SEMI-DISCRETE LINEAR WEINGARTEN SURFACES WITH WEIERSTRASS-TYPE REPRESENTATIONS AND THEIR SINGULARITIES

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Abstract

We establish what semi-discrete linear Weingarten surfaces with Weierstrass-type representations in 3-dimensional Riemannian and Lorentzian spaceforms are, confirming their required properties regarding curvatures and parallel surfaces, and then classify them. We then define and analyze their singularities. In particular, we discuss singularities of (1) semi-discrete surfaces with non-zero constant Gaussian curvature, (2) parallel surfaces of semi-discrete minimal and maximal surfaces, and (3) semi-discrete constant mean curvature 1 surfaces in de Sitter 3-space. We include comparisons with different previously known definitions of such singularities.

1. Introduction

Smooth (spacelike) linear Weingarten surfaces in 3-dimensional Riemannian or Lorentzian spaceforms are those for which the Gaussian and mean curvatures K and H satisfy an affine linear relation

$$\alpha K + 2\beta H + \gamma = 0$$

for constants α, β and γ not all zero, and generally these surfaces will have singularities. There are special cases of these surfaces that admit Weierstrass-type representations:

(1) minimal surfaces in 3-dimensional Euclidean space $\mathbb{R}^3$ and their parallel surfaces,

(2) maximal surfaces in 3-dimensional Minkowski space $\mathbb{R}^{2,1}$ and their parallel surfaces,

(3) surfaces in 3-dimensional hyperbolic space $\mathbb{H}^3$ such that $\alpha = 1 - \beta$ and $\gamma = -1 - \beta$, referred to here as linear Weingarten surfaces of Bryant type, or BrLW surfaces for short (note that flat surfaces occur when $\beta = 0$),

(4) surfaces in 3-dimensional de Sitter space $\mathbb{S}^{2,1}$ such that $\alpha = -1 - \beta$ and $\gamma = 1 - \beta$, referred to here as linear Weingarten surfaces of Bianchi type, or BiLW surfaces for short.

The case of fully discrete surfaces with Weierstrass-type representations was considered in [19], and the semi-discrete case is considered here. The semi-discrete case incorporates

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properties of both the smooth and fully discrete cases, and is informative for understanding differences between those two cases, and in this work we elucidate some aspects of this, particularly with regard to properties of singularities of these surfaces. Singularities of smooth surfaces with Weierstrass representations have been carefully examined recently, see, for example, [6], [8], [11], [15], [21].

Amongst our results, we establish the next two facts (see Section 4), which are important for confirming that our choices for Weierstrass-type representations for semi-discrete surfaces are correct.

Fact 1. *Semi-discrete surfaces with Weierstrass representations satisfy the same affine linear relations between the Gaussian and mean curvatures as both smooth and fully discrete surfaces with Weierstrass representations do.*

In the smooth case, as mentioned in [16], parallel surfaces of BrLW surfaces in $\mathbb{H}^3$, resp. BiLW surfaces in $\mathbb{S}^{2,1}$, are also BrLW surfaces, resp. BiLW surfaces, and these surfaces are classified into three types. Including minimal and maximal surfaces, there are then five types, like as listed in Fact 2 below. The same is true of fully discrete surfaces with Weierstrass-type representations, see [19].

In this paper we investigate semi-discrete linear Weingarten surfaces with Weierstrass-type representations. As will be seen later, together with explanations of the terminologies used, semi-discrete linear Weingarten surfaces are classified as in Fact 2 below.

Fact 2. *Semi-discrete surfaces with Weierstrass-type representations can be classified into the following five types:*

(1) *minimal surfaces and their parallel surfaces in $\mathbb{R}^3$,*
(2) *maximal surfaces and their parallel surfaces in $\mathbb{R}^{2,1}$,*
(3) *flat surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$,*
(4) *linear Weingarten surfaces of hyperbolic type in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$,*
(5) *linear Weingarten surfaces of de Sitter type in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$.*

Parallel surfaces of each type belong again to the same type.

Singularities on semi-discrete surfaces. In the smooth case, linear Weingarten surfaces with Weierstrass-type representations as listed in Fact 2 above might have singularitites. So it is natural to expect that semi-discrete linear Weingarten surfaces with Weierstrass-type representations also have some notion of "singularities" (for the fully discrete case, see [14], [19], [22]). The main purpose in this paper is to clarify and characterize such singularities.

Let us remark on two previous works on singularities of semi-discrete surfaces:

(1) In [23], the first author described semi-discrete maximal surfaces in $\mathbb{R}^{2,1}$ and analyzed their singularities. Singularities of semi-discrete maximal surfaces were defined on the set of edges and so are called *singular edges*, and they reflect the property of non-spacelikeness of tangent planes of smooth maximal surfaces at singular points (see [23] and Definition 5.5 here).

(2) Though singular edges can appear on semi-discrete spacelike surfaces with Weierstrass-type representations in Lorentzian spaceforms, they do not appear on such semi-discrete surfaces in Riemannian spaceforms. In [24], in order to consider singularities of general semi-discrete surfaces in Riemannian spaceforms as well,
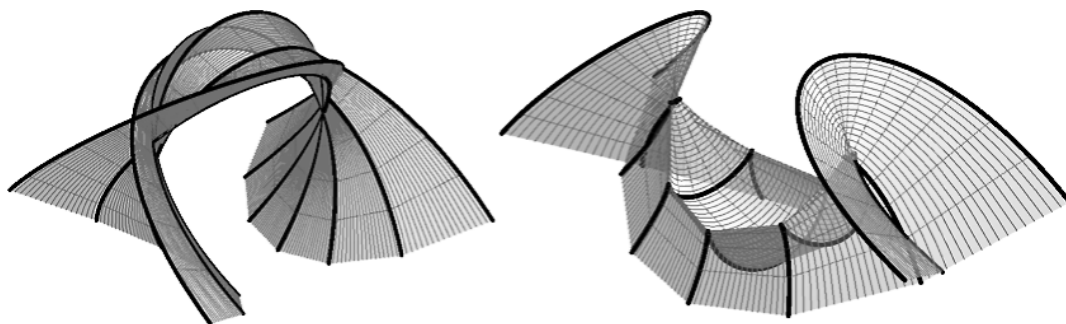
Fig. 1. Two different parallel surfaces of a semi-discrete Enneper minimal surface in $\mathbb{R}^3$, with singularities (which have the appearance of cuspidal edges and swallowtails).

points along the smooth curves of the semi-discrete surfaces that could be singular were introduced, which were called *flat-parabolic-singular (FPS, for short) points*. FPS points are directly related to behaviors of the principal curvatures of semi-discrete surfaces. Applying this, singularities of particular semi-discrete surfaces were analyzed.

However, as already mentioned in [24], those FPS points did not identify certain possible singularities that we would like to consider. So we need to modify the definition of FPS points of semi-discrete surfaces (see Definition 5.1 here). This enables us to analyze possible singularities that we could not analyze before.

The semi-discrete case has some uniquely interesting singular behaviors, since it combines elements from both the smooth and fully discrete cases. In the final section, we establish a definition of singularities on semi-discrete surfaces which takes into account that singularities can occur with respect to either the smooth parameter or the discrete parameter for the surface. Because, like in the fully discrete case, this definition incorporates sign changes in the principal curvatures, we need to also include the possibilities of flat and parabolic points. Examples of such singularities can be seen in Figure 1.

