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AN ENERGY-THEORETIC APPROACH TO THE HITCHIN-KOBAYASHI CORRESPONDENCE FOR MANIFOLDS, II

Dedicated to Professor Eugenio Calabi on his eightieth birthday

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Abstract

Recently, Donaldson proved asymptotic stability for a polarized algebraic manifold M with polarization class admitting a Kähler metric of constant scalar curvature, essentially when the linear algebraic part H of $\mathrm{Aut}^0(M)$ is semisimple. The purpose of this paper is to give a generalization of Donaldson's result to the case where the polarization class admits an extremal Kähler metric, even when H is not semisimple.

0. Introduction

For a connected polarized algebraic manifold (M, L) with an extremal Kähler metric in the polarization class $c_1(L)_{\mathbb{R}}$, we consider the Kodaira embedding

$$\Phi_m = \Phi_{|L^{\otimes m}|} \colon M \hookrightarrow \mathbb{P}^*(V_m), \quad m \gg 1,$$

where $\mathbb{P}^*(V_m)$ denotes the set of all hyperplanes in $V_m := H^0(M, \mathcal{O}(L^m))$ through the origin. For the identity component $\mathrm{Aut}^0(M)$ of the group of holomorphic automorphisms of M, let H denote its maximal connected linear algebraic subgroup. Replacing the ample holomorphic line bundle L by some positive integral multiple of L if necessary, we may fix an H-linearization of L, i.e., a lift to L of the H-action on M such that H acts on L as bundle isomorphisms covering the H-action on M, and may further assume that the natural H-equivariant maps

$$\mathrm{pr}_m \colon \bigotimes^m V_1 \to V_m, \quad m = 1, 2, \dots,$$

are surjective (cf. [18], Theorem 3). In this paper, applying a method in [15], we shall generalize a result in Donaldson [3] about stability to extremal Kähler cases:

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Main Theorem. *For a polarized algebraic manifold* (M, L) *as above with an extremal Kähler metric in the polarization class, there exists an algebraic torus* T *in* H *such that the image* $\Phi_m(M)$ *in* $\mathbb{P}^*(V_m)$ *is stable relative to* T (*cf.* Section 2 *and* [14]) *for* $m \gg 1$.

In particular in [16], by an argument as in [3], an extremal Kähler metric in a fixed integral Kähler class on a projective algebraic manifold M will be shown to be unique[1] up to the action of the group H.

Fix once for all an extremal Kähler metric ω_0 in the polarization class in Main Theorem. By a result of Calabi [1], the identity component K of the group of isometries of (M, ω_0) is a maximal compact connected subgroup of H. For the identity component Z of the center of K, we consider the complexification $Z^{\mathbb{C}}$ of Z in H. Then we shall see that Main Theorem is true for $T = Z^{\mathbb{C}}$ (cf. Section 1).

One may ask why relative stability in place of ordinary stability has to be considered in our study. The reason why we choose relative stability is because, in general, the obstruction in [13] to asymptotic semistability does not vanish (cf. [17]). Thus, as to the group action on V_m related to stability, we must replace the full special linear group $\mathrm{SL}(V_m)$ of V_m by its subgroup $G_m(T)$ (see (1.3)), where the algebraic torus T in $Z^{\mathbb{C}}$ is chosen in such a way that the obstruction vanishes when restricted to $G_m(T)$, i.e., $G'_m(T)$ fixes $\hat{M}_m$ (cf. Section 1). Note also that $G_m(T)$ is a direct product of special linear groups. To see why we choose such a group $G_m(T)$ in place of $\mathrm{SL}(V_m)$, we may compare our stability with that of holomorphic vector bundles. Recall that a holomorphic vector bundle splitting into a direct sum of stable vector bundles often appears in the boundary of a compactified moduli space of stable vector bundles. Similarly for our stability of manifolds, a splitting phenomenon occurs for V_m in (1.2). Roughly speaking, we consider the moduli space of all M's with fixed decomposition data (1.2), where same type of construction of moduli spaces occurs typically for the Hodge decomposition in the variation of Hodge structures.

We now explain the difficulty which we encounter in applying the method of [15]. Such a difficulty comes up when we use the estimate of Phong and Sturm [21]. By applying a stability criterion in [15] of Hilbert-Mumford's type, we write the vector space $\mathfrak{p}_m$ as an orthogonal direct sum

$$\mathfrak{p}_m = \mathfrak{p}'_m \oplus \mathfrak{p}''_m \quad \text{(cf. Section 3)},$$

and then check the stability of $\hat{M}_m$ along the orbits of the one-parameter subgroups in $G_m(T)$ generated by elements of $\mathfrak{p}''_m$. Though $\mathfrak{p}$ and $\mathfrak{p}''_m$ are transversal by the equality $\mathfrak{p}'_m = \mathfrak{p}_m \cap \mathfrak{p}$, we further need the orthogonality of $\mathfrak{p}$ and $\mathfrak{p}''_m$ in order to apply directly the estimate in [21]. Since such an orthogonality does not generally hold, we are in

[1]For uniqueness of extremal Kähler metrics, Chen and Tian recently obtain a more general result without any projectivity condition.

trouble, but still the situation is not so bad (see (3.17), (3.18)), and this overcomes the difficulty.

1. Reduction of Main Theorem

In this section, by introducing necessary notation, we reduce the proof of Main Theorem to showing Theorems A and B below. Throughout this paper, we fix once for all a pair (M, L) of a connected projective algebraic manifold M and an ample holomorphic line bundle L over M as in the introduction. For V_m in the introduction, we put $N_m := \dim_{\mathbb{C}} V_m - 1$, where the positive integer m is such that L^m is very ample. Let n and d be respectively the dimension of M and the degree of the image $M_m := \Phi_m(M)$ in the projective space $\mathbb{P}^*(V_m)$. Fixing an H-linearization of L as in the introduction, we consider the associated representation: $H \to \mathrm{PGL}(V_m)$. Pulling it back by the finite unramified cover: $\mathrm{SL}(V_m) \to \mathrm{PGL}(V_m)$, we obtain an isogeny

$$\iota\colon \tilde{H} \to H, \tag{1.1}$$

where $\tilde{H}$ is an algebraic subgroup of $\mathrm{SL}(V_m)$. On the other hand, for an algebraic torus T in H, the H-linearization of L naturally induces a faithful representation

$$H \to \mathrm{GL}(V_m),$$

and this gives a T-action on V_m for each m. Then we have a finite subset $\Gamma_m = \{\chi_1, \chi_2, \dots, \chi_{\nu_m}\}$ of the free Abelian group $\mathrm{Hom}(T, \mathbb{C}^*)$ of all characters of T such that the vector space $V_m = H^0(M, \mathcal{O}(L^m))$ is uniquely written as a direct sum

$$V_m = \bigoplus_{k=1}^{\nu_m} V_T(\chi_k), \tag{1.2}$$

where for each $\chi \in \mathrm{Hom}(T, \mathbb{C}^*)$, we set $V_T(\chi) := \{s \in V_m;\ t \cdot s = \chi(t)s \text{ for all } t \in T\}$. Define an algebraic subgroup $G_m = G_m(T)$ of $\mathrm{SL}(V_m)$ by

$$G_m := \prod_{k=1}^{\nu_m} \mathrm{SL}(V_T(\chi_k)), \tag{1.3}$$

and the associated Lie subalgebra of $\mathrm{sl}(V_m)$ will be denoted by $\mathfrak{g}_m$. Here, G_m and $\mathfrak{g}_m$ possibly depend on the choice of the algebraic torus T, and if necessary, we denote these by $G_m(T)$ and $\mathfrak{g}_m(T)$, respectively. The T-action on V_m is, more precisely, a right action, while the G_m-action on V_m is a left action. Since T is Abelian, this T-action on V_m can be regarded also as a left action. Note that the group G_m acts diagonally on V_m in such a way that, for each k, the k-th factor $\mathrm{SL}(V_T(\chi_k))$ of G_m acts just on the k-th factor $V_T(\chi_k)$ of V_m. We now put

$$W_m := \{S^d(V_m)\}^{\otimes n+1},$$

where $S^d(V_m)$ denotes the d-th symmetric tensor product of V_m. To the image M_m of M, we can associate a nonzero element $\hat{M}_m$ in W_m^* such that the corresponding element $[\hat{M}_m]$ in $\mathbb{P}^*(W_m)$ is the Chow point of the irreducible reduced algebraic cycle M_m on $\mathbb{P}^*(V_m)$. Note that the G_m-action on V_m naturally induces a G_m-action on W_m and also on W_m^*. As in [14], the subvariety M_m of $\mathbb{P}^*(V_m)$ is said to be *stable relative to* T or *semistable relative to* T, according as the orbit $G_m \cdot \hat{M}_m$ is closed in W_m^* or the closure of $G_m \cdot \hat{M}_m$ in W_m^* does not contain the origin of W_m^*.

Let Δ be the set of all algebraic subtori T of $Z^{\mathbb{C}}$. Take a Hermitian metric h_0 for L such that $c_1(L; h_0)$ is the extremal Kähler metric ω_0 in the Main Theorem. Let E be the extremal Kähler vector field for (M, ω_0), and let $\mathfrak{k}$ be the Lie algebra of K. Let $K^{\mathbb{C}}$ be the complexification of K in H. For ω_0 above, we further define $\Delta_{\min}$ as the set of all $T \in \Delta$ for which the statement of Theorem B in [14] is valid. Then, as the procedure in Section 6 of [14] shows, there exists a unique minimal element[2], denoted by T_0, of $\Delta_{\min}$ such that

$$\Delta_{\min} = \{T \in \Delta;\, T_0 \subset T\}.$$

For each $T \in \Delta_{\min}$, we put $\tilde{T} := \iota^{-1}(T)$ and $\tilde{Z}^{\mathbb{C}} := \iota^{-1}(Z^{\mathbb{C}})$, and let $G'_m(T)$ and $Z'_m(T)$ be the identity components of $G_m(T) \cap \tilde{H}$ and $G_m(T) \cap \tilde{Z}^{\mathbb{C}}$, respectively.

