ldots, 0, \sqrt{-1}e^{\sqrt{-1}\frac{k\pi-\theta}{n-1}}, 0, \ldots, 0\right) \qquad (q : \text{even}),$$
$$\frac{\sigma}{\|\sigma\|} \to \frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}\frac{(2k+1)\pi-2\theta}{2(n-1)}}, 0, \ldots, 0, \sqrt{-1}e^{\sqrt{-1}\frac{(2k+1)\pi-2\theta}{2(n-1)}}, 0, \ldots, 0\right) \qquad (q : \text{odd}).$$

as $\xi_1 \to \infty$.

As $\tau$ approaches to 0, $\cos\tau$ and $\sin\tau$ are asymptotic to

$$\cos\tau \longrightarrow 1, \qquad \sin\tau \longrightarrow \tau.$$

Then (3.2) is asymptotic to

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\tau'\tau^{q-1}\right) = 0,$$

and it is equivalent to the equation

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\tau^{q}\right) = c$$

for some $c \in \mathbb{R}$. In particular, when $c = 0$ solutions of the above equation are the following half-lines:

$$\left\{\tau = t + \sqrt{-1}\xi_1 \;\middle|\; \arg(\tau) = \frac{k\pi - \theta}{q}\right\} \qquad (k = 0, 1, 2, \ldots, 2q-1).$$

Therefore the solution of (3.2) branches to $2q$ curves at 0, and these curves are asymptotic to the above half-lines around 0. The orbit of the action of $SO(p) \times SO(q)$ through $z = (1, 0, \ldots, 0)$ is a singular orbit, which is diffeomorphic to $S^{p-1}$.

As $\tau \to \pi/2$, $\cos\tau$ and $\sin\tau$ are asymptotic to

$$\cos\tau \longrightarrow \frac{\pi}{2} - \tau, \qquad \sin\tau \longrightarrow 1.$$

Then (3.2) is asymptotic to

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\tau'\left(\frac{\pi}{2} - \tau\right)^{p-1}\right) = 0,$$

and it is equivalent to the equation

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\left(\tau - \frac{\pi}{2}\right)^{p}\right) = c$$

for some $c \in \mathbb{R}$. In particular, when $c = 0$ solutions of the above equation are the following half-lines:

$$\left\{\tau = t + \sqrt{-1}\xi_1 \;\middle|\; \arg(\tau - \frac{\pi}{2}) = \frac{k\pi - \theta}{p}\right\} \qquad (k = 0, 1, 2, \ldots, 2p-1).$$

Therefore the solution of (3.2) branches to $2p$ curves at $\pi/2$, and these curves are asymptotic to the above half-lines around $\pi/2$. The orbit of the action of $SO(p) \times SO(q)$ through

$$z = (0, \ldots, 0, \underbrace{1}_{p+1}, 0, \ldots, 0)$$

is a singular orbit, which is diffeomorphic to $S^{q-1}$.

Consequently we obtain the following observations.

**Proposition 5.1.** *In the case of $3 \leq p \leq q$, cohomogeneity one special Lagrangian submanifolds $L$ invariant under $SO(p) \times SO(q)$ are diffeomorphic to $I \times S^{p-1} \times S^{q-1}$ and embedded in $T^*S^n \cong Q^n$ generically.*

(1) *Two ends of $L$ in $Q^n$ are asymptotic to special Lagrangian cones in $Q_0^n$ which are the cones over the orbits through*

$$\frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}\frac{k\pi-\theta}{n-1}}, 0, \dots, 0, \sqrt{-1}e^{\sqrt{-1}\frac{k\pi-\theta}{n-1}}, 0, \dots, 0\right) \quad (k \in \mathbb{Z}) \qquad (q : \text{even})$$

$$\frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}\frac{(2k+1)\pi-2\theta}{2(n-1)}}, 0, \dots, 0, \sqrt{-1}e^{\sqrt{-1}\frac{(2k+1)\pi-2\theta}{2(n-1)}}, 0, \dots, 0\right) \quad (k \in \mathbb{Z}) \qquad (q : \text{odd})$$

*by the action of $SO(p) \times SO(q)$.*

(2) *When the curve $\tau$ passes through $0$, the map $\Psi : I \times S^{p-1} \times S^{q-1} \to Q^n$ degenerates, and $q$ special Lagrangian submanifolds of $Q^n$ meet at the singular set $S^{p-1}$.*

(3) *When the curve $\tau$ passes through $\pi/2$, the map $\Psi : I \times S^{p-1} \times S^{q-1} \to Q^n$ degenerates, and $p$ special Lagrangian submanifolds of $Q^n$ meet at the singular set $S^{q-1}$.*

Furthermore we observe the following.