We thus find ourselves in a situation where we have two independent notions of potential singularities of semi-discrete surfaces, one defined on edges and the other defined at points in the smooth curves of the surfaces. It is natural to look for relations between these two notions, and this is the purpose of Theorems 5.3 and 5.5 here. Specifically, we prove that singular points on semi-discrete maximal surfaces in $\mathbb{R}^{2,1}$ and semi-discrete CMC 1 surfaces in $\mathbb{S}^{2,1}$ as defined in this paper imply existence of neighboring singular edges (see Theorems 5.3 and 5.5). Finally we give criteria for singular edges of semi-discrete CMC 1 surfaces in $\mathbb{S}^{2,1}$ and prove the analogous result as in Theorem 1.2 in [23] for this case as well (see Theorem 5.4). With these two theorems we see strong correspondence between the two notions of potential singularities, giving us further confidence in the usefulness of these two notions.

Along the way, we give criteria for determining singularities on parallel surfaces of semi-discrete minimal and maximal surfaces (see Theorem 5.1), as well as on semi-discrete surfaces of Bryant and Bianchi types (see Theorem 5.2).

2. Semi-discrete Legendre immersions in 3-dimensional spaceforms

Let M^3 be a 3-dimensional Riemannian or Lorentzian spaceform that is a quadric in a 4-dimensional Riemannian or Lorentzian vector space V^4. A semi-discrete map is a map

$$x(k, t) : \mathbb{D} \to M^3 ,$$

where $\mathbb{D}$ is a subdomain of $\mathbb{Z} \times \mathbb{R}$. We define derivatives and differences of x by

$$x = x(k, t), \; x_1 = x(k + 1, t), \; \partial x = \frac{dx}{dt}, \; \Delta x = x_1 - x.$$

We will assume x is a conjugate net, that is, ∂x, ∂x_1 and Δx lie in a 2-plane in V^4, called the *tangent plane* of the surface at the edge $[x, x_1]$ with endpoints x and x_1.

Definition 2.1. *This map $x(k, t)$, together with a unit normal map $n(k, t)$, is called a* semi-discrete Legendre immersion

$$\mathbb{D} \ni (k, t) \to (x, n) \in T_1 M^3$$

if it satisfies the following conditions:
   (1) *∂n, ∂n_1 and Δn all lie in the tangent plane of the surface at the edge $[x, x_1]$,*
   (2) *Δx, n_1 and n all lie in one 2-dimensional plane in V^4,*
   (3) *n is perpendicular to ∂x.*

Like for the fully discrete case, the curvature line condition in the discrete direction is partially built into condition (2) above, but we would additionally require that Δn is parallel to Δx. The curvature line condition in the smooth direction is simply that ∂x and ∂n are parallel, as in the next definition. Existence of a curvature-line parametrization in the case of smooth surfaces rules out most types of umbilic points, and so in the following definition we are implicitly ruling out any semi-discrete surface with some notion of umbilic point.

Definition 2.2. *If $\Delta n \| \Delta x$ and $\partial x \| \partial n$ and the tangent cross ratio*

$$cr(x, x_1) := \partial x \cdot (\Delta x)^{-1} \cdot \partial x_1 \cdot (\Delta x)^{-1}$$

satisfies that

$$cr(x, x_1) \in \mathbb{R} \quad and \quad cr(x, x_1) < 0 ,$$

we say that x is curvature-line parametrized.

To define the tangent cross ratio $cr(x, x_1)$ above requires that we multiply and invert points in M^3, which can be done as follows: we set

$$\mathbb{R}^3 := \{(z_1, z_2, z_3, 0) \,|\, z_j \in \mathbb{R}\} \subseteq V = \mathbb{R}^4 := \{(z_1, z_2, z_3, z_4) \,|\, z_j \in \mathbb{R}\}$$

with standard Euclidean metric

$$(z_1, z_2, z_3, z_4) \circ (w_1, w_2, w_3, w_4) = z_1 w_1 + z_2 w_2 + z_3 w_3 + z_4 w_4$$

on V, and set $\mathbb{H}^3 = \mathbb{H}^3_+ \cup \mathbb{H}^3_-$, with

$$\mathbb{H}^3_+ := \{(z_1, z_2, z_3, z_0) \,|\, z_j \in \mathbb{R}, \, z_1^2 + z_2^2 + z_3^2 - z_0^2 = -1, \, z_0 > 0\} ,$$

$$\mathbb{H}^3_- := \{(z_1, z_2, z_3, z_0) \,|\, z_j \in \mathbb{R},\ z_1^2 + z_2^2 + z_3^2 - z_0^2 = -1,\ z_0 < 0\},$$

and

$$\mathbb{S}^{2,1} = \{(z_1, z_2, z_3, z_0) \,|\, z_j \in \mathbb{R},\ z_1^2 + z_2^2 + z_3^2 - z_0^2 = 1\},$$

all lying in

$$V = \mathbb{R}^{3,1} = \{(z_1, z_2, z_3, z_0) \,|\, z_j \in \mathbb{R}\}$$

with the Minkowski metric

$$(z_1, z_2, z_3, z_0) \circ (w_1, w_2, w_3, w_0) = z_1 w_1 + z_2 w_2 + z_3 w_3 - z_0 w_0 \ .$$

The relevant 4-dimensional spaces are only $\mathbb{R}^4$ and $\mathbb{R}^{3,1}$, and we can identify points in those two spaces with 2 by 2 matrices as follows:

$$\mathbb{R}^4 \ni (z_1, z_2, z_3, z_4) \mapsto \begin{pmatrix} z_1 + iz_2 & z_3 + iz_4 \\ -z_3 + iz_4 & z_1 - iz_2 \end{pmatrix}, \quad \mathbb{R}^{3,1} \ni (z_1, z_2, z_3, z_0) \mapsto \begin{pmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix}.$$

We then can regard multiplication and inversion of points in M^3 as multiplication and inversion of matrices. For example, in the case of $\mathbb{R}^{3,1}$,

$$(z_1, z_2, z_3, z_0) \circ (w_1, w_2, w_3, w_0) =$$
$$\frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} w_0 + w_3 & w_1 - iw_2 \\ w_1 + iw_2 & w_0 - w_3 \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \ .$$

Note that the tangent cross ratio being real, which means it is a real scalar multiple of the 2 by 2 identity matrix and then we can regard that scalar multiple as the tangent cross ratio itself, implies the circularity condition, that is, there is a circle through x and x_1 which is tangent to ∂x at x and ∂x_1 at x_1.

We can then define semi-discrete isothermic surface as follows:

Definition 2.3. *A semi-discrete x in M^3 is* semi-discrete isothermic *if the equation*

$$cr(x(k,t), x(k+1,t)) = \frac{\tau(t)}{\sigma(k)} < 0$$

holds, where $\tau = \tau(t) \in \mathbb{R}$ depends only on t and $\sigma = \sigma(k) \in \mathbb{R}$ depends only on k.