DEFINITION 1.4. For an algebraic torus T in $\Delta_{\min}$, we say that T is *irredundant,* if for all sufficiently large positive integers m, $\dim_{\mathbb{C}} K^{\mathbb{C}} = \dim_{\mathbb{C}} G'_m(T) + \dim_{\mathbb{C}} T$ (or equivalently $\dim_{\mathbb{C}} Z^{\mathbb{C}} = \dim_{\mathbb{C}} Z'_m(T) + \dim_{\mathbb{C}} T$).

For instance, $Z'_m(T) = \{1\}$ if $T = Z^{\mathbb{C}}$. In particular, $Z^{\mathbb{C}}$ is irredundant. We now define subsets Δ_0 and Δ_1 of $\Delta_{\min}$ by

$$\Delta_0\colon \text{the set of all irredundant elements in } \Delta_{\min},$$
$$\Delta_1 := \{T \in \Delta_{\min};\; G'_m(T) \cdot \hat{M}_m = \hat{M}_m \text{ for all } m \gg 1\}.$$

DEFINITION 1.5. Let Δ_L denote the set of all algebraic subtori T of $Z^{\mathbb{C}}$ for which the statement of Main Theorem is valid.

Note that, if T' and T'' are algebraic subtori of $Z^{\mathbb{C}}$ with $T' \subset T''$ and $T' \in \Delta_L$, then the stability criterion of Hilbert-Mumford type (cf. [14], Theorem 3.2) shows that T'' also belongs to Δ_L. We now pose the following:

Theorem A. *The algebraic torus $Z^{\mathbb{C}}$ belongs to Δ_1.*

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[2] The algebraic torus T_0 is actually the closure in $Z^{\mathbb{C}}$ of the complex Lie subgroup generated by the vector fields E, F_k, $k = 1, 2, \ldots$ which appear in the asymptotic approximation (cf. (3.1) below) of the weighted analogues (cf. [14], 2.6) of balanced metrics.

Theorem B. $\Delta_L \cap \Delta_0 = \Delta_0 \cap \Delta_1$.

Once these theorems are proved, then by Theorem A, we have $Z^{\mathbb{C}} \in \Delta_0 \cap \Delta_1$. This together with Theorem B implies that $Z^{\mathbb{C}} \in \Delta_L$, completing the proof of Main Theorem.

If the extremal Kähler metric ω_0 above has a constant scalar curvature, and if the obstruction as in [13] vanishes, then we have both $\Delta_1 = \Delta_{\min}$ and $\{1\} \in \Delta_0$. Hence in this case, Theorem B shows that $\{1\} \in \Delta_L$. This then proves the main theorem in [15].

It is very likely that the set Δ_L has a natural minimal element closely related to the algebraic torus T_0. To see this, let us consider the case where M is an extremal Kähler toric Fano surface polarized by $L = K_M^{-1}$. Then M is possibly a complex projective plane blown up at r points with $r \leq 3$. If $r = 0$ or 3, then M admits a Kähler-Einstein metric, and Δ_L has the unique minimal element $\{1\}$ $(= T_0)$. On the other hand, if $r = 1$, then T_0 coincides with $Z^{\mathbb{C}}$, and is the one-dimensional algebraic torus generated by the extremal Kähler vector field. Hence, in this case, Δ_L has the unique minimal element T_0. Finally for $r = 2$, the involutive holomorphic symmetry of M switching the blown-up points allows us to regard T_0 as the one-dimensional algebraic torus generated by the extremal Kähler vector field. It then follows that T_0 again has to be a minimal element of Δ_L.

2. Proof of Theorem A

In this section, we first prove Theorem A, and then make several definitions with a lemma added. Put $\tilde{K}^{\mathbb{C}} := \iota^{-1}(K^{\mathbb{C}})$.

Proof of Theorem A. In this proof, let $T = Z^{\mathbb{C}}$, and consider the associated set $\Gamma_m = \{\chi_1, \chi_2, \dots, \chi_{\nu_m}\}$ of characters for $m \gg 1$. Since $K^{\mathbb{C}}$ commutes with $Z^{\mathbb{C}}$, we have

$$\tilde{K}^{\mathbb{C}} \subset \mathrm{SL}(V_m) \cap \prod_{k=1}^{\nu_m} \mathrm{GL}(V_{Z^{\mathbb{C}}}(\chi_k)). \tag{2.1}$$

Recall that the extremal Kähler vector field E belongs to the Lie algebra of T_0. Hence, a theorem of Calabi [1] shows that $G'_m(Z^{\mathbb{C}}) \subset G'_m(T_0) \subset \tilde{K}^{\mathbb{C}}$. Hence,

$$G'_m(Z^{\mathbb{C}}) \cdot \tilde{Z}^{\mathbb{C}} \subset \tilde{K}^{\mathbb{C}}. \tag{2.2}$$

To complete the proof of Theorem A, we compare two groups $[\tilde{K}^{\mathbb{C}}, \tilde{K}^{\mathbb{C}}]$ and $G'_m(Z^{\mathbb{C}})$. By (2.1), we obviously have $[\tilde{K}^{\mathbb{C}}, \tilde{K}^{\mathbb{C}}] \subset G'_m(Z^{\mathbb{C}})$. On the other hand,

$$\dim_{\mathbb{C}}[\tilde{K}^{\mathbb{C}}, \tilde{K}^{\mathbb{C}}] = \dim_{\mathbb{C}} \tilde{K}^{\mathbb{C}} - \dim_{\mathbb{C}} \tilde{Z}^{\mathbb{C}} \geq \dim_{\mathbb{C}} G'_m(Z^{\mathbb{C}}),$$

where the last inequality follows from (2.2) in view of the fact that the intersection of $G'_m(Z^{\mathbb{C}})$ and $\tilde{Z}^{\mathbb{C}}$ is a finite group. It now follows that $G'_m(Z^{\mathbb{C}})$ coincides with $[\tilde{K}^{\mathbb{C}}, \tilde{K}^{\mathbb{C}}]$. Hence $G'_m(Z^{\mathbb{C}}) \cdot \hat{M}_m = \hat{M}_m$. Then by $T_0 \subset Z^{\mathbb{C}}$, we now obtain $Z^{\mathbb{C}} \in \Delta_1$, as required. □

Let h be a Hermitian metric for L such that $\omega := c_1(L; h)$ is a K-invariant Kähler metric on M. Define a Hermitian metric ρ_h on V_m by

$$\rho_h(s, s') := \int_M (s, s')_{h^m}\, \omega^n, \quad s, s' \in V_m, \tag{2.3}$$

where $(s, s')_{h^m}$ denotes the function on M obtained as the the pointwise inner product of s, s' by h^m. Let $\mathcal{S} := \{s_0, s_1, \dots, s_{N_m}\}$ be an orthonormal basis for V_m satisfying

$$\rho_h(s_i, s_j) = \delta_{ij}.$$

Let $T \in \Delta_0 \cap \Delta_1$. Then we say that $\mathcal{S}$ is *T-admissible*, if each $V_T(\chi_k)$, $k = 1, 2, \dots$, admits a basis $\{s_{k,i};\ i = 1, 2, \dots, n_k\}$ such that

$$s_{l(k,i)} = s_{k,i}, \quad i = 1, 2, \dots, n_k;\ k = 1, 2, \dots, \nu_m, \tag{2.4}$$

where $n_k := \dim_{\mathbb{C}} V_T(\chi_k)$, and $l(k, i) := (i - 1) + \sum_{k'=1}^{k-1} n_{k'}$ for all k and i (cf. [14]). Let $\mathfrak{t}_c := \mathrm{Lie}(T_c)$ denote the Lie algebra of the maximal compact subgroup T_c of T. Put $q := 1/m$ and $\mathfrak{t}_{\mathbb{R}} := \sqrt{-1}\mathfrak{t}_c$. For each $F \in \mathfrak{t}_{\mathbb{R}}$, we define

$$B_q(\omega, F) := \frac{n!}{m^n} \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} e^{-d\chi_k(F)} |s_{k,i}|^2_{h^m}, \tag{2.5}$$

where $|s|^2_{h^m} := (s, s)_{h^m}$ for all $s \in V_m$, and $d\chi_k \colon \mathfrak{t}_{\mathbb{R}} \to \mathbb{R}$ denotes the restriction to $\mathfrak{t}_{\mathbb{R}}$ of the differential at $t = 1$ for the character $\chi_k \in \mathrm{Hom}(T, \mathbb{C}^*)$.

As a final remark in this section, we give an upper bound for degrees of the characters in Γ_m. Let T be an algebraic torus sitting in $Z^{\mathbb{C}}$. By setting $r := \dim_{\mathbb{C}} T$, we identify T with the multiplicative group $(\mathbb{C}^*)^r := \{t = (t_1, t_2, \dots, t_r);\ t_j \in \mathbb{C}^* \text{ for all } j\}$. Since each χ_k in (1.2) may depend on m, the character χ_k will be rewritten as $\chi_{m;k}$ until the end of this section. Then for each $k \in \{1, 2, \dots, \nu_m\}$,

$$\chi_{m;k}(t) = \prod_{i=1}^{r} t_i^{\alpha(m,k,i)}, \quad t = (t_1, t_2, \dots, t_r) \in T,$$

for some integers $\alpha(m, k, i)$ independent of the choice of t. Define a nonnegative integer α_m by $\alpha_m := \sup_{k=1}^{\nu_m} \sum_{i=1}^{r} |\alpha(m, k, i)|$. Then we have the following upper bound for α_m:

Lemma 2.6. *For all positive integers m, the inequality $\alpha_m \le m\alpha_1$ holds.*