**Remark 5.2.** A smooth solution of (3.2) approaches to a singular one as $c \to 0$. This implies that a smooth special Lagrangian submanifold is deformed to a singular one. In other words, a branched special Lagrangian submanifold can be deformed to be smooth.

**Example.** In the case of $n = 6, p = 3, q = 4$, the differential equation (3.2) is

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\tau'(\cos\tau)^2(\sin\tau)^3\right) = 0.$$

The following figures shows solution curves of this ODE, when $\theta = 0, \pi/4$ and $\pi/2$. Each solution curve corresponds to a special Lagrangian submanifold in $Q^n$.

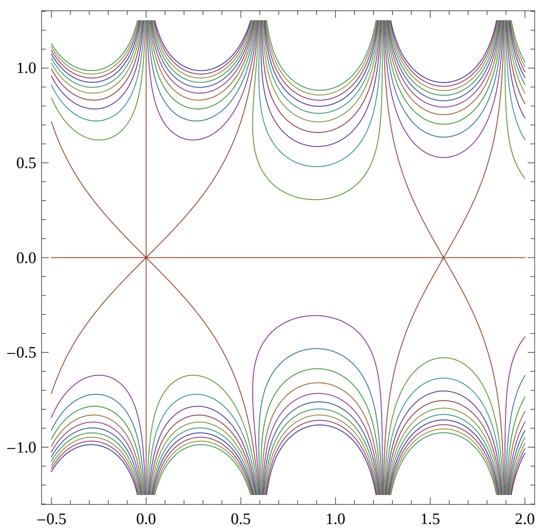


FIGURE 1. $\theta = 0$

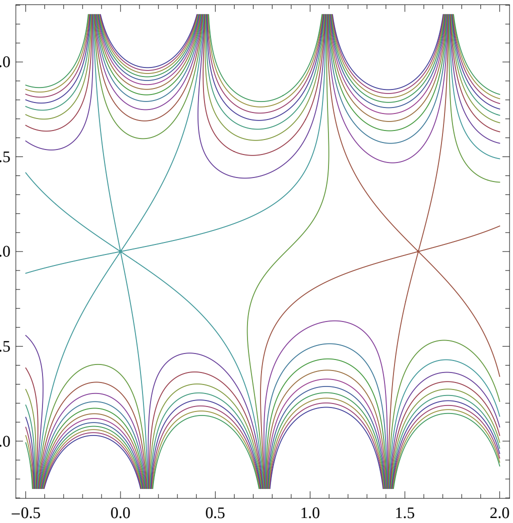


FIGURE 2. $\theta = \pi/4$

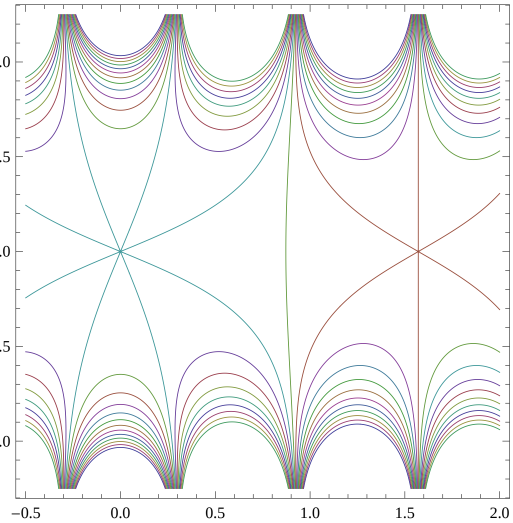


FIGURE 3. $\theta = \pi/2$

5.2. **Case of $p = 1, q \geq 3$.** In the phase space $\mathbb{C}$, the orbit space of $G$-action on $\mu^{-1}(0)$ can be reduced to

$$\{\tau = t + \sqrt{-1}\xi_1 \mid 0 \leq t \leq \pi,\ \xi_1 \in \mathbb{R}\}.$$

In this area, (3.3) has singularities at 0 and $\pi$. When $\theta = 0$, the real segment $[0, \pi]$ is a trivial solution, and its corresponding special Lagrangian submanifold is the zero-section $S^n$ of $T^*S^n$.