3. Curvatures of curvature-line parametrized semi-discrete surfaces

First we define the principal curvatures:

Definition 3.1. *For a semi-discrete Legendre map (x, n), the scalar functions $\kappa_k(t)$, $\kappa_{k,k+1}(t)$ given by*

$$\partial n = -\kappa_k(t)\partial x, \quad \Delta n = -\kappa_{k,k+1}(t)\Delta x \ ,$$

are called the principal curvatures *of x. Here we abbreviate*

$$\kappa = \kappa_k(t), \quad \kappa_1 = \kappa_{k+1}(t), \quad \kappa_{01} = \kappa_{k,k+1}(t) \quad (\text{also } \kappa_{-10} = \kappa_{k-1,k}(t)).$$

The following Definition 3.2 in the case of $M^3 = \mathbb{R}^3$ was given in [13], then in $M^3 = \mathbb{R}^{2,1}$ in [23]. The definition of H for semi-discrete surfaces in general 3-dimensional spaceforms

M^3 was given in [3], and here we also give the definition of K for general M^3. For this definition we use the mixed area formulation found in [3].

For two semi-discrete conjugate surfaces $x, y : \mathbb{D} \to V^4$ satisfying parallelity conditions $\partial x \parallel \partial y$ and $\Delta x \parallel \Delta y$, we define the mixed area element

$$A(x, y) := \frac{1}{4}((\partial x + \partial x_1) \wedge \Delta y + (\partial y + \partial y_1) \wedge \Delta x),$$

where the operator $\wedge$ is defined by

$$(a \wedge b)c := (a \circ c)b - (b \circ c)a.$$

Definition 3.2. *Let $(x, n) : \mathbb{Z} \times \mathbb{R} \to T_1 M^3$ be a semi-discrete curvature-line parametrized surface. Then the Gaussian curvature K and mean curvature H of x are defined, as functions on the set of edges $[x, x_1]$, by*

$$A(n, n) = K \cdot A(x, x), \quad A(x, n) = -H \cdot A(x, x).$$

Similarly, the Gaussian and mean curvatures of n can be defined, regarding x as the normal vector of n.

Similarly to the arguments in [1] and [24], we have the following proposition:

Proposition 3.1. *Let x be a semi-discrete curvature-line parametrized surface with Gauss map n so that (x, n) is a Legendre immersion. Let κ, κ_1, κ_{01}, K, H be the resulting principal, Gaussian and mean curvatures. Then*

$$K = \frac{\kappa_{01}(2\kappa\kappa_1 - \kappa\kappa_{01} - \kappa_1\kappa_{01})}{\kappa_1 + \kappa - 2\kappa_{01}}, \quad H = \frac{\kappa\kappa_1 - \kappa_{01}^2}{\kappa_1 + \kappa - 2\kappa_{01}}.$$

Example 3.1. Like as seen in [18], where semi-discrete catenoids in $\mathbb{R}^3$ with smooth profile curves were shown to have the same profile curves as smooth catenoids, one can now check here that, more generally, semi-discrete linear Weingarten surfaces in spaceforms with smooth profile curves have the same profile curves as their smooth counterparts.

4. Semi-discrete surfaces with Weierstrass representations

4.1. The cases of $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$. Let g be a semi-discrete holomorphic function, that is, a semi-discrete isothermic map into the plane, with tangent cross ratio factorizing functions τ, σ. We assume the semi-discrete analog of a smooth holomorphic function having a nonzero derivative, that is, ∂g and Δg are never zero, and we now state the Weierstrass-type representations for semi-discrete isothermic minimal and maximal surfaces, i.e. those with $H = 0$ in $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$:

Proposition 4.1 ([18], [23]). *Any semi-discrete minimal (resp. maximal) surface in $\mathbb{R}^3$ (resp. $\mathbb{R}^{2,1}$) can be piecewise represented using a semi-discrete holomorphic function g by solving*

$$(1) \qquad \partial x = \mathrm{Re}\left(\frac{\tau}{2\partial g}\begin{pmatrix} 1 - \epsilon g^2 \\ i(1 + \epsilon g^2) \\ 2\epsilon g \end{pmatrix}\right), \quad \Delta x = \mathrm{Re}\left(\frac{\sigma}{2\Delta g}\begin{pmatrix} 1 - \epsilon gg_1 \\ i(1 + \epsilon gg_1) \\ \epsilon(g + g_1) \end{pmatrix}\right),$$

with $\epsilon = 1$ (resp. $\epsilon = -1$), and the normal field is

$$n = \frac{1}{1 + \epsilon|g|^2} \begin{pmatrix} 2\epsilon\mathrm{Re}(g) \\ 2\epsilon\mathrm{Im}(g) \\ 1 - \epsilon|g|^2 \end{pmatrix}.$$

Direct computation shows the following lemma.

Lemma 4.1. *For any semi-discrete minimal (resp. maximal) surface, the κ and κ_{01} in Definition 3.1 satisfy*

$$(2) \qquad \kappa = \frac{-4|\partial g|^2}{\tau(1 + \epsilon|g|^2)^2}, \quad \kappa_{01} = \frac{-4|\Delta g|^2}{\sigma(1 + \epsilon|g|^2)(1 + \epsilon|g_1|^2)}.$$

One can also confirm this corollary:

Corollary 4.1. *For any choice of $\theta \in \mathbb{R}$, the parallel surface*

$$x_\theta := x + \theta \cdot n$$

satisfies the circularity condition, with Gaussian and mean curvatures

$$K_\theta^x = \frac{K_0}{1 - 2\theta \cdot H_0 + \theta^2 \cdot K_0}, \quad H_\theta^x = \frac{H_0 - \theta K_0}{1 - 2\theta \cdot H_0 + \theta^2 \cdot K_0}$$

satisfying

$$\frac{H_\theta^x}{K_\theta^x} = -\theta .$$

Also, the principal curvatures for x_θ satisfy

$$\kappa_\theta = \frac{\kappa}{1 - \theta \cdot \kappa}, \quad \kappa_{01.\theta} = \frac{\kappa_{01}}{1 - \theta \cdot \kappa_{01}} .$$

4.2. The cases of $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$. Taking the same g as in Subsection 4.1, we make the genericity assumption

$$\mathcal{T} := 1 + sg\bar{g} \neq 0$$

for some chosen constant $s \in \mathbb{R}$. Take $\lambda \in \mathbb{R}$ to be any non-zero constant so that $1 - \lambda\sigma \neq 0$. Solving, for $E \in \mathrm{GL}_2\mathbb{C}$,

$$(3) \qquad E^{-1}\Delta E = \begin{pmatrix} 0 & \Delta g \\ \frac{\lambda\sigma}{\Delta g} & 0 \end{pmatrix}, \quad E^{-1}\partial E = \begin{pmatrix} 0 & \partial g \\ \frac{\lambda\tau}{\partial g} & 0 \end{pmatrix},$$

and defining

$$(4) \qquad L = \begin{pmatrix} 0 & \sqrt{\mathcal{T}} \\ \frac{-1}{\sqrt{\mathcal{T}}} & \frac{-s\bar{g}}{\sqrt{\mathcal{T}}} \end{pmatrix},$$

and the surface x and its normal n by

$$(5) \qquad x = \frac{\mathrm{sgn}(\mathcal{T})}{\det E} EL(\overline{EL})^t , \quad n = \frac{\mathrm{sgn}(\mathcal{T})}{\det E} EL \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\overline{EL})^t ,$$

we will see that these are discrete BrLW surfaces and BiLW surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$, respectively. First, analogous to the discrete case, we have the following proposition.