Proof. Put $S := \mathrm{Ker}\, \mathrm{pr}_m$. Since the subspace S of $\bigotimes^m V_1$ is preserved by the T-action, we have a T-invariant subspace, denoted by $S^{\perp}$, of $\bigotimes^m V_1$ such that the vector space $\bigotimes^m V_1$ is written as a direct sum

$$\bigotimes^m V_1 = S \oplus S^{\perp}.$$

Then the restriction of pr_m to $S^{\perp}$ defines a T-equivariant isomorphism $S^{\perp} \cong V_m$. On the other hand, the characters of T appearing in the T-action on $\bigotimes^m V_1$ are

$$\chi_{\vec{k}}(t) := t_1^{\sum_{j=1}^m \alpha(1,k_j,1)} t_2^{\sum_{j=1}^m \alpha(1,k_j,2)} \cdots t_r^{\sum_{j=1}^m \alpha(1,k_j,r)}, \quad \vec{k} = (k_1, k_2, \ldots, k_m) \in I^m,$$

where I^m is the Cartesian product of m-pieces of $I := \{1, 2, \ldots, \nu_m\}$. Since $S^{\perp}$ ($\cong V_m$) is a subspace of $\bigotimes^m V_1$, we now obtain

$$\alpha_m \leq \max_{\vec{k}\in I^m} \sum_{i=1}^{r} \left| \sum_{j=1}^{m} \alpha(1, k_j, i) \right| \leq \max_{\vec{k}\in I^m} \sum_{j=1}^{m} \sum_{i=1}^{r} |\alpha(1, k_j, i)| \leq m\alpha_1,$$

as required. □

3. Proof of Theorem B

Fix an arbitrary element T of $\Delta_0 \cap \Delta_1$. Let $m \gg 1$. Then by [14], Theorem B, there exist $F_k \in \mathfrak{t}_{\mathbb{R}}$, real numbers $\alpha_k \in \mathbb{R}$, and smooth real-valued K-invariant functions φ_k, $k = 1, 2, \ldots$, on M such that, for each $\ell \in \mathbb{Z}_{\geq 0}$, we have

$$B_q(\omega(\ell), F(\ell)) = C_{q,\ell} + 0(q^{\ell+2}), \quad m \gg 1, \tag{3.1}$$

where $F(\ell) := (\sqrt{-1}E/2)q^2 + \sum_{j=1}^{\ell} q^{j+2} F_j$, $h(\ell) := h_0 \exp\left(-\sum_{k=1}^{\ell} q^j \varphi_j\right)$, $C_{q,\ell} := 1 + \sum_{j=0}^{\ell} \alpha_j q^{j+1}$, and $\omega(\ell) := c_1(L; h(\ell))$. Let us now fix an arbitrary positive integer ℓ. To each T-admissible orthonormal basis $\mathcal{S} := \{s_0, s_1, \ldots, s_{N_m}\}$ for $(V_m; \rho_{h(\ell)})$, we associate a basis $\tilde{\mathcal{S}} := \{\tilde{s}_0, \tilde{s}_1, \ldots, \tilde{s}_{N_m}\}$ for V_m by

$$\tilde{s}_{k,i} = e^{-d\chi_k(F(\ell))/2} s_{k,i}, \quad i = 1, 2, \ldots, n_k;\ k = 1, 2, \ldots, \nu_m, \tag{3.2}$$

where we put $s_{l(k,i)} = s_{k,i}$ and $\tilde{s}_{l(k,i)} = \tilde{s}_{k,i}$ by using the notation in (2.4).

REMARK. Lemma 2.6 above implies that $|d\chi_{m;k}(F(\ell))| \leq C\alpha_1 q$ for some positive real constant C independent of the choice of m and k, where $\chi_{m;k}$ is as in the last section. Hence in (3.1) above, for each fixed nonnegative integer ℓ, there exists a positive constant C' independent of m and k such that

$$|e^{-d\chi_k(F(\ell))} - 1| \leq C'q, \quad k = 1, 2, \ldots, \nu_m.$$

In particular, in (3.2) above, the integral $\int_M \|\tilde{s}_{k,i}\|^2_{h^m} \omega(\ell)^n$ ($= e^{-d\chi_k(F(\ell))}$) converges to 1, uniformly in k, as $m \to \infty$.

We now consider the Kodaira embedding $\Phi_m\colon M \to \mathbb{P}^*(V_m)$ defined by

$$\Phi_m(x) := (\tilde{s}_0(x) : \tilde{s}_1(x) : \cdots : \tilde{s}_{N_m}(x)), \quad x \in M,$$

where $\mathbb{P}^*(V_m)$ is identified with $\mathbb{P}^{N_m}(\mathbb{C}) = \{(z_0 : z_1 : \cdots : z_{N_m})\}$ by the basis $\tilde{\mathcal{S}}$. Put $M_m := \Phi_m(M)$. Since $\Delta_L \cap \Delta_0$ is a subset of $\Delta_1 \cap \Delta_0$ (cf. [13], Section 3), the proof of Theorem B (and Main Theorem also) is reduced to showing the following assertion:

Assertion. *The orbit $G_m(T) \cdot \hat{M}_m$ is closed in W_m^*.*

In the Hermitian vector space $(V_m; \rho_{h(\ell)})$, the subspaces $V_T(\chi_k)$, $k = 1, 2, \ldots, \nu_m$, are mutually orthogonal. Put

$$K_m := \prod_{k=1}^{\nu_m} \mathrm{SU}(V_T(\chi_k); \rho_{h(\ell)}), \quad \mathfrak{k}_m := \bigoplus_{k=1}^{\nu_m} \mathfrak{su}(V_T(\chi_k); \rho_{h(\ell)}).$$

Since T belongs to Δ_1, the group $G'_m = G'_m(T)$ coincides with the isotropy subgroup of G_m at $\hat{M}_m \in W_m^*$. Consider the Lie algebra $\mathfrak{g}'_m := \mathrm{Lie}(G'_m)$ of G'_m. Put $\mathfrak{h} := \mathrm{Lie}(H) = \mathrm{Lie}(\tilde{H})$. Then by $G'_m \subset \tilde{H} \subset \mathrm{SL}(V_m)$, we have the inclusions

$$\mathfrak{g}'_m \hookrightarrow \mathfrak{h} \hookrightarrow \mathfrak{sl}(V_m).$$

Put $\mathfrak{k}'_m := \mathrm{Lie}(K'_m)$, where K'_m is the isotropy subgroup of K_m at the point $\hat{M}_m \in W_m^*$. Then $\mathfrak{g}_m$ and $\mathfrak{g}'_m$ are the complexifications of $\mathfrak{k}_m$ and $\mathfrak{k}'_m$, respectively (cf. [1]). Put $\mathfrak{p}_m := \sqrt{-1}\mathfrak{k}_m$ and $\mathfrak{p}'_m := \sqrt{-1}\mathfrak{k}'_m$. We further define

$$\mathfrak{K}_m := \left\{\bigoplus_{k=1}^{\nu_m} \mathfrak{u}(V_T(\chi_k); \rho_{h(\ell)})\right\} \cap \mathfrak{su}(V_m; \rho_{h(\ell)}), \quad \mathfrak{P}_m := \sqrt{-1}\mathfrak{K}_m.$$

By the above inclusions of Lie algebras (see also (2.1)), we can regard $\mathfrak{p} := \sqrt{-1}\mathfrak{k}$ as a Lie subalgebra of $\mathfrak{P}_m$. Let ω_{FS} be the Fubini-Study metric on $\mathbb{P}^*(V_m)$ defined by

$$\omega_{\mathrm{FS}} := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{\alpha=0}^{N_m} |z_\alpha|^2\right).$$

For each $Q \in \mathfrak{P}_m$, let $\mathcal{Q}$ be the associated holomorphic vector field on $\mathbb{P}^*(V_m)$. By the notation for $t = 0$ in Step 1 later in the proof of Assertion, we obtain a vector field $\mathcal{Q}_{TM_m}$ on M_m via the orthogonal projection of $\mathcal{Q}$ along M_m to tangential directions. Then we have a unique real-valued function φ_Q on $\mathbb{P}^*(V_m)$ satisfying both $\int_{\mathbb{P}^*(V_m)} \varphi_Q\, \omega_{\mathrm{FS}}^{N_m} = 0$ and

$$i_{\mathcal{Q}}\left(\frac{\omega_{\mathrm{FS}}}{m}\right) = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\varphi_Q.$$

Let $\Box_{M,\mathrm{FS}} := -\bar\partial^*\bar\partial$ denote the Laplacian on functions on the Kähler manifold $(M, \Phi_m^*\omega_{\mathrm{FS}})$. Define a positive semidefinite K_m'-invariant inner product $(\ ,\ )$ on $\mathfrak{P}_m$ by setting

$$\begin{aligned}(Q_1, Q_2) &:= \frac{1}{m^2}\int_{M_m} ((\mathcal{Q}_1)_{TM_m}, (\mathcal{Q}_2)_{TM_m})_{\omega_{\mathrm{FS}}}\, \omega_{\mathrm{FS}}^n \\ &= \frac{\sqrt{-1}}{2\pi}\int_{M_m} \partial\varphi_{Q_2}\wedge\bar\partial\varphi_{Q_1}\wedge n\,\omega_{\mathrm{FS}}^{n-1} = \int_{M_m} (\bar\partial\varphi_{Q_1}, \bar\partial\varphi_{Q_2})_{\omega_{\mathrm{FS}}}\, \omega_{\mathrm{FS}}^n \\ &= -\int_M \varphi_{Q_1}(\Box_{M,\mathrm{FS}}\varphi_{Q_2})\Phi_m^*\,\omega_{\mathrm{FS}}^n \in \mathbb{R}\end{aligned}$$

for all $Q_1, Q_2 \in \mathfrak{P}_m$. Restrict this inner product to $\mathfrak{p}_m$. Then the inner product $(\ ,\ )$ on $\mathfrak{p}_m$ is positive definite on $\mathfrak{p}$ and hence on $\mathfrak{p}_m'$. As vector spaces, $\mathfrak{P}_m$ and $\mathfrak{p}_m$ are written respectively as orthogonal direct sums

$$\mathfrak{P}_m = \mathfrak{p}\oplus\mathfrak{p}^\perp, \quad \mathfrak{p}_m = \mathfrak{p}_m'\oplus\mathfrak{p}_m'',$$