Similarly with the previous case, we see that solution curves of (3.3) are asymptotic to the following lines:

$$\left\{\tau = t + \sqrt{-1}\xi_1 \;\middle|\; t = \frac{\theta - k\pi}{n-1},\ \xi_1 \in \mathbb{R}\right\} \quad (k \in \mathbb{Z}) \qquad (n : \text{even})$$

$$\left\{\tau = t + \sqrt{-1}\xi_1 \;\middle|\; t = \frac{2\theta - (2k+1)\pi}{2(n-1)},\ \xi_1 \in \mathbb{R}\right\} \quad (k \in \mathbb{Z}) \qquad (n : \text{odd})$$

as $\xi_1 \to \infty$.

The solution of (3.3) branches to $2n$ curves at 0 and $\pi$, and these curves are asymptotic to the following half-lines:

$$\left\{\tau = t + \sqrt{-1}\xi_1 \;\middle|\; \arg(\tau) = \frac{k\pi - \theta}{n}\right\}$$

$$\left\{\tau = t + \sqrt{-1}\xi_1 \;\middle|\; \arg(\tau - \pi) = \frac{k\pi - \theta}{n}\right\} \qquad (k = 0, 1, 2, \ldots, 2n-1).$$

around 0 and $\pi$, respectively. The orbits of the action of $SO(n)$ through $z = (\pm 1, 0, \ldots, 0)$ are singular orbits, that is, fixed orbits.

Therefore we obtain the following observations.

**Proposition 5.3.** *In the case of $p = 1, q \geq 3$, cohomogeneity one special Lagrangian submanifolds $L$ invariant under $SO(n)$ are diffeomorphic to $I \times S^{n-1}$ and embedded in $T^*S^n \cong Q^n$ generically.*

(1) *Two ends of $L$ in $Q^n$ are asymptotic to special Lagrangian cones in $Q_0^n$ which are the cones over the orbit through*

$$\frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}\frac{k\pi-\theta}{n-1}}, \sqrt{-1}e^{\sqrt{-1}\frac{k\pi-\theta}{n-1}}, 0, \dots, 0\right) \quad (k \in \mathbb{Z}) \qquad (n : \text{even})$$

$$\frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}\frac{(2k+1)\pi-2\theta}{2(n-1)}}, \sqrt{-1}e^{\sqrt{-1}\frac{(2k+1)\pi-2\theta}{2(n-1)}}, 0, \dots, 0\right) \quad (k \in \mathbb{Z}) \qquad (n : \text{odd})$$

*by the action of $SO(n)$.*

(2) *When the curve $\tau$ passes through 0 or $\pi$, the map $\Psi : I \times S^{n-1} \to Q^n$ degenerates, and $n$ special Lagrangian submanifolds of $Q^n$ meet at the singular point $z = (\pm 1, 0, \dots, 0)$.*

**Example.** In the case of $n = 4, p = 1, q = 4$, the differential equation (3.3) is

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\tau'(\sin\tau)^3\right) = 0.$$

The following figures shows solution curves of this ODE, when $\theta = 0, \pi/4$ and $\pi/2$.