Proposition 4.2. *Semi-discrete BrLW surfaces in $\mathbb{H}^3$ and BiLW surfaces in $\mathbb{S}^{2,1}$ with Weierstrass-type representations as in Equations* (3), (4), (5) *are circular nets.*

Proof. Let x be a BrLW surface in $\mathbb{H}^3$ described by a semi-discrete holomorphic function g. Observing that $\det E$ does not depend on the smooth parameter t, we have

$$\partial x = c_1 E \begin{pmatrix} 0 & \partial g(1 + s|g|^2) \\ \partial \bar{g}(1 + s|g|^2) & -s(\bar{g}\partial g + g\partial \bar{g}) \end{pmatrix} \overline{E}^t,$$

$$\partial x_1 = c_2 E \begin{pmatrix} 1 & \Delta g \\ \frac{\lambda\sigma}{\Delta g} & 1 \end{pmatrix} \begin{pmatrix} 0 & \partial g_1(1 + s|g_1|^2) \\ \partial \bar{g}_1(1 + s|g_1|^2) & -s(\bar{g}_1\partial g_1 + g_1\partial \bar{g}_1) \end{pmatrix} \begin{pmatrix} 1 & \frac{\lambda\sigma}{\Delta g} \\ \overline{\Delta g} & 1 \end{pmatrix} \overline{E}^t,$$

$$\Delta x = c_3 E \begin{pmatrix} |\Delta g|^2(1 + s|g|^2) & \Delta g(1 + s|g|^2) \\ \overline{\Delta g}(1 + s|g|^2) & \lambda\sigma(1 + s|g_1|^2) - s(|g_1|^2 - |g|^2) \end{pmatrix} \overline{E}^t,$$

where

$$c_1 := \frac{(1-s)|\partial g|^2 + \lambda\tau(1 + s|g|^2)^2}{|\partial g|^2(1 + s|g|^2)^2 \det E}, \quad c_2 := \frac{(1-s)|\partial g_1|^2 + \lambda\tau(1 + s|g_1|^2)^2}{|\partial g_1|^2(1 + s|g_1|^2)^2(1 - \lambda\sigma) \det E},$$

$$c_3 := \frac{(1-s)|\Delta g|^2 + \lambda\sigma(1 + s|g|^2)(1 + s|g_1|^2)}{|\Delta g|^2(1 + s|g|^2)(1 + s|g_1|^2)(1 - \lambda\sigma) \det E}.$$

By an isometry of $\mathbb{R}^{3,1}$, without loss of generality, we can assume that $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ at one point. Using the tangent cross ratio condition $cr(g, g_1) = \frac{\tau}{\sigma}$, by a calculation, we have

$$\partial x \cdot (\Delta x)^{-1} \cdot \partial x_1 \cdot (\Delta x)^{-1} = cr(x, x_1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with

$$cr(x, x_1) = \frac{\sigma(1 - \lambda\sigma)}{\tau} \cdot \frac{\{(1-s)|\partial g|^2 + \lambda\tau(1 + s|g|^2)^2\}\{(1-s)|\partial g_1|^2 + \lambda\tau(1 + s|g_1|^2)^2\}}{\{(1-s)|\Delta g|^2 + \lambda\sigma(1 + s|g|^2)(1 + s|g_1|^2)\}^2} .$$

Thus x is a circular net. Note that x is not generically semi-discrete isothermic.

A proof that n is a semi-discrete circular net will be given just after Lemma 4.2. $\square$

Direct computations confirm this lemma:

Lemma 4.2. *For any allowed choice of s, we have the following:*
- $\partial x \parallel \partial n$, $\Delta x \parallel \Delta n$ *in* $\mathbb{R}^{3,1}$, *and the principal curvatures in Definition 3.1 satisfy*

(6) $\quad \kappa = \dfrac{|\partial g|^2(-1 - s) + (1 + s|g|^2)^2\lambda\tau}{|\partial g|^2(1 - s) + (1 + s|g|^2)^2\lambda\tau}$, $\kappa_{01} = \dfrac{|\Delta g|^2(-1 - s) + (1 + s|g|^2)(1 + s|g_1|^2)\lambda\sigma}{|\Delta g|^2(1 - s) + (1 + s|g|^2)(1 + s|g_1|^2)\lambda\sigma}$.

- $1 + s|g|^2 > 0$, *resp.* $1 + s|g|^2 < 0$, *if and only if x lies in* $\mathbb{H}^3_+$, *resp.* $\mathbb{H}^3_-$.
- $\Delta x, \partial x, \partial x_1$ *lie in a plane (that is generically spacelike) in* $\mathbb{R}^{3,1}$, *and thus x satisfies the circularity condition.*

Now we show the semi-discrete circularity of semi-discrete BiLW surfaces in $\mathbb{S}^{2,1}$. Let n be a semi-discrete BiLW surface described by a semi-discrete holomorphic function g. By Lemma 4.2, we have

$$\partial n \cdot (\Delta n)^{-1} \cdot \partial n_1 \cdot (\Delta n)^{-1} = \frac{\kappa \kappa_1}{\kappa_{01}^2} \partial x \cdot (\Delta x)^{-1} \cdot \partial x_1 \cdot (\Delta x)^{-1}$$

$$= \frac{\sigma(1 - \lambda\sigma)}{\tau} \cdot \frac{\{(-1 - s)|\partial g|^2 + \lambda\tau(1 + s|g|^2)^2\}\{(1 - s)|\partial g_1|^2 + \lambda\tau(1 + s|g_1|^2)^2\}}{\{(-1 - s)|\Delta g|^2 + \lambda\sigma(1 + s|g|^2)(1 + s|g_1|^2)\}^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus n is also semi-discrete circular, proving the last part of Proposition 4.2.

Furthermore, combining Proposition 4.2 and Lemma 4.2, we can show the following curvature properties of semi-discrete BrLW and BiLW surfaces, which also imply Fact 1 in the introduction:

Proposition 4.3. *A semi-discrete BrLW surface x in $\mathbb{H}^3$ and a semi-discrete BiLW surface n in $\mathbb{S}^{2,1}$ described by a semi-discrete holomorphic function g via Equations (3), (4), (5) satisfy the following curvature conditions:*

$$(7) \qquad 2s(H^x - 1) + (1 - s)(K^x - 1) = 0, \quad 2s(H^n - 1) - (1 + s)(K^n - 1) = 0,$$

where H^x and K^x are the mean and Gaussian curvatures of x and H^n and K^n are the mean and Gaussian curvatures of n.

