A Payne-Rayner type inequality for the Robin problem on arbitrary minimal surfaces in $\mathbb{R}^N$

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Abstract. We prove a Payne-Rayner type inequality for the first eigenfunction of the Laplacian with Robin boundary condition on *any* compact minimal surface with boundary in $\mathbb{R}^N$. We emphasize that no topological condition is necessary on the boundary.

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1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$, and let $\lambda_1(\Omega)$ and ψ denote the first eigenvalue and the corresponding first eigenfunction, respectively, to the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In [7], Payne and Rayner proved the following inequality

$$\left(\int_\Omega \psi^2 dx \right) \le \frac{\lambda_1(\Omega)}{4\pi} \left(\int_\Omega \psi \, dx \right)^2 .$$

A remarkable point of this inequality is that it gives an exact lower-bound of the first eigenvalue by means of some integral-norms of the first eigenfunction, on one hand, and on the other hand, it also says that the first eigenfunction satisfies a reverse Hölder type inequality. Actually, the L^2 norm of ψ is bounded by the L^1 norm of ψ.

In this paper, we extend the above result, known to hold on a flat domain with the Dirichlet boundary condition, to a more general setting. Namely, let Σ be

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a compact minimal surface in $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial\Sigma$. We consider the following eigenvalue problem with the Robin boundary condition:

$$\begin{cases} -\Delta_\Sigma u = \lambda u & \text{in } \Sigma, \\ \frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Sigma, \end{cases} \tag{1.1}$$

where Δ_Σ is the Laplace-Beltrami operator on Σ, β is a positive constant and ν is the outer unit normal to $\partial\Sigma$. Let $\lambda_1^\beta(\Sigma)$ denote the first eigenvalue of (1.1), given by the variational formula

$$\lambda_1^\beta(\Sigma) = \min_{u \in H^1(\Sigma)} \frac{\int_\Sigma |\nabla_\Sigma u|^2 d\mathcal{H}^2 + \beta \int_{\partial\Sigma} u^2 d\mathcal{H}^1}{\int_\Sigma u^2 d\mathcal{H}^2},$$

where ∇_Σ is the gradient operator on Σ and $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure in $\mathbb{R}^N$. It is well known that $\lambda_1^\beta(\Sigma)$ is simple and isolated, and the corresponding eigenfunction ψ_β is smooth, positive, and unique up to multiplication by constants. (see, for example, [3]).

Now, let us consider the auxiliary problem

$$\begin{cases} \Delta_\Sigma f = 2 & \text{in } \Sigma, \\ f = 0 & \text{on } \partial\Sigma. \end{cases} \tag{1.2}$$

Our main result is the following Payne-Rayner type inequality.

Theorem 1.1. *Let $\lambda_1^\beta(\Sigma)$ be the first eigenvalue of (1.1) and ψ_β be the eigenfunction corresponding to $\lambda_1^\beta(\Sigma)$. Then*

$$\int_\Sigma \psi_\beta^2 \, d\mathcal{H}^2 \leq \frac{\lambda_1^\beta(\Sigma)}{\sqrt{2}\pi} \left(\int_\Sigma \psi_\beta \, d\mathcal{H}^2 \right)^2 + \frac{1}{2} \int_{\partial\Sigma} \psi_\beta^2 \left(\frac{\partial f_\Sigma}{\partial \nu} \right) \, d\mathcal{H}^1 + \frac{1}{\sqrt{2}\pi} \mathcal{H}^1(\partial\Sigma)^2 (M^2 - m_*^2)$$

holds, where $M = \max_{\partial\Sigma} \psi_\beta$, $m_* = \min_{\Sigma \cup \partial\Sigma} \psi_\beta$, and f_Σ is the unique solution to the problem (1.2)

As for the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta_\Sigma u = \lambda u & \text{in } \Sigma, \\ u = 0 & \text{on } \partial\Sigma, \end{cases} \tag{1.3}$$

the same proof of Theorem 1.1 works well and we obtain

Theorem 1.2. *Let $\lambda_1^D(\Sigma)$ be the first eigenvalue of (1.3) and ψ_D be the eigenfunction corresponding to $\lambda_1^D(\Sigma)$. Then we have*

$$\int_\Sigma \psi_D^2 \, d\mathcal{H}^2 \leq \frac{\lambda_1^D(\Sigma)}{2\sqrt{2}\pi} \left(\int_\Sigma \psi_D \, d\mathcal{H}^2 \right)^2.$$