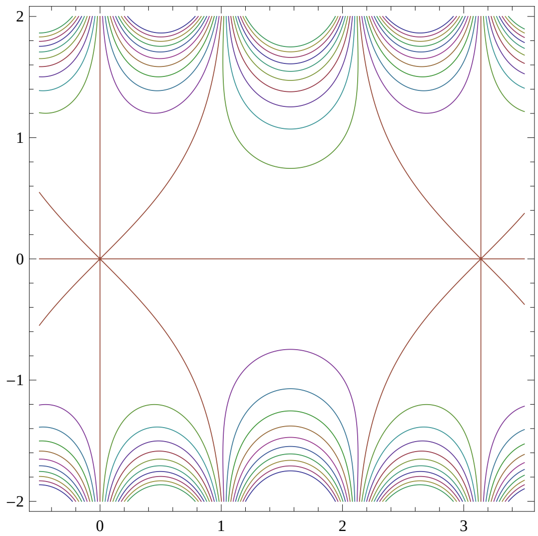


FIGURE 4. $\theta = 0$

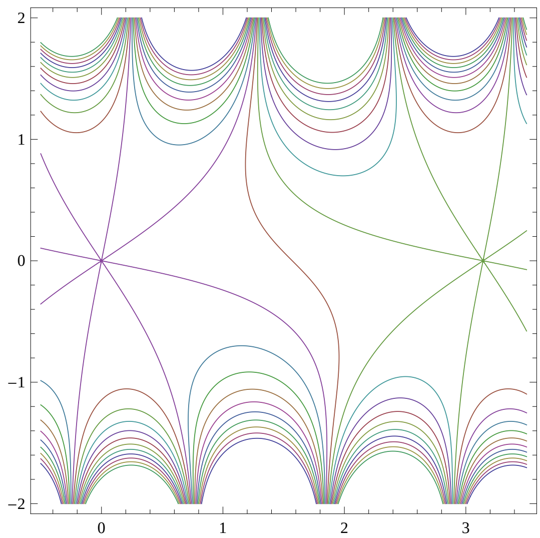


FIGURE 5. $\theta = \pi/4$

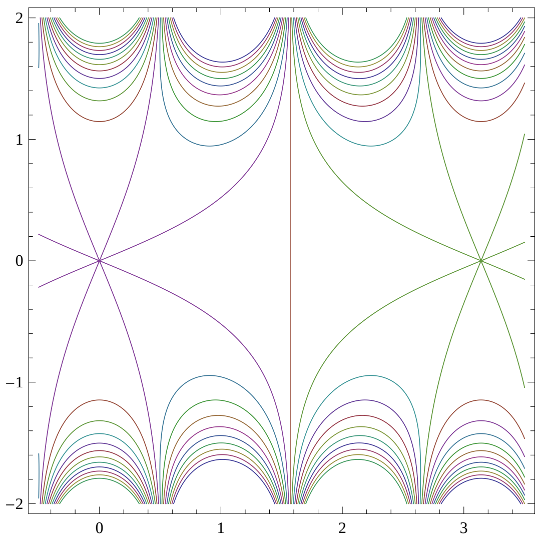


FIGURE 6. $\theta = \pi/2$

5.3. **Case of $p = 2, q \geq 3$.** We express $z \in \Phi(\Sigma)$ as

$$z = (z_1, z_2, z_3, 0, \ldots, 0),$$

where

$$\begin{aligned} z_1 &= \cos t \cosh \rho - \sqrt{-1}\frac{\xi_1}{\rho} \sin t \sinh \rho, \\ z_2 &= \sqrt{-1}\frac{\xi_2}{\rho} \sinh \rho, \\ z_3 &= \sin t \cosh \rho + \sqrt{-1}\frac{\xi_1}{\rho} \cos t \sinh \rho. \end{aligned}$$

Then the condition to be $z \in \mu^{-1}(c_1\theta_{12})$ is

$$c_1 = -u'(\cosh(2\rho))\frac{\xi_2}{\rho} \cos t \sinh(2\rho).$$

This equation approaches to the condition to be $z \in \mu^{-1}(0)$ as $\rho \to \infty$. Thus $\mu^{-1}(c_1\theta_{12}) \cap \Phi(\Sigma)$ is asymptotic to $\mu^{-1}(0) \cap \Phi(\Sigma)$ as $\rho \to \infty$. Therefore, we shall describe the asymptotic behavior of special Lagrangian submanifolds in the case of $c_1 = 0$.

When $c_1 = 0$, the orbit space $\mu^{-1}(0)/G$ of $G$-action on $\mu^{-1}(0)$ is parametrized as

$$\mu^{-1}(0) \cap \Phi(\Sigma) = \{(\cos \tau, 0, \sin \tau, 0, \ldots, 0) \mid \tau = t + \sqrt{-1}\xi_1 \ (t, \xi_1 \in \mathbb{R})\}.$$

Let $\tau$ be a regular curve in the complex plane $\mathbb{C}$. We define a curve $\sigma$ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by

$$\sigma(s) = (\cos \tau(s), 0, \sin \tau(s), 0, \ldots, 0).$$

Then the $G$-orbit $L = G \cdot \sigma$ through $\sigma$ is a Lagrangian submanifold in $Q^n$. For a curve $\tau$, $L$ coincides with the image of the following map:

$$\begin{aligned} \Psi_0 : I \times S^1 \times S^{n-2} &\longrightarrow Q^n \\ (s, x, y) &\longmapsto (\cos \tau(s)x_1, \cos \tau(s)x_2, \sin \tau(s)y_1, \ldots, \sin \tau(s)y_{n-1}). \end{aligned}$$