where $\mathfrak{p}^\perp$ is the orthogonal complement of $\mathfrak{p}$ in $\mathfrak{P}_m$, and moreover $\mathfrak{p}_m''$ is the orthogonal complements of $\mathfrak{p}_m'$ in $\mathfrak{p}_m$ (cf. [15]). Hence if $Q\in\mathfrak{p}^\perp$, then for any holomorphic vector field $\mathcal{W}$ on M_m, we have

$$\begin{aligned}&\frac{1}{m^2}\int_{M_m} (\mathcal{Q}_{TM_m}, \mathcal{W}_{TM_m})_{\omega_{\mathrm{FS}}}\, \omega_{\mathrm{FS}}^n \\ &= \int_{M_m} (\bar\partial\varphi_Q, \bar\theta_0 + \bar\partial(\varphi_{W^1} + \sqrt{-1}\varphi_{W^2}))_{\omega_{\mathrm{FS}}}\, \omega_{\mathrm{FS}}^n \\ &= \frac{\sqrt{-1}}{2\pi}\int_{M_m} \{\theta_0\wedge\bar\partial\varphi_Q + \partial(\varphi_{W^1} - \sqrt{-1}\varphi_{W^2})\wedge\bar\partial\varphi_Q\}\wedge n\,\omega_{\mathrm{FS}}^{n-1} = 0,\end{aligned}$$

where $i_{\mathcal{W}}(\omega_{\mathrm{FS}}/m)$ on M_m is known to be expressible as $\bar\theta_0 + \bar\partial(\varphi_{W^1} + \sqrt{-1}\varphi_{W^2})$ for some holomorphic 1-form θ_0 on M_m and elements W^1, W^2 in $\mathfrak{p}$. We consider the open neighbourhood (cf. [15])

$$U_m := \{X\in\mathfrak{p}_m'';\ \zeta(\mathrm{ad}\,X)\mathfrak{p}_m'\cap\mathfrak{p}_m'' = \{0\}\}$$

of the origin in $\mathfrak{p}_m''$, where $\zeta:\mathbb{R}\to\mathbb{R}$ is a real analytic function defined by $\zeta(x) := x(e^x + e^{-x})/(e^x - e^{-x})$, $x\neq 0$, and $\zeta(0) = 0$. By operating $(\sqrt{-1}/2\pi)\,\partial\bar\partial\log$ on both sides of (3.1), we obtain

$$\Phi_m^*\,\omega_{\mathrm{FS}} \equiv m\,\omega(\ell), \quad \mod q^{\ell+2}. \tag{3.3}$$

For an element X of $\mathfrak{P}_m$ (later we further assume $X\in\mathfrak{p}_m''$), there exists a T-admissible orthonormal basis $\mathcal{T} := \{\tau_0, \tau_1, \dots, \tau_{N_m}\}$ for $(V_m, \rho_{h(\ell)})$ such that the infinitesimal action of X on V_m can be diagonalized in the form

$$X\cdot\tau_\alpha = \gamma_\alpha(X)\tau_\alpha$$

for some real constants $\gamma_\alpha = \gamma_\alpha(X)$, $\alpha = 0, 1, \dots, N_m$, satisfying $\sum_{\alpha=0}^{N_m} \gamma_\alpha(X) = 0$. As in (3.2), we consider the associated basis $\tilde{\mathcal{T}} = \{\tilde\tau_0, \tilde\tau_1, \dots, \tilde\tau_{N_m}\}$ for V_m, where $\tilde\tau_{k,i} := e^{-d\chi_k(F(\ell))/2}\tau_{k,i}$. By setting

$$\lambda_X(e^t) := \exp(tX), \quad t \in \mathbb{R},$$

we consider the one-parameter group $\lambda_X\colon \mathbb{R}_+ \to \left\{\prod_{k=1}^{\nu_m} \mathrm{GL}(V_T(\chi_k))\right\} \cap \mathrm{SL}(V_m)$ associated to X. Then $\lambda_X(e^t)\cdot\tau_\alpha = e^{t\gamma_\alpha}\tau_\alpha$ for all α and all $t \in \mathbb{R}$. Moreover,

$$\Phi_m^*\varphi_X = \frac{\sum_{\alpha=0}^{N_m} \gamma_\alpha(X)|\tilde\tau_\alpha|^2}{m\sum_{\alpha=0}^{N_m} |\tilde\tau_\alpha|^2}, \quad X \in \mathfrak{P}_m. \tag{3.4}$$

Let η_m be the Kähler form on M defined by $\eta_m := (1/m)\Phi_m^*\,\omega_{\mathrm{FS}}$. To each $X \in \mathfrak{P}_m$, we can associate a real constant c_X such that $\phi_X := c_X + \Phi_m^*\varphi_X$ on M satisfies

$$\int_M \phi_X\,\eta_m^n = 0.$$

Proof of Assertion. Fix an arbitrary element $0 \neq X$ of $\mathfrak{p}_m''$, and define a real-valued function $f_{X,m}(t)$ on $\mathbb{R}$ by

$$f_{X,m}(t) := \log\|\lambda_X(e^t)\cdot\hat{M}_m\|_{\mathrm{CH}(\rho_{h(\ell)})}.$$

For this X, we consider the associated $\gamma_\alpha(X)$, $\alpha = 0, 1, \dots, N_m$, defined in the above. From now on, X regarded as a holomorphic vector field on $\mathbb{P}^*(V_m)$ will be denoted by $\mathcal{X}$. By [26] (see also [14], 4.5), we have $\ddot f_{X,m}(t) \geq 0$ for all t. Then by [15], Lemma 3.4, it suffices to show the existence of a real number $t_X^{(m)}$ such that

$$\dot f_{X,m}(t_X^{(m)}) = 0 < \ddot f_{X,m}(t_X^{(m)}) \quad \text{and} \quad t_X^{(m)}\cdot X \in U_m. \tag{3.5}$$

In the below, real numbers C_i, $i = 1, 2, \dots$, always mean positive real constants independent of the choice of m and X. Moreover by abuse of terminology, we write $m \gg 1$, if m satisfies $m \geq m_0$ for a sufficiently large m_0 independent of the choice of X. Then the proof of Assertion will be divided into the following eight steps:

STEP 1. Put $\lambda_t := \lambda_X(e^t)$ and $M_{m,t} := \lambda_t(M_m)$ for each $t \in \mathbb{R}$. Metrically, we identify the normal bundle of $M_{m,t}$ in $\mathbb{P}^*(V_m)$ with the subbundle $TM_{m,t}^\perp$ of $T\mathbb{P}^*(V_m)_{|M_{m,t}}$ obtained as the orthogonal complement of $TM_{m,t}$ in $T\mathbb{P}^*(V_m)_{|M_{m,t}}$. Hence, $T\mathbb{P}^*(V_m)_{|M_{m,t}}$ is differentiably written as the direct sum $TM_{m,t} \oplus TM_{m,t}^\perp$. Associated to this, the restriction $\mathcal{X}_{|M_{m,t}}$ of $\mathcal{X}$ to $M_{m,t}$ is written as

$$\mathcal{X}_{|M_{m,t}} = \mathcal{X}_{TM_{m,t}} \oplus \mathcal{X}_{TM_{m,t}^\perp}$$

for some smooth sections $\mathcal{X}_{TM_{m,t}}$ and $\mathcal{X}_{TM_{m,t}^{\perp}}$ of $TM_{m,t}$ and $TM_{m,t}^{\perp}$, respectively. Then the second derivative $\ddot{f}_{X,m}(t)$ is (see for instance [14], [21]) given by

$$\ddot{f}_{X,m}(t) = \int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}^{\perp}}|^2_{\omega_{\mathrm{FS}}}\, \omega_{\mathrm{FS}}^n \geq 0. \tag{3.6}$$

Since the Kodaira embedding $\Phi^{\tilde{\mathcal{T}}} : M \to \mathbb{P}^{N_m}(\mathbb{C})$ defined by

$$\Phi^{\tilde{\mathcal{T}}}(p) := (\tilde{\tau}_0(p) : \tilde{\tau}_1(p) : \cdots : \tilde{\tau}_{N_m}(p))$$

coincides with Φ_m above up to an isometry of $(V_m, \rho_{h(\ell)})$, we may assume without loss of generality that $\Phi^{\tilde{\mathcal{T}}}$ is chosen as Φ_m.

STEP 2. In view of the orthogonal decomposition $\mathfrak{P}_m = \mathfrak{p}^{\perp} \oplus \mathfrak{p}$, we can express X as an othogonal sum

$$X = X' + X''$$

for some $X' \in \mathfrak{p}$ and $X'' \in \mathfrak{p}^{\perp}$. Since $\omega(\ell)$ is K-invariant (cf. [14]), the group K acts isometrically on $(V_m, \rho_{h(\ell)})$. Now, there exists a T-admissible orthonormal basis $\mathcal{B} := \{\beta_0, \beta_1, \dots, \beta_{N_m}\}$ for V_m such that the infinitesimal action of X'' on V_m is written as

$$X'' \cdot \beta_\alpha = \gamma_\alpha(X'')\beta_\alpha, \quad \alpha = 0, 1, \dots, N_m,$$

for some real constants $\gamma_\alpha(X'')$, $\alpha = 0, 1, \dots, N_m$, satisfying $\sum_{\alpha=0}^{N_m} \gamma_\alpha(X'') = 0$. By the notation as in (3.2), we consider the associated basis $\tilde{\mathcal{B}} := \{\tilde{\beta}_0, \tilde{\beta}_1, \dots, \tilde{\beta}_{N_m}\}$ for V_m. Then

$$\phi_{X''} = \frac{\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')|\tilde{\beta}_\alpha|^2}{m \sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2}, \tag{3.7}$$

where $\hat{\gamma}_\alpha(X'') := \gamma_\alpha(X'') + mc_{X''}$. Now, X' and X'' regarded as holomorphic vector fields on $\mathbb{P}^*(V_m)$ will be denoted by $\mathcal{X}'$ and $\mathcal{X}''$, respectively. Associated to the expression $T\mathbb{P}^*(V_m)_{|M_{m,t}} = TM_{m,t} \oplus TM_{m,t}^{\perp}$ as differentiable vector bundles, the restrictions $\mathcal{X}'_{|M_{m,t}}$, $\mathcal{X}''_{|M_{m,t}}$ of $\mathcal{X}'$ and $\mathcal{X}''$ to $M_{m,t}$ are respectively written as

$$\mathcal{X}'_{|M_{m,t}} = \mathcal{X}'_{TM_{m,t}} \oplus \mathcal{X}'_{TM_{m,t}^{\perp}}$$

and

$$\mathcal{X}''_{|M_{m,t}} = \mathcal{X}''_{TM_{m,t}} \oplus \mathcal{X}''_{TM_{m,t}^{\perp}},$$

where $\mathcal{X}'_{TM_{m,t}}$, $\mathcal{X}''_{TM_{m,t}}$ are smooth sections of $TM_{m,t}$, and $\mathcal{X}'_{TM_{m,t}^{\perp}}$, $\mathcal{X}''_{TM_{m,t}^{\perp}}$ are smooth sections of $TM_{m,t}^{\perp}$. Then by $X' \in \mathfrak{p}$, we have

$$\mathcal{X}'_{TM_{m,t}^{\perp}} = 0, \quad \text{i.e.,} \quad \mathcal{X}_{TM_{m,t}^{\perp}} = \mathcal{X}''_{TM_{m,t}^{\perp}}. \tag{3.8}$$