Proof. The curvature condition for x can be obtained by a direct but tedious calculation, which we omit here. We now see the curvature condition for n from the curvature condition for x: The surfaces satisfy $K^x = \dfrac{1}{K^n}$, $H^x = \dfrac{H^n}{K^n}$. Substituting K^x, H^x into the curvature condition for x, we have the curvature condition for n, proving the proposition. □

Like in the smooth and fully discrete cases, we define types of semi-discrete BrLW and BiLW surfaces as follows:

Definition 4.1. *The surfaces x and n are said to be of* hyperbolic type *if $s > 0$, and of* de Sitter type *if $s < 0$.*

Let x be a semi-discrete BrLW surface in $\mathbb{H}^3$ and let n be a semi-discrete BiLW surface in $\mathbb{S}^{2,1}$ described by a single choice of g and s. Then we define the parallel surface x_θ of x at distance θ ($\theta \in \mathbb{R}$) as

$$x_\theta := \cosh\theta \cdot x + \sinh\theta \cdot n \in \mathbb{H}^3.$$

One can confirm the following proposition, which proves Fact 2 in the introduction:

Proposition 4.4. *For any choices of s and $\theta \in \mathbb{R}$, the parallel surface x_θ of a semi-discrete circular surface x in $\mathbb{H}^3$ with unit normal vector field*

$$n_\theta := \sinh\theta \cdot x + \cosh\theta \cdot n \in \mathbb{S}^{2,1}$$

satisfies the circularity condition, and

$$K_\theta^x = \frac{K_0^x \cosh^2\theta - H_0^x \sinh(2\theta) + \sinh^2\theta}{\cosh^2\theta - H_0^x \sinh(2\theta) + K_0^x \sinh^2\theta}, \quad H_\theta^x = \frac{-(K_0^x + 1)\sinh(2\theta) + 2H_0^x \cosh(2\theta)}{2\{\cosh^2\theta - H_0^x \sinh(2\theta) + K_0^x \sinh^2\theta\}}.$$

In particular, if x is a semi-discrete BrLW surface in $\mathbb{H}^3$, x_θ is also of Bryant type satisfying

$$(8) \qquad\qquad 2s_\theta(H_\theta^x - 1) + (1 - s_\theta)(K_\theta^x - 1) = 0,$$

where $s_\theta = e^{-2\theta}s$, and x_θ can be also obtained from the Weierstrass-type representation.

Moreover, the normal n_θ also satisfies the circularity condition, and is of Bianchi type satisfying

$$(9) \qquad 2s_\theta(H_\theta^x - 1) - (1 + s_\theta)(K_\theta^x - 1) = 0,$$

and n_θ can be also obtained from the Weierstrass-type representation.

Proof. Combining Lemma 4.2 and the definition of the tangent cross ratio, we can easily show the circularity conditions for x_θ and n_θ. Next we determine the mean and Gaussian curvatures of a parallel surface of a semi-discrete circular surface in $\mathbb{H}^3$. Let K_0^x, H_0^x (resp. K_θ^x, H_θ^x) be the Gaussian and mean curvatures of x (resp. x_θ). By a calculation, we have

$$A(x_\theta, x_\theta) = \cosh^2\theta \cdot A(x, x) + \sinh(2\theta) \cdot A(x, n) + \sinh^2\theta \cdot A(n, n)$$
$$= \{\cosh^2\theta - H_0^x \sinh(2\theta) + K_0^x \sinh^2\theta\} \cdot A(x, x).$$

Similarly, we have

$$A(n_\theta, n_\theta) = \{\sinh^2\theta - H_0^x \sinh(2\theta) + K_0^x \cosh^2\theta\} \cdot A(x, x),$$

$$A(x_\theta, n_\theta) = \frac{1}{2}\{(K_0^x + 1)\sinh(2\theta) - 2H_0^x \cosh(2\theta)\} \cdot A(x, x).$$

Thus we have K_θ^x, H_θ^x of the forms as in Proposition 4.4.

Here we assume that (x, n) is a pair of a semi-discrete BrLW surface in $\mathbb{H}^3$ and a semi-discrete BiLW surface in $\mathbb{S}^{2,1}$ described by a single choice of g and s. Note that x is a parallel surface of x_θ with distance $-\theta$, that is, $x = (x_\theta)_{-\theta}$. Then we have

$$K_0^x = \frac{K_\theta^x \cosh^2\theta + H_\theta^x \sinh(2\theta) + \sinh^2\theta}{\cosh^2\theta + H_\theta^x \sinh(2\theta) + K_\theta^x \sinh^2\theta}, \qquad H_0^x = \frac{(K_\theta^x + 1)\sinh(2\theta) + 2H_\theta^x \cosh(2\theta)}{2\{\cosh^2\theta + H_\theta^x \sinh(2\theta) + K_\theta^x \sinh^2\theta\}}.$$

Substituting these into Equation (7), we have the relation (8). By a similar argument as in the proof of Proposition 4.3, we have Equation (9).

Finally, we show that any parallel surfaces of semi-discrete BrLW and BiLW surfaces with Weierstrass-type representations can be also described by Weierstrass-type representations. First we consider parallel surfaces of semi-discrete BrLW surfaces in $\mathbb{H}^3$. Let x be a semi-discrete BrLW surface in $\mathbb{H}^3$ and x_θ be a parallel surface of x with distance θ. Here we assume that $\mathcal{T} > 0$ (even when $\mathcal{T} < 0$, the conclusion is the same). Then

$$x_\theta = \frac{1}{\det E} EL \begin{pmatrix} e^\theta & 0 \\ 0 & e^{-\theta} \end{pmatrix} \overline{(EL)}^t = \frac{1}{\det E} E \begin{pmatrix} e^{-\theta}(1 + sg\bar{g}) & -se^{-\theta}g \\ -se^{-\theta}\bar{g} & \frac{e^\theta(1+s^2 e^{-2\theta}g\bar{g})}{1+sg\bar{g}} \end{pmatrix} \overline{E}^t$$

$$= \frac{1}{\det E} \left(E \begin{pmatrix} e^{-\theta/2} & 0 \\ 0 & e^{\theta/2} \end{pmatrix} \right) \begin{pmatrix} 1 + sg\bar{g} & -se^{-\theta}g \\ -se^{-\theta}\bar{g} & \frac{1+s^2 e^{-2\theta}g\bar{g}}{1+sg\bar{g}} \end{pmatrix} \overline{\left(E \begin{pmatrix} e^{-\theta/2} & 0 \\ 0 & e^{\theta/2} \end{pmatrix} \right)}^t = \frac{1}{\det \tilde{E}} \tilde{E}\tilde{L}\overline{(\tilde{E}\tilde{L})}^t,$$

where $\tilde{E} := E \begin{pmatrix} e^{-\theta/2} & 0 \\ 0 & e^{\theta/2} \end{pmatrix}$ and $\tilde{L} := \begin{pmatrix} 1 + \tilde{s}\tilde{g}\bar{\tilde{g}} & -\tilde{s}\tilde{g} \\ -\tilde{s}\bar{\tilde{g}} & \frac{-\tilde{s}\bar{\tilde{g}}}{1+\tilde{s}\tilde{g}\bar{\tilde{g}}} \end{pmatrix}$ ($\tilde{g} = e^\theta g$, $\tilde{s} = se^{-2\theta}$). By the definition of E, $\tilde{E}$ is a solution of

$$\partial\tilde{E} = \tilde{E} \begin{pmatrix} 0 & \partial\tilde{g} \\ \frac{\lambda\tau}{\partial\tilde{g}} & 0 \end{pmatrix}, \qquad \Delta\tilde{E} = \tilde{E} \begin{pmatrix} 0 & \Delta\tilde{g} \\ \frac{\lambda\sigma}{\Delta\tilde{g}} & 0 \end{pmatrix}.$$

Thus x_θ can be obtained via the Weierstrass-type representation by replacing g in Equations

(3), (4), (5) with $\tilde{g}$ and choosing $\tilde{s}$. Similarly, we consider semi-discrete BiLW surfaces in $\mathbb{S}^{2,1}$. Let n be a semi-discrete BiLW surface in $\mathbb{S}^{2,1}$ and let n_θ be a parallel surface of n at distance θ. Then n_θ can be obtained via the Weierstrass-type representation by replacing g in Equations (3), (4), (5) with $\tilde{g}$ and choosing $\tilde{s}$, proving the proposition. $\qquad\square$

We have thus arrived at Facts 1 and 2 in the introduction. The smooth and fully discrete cases can be found in [10], [16], [19].