When $\tau$ passes through $m\pi/2$ ($m \in \mathbb{Z}$), the map $\Psi_0$ degenerates at that point. If $\tau$ does not pass through $m\pi/2$ ($m \in \mathbb{Z}$), then $L$ is diffeomorphic to $I \times S^1 \times S^{n-2}$ and immersed in $Q^n$ by the map $\Psi_0$. Moreover, $L$ is a special Lagrangian submanifold of phase $\theta$ if and only if $\tau$ satisfies

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\tau' \cos \tau (\sin \tau)^{n-2}\right) = 0. \tag{5.2}$$

This condition is equivalent to the equation

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}(\sin \tau)^{n-1}\right) = c_2$$

for some $c_2 \in \mathbb{R}$. In the phase space $\mathbb{C}$, the orbit space of $G$-action on $\mu^{-1}(0)$ can be reduced to

$$\{\tau = t + \sqrt{-1}\xi_1 \mid 0 \leq t \leq \frac{\pi}{2},\ \xi_1 \in \mathbb{R}\}.$$

In this area, (5.2) has singularities at 0 and $\pi/2$. When $\theta = 0$, the real segment $[0, \pi/2]$ is a trivial solution, and its corresponding special Lagrangian submanifold is the zero-section $S^n$ of $T^*S^n$.

Then, similarly with the previous cases, we obtain the following observations.

**Proposition 5.4.** *In the case of $p = 2, q \geq 3$, cohomogeneity one special Lagrangian submanifolds $L$ invariant under $SO(2) \times SO(n-2)$ are diffeomorphic to $I \times S^1 \times S^{n-2}$ and embedded in $T^*S^n \cong Q^n$ generically.*

(1) *Two ends of $L$ in $Q^n$ are asymptotic to special Lagrangian cones in $Q^n_0$ which are the cones over the orbits through*

$$\frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}\frac{k\pi-\theta}{n-1}}, 0, \sqrt{-1}e^{\sqrt{-1}\frac{k\pi-\theta}{n-1}}, 0, \dots, 0\right) \quad (k \in \mathbb{Z}) \qquad (n : \text{odd})$$

$$\frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}\frac{(2k+1)\pi-2\theta}{2(n-1)}}, 0, \sqrt{-1}e^{\sqrt{-1}\frac{(2k+1)\pi-2\theta}{2(n-1)}}, 0, \dots, 0\right) \quad (k \in \mathbb{Z}) \qquad (n : \text{even})$$

*by the action of $SO(2) \times SO(n-1)$.*

(2) *When the curve $\sigma$ in $\mu^{-1}(c_1\theta_{12}) \cap \Phi(\Sigma)$ passes through $z = (\pm\cosh(\xi_2), \sqrt{-1}\sinh(\xi_2), 0, \dots, 0)$, the map $\Psi : I \times S^1 \times S^{n-2} \to Q^n$ degenerates at that point. Especially when $\sigma$ passes through $z = (\pm 1, 0, \dots, 0)$, then $(n-1)$ special Lagrangian submanifolds of $Q^n$ meet at the singular set $S^1$.*

(3) *When the curve $\sigma$ passes through $z = (0, 0, \pm 1, 0, \dots, 0)$, the map $\Psi : I \times S^1 \times S^{n-2} \to Q^n$ degenerates, and 2 special Lagrangian submanifolds of $Q^n$ meet at the singular set $S^{n-2}$.*

5.4. **Case of $p = q = 2$.** We express $z \in \Phi(\Sigma)$ as

$$z = (z_1, z_2, z_3, z_4),$$

where

$$z_1 = \cos t \cosh\rho - \sqrt{-1}\frac{\xi_1}{\rho}\sin t \sinh\rho,$$
$$z_2 = \sqrt{-1}\frac{\xi_2}{\rho}\sinh\rho,$$
$$z_3 = \sin t \cosh\rho + \sqrt{-1}\frac{\xi_1}{\rho}\cos t \sinh\rho,$$
$$z_4 = \sqrt{-1}\frac{\xi_3}{\rho}\sinh\rho.$$

Then the conditions to be $z \in \mu^{-1}(c_1\theta_{12} + c_2\theta_{34})$ are

$$c_1 = -u'(\cosh(2\rho))\frac{\xi_2}{\rho}\cos t \sinh(2\rho),$$
$$c_2 = -u'(\cosh(2\rho))\frac{\xi_3}{\rho}\sin t \sinh(2\rho).$$

These equations approach to the condition to be $z \in \mu^{-1}(0)$ as $\rho \to \infty$. Thus $\mu^{-1}(c_1\theta_{12}+c_2\theta_{34}) \cap \Phi(\Sigma)$ is asymptotic to $\mu^{-1}(0) \cap \Phi(\Sigma)$ as $\rho \to \infty$. Therefore, we shall describe the asymptotic behavior of special Lagrangian submanifolds in the case of $c_1 = c_2 = 0$.