Under the assumption that the boundary $\partial\Sigma$ is *weakly connected* (see Li-Schoen-Yau [6]), Wang and Xia [8] recently proved the sharp inequality

$$\int_\Sigma \psi_D^2 \, d\mathcal{H}^2 \leq \frac{\lambda_1^D(\Sigma)}{4\pi} \left(\int_\Sigma \psi_D \, d\mathcal{H}^2 \right)^2$$

for the first eigenfunction to (1.3), with the equality holds if and only if Σ is a flat disc on an affine 2-plane in $\mathbb{R}^N$.

Our method of proof is strongly related to that of [8], which in turn goes back to the work [7]. However, in our case, we cannot apply the sharp isoperimetric inequality by Li-Schoen-Yau [6] directly to level sets of the first eigenfunction, since we put no topological assumptions on the boundary. Instead, we use a weaker version of the isoperimetric inequality due to A. Stone ([1]: Lemma 4.3):

Let Σ be a compact minimal surface in $\mathbb{R}^N$ with boundary $\partial\Sigma$. Let A denote the area of Σ and L the length of $\partial\Sigma$. Then the inequality

$$2\sqrt{2}\pi A \leq L^2 \tag{1.4}$$

holds.

Though the constant $2\sqrt{2}\pi$ in front of A is not the best possible value 4π, this weaker inequality is valid for *any* compact minimal surface in $\mathbb{R}^N$ with boundary. Thanks to this, we do not need any topological assumption such as weak connectedness on the boundary in Theorem 1.1 and Theorem 1.2.

In case $\Sigma = \Omega \subset \mathbb{R}^2$ is a bounded smooth domain in (1.1), we can appeal to the classical sharp isoperimetric inequality $4\pi A \leq L^2$ on the plane, then we obtain

Theorem 1.3. *Let $\Sigma = \Omega$ is a smooth bounded domain in $\mathbb{R}^2$. Then we have*

$$\int_\Omega \psi_\beta^2 \, dx \leq \frac{\lambda_1^\beta(\Omega)}{2\pi} \left(\int_\Omega \psi_\beta \, dx \right)^2 + \frac{1}{2} \int_{\partial\Omega} \psi_\beta^2 \left(\frac{\partial f_\Omega}{\partial \nu} \right) d\mathcal{H}^1 + \frac{1}{2\pi} \mathcal{H}^1(\partial\Sigma)^2 (M^2 - m_*^2)$$

We do not repeat the proof of Theorem 1.2 and Theorem 1.3 here, since it needs only a trivial change in the proof of Theorem 1.1.

2. Proof of Theorem 1.1

First, we set

$$U(t) = \{x \in \Sigma : \psi_\beta(x) > t\},$$
$$S(t) = \Sigma \cap \partial U(t),$$
$$\Gamma(t) = \partial\Sigma \cap \partial U(t)$$

for $t > 0$. Then $\partial U(t) = S(t) \cup \Gamma(t)$ is a disjoint union. Since ψ_β is smooth up to the boundary ([5]), Sard's lemma implies that $|\nabla_\Sigma \psi_\beta| \neq 0$ on $S(t)$, $S(t)$ is a smooth hypersuraface and can be written as $S(t) = \{x \in \Sigma : \psi_\beta(x) = t\}$ for a.e. $t > 0$. Recall $M = \max_{\partial\Sigma} \psi_\beta$ and $m_* = \min_{\Sigma\cup\partial\Sigma} \psi_\beta$. We claim that $\min_{\partial\Sigma} \psi_\beta > 0$. Indeed, if $\psi_\beta(x_0) = 0$ for some $x_0 \in \partial\Sigma$, then the boundary condition implies that $\frac{\partial\psi_\beta}{\partial\nu}(x_0) = 0$ also holds. On the other hand, by the positivity of ψ_β and Hopf's lemma, we have $\frac{\partial\psi_\beta}{\partial\nu}(x_0) < 0$, which is a contradiction. Since ψ_β is positive on Σ, the above claim yields $m_* > 0$, and then $U(t) = \Sigma$ for any $0 < t < m_*$. Also we note that $\Gamma(t) = \phi$ if $t > M$.