5. Singularities of semi-discrete surfaces with Weierstrass representations

Influenced by definitions of singularities in the smooth and fully discrete cases, we make the following definition, refining the definition in [24] (recall that flat points on smooth surfaces are those for which both principal curvatures are zero, and parabolic points are those for which exactly one principal curvature is zero):
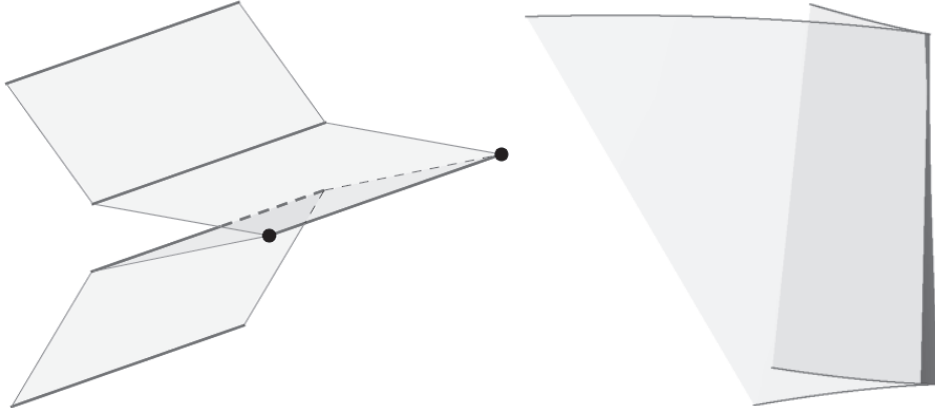

Fig. 2. Left: A typical example of the first part of item (1) in Definition 5.1. Right: A typical example of the first part of item (2).

Definition 5.1. *We say that a point (k_0, t_0), also its image $x(k_0, t_0)$, is a flat (F) or parabolic (P) or singular (S) point of the semi-discrete surface $x(k, t)$, with respect to either the discrete direction represented by changing k (see the left-hand side of* Figure 2*) or the smooth direction represented by changing t (see the right-hand side of* Figure 2*), as follows:*

(1) *$x(k_0, t_0)$ is an FPS point with respect to the discrete direction if*

$$\kappa_{k_0-1, k_0}(t_0) \cdot \kappa_{k_0, k_0+1}(t_0) < 0, \text{ or at least one of } \kappa_{k_0-1, k_0}(t_0), \kappa_{k_0, k_0+1}(t_0) \text{ is infinite.}$$

(2) *$x(k_0, t_0)$ is an FPS point with respect to the smooth direction if*

$$\kappa_{k_0-1}(t_0) \cdot \kappa_{k_0}(t_0) < 0 \text{ or } \kappa_{k_0}(t_0) \cdot \kappa_{k_0+1}(t_0) < 0 \text{ or}$$
$$\text{at least one of } \kappa_{k_0-1}(t_0), \kappa_{k_0}(t_0), \kappa_{k_0+1}(t_0) \text{ is infinite.}$$

In the latter cases of either (1) *or* (2) *above, where infinite values occur, we can say that $x(k_0, t_0)$ is a singular (S) point.*

Like in [19], we are interested in cases where we can differentiate between FP and S vertices. This is the purpose of the next two definitions, which are independent of whether

the surface has a Weierstrass representation.

Definition 5.2. *We say that a semi-discrete circular surface x is* embedded *at a given edge $[x, x_1]$ if ∂x and ∂x_1 lie to the same side of the line through Δx within the tangent plane. Embeddedness of the Gauss map n is similarly defined.*

Generically, in the (non-umbilic) smooth case, rank 1 singularities of a surface correspond to flat or parabolic points of the surface's unit normal field, and vice versa. When x or n is not locally embedded, we certainly have a singular point, and this motivates the next definition.

Definition 5.3. *Let x be a semi-discrete circular surface with bounded principal curvatures and spacelike tangent planes on its edges. Suppose that $x(k_0, t_0)$ is an FPS point with respect to just the smooth direction, and that precisely one of $\ell_{-10} := \kappa_{k_0-1}(t_0) \cdot \kappa_{k_0}(t_0)$ and $\ell_{10} := \kappa_{k_0}(t_0) \cdot \kappa_{k_0+1}(t_0)$ is negative and the other is positive. Then we call $x(k_0, t_0)$ an FP point (i.e. non-singular), resp. a singular (S) point, if the Gauss map is not embedded, resp. is embedded, on the edge corresponding to the ℓ_{*0} that is negative at $t = t_0$ ($* = -1$ or 1).*

REMARK. Suppose $\kappa_{k_0-1}(t_0) \cdot \kappa_{k_0}(t_0)$ is negative. Then n is embedded on the edge $[x_{k_0-1}, x_{k_0}]$ at $t = t_0$ if and only if $\mathrm{cr}(n_{k_0-1}, n_{k_0}) < 0$, since the tangent plane is spacelike. By definition, $\mathrm{cr}(x_{k_0-1}, x_{k_0}) = \kappa_{k_0-1,1}^2 \kappa_{k_0-1}^{-1} \kappa_{k_0}^{-1} \mathrm{cr}(n_{k_0-1}, n_{k_0})$, so n is embedded if and only if $\mathrm{cr}(x_{k_0-1}, x_{k_0}) > 0$.

It was pointed out in [19] that parallel surfaces of minimal and maximal surfaces never have flat or parabolic points, and this was used in that work to justify the analog of the definition below for the fully discrete surface case. Similarly, the definition just below is also justified in the semi-discrete case. In fact, we can see from the formulas for κ_θ and $\kappa_{01,\theta}$ in Corollary 4.1 that these principal curvatures are never zero.

Definition 5.4. *On any parallel surface of a semi-discrete minimal or maximal surface, all FPS points are called simply* singular (S) *points.*

Lemma 4.2 now provides proofs of the following two theorems.

Theorem 5.1. *For a parallel surface x_θ of a semi-discrete minimal or maximal surface at oriented distance θ (in the maximal case $\epsilon = -1$ we assume $|g_{-1}|$, $|g|$, $|g_1|$ are all not 1), the condition for $\kappa_{-10} \cdot \kappa_{01}$ to be nonpositive – that is, $x_\theta(k_0, t_0)$ is FPS (and in fact singular, by* Definition 5.1*) with respect to the discrete direction – is*

$$\theta \in [\min(a_{-1}, a_1), \max(a_{-1}, a_1)],$$

where

$$a_* = \frac{-\sigma(1 + \epsilon|g|^2)(1 + \epsilon|g_*|^2)}{4|\Delta g_{0*}|^2}, \quad * \in \{-1, 1\},$$

and the condition for $\kappa \cdot \kappa_1$ to be nonpositive – that is, $x_\theta(k_0, t_0)$ is FPS (and in fact singular, by Definition 5.1*) with respect to the smooth direction (with regard to the edge $[x_\theta(k_0, t_0), x_\theta(k_0+1, t_0)]$) – is*

$$\theta \in [\min(b, b_1), \max(b, b_1)],$$

where

$$b = \frac{-\tau(1 + \epsilon|g|^2)^2}{4|\partial g|^2} \ , \quad b_1 = \frac{-\tau(1 + \epsilon|g_1|^2)^2}{4|\partial g_1|^2} \ .$$

One can analogously give a condition for $x_\theta(k_0, t_0)$ to be singular in the smooth direction with regard to the edge $[x_\theta(k_0 - 1, t_0), x_\theta(k_0, t_0)]$.