STEP 3. Since T is irredundant, we have $\mathfrak{g}'_m(T) + \mathfrak{t} = \mathfrak{k}^{\mathbb{C}}$, i.e., $\mathfrak{p}'_m + \sqrt{-1}\mathfrak{t}_c = \mathfrak{p}$, where these are equalities as Lie subalgebras of $\mathfrak{h}$. From now on until the end of this step, as in the preceding steps, we regard both $\mathfrak{p}'_m$ and $\mathfrak{k}^{\mathbb{C}}$ as Lie subalgebras of $\mathfrak{sl}(V_m)$. Hence, as Lie subalgebras of $\mathfrak{sl}(V_m)$, we have

$$\mathfrak{p} = \mathfrak{p}'_m + \sqrt{-1}\,\tilde{\mathfrak{t}}_c,$$

where we put $\tilde{\mathfrak{t}}_c := \mathrm{Lie}(\tilde{T}_c)$ for the maximal compact subgroup $\tilde{T}_c$ of $\tilde{T} := \iota^{-1}(T)$. Then we can write $X' \in \mathfrak{p}$ as a sum

$$X' = Y + W$$

for some $Y \in \mathfrak{p}'_m$ and some $W \in \sqrt{-1}\,\tilde{\mathfrak{t}}_c$. Note that the holomorphic vector fields $\mathcal{Y}$ and $\mathcal{W}$ on $\mathbb{P}^*(V_m)$ induced by Y and W, respectively, are tangent to M. By $[Y, W] = 0$, there exists a T-admissible orthonormal basis $\{\sigma_0, \sigma_1, \dots, \sigma_{N_m}\}$ for V_m such that

$$\begin{cases} Y \cdot \sigma_\alpha = \gamma_\alpha(Y)\sigma_\alpha, & \alpha = 0, 1, \dots, N_m; \\ W \cdot \sigma_{k,i} = b_k \sigma_{k,i}, & k = 1, 2, \dots, \nu_m, \end{cases}$$

for some real constants $\gamma_\alpha(Y)$ and b_k, where in the last equality, we put $\sigma_{k,i} := \sigma_{l(k,i)}$ by using the same notation $l(k, i)$ as in (2.4). By setting $\tilde{\sigma}_{k,i} = e^{-d\chi_k(F(\ell))/2}\sigma_{k,i}$, we later consider the basis $\{\tilde{\sigma}_0, \tilde{\sigma}_1, \dots, \tilde{\sigma}_{N_m}\}$ for V_m. Note that $\sum_{\alpha=0}^{N_m} \gamma_\alpha(Y) = \sum_{k=1}^{\nu_m} n_k b_k = 0$. Since both X and Y belong to $\mathfrak{p}_m$, it follows from $X = X' + X'' = Y + W + X''$ that

$$\sum_{i=1}^{n_k} \gamma_{k,i}(X) = \sum_{i=1}^{n_k} \gamma_{k,i}(Y) = 0 \quad \text{and} \quad n_k b_k = -\sum_{i=1}^{n_k} \gamma_{k,i}(X'') \quad \text{for all} \quad k,$$

where $\gamma_{k,i} := \gamma_{l(k,i)}$ with $l(k, i)$ as in (2.4). Note that $X \in \mathfrak{p}''_m$ and $X'' \in \mathfrak{p}^\perp$, where by $\mathfrak{p}'_m + \sqrt{-1}\mathfrak{t} = \mathfrak{p}$, the space $\mathfrak{p}^\perp$ is perpendicular to $\mathfrak{p}'_m$. Then by $Y \in \mathfrak{p}'_m$ and $X = Y + W + X''$, we have

$$(Y, Y) + (Y, W) = (Y, X) = 0$$

in terms of the inner product $(\ ,\ )$ on $\mathfrak{P}_m$. Hence $(Y, Y) = -(Y, W) \leq \sqrt{(Y, Y)(W, W)}$. It now follows that

$$\int_{M_m} |\mathcal{Y}_{|M_m}|^2_{\omega_{\mathrm{FS}}}\, \omega^n_{\mathrm{FS}} = m^2(Y, Y) \leq m^2(W, W) = \int_{M_m} |\mathcal{W}_{|M_m}|^2_{\omega_{\mathrm{FS}}}\, \omega^n_{\mathrm{FS}}. \tag{3.9}$$

The integral on the right-hand side is, for $m \gg 1$,

$$\int_M \frac{\left(\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|\tilde\sigma_{k,i}|^2_{h(\ell)}\right)\left(\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}b_k^2|\tilde\sigma_{k,i}|^2_{h(\ell)}\right)-\left(\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}b_k|\tilde\sigma_{k,i}|^2_{h(\ell)}\right)^2}{\left(\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|\tilde\sigma_{k,i}|^2_{h(\ell)}\right)^2} m^n\,\eta_m^n$$
$$\le m^n\int_M \frac{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}b_k^2|\tilde\sigma_{k,i}|^2_{h(\ell)}}{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|\tilde\sigma_{k,i}|^2_{h(\ell)}}\,\eta_m^n$$
$$\le \frac{n!}{2}\int_M \sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k} b_k^2|\tilde\sigma_{k,i}|^2_{h(\ell)}\,\eta_m^n \le C_1\sum_{k=1}^{\nu_m} n_k b_k^2$$

for some C_1, where in the last two inequalities, we used the Remark right after (3.2). Hence, by setting $\hat\gamma_\alpha(Y) := \gamma_\alpha(Y) + mc_Y$, we see from (3.9) that

$$\int_M \frac{\left(\sum_{\alpha=0}^{N_m}|\tilde\sigma_\alpha|^2_{h(\ell)}\right)\left(\sum_{\alpha=0}^{N_m}\hat\gamma_\alpha(Y)^2|\tilde\sigma_\alpha|^2_{h(\ell)}\right)-\left(\sum_{\alpha=0}^{N_m}\hat\gamma_\alpha(Y)|\tilde\sigma_\alpha|^2_{h(\ell)}\right)^2}{\left(\sum_{\alpha=0}^{N_m}|\tilde\sigma_\alpha|^2_{h(\ell)}\right)^2}\,\eta_m^n \le q^n C_1\sum_{k=1}^{\nu_m} n_k b_k^2. \tag{3.10}$$

Define real numbers f_1 and f_2 by

$$\begin{cases} f_1 := \displaystyle\int_M \left(\sum_{\alpha=0}^{N_m}|\tilde\sigma_\alpha|^2_{h(\ell)}\right)^{-1}\left(\sum_{\alpha=0}^{N_m}\hat\gamma_\alpha(Y)^2|\tilde\sigma_\alpha|^2_{h(\ell)}\right)\eta_m^n,\\ f_2 := \displaystyle\int_M \left\{\left(\sum_{\alpha=0}^{N_m}|\tilde\sigma_\alpha|^2_{h(\ell)}\right)^{-1}\left(\sum_{\alpha=0}^{N_m}\hat\gamma_\alpha(Y)|\tilde\sigma_\alpha|^2_{h(\ell)}\right)\right\}^2\eta_m^n. \end{cases}$$

If $f_1 \ge 2f_2$, then by (3.10) and the Remark right after (3.2), we have

$$\sum_{\alpha=0}^{N_m}\hat\gamma_\alpha(Y)^2 \le C_2\sum_{k=1}^{\nu_m} n_k b_k^2, \quad \text{if} \quad m \gg 1,$$

for some C_2. Next, assume $f_1 < 2f_2$. Then for

$$\phi_Y := \left(m\sum_{\alpha=0}^{N_m}|\tilde\sigma_\alpha|^2_{h(\ell)}\right)^{-1}\left(\sum_{\alpha=0}^{N_m}\hat\gamma_\alpha(Y)|\tilde\sigma_\alpha|^2_{h(\ell)}\right),$$

the left-hand side of (3.10) divided by $m \gg 1$ is written as

$$\int_{M_m} |\mathcal{Y}_{|M_m}|^2_{(\omega_{FS}/m)}\left(\frac{\omega_{FS}}{m}\right)^n = \int_M |\bar\partial\phi_Y|^2_{\eta_m}\,\eta_m^n \ge C_3\int_M \phi_Y^2\,\eta_m^n,$$

for some C_3, because the Kähler manifolds (M, η_m), $m \gg 1$, have bounded geometry (see also the Remark right after (3.2)). Hence, by $f_1 < 2f_2$ and (3.10), we see that, for $m \gg 1$,

$$q^n C_1 \sum_{k=1}^{\nu_m} n_k b_k^2 \geq m C_3 \int_M \phi_Y^2 \, \eta_m^n = C_3 q f_2$$
$$> \frac{C_3 q f_1}{2} \geq C_4 q^{n+1} \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(Y)^2$$

for some C_4, where in the last inequality, we used the Remark right after (3.2). By $\sum_{\alpha=0}^{N_m} \gamma_\alpha(Y) = 0$, we here observe that $\sum_{\alpha=0}^{N_m} \gamma_\alpha(Y)^2 \leq \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(Y)^2$. Hence, whether $f_1 \geq 2f_2$ or not, there always exists C_5 such that, for $m \gg 1$,

$$q \sum_{\alpha=0}^{N_m} \gamma_\alpha(Y)^2 \leq C_5 \sum_{k=1}^{\nu_m} n_k b_k^2. \tag{3.11}$$