As in the proof of [2], [3], [8], our main tool is the following co-area formula, asserting that for every $w \in L^1(\Sigma)$, it holds

$$\int_{U(t)} w d\mathcal{H}^2 = \int_t^\infty \int_{S(\tau)} \frac{w}{|\nabla_\Sigma \psi_\beta|} d\mathcal{H}^1 d\tau,$$

$$\frac{d}{dt} \int_{U(t)} w d\mathcal{H}^2 = - \int_{S(t)} \frac{w}{|\nabla_\Sigma \psi_\beta|} d\mathcal{H}^1.$$

See, for instance, [4]. Note that in the right hand side, the integral over $\Gamma(t)$ does not appear.

We define the following two functions g and h as

$$\begin{aligned} g(t) &= \int_{U(t)} \psi_\beta \, d\mathcal{H}^2 = \int_t^\infty \int_{S(\tau)} \frac{\psi_\beta}{|\nabla_\Sigma \psi_\beta|} d\mathcal{H}^1 d\tau, \\ h(t) &= - \int_{U(t)} \left\langle \nabla_\Sigma \left(\frac{1}{2} \psi_\beta^2 \right), \nabla_\Sigma f \right\rangle d\mathcal{H}^2 \\ &= - \int_t^\infty \int_{S(\tau)} \frac{\psi_\beta \langle \nabla_\Sigma \psi_\beta, \nabla_\Sigma f \rangle}{|\nabla_\Sigma \psi_\beta|} d\mathcal{H}^1 ds, \end{aligned}$$

where f is the unique solution of the problem (1.2).

Differentiating g and h, we have

$$g'(t) = -t \int_{S(t)} \frac{1}{|\nabla_\Sigma \psi_\beta|} \, d\mathcal{H}^1, \tag{2.1}$$

$$\begin{aligned} h'(t) &= t \int_{S(t)} \frac{\langle \nabla_\Sigma \psi_\beta, \nabla_\Sigma f \rangle}{|\nabla_\Sigma \psi_\beta|} \, d\mathcal{H}^1 = -t \int_{S(t)} \langle \nabla_\Sigma f, \nu \rangle \, d\mathcal{H}^1 \\ &= -t \int_{S(t)} \frac{\partial f}{\partial \nu} d\mathcal{H}^1 \end{aligned} \tag{2.2}$$

for a.e. $t > 0$, since $-\frac{\nabla_\Sigma \psi_\beta}{|\nabla_\Sigma \psi_\beta|}\Big|_{S(t)}$ is outward unit normal vector field ν of $S(t)$.

On the other hand, integrating both sides of $-\Delta_\Sigma \psi_\beta = \lambda_1^\beta(\Sigma)\psi_\beta$ over $U(t)$, we have

$$\begin{aligned} &\lambda_1^\beta(\Sigma) g(t) = \lambda_1^\beta(\Sigma) \int_{U(t)} \psi_\beta d\mathcal{H}^2 = - \int_{U(t)} \Delta_\Sigma \psi_\beta d\mathcal{H}^2 \\ &= \int_{S(t)} |\nabla_\Sigma \psi_\beta| \, d\mathcal{H}^1 - \int_{\Gamma(t)} \frac{\partial \psi_\beta}{\partial \nu} d\mathcal{H}^1 \\ &= \int_{S(t)} |\nabla_\Sigma \psi_\beta| \, d\mathcal{H}^1 + \beta \int_{\Gamma(t)} \psi_\beta d\mathcal{H}^1 \\ &\geq \int_{S(t)} |\nabla_\Sigma \psi_\beta| \, d\mathcal{H}^1, \end{aligned} \tag{2.3}$$

since $-\frac{\partial \psi_\beta}{\partial \nu} = \beta \psi_\beta > 0$ on $\Gamma(t) \subset \partial \Sigma$.