For notational simplicity, set

$$\alpha_{-1}(s) := 1 + s|g_{-1}|^2, \quad \alpha(s) := 1 + s|g|^2, \quad \alpha_1(s) := 1 + s|g_1|^2.$$

We then have the next theorem.

Theorem 5.2. *Let x be a semi-discrete BrLW surface with Gauss map n a semi-discrete BiLW surface. The condition for $\kappa^x_{-10} \cdot \kappa^x_{01}$ (equivalently, $\kappa^n_{-10} \cdot \kappa^n_{01}$) to be nonpositive – that is, we have an FPS point with respect to the discrete direction – is*

$$\begin{cases} \{|\Delta g_{-1}|^2(1 + s) - \lambda\sigma_{-1}\alpha_{-1}(s)\alpha(s)\} \cdot \{|\Delta g|^2(1 + s) - \lambda\sigma\alpha(s)\alpha_1(s)\} > 0, \\ \{|\Delta g_{-1}|^2(1 - s) + \lambda\sigma_{-1}\alpha_{-1}(s)\alpha(s)\} \cdot \{|\Delta g|^2(1 - s) + \lambda\sigma\alpha(s)\alpha_1(s)\} < 0, \end{cases}$$

or

$$\begin{cases} \{|\Delta g_{-1}|^2(1 + s) - \lambda\sigma_{-1}\alpha_{-1}(s)\alpha(s)\} \cdot \{|\Delta g|^2(1 + s) - \lambda\sigma\alpha(s)\alpha_1(s)\} < 0, \\ \{|\Delta g_{-1}|^2(1 - s) + \lambda\sigma_{-1}\alpha_{-1}(s)\alpha(s)\} \cdot \{|\Delta g|^2(1 - s) + \lambda\sigma\alpha(s)\alpha_1(s)\} > 0, \end{cases}$$

and the condition for $\kappa^x \cdot \kappa^x_1$ (equivalently, $\kappa^n \cdot \kappa^n_1$) to be nonpositive – that is, we have an FPS point with respect to the smooth direction – is

$$\begin{cases} \{|\partial g|(1 + s) - \lambda\tau\alpha(s)^2\} \cdot \{|\partial g_1|(1 + s) - \lambda\tau\alpha_1(s)^2\} > 0, \\ \{|\partial g|(1 - s) + \lambda\tau\alpha(s)^2\} \cdot \{|\partial g_1|(1 - s) + \lambda\tau\alpha_1(s)^2\} < 0, \end{cases}$$

or

$$\begin{cases} \{|\partial g|(1 + s) - \lambda\tau\alpha(s)^2\} \cdot \{|\partial g_1|(1 + s) - \lambda\tau\alpha_1(s)^2\} < 0, \\ \{|\partial g|(1 - s) + \lambda\tau\alpha(s)^2\} \cdot \{|\partial g_1|(1 - s) + \lambda\tau\alpha_1(s)^2\} > 0, \end{cases}$$

In [23], the second author established a notion of singular edges for semi-discrete surfaces in Lorentzian spaceforms:

Definition 5.5. *An edge $[x, x_1]$ of a semi-discrete surface is said to be* singular *if the tangent plane at this edge is not spacelike.*

With this definition in hand, the second author proved this in [23]: Let g be a semi-discrete holomorphic function and let x be a semi-discrete maximal surface determined from g by Equation (1). Then an edge $[x, x_1]$ is singular if and only if the tangent circle C at g, g_1 intersects the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

We now prove the following relationship between singular points and singular edges:

Theorem 5.3. *At any singular point $x(k, t)$ of a semi-discrete maximal surface with respect to the discrete direction such that $\kappa_{k-1,k}$ and $\kappa_{k,k+1}$ are both finite, at least one of the two adjacent edges is singular.*

At any singular point with respect to the smooth direction, the only possibility is that κ is infinite there and the corresponding image of g lies in $\mathbb{S}^1$ there.

Proof. Proof of the first paragraph:

$$0 > \kappa_{-10} \cdot \kappa_{01} = \frac{16|\Delta g_{-10}|^2|\Delta g_{01}|^2}{\sigma_{-10}\sigma_{01}(1 - |g_{-1}|^2)(1 - |g|^2)^2(1 - |g_1|^2)} ,$$

and thus $0 > (1 - |g_{-1}|^2)(1 - |g_1|^2)$, which implies exactly one of g_{-1} and g_1 lies inside $\mathbb{S}^1$.

Proof of the second paragraph: By Equation (2), κ cannot change sign, and the result follows.   □

We have the analogous definition for singular edges of semi-discrete CMC 1 surfaces in $\mathbb{S}^{2,1}$, as was given for semi-discrete maximal surfaces in $\mathbb{R}^{2,1}$ in [23]:

Definition 5.6. *Let n be a semi-discrete CMC 1 surface in $\mathbb{S}^{2,1}$. Then $[n, n_1]$ for some $(k, t) \in \mathbb{Z} \times \mathbb{R}$ is a* singular edge *if the plane $\mathcal{P}(n, n_1)$ spanned by $\{\partial n, \Delta n, \partial\Delta n\}$ is not spacelike.*

Now, similar to Theorem 1.2 in [23], we have the following proposition, which is preparatory for proving Theorem 5.5 below.

Theorem 5.4. *Let n be a semi-discrete CMC 1 surface in $\mathbb{S}^{2,1}$. Then $[n, n_1]$ for some $(k, t) \in \mathbb{Z} \times \mathbb{R}$ is a* singular edge *for all λ sufficiently close to zero if and only if the circle tangent to ∂g and ∂g_1 at g and g_1, respectively, intersects $\mathbb{S}^1$ transversely.*

Proof. By Lemma 4.2, $\partial(x + n) \parallel \partial n$ and $\Delta(x + n) \parallel \Delta n$, implying that each tangent plane of $x + n$ at the edge $[x + n, x_1 + n_1]$ is parallel to the one for x. Thus checking the causality of a tangent plane of x at $[x, x_1]$ is equivalent to checking causality of a tangent plane of $x + n$ at the edge $[x + n, x_1 + n_1]$.