STEP 4. Put $P := W + X''$. In view of [26], Theorem 1.6, a weighted version of (3.4.2) in [26] is true (cf. [14], [19]). Hence by $T \in \Delta_1$, we obtain

$$\dot{f}_{X,m}(0) = \dot{f}_{P,m}(0) = (n+1) \int_M \frac{\sum_{\alpha=0}^{N_m} \gamma_\alpha(P) |\tilde{\beta}_\alpha|_{h(\ell)}^2}{\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|_{h(\ell)}^2} \Phi_m^* \, \omega_{\mathrm{FS}}^n, \tag{3.12}$$

where $\gamma_{k,i}(P) := b_k + \gamma_{k,i}(X'')$ $(= \gamma_{l(k,i)}(P))$. Let $C_{q,\ell} = 1 + \sum_{j=0}^{\ell} \alpha_j q^{j+1}$ be as in (3.1). Then by (3.1) and (3.3), there exist a function $u_{m,\ell}$ and a 2-form $\theta_{m,\ell}$ on M such that

$$\begin{cases} B_q(\omega(\ell), F(\ell)) = \dfrac{n!}{m^n} \displaystyle\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|_{h(\ell)}^2 = C_{q,\ell} + u_{m,\ell} q^{\ell+2}; \\ \eta_m = \dfrac{1}{m} \Phi_m^* \, \omega_{\mathrm{FS}} = \omega(\ell) + \theta_{m,\ell} q^{\ell+2}, \end{cases} \tag{3.13}$$

where we have the inequalities $\|u_{m,\ell}\|_{C^0(M)} \leq C_6$ and $\|\theta_{m,\ell}\|_{C^0(M,\omega_0)} \leq C_7$ for some C_6 and C_7 (cf. Remark 2.11; see also [25], [14]). Hence, if $m \gg 1$,

$$\frac{|\dot{f}_{X,m}(0)|}{(n+1)!}$$
$$\leq \int_M \frac{\left(\sum_{\alpha=0}^{N_m} \gamma_\alpha(P) |\tilde{\beta}_\alpha|_{h(\ell)}^2\right) \left\{1 + \sum_{i=1}^{\infty} (-u_{m,\ell} q^{\ell+2}/C_{q,\ell})^i\right\} \{\omega(\ell) + \theta_{m,\ell} q^{\ell+2}\}^n}{C_{q,\ell}}.$$

Here, $\left\{1 + \sum_{i=1}^{\infty} (-u_{m,\ell} q^{\ell+2}/C_{q,\ell})^i\right\} \{\omega(\ell) + \theta_{m,\ell} q^{\ell+2}\}^n$ is written as $(1 + w_{m,\ell})\, \omega(\ell)^n$ for some function $w_{m,\ell}$ on M such that the inequality $\|w_{m,\ell}\|_{C^0(M)} \leq C_8$ holds for some

C_8. Then by $\int_M \{\sum_{\alpha=0}^{N_m} \gamma_\alpha(P)|\tilde{\beta}_\alpha|^2_{h(\ell)}\}\, \omega(\ell)^n = \sum_{k=1}^{\nu_m}\{e^{d\chi_k(F(\ell))}\sum_{i=1}^{n_k}(b_k + \gamma_{k,i}(X''))\} = 0$, we have

$$|\dot{f}_{X,m}(0)| \leq (n+1)!\, q^{\ell+2}\int_M \left|\frac{\sum_{\alpha=0}^{N_m}\gamma_\alpha(P)|\tilde{\beta}_\alpha|^2_{h(\ell)}}{C_{q,\ell}}\right| \cdot |w_{m,\ell}|\, \omega(\ell)^n$$
$$= (n+1)!\, q^{\ell+2-n}\int_M \left(1 + \frac{u_{m,\ell}q^{\ell+2}}{C_{q,\ell}}\right)\left|\frac{\sum_{\alpha=0}^{N_m}\gamma_\alpha(P)|\tilde{\beta}_\alpha|^2_{h(\ell)}}{\sum_{\alpha=0}^{N_m}|\tilde{\beta}_\alpha|^2_{h(\ell)}}\right| \cdot \left|\frac{w_{m,\ell}}{n!}\right| \omega(\ell)^n.$$

In view of (3.4), by setting $\hat{\phi} := \left(m\sum_{\alpha=0}^{N_m}|\tilde{\beta}_\alpha|^2\right)^{-1}\left(\sum_{\alpha=0}^{N_m}\gamma_\alpha(P)|\tilde{\beta}_\alpha|^2\right)$, there exist C_9 and C_{10} such that, for $m \gg 1$,

$$|\dot{f}_{X,m}(0)| \leq q^{\ell+1-n}C_9\|\hat{\phi}\|_{L^1(M,\omega(\ell))} \leq q^{\ell+1-n}C_{10}\|\hat{\phi}\|_{L^2(M,\omega(\ell))}. \tag{3.14}$$

STEP 5. Note that $0 \leq n_k^2 b_k^2 = \left\{\sum_{i=1}^{n_k}\gamma_{k,i}(X'')\right\}^2 \leq n_k\sum_{i=1}^{n_k}\gamma_{k,i}(X'')^2$ holds for all k by the Cauchy-Schwarz inequality. Hence

$$\sum_{k=1}^{\nu_m} n_k b_k^2 \leq \sum_{\alpha=0}^{N_m}\gamma_\alpha(X'')^2. \tag{3.15}$$

From $\sum_{\alpha=0}^{N_m}\gamma_\alpha(X'') = 0$ and $\hat{\gamma}_\alpha(X'') = \gamma_\alpha(X'') + mc_{X''}$, it follows that $\sum_{\alpha=0}^{N_m}\hat{\gamma}_\alpha(X'')^2 = (N_m+1)(mc_{X''})^2 + \sum_{\alpha=0}^{N_m}\gamma_\alpha(X'')^2$. In particular,

$$\sum_{\alpha=0}^{N_m}\gamma_\alpha(X'')^2 \leq \sum_{\alpha=0}^{N_m}\hat{\gamma}_\alpha(X'')^2. \tag{3.16}$$

Since $\gamma_{k,i}(P) = b_k + \gamma_{k,i}(X'')$, (3.15) and (3.16) above imply that

$$\sum_{\alpha=0}^{N_m}\gamma_\alpha(P)^2 \leq 2\left\{\sum_{k=1}^{\nu_m}n_k b_k^2 + \sum_{\alpha=0}^{N_m}\gamma_\alpha(X'')^2\right\} \leq 4\sum_{\alpha=0}^{N_m}\hat{\gamma}_\alpha(X'')^2. \tag{3.17}$$

By $X = Y + P$, we have $\sum_{\alpha=0}^{N_m}\gamma_\alpha(X)^2 \leq 2\{\sum_{\alpha=0}^{N_m}\gamma_\alpha(Y)^2 + \sum_{\alpha=0}^{N_m}\gamma_\alpha(P)^2\}$, because $\gamma_\alpha(X) = \gamma_\alpha(Y) + \gamma_\alpha(P)$. Hence, by (3.11), (3.15), (3.16) and (3.17), we obtain

$$\begin{aligned} q\sum_{\alpha=0}^{N_m}\gamma_\alpha(X)^2 &\leq 2q\sum_{\alpha=0}^{N_m}\gamma_\alpha(Y)^2 + 2q\sum_{\alpha=0}^{N_m}\gamma_\alpha(P)^2 \\ &\leq 2C_5\sum_{k=1}^{\nu_m}n_k b_k^2 + \sum_{\alpha=0}^{N_m}\gamma_\alpha(P)^2 \leq C_{11}\sum_{\alpha=0}^{N_m}\hat{\gamma}_\alpha(X'')^2 \end{aligned} \tag{3.18}$$

for $m \gg 1$, where we put $C_{11} := 4 + 2C_5$. Fix a positive real number ℓ_0 independent of the choice of m and X. Put $\delta_0 := q^{1/2+\ell_0}\Big/\sqrt{\sum_{\alpha=0}^{N_m}\hat{\gamma}_\alpha(X'')^2}$. Then by (3.18), we have

$0 < \delta_0 < \sqrt{C_{11}}/\bar{\gamma}$, where $\bar{\gamma} := \max\{|\gamma_\alpha(X)|;\ \alpha = 0, 1, \dots, N_m\}$. In view of Step 1 of [15], Section 4, by assuming $|t| \leq \delta_0$, we see that the family of Kähler manifolds $(M, q\Phi_m^*\lambda_t^*\,\omega_{\mathrm{FS}})$ have bounded geometry.