Also, we see

$$\begin{aligned} 2\mathcal{H}^2(U(t)) &= \int_{U(t)} 2 d\mathcal{H}^2 = \int_{U(t)} \Delta f d\mathcal{H}^2 = \int_{\partial U(t)} \frac{\partial f}{\partial \nu} d\mathcal{H}^1 \\ &= \int_{S(t)} \frac{\partial f}{\partial \nu} d\mathcal{H}^1 + \int_{\Gamma(t)} \frac{\partial f}{\partial \nu} d\mathcal{H}^1 \\ &\geq \int_{S(t)} \frac{\partial f}{\partial \nu} d\mathcal{H}^1 = \frac{-1}{t} h'(t) \end{aligned} \tag{2.4}$$

by (2.2). The last inequality follows by the fact $\frac{\partial f}{\partial \nu} > 0$ on $\Gamma(t) \subset \partial\Sigma$, which in turn is assured by the Hopf lemma.

From the weak isoperimetric inequality (1.4) applied to $U(t)$, we have

$$\begin{aligned} 2\sqrt{2}\pi\mathcal{H}^2(U(t)) &\leq \mathcal{H}^1(\partial U(t))^2 \\ &\leq \left(\mathcal{H}^1(S(t)) + \mathcal{H}^1(\Gamma(t))\right)^2 \\ &\leq 2\mathcal{H}^1(S(t))^2 + 2\mathcal{H}^1(\Gamma(t))^2. \end{aligned} \tag{2.5}$$

Now, Schwarz's inequality, (2.1) and (2.3) imply

$$\begin{aligned} \mathcal{H}^1(S(t))^2 = \left(\int_{S(t)} 1 \, d\mathcal{H}^1\right)^2 &\leq \left(\int_{S(t)} |\nabla_\Sigma \psi_\beta| \, d\mathcal{H}^1\right)\left(\int_{S(t)} \frac{1}{|\nabla_\Sigma \psi_\beta|} \, d\mathcal{H}^1\right) \\ &\leq \lambda_1^\beta(\Sigma) g(t) \cdot \left(-\frac{g'(t)}{t}\right). \end{aligned}$$

Therefore, by (2.4) and (2.5), we obtain

$$-\frac{\sqrt{2}\pi}{t} h'(t) \leq 2\sqrt{2}\pi\mathcal{H}^2(U(t)) \leq 2\lambda_1^\beta(\Sigma) g(t) \cdot \left(-\frac{g'(t)}{t}\right) + 2\mathcal{H}^1(\Gamma(t))^2,$$

or equivalently,

$$\frac{d}{dt}\left\{\lambda_1^\beta(\Sigma) g(t)^2 - \sqrt{2}\pi h(t) - \int_0^t 2\tau\mathcal{H}^1(\Gamma(\tau))^2 d\tau\right\} \leq 0. \tag{2.6}$$

for a.e $t > 0$. Note that the function $l(t) = 2t\mathcal{H}^1(\Gamma(t))^2$ is integrable on the interval $t \in (0, \|\psi_\beta\|_{L^\infty(\partial\Sigma)})$, and thus $l(t) = \frac{d}{dt}\int_0^t l(\tau) d\tau$.

Fix $\varepsilon > 0$ so small such that $\varepsilon < m_*$. Integrating (2.6) from $m_\varepsilon = m_* - \varepsilon$ to t, we have

$$\lambda_1^\beta(\Sigma) g(t)^2 - \sqrt{2}\pi h(t) - \int_0^t 2\tau\mathcal{H}^1(\Gamma(\tau))^2 d\tau \leq \lambda_1^\beta(\Sigma) g(m_\varepsilon)^2 - \sqrt{2}\pi h(m_\varepsilon) - \int_0^{m_\varepsilon} 2\tau\mathcal{H}^1(\Gamma(\tau))^2 d\tau,$$

which implies

$$\sqrt{2}\pi h(m_\varepsilon) \leq \lambda_1^\beta(\Sigma) g(m_\varepsilon)^2 - \lambda_1^\beta(\Sigma) g(t)^2 + \sqrt{2}\pi h(t) + \int_{m_\varepsilon}^t 2\tau\mathcal{H}^1(\Gamma(\tau))^2 d\tau.$$