Defining $F := E \begin{pmatrix} 1 & -g \\ 0 & 1 \end{pmatrix}$, we have the following forms:

$$x + n = \frac{F}{\det F}\left(\frac{2}{1 - |g|^2}\begin{pmatrix} |g|^2 & g \\ \bar{g} & 1 \end{pmatrix}\right)\overline{F}^t,$$

$$x_1 + n_1 = \frac{F}{\det F}\left(\frac{2}{(1 - \lambda\sigma)(1 - |g_1|^2)}\begin{pmatrix} |g_1|^2 & g_1 \\ \bar{g_1} & 1 \end{pmatrix}\right)\overline{F}^t,$$

where F and g satisfy

$$\partial F = F\begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\frac{\lambda\tau}{\partial g}, \quad \Delta F = F\begin{pmatrix} g & -gg_1 \\ 1 & -g_1 \end{pmatrix}\frac{\lambda\sigma}{\Delta g}, \quad \frac{|\partial g||\partial g_1|}{|\Delta g|^2} = -\frac{\tau}{\sigma}.$$

Then

$$\partial(x + n) = \frac{2}{(1 - |g|^2)^2 \det F}FX_1\overline{F}^t, \quad \Delta(x + n) = \frac{2}{(1 - \lambda\sigma)(1 - |g|^2)(1 - |g_1|^2)\det F}FX_2\overline{F}^t,$$

where

$$X_1 := \begin{pmatrix} \partial g \cdot \bar{g} + g \cdot \partial\bar{g} & \partial g + g^2 \cdot \partial\bar{g} \\ \partial\bar{g} + \bar{g}^2\partial g & \partial g \cdot \bar{g} + g \cdot \partial\bar{g} \end{pmatrix},$$

$$X_2 := \begin{pmatrix} |g_1|^2 - |g|^2 & \Delta g + gg_1\Delta\bar{g} \\ \Delta\bar{g} + \bar{g}\bar{g_1}\Delta g & |g_1|^2 - |g|^2 \end{pmatrix} + \lambda\sigma(1 - |g_1|^2)\begin{pmatrix} |g|^2 & g \\ \bar{g} & 1 \end{pmatrix}.$$

Because $\partial(x + n)$, $\Delta(x + n)$ and $\partial(x_1 + n_1)$ are coplanar, our task is to find a condition,

call it condition (C), for the span of X_1 and X_2 to be non-spacelike for all λ close to zero, i.e. $\langle X_1, X_1 \rangle \langle X_2, X_2 \rangle - \langle X_1, X_2 \rangle^2 < 0$ (for all λ close to 0), and show that this condition (C) is equivalent to the circle tangent to ∂g and ∂g_1 at g and g_1, respectively, intersecting $\mathbb{S}^1$ transversely. When $|g| = 1$, respectively $|g_1| = 1$, we know that $\partial(x + n)$, respectively $\partial(x_1 + n_1)$, itself is lightlike for all $\lambda \in \mathbb{R} \setminus \{0\}$, so the tangent plane will certainly not be spacelike. Thus, it remains only to find the condition (C) when $|g|$ and $|g_1|$ are both not 1, and in this case it is

$$4|\Delta g|^2 |\partial g|^2 (1 - |g|^2)(1 - |g_1|^2) < \{(1 - \bar{g}g_1)\overline{\Delta g} \cdot \partial g + (1 - g\overline{g_1})\Delta g \cdot \partial \bar{g}\}^2 \, .$$

A direct computation verifies that this condition (C) is precisely the condition that the circle tangent to ∂g and ∂g_1 at g and g_1, respectively, intersects $\mathbb{S}^1$ transversely.   □

We now prove the theorem we have been aiming towards:

Theorem 5.5. *Consider a point (k, t) in the domain of a semi-discrete CMC 1 surface in $\mathbb{S}^{2,1}$. Suppose this point is a singular point with respect to the discrete direction such that $\kappa_{k-1,k}$ and $\kappa_{k,k+1}$ are both finite, for all λ sufficiently close to 0. Then at least one of the two adjacent edges is also singular for all λ sufficiently close to 0.*

Proof. From the Weierstrass-type representation, we can write the surface as n in $\mathbb{S}^{2,1}$ given by some semi-discrete holomorphic function g with $s = -1$, with corresponding HMC 1 surface x in $\mathbb{H}^3$ using the same g and same value of s. The assumptions regarding (k, t) being a singular point for all λ close to 0 imply that none of $|g|$, $|g_1|$ and $|g_{-1}|$ are 1, and also that g_1 and g_{-1} lie on opposite sides of $\mathbb{S}^1$.

By Theorem 5.4, the edge $[n(k, t), n(k + 1, t)]$ is singular for all λ close to zero if and only if the circle tangent to ∂g and ∂g_1 at g and g_1, respectively, intersects $\mathbb{S}^1$ transversely. From the above properties of g, g_1 and g_{-1}, it is clear that at least one of the two edges $[n(k, t), n(k + 1, t)]$ and $[n(k - 1, t), n(k, t)]$ is then singular for all λ close to zero.   □
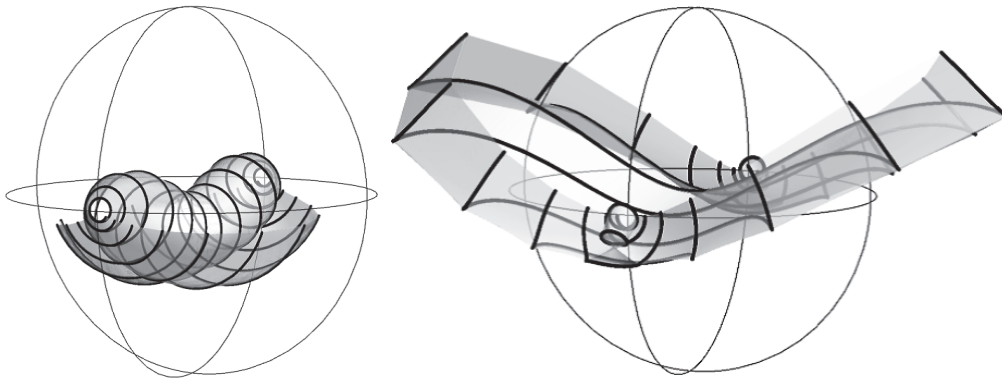

Fig.3. Left-hand side: a semi-discrete CMC 1 Enneper cousin in $\mathbb{H}_+^3$, right-hand side: a semi-discrete HMC 1 surface in $\mathbb{H}_+^3 \cup \mathbb{H}_-^3$. The hyperbolic 3-spaces $\mathbb{H}_+^3$ and $\mathbb{H}_-^3$ are visualized here by stereographically projecting within 4-dimensional Minkowski space to a horizontal 3-dimensional spacelike vector subspace.
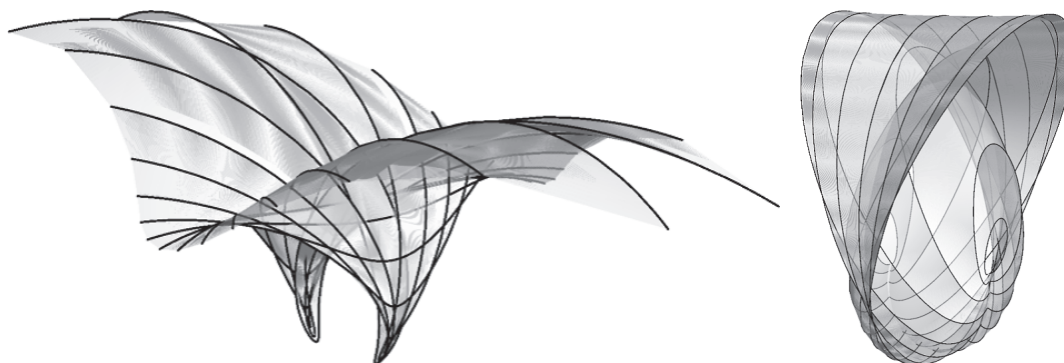
Fig.4. Left-hand side: a semi-discrete CMC 1 Enneper cousin in $\mathbb{S}^{2,1}$, right-hand side: a semi-discrete HMC 1 surface in $\mathbb{S}^{2,1}$. The de Sitter 3-space $\mathbb{S}^{2,1}$ is visualized here using the hollow ball model (see [7]).

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