STEP 6. At the beginning of this step, we shall show the inequality (3.19) below as an analogue of [21], (5.9), by proving that an argument of Phong and Sturm [21] for $\dim H = 0$ is valid also for $\dim H > 0$. To see this, we consider the following exact sequence of holomorphic vector bundles

$$0 \to TM_{m,t} \to T\mathbb{P}^*(V_m)_{|M_{m,t}} \to TM_{m,t}^{\perp} \to 0,$$

where $TM_{m,t}^{\perp}$ is regarded as the normal bundle of $M_{m,t}$ in $\mathbb{P}^*(V)$. The pointwise estimate (cf. [21], (5.16)) of the second fundamental form for this exact sequence has nothing to do with $\dim H$, and as in [21], (5.15), it gives the inequality

$$\int_{M_{m,t}} |\mathcal{X}''_{TM_{m,t}^{\perp}}|^2_{\omega_{\mathrm{FS}}}\,\omega_{\mathrm{FS}}^n \geq C_{12}\int_{M_{m,t}} |\bar\partial\mathcal{X}''_{TM_{m,t}^{\perp}}|^2_{\omega_{\mathrm{FS}}}\,\omega_{\mathrm{FS}}^n,$$

for some C_{12}. Let $\mathcal{A}^{0,p}(T_M)$, $p = 0, 1$, denote the sheaf of germs of smooth $(0, p)$-forms on M with values in the holomorphic tangent bundle TM of M, and endow M with the Kähler metric $(1/m)\Phi_m^*\lambda_t^*\,\omega_{\mathrm{FS}}$. We then consider the operator $\Box_{TM} := -\bar\partial^{\#}\bar\partial$ on $\mathcal{A}^{0,0}(T_M)$, where $\bar\partial^{\#} : \mathcal{A}^{0,1}(T_M) \to \mathcal{A}^{0,0}(T_M)$ is the formal adjoint of $\bar\partial$: $\mathcal{A}^{0,0}(T_M) \to \mathcal{A}^{0,1}(T_M)$. Since by Step 1, the Kähler metrics $q\Phi_m^*\lambda_t^*\,\omega_{\mathrm{FS}}$ has bounded geometry, the first positive eigenvalue of the operator $-\Box_{TM}$ on $\mathcal{A}^{0,0}(T_M)$ is bounded from below by C_{13}. Hence, by $X'' \in \mathfrak{p}^{\perp}$,

$$\int_{M_{m,t}} |\bar\partial\mathcal{X}''_{TM_{m,t}}|^2_{(\omega_{\mathrm{FS}}/m)}\left(\frac{\omega_{\mathrm{FS}}}{m}\right)^n \geq C_{13}\int_{M_{m,t}} |\mathcal{X}''_{TM_{m,t}}|^2_{(\omega_{\mathrm{FS}}/m)}\left(\frac{\omega_{\mathrm{FS}}}{m}\right)^n.$$

Since $\bar\partial\mathcal{X}''_{TM_{m,t}^{\perp}} = -\bar\partial\mathcal{X}''_{TM_{m,t}}$, it now follows that

$$\begin{aligned}\ddot f_{X,m}(t) &= \int_{M_{m,t}} |\mathcal{X}''_{TM_{m,t}^{\perp}}|^2_{\omega_{\mathrm{FS}}}\,\omega_{\mathrm{FS}}^n \\ &\geq C_{12}C_{13}q\int_{M_{m,t}} |\mathcal{X}''_{TM_{m,t}}|^2_{\omega_{\mathrm{FS}}}\,\omega_{\mathrm{FS}}^n.\end{aligned} \tag{3.19}$$

In view of the equality $|\mathcal{X}''_{TM_{m,t}}|^2_{\omega_{\mathrm{FS}}} + \left|\mathcal{X}''_{TM_{m,t}^{\perp}}\right|^2_{\omega_{\mathrm{FS}}} = |\mathcal{X}''_{|M_{m,t}}|^2_{\omega_{\mathrm{FS}}}$, by adding the integral $C_{12}C_{13}q\int_{M_{m,t}}\left|\mathcal{X}''_{TM_{m,t}^{\perp}}\right|^2_{\omega_{\mathrm{FS}}}\omega_{\mathrm{FS}}^n$ to both sides of (3.19) and by dividing the resulting inequality by $(1 + C_{12}C_{13}q)$, we see that, for some C_{14} and C_{15},

$$
\begin{aligned}
\ddot{f}_{X,m}(t) &= \int_M \Phi_m^* \lambda_t^* \big(|\mathcal{X}''_{TM_{m,t}^{\perp}}|^2_{\omega_{\mathrm{FS}}}\, \omega_{\mathrm{FS}}^n \big) \\
&\geq C_{14} q \int_M \Phi_m^* \lambda_t^* (|\mathcal{X}''_{|M_{m,t}}|^2_{\omega_{\mathrm{FS}}}\, \omega_{\mathrm{FS}}^n) \\
&\geq C_{15} q \int_{M_m} |\mathcal{X}''_{|M_m}|^2_{\omega_{\mathrm{FS}}}\, \omega_{\mathrm{FS}}^n \geq C_{15} q \int_M \Theta \Phi_m^*\, \omega_{\mathrm{FS}}^n,
\end{aligned}
\tag{3.20}
$$

where $\Theta := \big(\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2\big)^{-2} \big\{ \big(\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2\big)\big(\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 |\tilde{\beta}_\alpha|^2\big) - \big(\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')|\tilde{\beta}_\alpha|^2\big)^2 \big\}$ is nonnegative everywhere on M. Then by (3.14) and (3.20),

$$
\begin{cases}
\dot{f}_{X,m}(\delta_0) \geq \dot{f}_{X,m}(0) + C_{15}\delta_0 q \displaystyle\int_M \Theta \Phi_m^*\, \omega_{\mathrm{FS}}^n \\
\qquad \geq -q^{\ell+1-n} C_{10} \|\hat{\phi}\|_{L^2(M,\omega(\ell))} + C_{15}\delta_0 q \displaystyle\int_M \Theta \Phi_m^*\, \omega_{\mathrm{FS}}^n, \\
\dot{f}_{X,m}(-\delta_0) \leq \dot{f}_{X,m}(0) - C_{15}\delta_0 q \displaystyle\int_M \Theta \Phi_m^*\, \omega_{\mathrm{FS}}^n \\
\qquad \leq q^{\ell+1-n} C_{10} \|\hat{\phi}\|_{L^2(M,\omega(\ell))} - C_{15}\delta_0 q \displaystyle\int_M \Theta \Phi_m^*\, \omega_{\mathrm{FS}}^n.
\end{cases}
\tag{3.21}
$$

By (3.20) and [15], Lemma 3.4, the proof of Main Theorem is reduced to showing the following three conditions for all $m \gg 1$:

$$
\text{i)}\ \ \dot{f}_{X,m}(\delta_0) > 0 > \dot{f}_{X,m}(-\delta_0), \qquad \text{ii)}\ \ \int_M \Theta \Phi_m^*\, \omega_{\mathrm{FS}}^n > 0, \qquad \text{iii)}\ \ t_X^{(m)} \cdot X \in U_m.
$$

Since iii) follows from Remark 3.31 below, we have only to prove i) and ii). Then by (3.21), it suffices to show the following for all $m \gg 1$:

$$
C_{15}\delta_0 q \int_M \Theta \Phi_m^*\, \omega_{\mathrm{FS}}^n - C_{10} q^{\ell+1-n} \|\hat{\phi}\|_{L^2(M,\omega(\ell))} > 0.
\tag{3.22}
$$

Let us define real numbers $\hat{e}_1$, $\hat{e}_2$, e_1, e_2 by setting

$$
\begin{aligned}
\hat{e}_1 &:= \int_M \frac{\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 |\tilde{\beta}_\alpha|^2}{\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2}\, \omega(\ell)^n, \quad &\hat{e}_2 &:= \int_M \left(\frac{\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'') |\tilde{\beta}_\alpha|^2}{\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2} \right)^2 \omega(\ell)^n, \\
e_1 &:= \int_M \frac{\sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 |\tilde{\beta}_\alpha|^2}{\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2}\, \omega(\ell)^n, \quad &e_2 &:= \int_M \left(\frac{\sum_{\alpha=0}^{N_m} \gamma_\alpha(P) |\tilde{\beta}_\alpha|^2}{\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2} \right)^2 \omega(\ell)^n.
\end{aligned}
$$

By the Cauchy-Schwarz inequality, we always have $\hat{e}_1 \geq \hat{e}_2$ and $e_1 \geq e_2$. Now, the following cases are possible:

CASE 1: $\hat{e}_1 > 2\hat{e}_2$.

CASE 2: $\hat{e}_1 \leq 2\hat{e}_2$.

In view of the identities in (3.13), we can write

$$\hat{e}_1 = q^n n! \int_M \frac{\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 |\tilde{\beta}_\alpha|^2_{h(\ell)}}{1 + \sum_{\alpha=0}^{\ell} \alpha_k q^{k+1} + u_{m,\ell} q^{\ell+2}} \, \omega(\ell)^n,$$

$$e_1 = q^n n! \int_M \frac{\sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 |\tilde{\beta}_\alpha|^2_{h(\ell)}}{1 + \sum_{\alpha=0}^{\ell} \alpha_k q^{k+1} + u_{m,\ell} q^{\ell+2}} \, \omega(\ell)^n,$$

$$\int_M \Theta \Phi_m^* \, \omega_{\mathrm{FS}}^n = m^n \int_M \Theta \{\omega(\ell) + \theta_{m,\ell} q^{\ell+2}\}^n,$$

and hence, given a positive real number $0 < \varepsilon \ll 1$, both $\hat{e}_1$ and $\int_M \Theta \Phi_m^* \, \omega_{\mathrm{FS}}^n$ above are estimated, for all $m \gg 1$, by

$$(1-\varepsilon) q^n \left\{ \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 \right\} \le \frac{\hat{e}_1}{n!} \le (1+\varepsilon) q^n \left\{ \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 \right\}, \tag{3.23}$$

$$(1-\varepsilon) q^n \left\{ \sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 \right\} \le \frac{e_1}{n!} \le (1+\varepsilon) q^n \left\{ \sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 \right\}, \tag{3.24}$$

$$(1-\varepsilon) q^{-n} \int_M \Theta \, \omega(\ell)^n \le \int_M \Theta \Phi_m^* \, \omega_{\mathrm{FS}}^n \le (1+\varepsilon) q^{-n} \int_M \Theta \, \omega(\ell)^n, \tag{3.25}$$

where we used the Remark right after (3.2). Moreover, we can write e_2 in the form

$$q^{-2} \|\hat{\phi}\|^2_{L^2(M,\omega(\ell))} = e_2 \le e_1. \tag{3.26}$$

STEP 7. We first consider Case 1. Then from (3.17), (3.23), (3.24), (3.25), (3.26), $\hat{e}_2 < \hat{e}_1/2$ and the definition of δ_0, it follows that

$$\begin{aligned}
&\text{L.H.S. of (3.22)} \\
&\ge (1-\varepsilon) C_{15} \delta_0 q^{1-n} \int_M \Theta \, \omega(\ell)^n - q^{\ell+1-n} C_{10} \|\hat{\phi}\|_{L^2(M,\omega(\ell))} \\
&\ge (1-\varepsilon) C_{15} q^{1-n} \delta_0 (\hat{e}_1 - \hat{e}_2) - q^{\ell+2-n} C_{10} \sqrt{e_1} \\
&\ge \frac{(1-\varepsilon) C_{15} \delta_0 q^{1-n} \hat{e}_1}{2} - (1+\varepsilon)^{1/2} C_{10} q^{\ell+2-n/2} \left\{ n! \sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 \right\}^{1/2} \\
&\ge \frac{(1-\varepsilon)^2 C_{15} \delta_0 q \left\{ \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 \right\} n!}{2} - 2(1+\varepsilon)^{1/2} C_{10} q^{\ell+2-n/2} \left\{ n! \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 \right\}^{1/2} \\
&\ge \left\{ n! \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 \right\}^{1/2} \left\{ \frac{\sqrt{n!}(1-\varepsilon)^2 C_{15} q^{\ell_0+3/2}}{2} - 2(1+\varepsilon)^{1/2} C_{10} q^{\ell+2-n/2} \right\},
\end{aligned}$$

for $m \gg 1$. Now we see that, if $\ell > (n-1)/2 + \ell_0$, then $q^{\ell+2-n/2}/q^{\ell_0+3/2}$ converges to

0 as $m \to \infty$. Thus, if $m \gg 1$, then by choosing ℓ such that $\ell > (n-1)/2 + \ell_0$, we now see from the computation above that L.H.S. of (3.22) is positive, as required.