We easily see that

$$\int_{m_\varepsilon}^{t} 2\tau\mathcal{H}^1(\Gamma(\tau))^2 d\tau \le \mathcal{H}^1(\partial\Sigma)^2 \int_{m_\varepsilon}^{M} 2\tau d\tau = \mathcal{H}^1(\partial\Sigma)^2 \left(M^2 - m_\varepsilon^2\right)$$

for any $t > m_\varepsilon$. Letting $t \to +\infty$, and noting that $U(t)$ is empty for sufficiently large t, we obtain

$$h(m_\varepsilon) \le \frac{\lambda_1^\beta(\Sigma)}{\sqrt{2}\pi} g^2(m_\varepsilon) + \frac{1}{\sqrt{2}\pi}\mathcal{H}^1(\partial\Sigma)^2 \left(M^2 - m_\varepsilon^2\right).$$

$g(m_\varepsilon)$ and $h(m_\varepsilon)$ are given by

$$\begin{aligned} g(m_\varepsilon) &= \int_\Sigma \psi_\beta \, d\mathcal{H}^2, \\ h(m_\varepsilon) &= -\int_\Sigma \left\langle \nabla_\Sigma \left(\frac{1}{2}\psi_\beta^2\right), \nabla_\Sigma f \right\rangle d\mathcal{H}^2 \\ &= \int_\Sigma \frac{1}{2}\psi_\beta^2 \Delta f \, d\mathcal{H}^2 - \frac{1}{2}\int_{\partial\Sigma} \psi_\beta^2 \frac{\partial f}{\partial\nu} d\mathcal{H}^1. \end{aligned}$$

Since $\Delta_\Sigma f = 2$ by (1.2), we have

$$\int_\Sigma \psi_\beta^2 \, d\mathcal{H}^2 - \frac{1}{2}\int_{\partial\Sigma} \psi_\beta^2 \frac{\partial f}{\partial\nu} d\mathcal{H}^1 \le \frac{\lambda_1^\beta(\Sigma)}{\sqrt{2}\pi}\left(\int_\Sigma \psi_\beta \, d\mathcal{H}^2\right)^2 + \frac{1}{\sqrt{2}\pi}\mathcal{H}^1(\partial\Sigma)^2 \left(M^2 - m_\varepsilon^2\right).$$

Finally letting $\varepsilon \to 0$, we obtain the result. □

Remark 2.1. *In the case that* $\Omega = B_R \subset \mathbb{R}^2$ *is a disc of radius* R*, then the inequality in Theorem* 1.3 *becomes the equality*

$$\int_{B_R} \psi_\beta^2 \, dx = \frac{\lambda_1^\beta(\Omega)}{4\pi}\left(\int_{B_R} \psi_\beta \, dx\right)^2 + \frac{R}{2}\int_{\partial\Omega} \psi_\beta^2 \, d\mathcal{H}^1. \tag{2.7}$$

This is because, first, ψ_β is positive, radial and decreasing in the radial direction on B_R ([3]:Proposition 2.6). Therefore $\psi_\beta \equiv c > 0$ on ∂B_R and $U(c) = B_R$, $\partial U(t) = S(t)$ for any $t > c$. Also $|\nabla\psi_\beta|$ is constant on $S(t)$. Secondly, we can use the sharp isoperimetric inequality as the equality $4\pi\mathcal{H}^2(U(t)) = \mathcal{H}^1(S(t))^2$ in (2.5) in this case. Finally, the unique solution f_{B_R} of (1.2) is $f_{B_R} = \frac{1}{2}|x|^2 - \frac{1}{2}R^2$. By these reasons, we see all inequalities in the proof of Theorem 1.1 are equalities and we obtain

$$\frac{d}{dt}\left\{\lambda_1^\beta(B_R) g(t)^2 - 4\pi h(t)\right\} = 0$$

for a.e. $t > c$, instead of (2.6). Integrating this from $t = c$ to t, and letting $t \to \infty$, we obtain (2.7).

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