When $c_1 = c_2 = 0$, the orbit space $\mu^{-1}(0)/G$ of the $G$-action on $\mu^{-1}(0)$ is parametrized as

$$\mu^{-1}(0) \cap \Phi(\Sigma) = \left\{(\cos\tau, 0, \sin\tau, 0) \mid \tau = t + \sqrt{-1}\xi_1 \ (t, \xi_1 \in \mathbb{R})\right\}.$$

Let $\tau$ be a regular curve in the complex plane $\mathbb{C}$. We define a curve $\sigma$ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by

$$\sigma(s) = (\cos\tau(s), 0, \sin\tau(s), 0).$$

Then the $G$-orbit $L = G \cdot \sigma$ through $\sigma$ is a Lagrangian submanifold. For a curve $\tau$, $L$ coincides with the image of the following map:

$$\begin{array}{rcl} \Psi_0 : I \times S^1 \times S^1 & \longrightarrow & Q^3 \\ (s, x, y) & \longmapsto & (\cos\tau(s)x_1, \cos\tau(s)x_2, \sin\tau(s)y_1, \sin\tau(s)y_2). \end{array}$$

When $\tau$ passes through $m\pi/2$ $(m \in \mathbb{Z})$, the map $\Psi_0$ degenerates at that point. If $\tau$ does not pass through $m\pi/2$ $(m \in \mathbb{Z})$, then $L$ is diffeomorphic to $I \times S^1 \times S^1$ and immersed in $Q^3$ by the map $\Psi_0$. Moreover, $L$ is a special Lagrangian submanifold of phase $\theta$ if and only if $\tau$ satisfies

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\tau'\cos\tau\sin\tau\right) = 0. \tag{5.3}$$

This condition is equivalent to

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}(\sin\tau)^2\right) = c_3$$

for some $c_3 \in \mathbb{R}$. In the phase space $\mathbb{C}$, the orbit space of $G$-action on $\mu^{-1}(0)$ can be reduced to

$$\{\tau = t + \sqrt{-1}\xi_1 \mid 0 \leq t \leq \frac{\pi}{2},\ \xi_1 \in \mathbb{R}\}.$$

In this area, (5.3) has singularities at 0 and $\pi/2$. When $\theta = 0$, the real segment $[0, \pi/2]$ is a trivial solution, and its corresponding special Lagrangian submanifold is the zero-section $S^3$ of $T^*S^3$.

Then we obtain the following observations.

**Proposition 5.5.** *In the case of $p = q = 2$, cohomogeneity one special Lagrangian submanifolds $L$ invariant under $SO(2) \times SO(2)$ are diffeomorphic to $I \times S^1 \times S^1$ and embedded in $T^*S^3 \cong Q^3$ generically.*

(1) *Two ends of $L$ in $Q^3$ are asymptotic to special Lagrangian cones in $Q^3_0$ which are the cones over the orbits through*

$$\frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}\frac{k\pi-\theta}{2}}, 0, \sqrt{-1}e^{\sqrt{-1}\frac{k\pi-\theta}{2}}, 0\right) \quad (k \in \mathbb{Z})$$

*by the action of $SO(2) \times SO(2)$.*

(2) *When the curve $\sigma$ in $\mu^{-1}(c_1\theta_{12}+c_2\theta_{34}) \cap \Phi(\Sigma)$ passes through $z = (\pm\cosh(\xi_2), \sqrt{-1}\sinh(\xi_2), 0, 0)$ or $(0, 0, \pm\cosh(\xi_3), \sqrt{-1}\sinh(\xi_3))$, the map $\Psi : I \times S^1 \times S^1 \to Q^3$ degenerates at that point. Especially when $\sigma$ passes through $z = (\pm 1, 0, 0, 0)$ or $(0, 0, \pm 1, 0)$, then 2 special Lagrangian submanifolds of $Q^3$ meet at the singular set $S^1$.*

5.5. **Case of** $p = 1, q = 2$**.** We express $z \in \Phi(\Sigma)$ as

$$z = (z_1, z_2, z_3),$$

where

$$z_1 = \cos t \cosh\rho - \sqrt{-1}\frac{\xi_1}{\rho}\sin t \sinh\rho,$$

$$z_2 = \sin t \cosh\rho + \sqrt{-1}\frac{\xi_1}{\rho}\cos t \sinh\rho,$$

$$z_3 = \sqrt{-1}\frac{\xi_2}{\rho}\sinh\rho.$$

Then the condition to be $z \in \mu^{-1}(c_1\theta_{23})$ is

$$c_2 = -u'(\cosh(2\rho))\frac{\xi_2}{\rho}\sin t \sinh(2\rho).$$

This equation approaches to the condition to be $z \in \mu^{-1}(0)$ as $\rho \to \infty$. Thus $\mu^{-1}(c_1\theta_{23}) \cap \Phi(\Sigma)$ is asymptotic to $\mu^{-1}(0) \cap \Phi(\Sigma)$ as $\rho \to \infty$. Therefore, we shall describe the asymptotic behavior of special Lagrangian submanifolds in the case of $c_1 = 0$.

When $c_1 = 0$, the orbit space $\mu^{-1}(0)/G$ of the $G$-action on $\mu^{-1}(0)$ is parametrized as

$$\mu^{-1}(0) \cap \Phi(\Sigma) = \left\{(\cos\tau, \sin\tau, 0) \mid \tau = t + \sqrt{-1}\xi_1\ (t, \xi_1 \in \mathbb{R})\right\}.$$

Let $\tau$ be a regular curve in the complex plane $\mathbb{C}$. We define a curve $\sigma$ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by

$$\sigma(s) = (\cos\tau(s), \sin\tau(s), 0).$$

Then the $G$-orbit $L = G \cdot \sigma$ through $\sigma$ is a Lagrangian submanifold. For a curve $\tau$, $L$ coincides with the image of the following map:

$$\begin{array}{rcl} \Psi_0 : I \times S^1 & \longrightarrow & Q^2 \\ (s, y) & \longmapsto & (\cos\tau(s), \sin\tau(s)y_1, \sin\tau(s)y_2). \end{array}$$

When $\tau$ passes through $m\pi$ ($m \in \mathbb{Z}$), the map $\Psi_0$ degenerates at that point. If $\tau$ does not pass through $m\pi$ ($m \in \mathbb{Z}$), then $L$ is diffeomorphic to $I \times S^1$ and immersed in $Q^2$ by the map $\Psi_0$. Moreover, $L$ is a special Lagrangian submanifold of phase $\theta$ if and only if $\tau$ satisfies

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\tau' \sin\tau\right) = 0. \tag{5.4}$$

This condition is equivalent to

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\cos\tau\right) = c_2$$

for some $c_2 \in \mathbb{R}$. In the phase space $\mathbb{C}$, the orbit space of $G$-action on $\mu^{-1}(0)$ can be reduced to

$$\{\tau = t + \sqrt{-1}\xi_1 \mid 0 \le t \le \pi,\ \xi_1 \in \mathbb{R}\}.$$

In this area, (5.4) has singularities at 0 and $\pi$. When $\theta = 0$, the real segment $[0, \pi]$ is a trivial solution, and its corresponding special Lagrangian submanifold is the zero-section $S^2$ of $T^*S^2$.

Then we obtain the following observations.

**Proposition 5.6.** *In the case of $p = 1, q = 2$, cohomogeneity one special Lagrangian submanifolds $L$ invariant under $SO(2)$ are diffeomorphic to $I \times S^1$ and embedded in $T^*S^2 \cong Q^2$ generically.*

(1) *Two ends of $L$ in $Q^2$ are asymptotic to special Lagrangian cones in $Q_0^2$ which are the cones over the orbits through*

$$\frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}(k\pi-\theta)}, \sqrt{-1}e^{\sqrt{-1}(k\pi-\theta)}, 0\right) \quad (k \in \mathbb{Z})$$

*by the action of $SO(2)$.*

(2) *When the curve $\sigma$ in $\mu^{-1}(c_1\theta_{23}) \cap \Phi(\Sigma)$ passes through $z = (\pm 1, 0, 0)$, the map $\Psi : I \times S^1 \to Q^2$ degenerates, and 2 special Lagrangian submanifolds of $Q^2$ meet at the singular point $z = (\pm 1, 0, 0)$.*

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