STEP 8. Let us finally consider Case 2. For each fixed ℓ, the Kähler form η_m converges to ω_0, as $m \to \infty$, in $C^j(M)$-norm for all positive integers j (cf. [25], [14]; see also the Remark right after (3.2)). Note that $X'' \in \mathfrak{p}^{\perp}$. In view of $\int_M \phi_{X''}\, \eta_m^n = 0$, we see that

$$\|\phi_{X''}\|^2_{L^2(M,\eta_m)} \leq C_{16} \|\bar{\partial}\phi_{X''}\|^2_{L^2(M,\eta_m)} = C_{16} \|\Phi_m^* \mathcal{X}''_{TM_m}\|^2_{L^2(M,\eta_m)}, \tag{3.27}$$

for some C_{16}, where by abuse of terminology, the differential $(\Phi_m^{-1})_*\colon TM_m \to TM$ is denoted by Φ_m^*. Moreover, by (3.19) applied to $t = 0$, we obtain

$$\|\Phi_m^* \mathcal{X}''_{TM_m}\|^2_{L^2(M,\eta_m)} \leq (C_{12}C_{13})^{-1} q^{-1} \left\| \Phi_m^* \mathcal{X}''_{TM_m^{\perp}} \right\|^2_{L^2(M,\eta_m)}. \tag{3.28}$$

From now on until the end of this proof, we assume that $m \gg 1$. By (3.27) together with (3.28) and (3.7), there exist C_{17} and C_{18} such that

$$\begin{aligned} \|\Phi_m^* \mathcal{X}''_{TM_m^{\perp}}\|_{L^2(M,\eta_m)} &\geq C_{17} q^{1/2} \|\phi_{X''}\|_{L^2(M,\eta_m)} \\ &\geq C_{18} q^{1/2} \|\phi_{X''}\|_{L^2(M,\omega(\ell))} = C_{18} q^{3/2} \sqrt{\hat{e}_2}. \end{aligned} \tag{3.29}$$

We now observe the pointwise estimate $q^{1/2} |\mathcal{X}''_{|M_m}|_{\omega_{\mathrm{FS}}} = |\mathcal{X}''_{|M_m}|_{\eta_m} \geq |\mathcal{X}''_{TM_m^{\perp}}|_{\eta_m}$. Hence by (3.20) and (3.29), we obtain

$$\begin{aligned} \ddot{f}_{X,m}(t) &\geq C_{15} \int_{M_m} (q^{1/2} |\mathcal{X}''_{|M_m}|_{\omega_{\mathrm{FS}}})^2\, \omega_{\mathrm{FS}}^n \\ &\geq C_{15} q^{-n} \left\| \Phi_m^* \mathcal{X}''_{TM_m^{\perp}} \right\|^2_{L^2(M,\eta_m)} \geq C_{19} q^{3-n} \hat{e}_2, \end{aligned} \tag{3.30}$$

for some C_{19}. As in deducing (3.21) from (3.14) and (3.20), we obtain by (3.14) and (3.30) the inequalities

$$\dot{f}_{X,m}(\delta_0) \geq R$$

and

$$\dot{f}_{X,m}(-\delta_0) \leq -R,$$

where $R := -q^{\ell+1-n} C_{10} \|\hat{\phi}\|_{L^2(M,\omega(\ell))} + C_{19}\delta_0 q^{3-n} \hat{e}_2$. Hence, it suffices to show that $R > 0$. In view of the definition of $\hat{\phi}$ and e_2, we see from $\hat{e}_1 \leq 2\hat{e}_2$ and (3.26) that

$$R = C_{19}\delta_0 q^{3-n} \hat{e}_2 - C_{10} q^{\ell+2-n} \sqrt{e_2} \geq \frac{C_{19}\delta_0 q^{3-n} \hat{e}_1}{2} - C_{10} q^{\ell+2-n} \sqrt{e_1}.$$

Here by (3.23) and (3.24), we obtain

$$\begin{aligned}\frac{\delta_0 q^{3-n}\hat{e}_1}{q^{\ell+2-n}\sqrt{e_1}} &= q^{3/2+\ell_0-\ell}\sqrt{\frac{\hat{e}_1}{e_1}}\sqrt{\frac{\hat{e}_1}{\sum_{\alpha=0}^{N_m}\hat{\gamma}_\alpha(X'')^2}}\\ &\geq C_{20}q^{(3+n)/2+\ell_0-\ell}\sqrt{\frac{\sum_{\alpha=0}^{N_m}\hat{\gamma}_\alpha(X'')^2}{\sum_{\alpha=0}^{N_m}\gamma_\alpha(P)^2}} \geq \frac{C_{20}}{2}q^{(3+n)/2+\ell_0-\ell}\end{aligned}$$

for some C_{20}, where the last inequality follows from (3.17). Therefore, by choosing ℓ such that $\ell > (3+n)/2+\ell_0$, we now conclude that $R > 0$ for $m \gg 1$, as required. □

REMARK 3.31. In the above proof, it is easy to check the condition iii) in Step 6 as follows: In view of $|t_X^{(m)}| < \delta_0$, it suffices to show that, if $m \gg 1$, then

$$t\cdot X \in U_m, \quad \text{for all} \quad (t, X) \in \mathbb{R}\times\mathfrak{p}''_m \quad \text{with} \quad |t| < \delta_0. \tag{3.32}$$

For each $Q \in \mathfrak{p}$, let $u_Q \in C^\infty(M)_{\mathbb{R}}$ denote the Hamiltonian function for the holomorphic vector field Q on the Kähler manifold (M, ω_0) characterized by the equalities

$$i_Q\,\omega_0 = \frac{\sqrt{-1}}{2\pi}\,\bar{\partial}u_Q$$

and

$$\int_M u_Q\,\omega_0^n = 0.$$

Define compact subsets $\Sigma(\mathfrak{p}'_m)$, $\Sigma(\mathfrak{p})$ of $\mathfrak{p}$ by setting

$$\begin{cases}\Sigma(\mathfrak{p}'_m) := \{Q \in \mathfrak{p}'_m;\, \|\bar{\partial}u_Q\|_{L^2(M,\omega_0)} = 1\},\\ \Sigma(\mathfrak{p}) := \{Q \in \mathfrak{p};\, \|\bar{\partial}u_Q\|_{L^2(M,\omega_0)} = 1\}.\end{cases}$$

Choose an orthonormal basis $\mathcal{S} := \{s_0, s_1, \dots, s_{N_m}\}$ for the Hermitian vector space $(V_m, \rho_{h(\ell)})$. For the space $\mathcal{H}_m$ of all Hermitian matrices of order N_m+1, define a norm

$$\mathcal{H}_m \to \mathbb{R}_{\geq 0}, \quad A = (a_{\alpha\beta}) \mapsto \|A\|_m := \sqrt{\operatorname{tr} A^*A} = \sqrt{\sum_{\alpha,\beta}|a_{\alpha\beta}|^2}$$

on $\mathcal{H}_m$. Let $m \gg 1$. The infinitesimal action of $\mathfrak{p}_m$ on V_m is given by

$$Q\cdot s_\beta = \sum_{\alpha=0}^{N_m} s_\alpha\gamma_{\alpha\beta}(Q), \quad Q \in \mathfrak{p}_m,$$

where $\gamma_Q = (\gamma_{\alpha\beta}(Q)) \in \mathcal{H}_m$ denotes the representation matrix of Q on V_m with respect to $\mathcal{S}$. Let $X \in \mathfrak{p}''_m$, and let δ_0 be as in Step 5 above. For $t \in \mathbb{R}$ with $|t| < \delta_0$, we put $\tilde{X} := tX$. In order to prove (3.32) above, it suffices to show

$$\zeta(\mathrm{ad}\,\tilde{X})Q \notin \mathfrak{p}''_m \quad \text{for all} \quad Q \in \Sigma(\mathfrak{p}'_m). \tag{3.33}$$

Let $Q \in \Sigma(\mathfrak{p}'_m)$. For a suitable choice of a basis $\mathcal{S}$ as above, we may assume that the representation matrix γ_Q of Q is a real diagonal matrix. Note also that $\mathrm{tr}\,\gamma_Q = 0$. Let $\Phi_m : M \to \mathbb{P}^{N_m}(\mathbb{C})$ be the Kodaira embedding of M defined by (cf. (3.2))

$$\Phi_m(p) := (\tilde{s}_0(p) : \tilde{s}_1(p) : \cdots : \tilde{s}_{N_m}(p)).$$

In view of the definition $\eta_m := \Phi_m^* \, \omega_{\mathrm{FS}}/m$ of η_m, the Hamiltonian function ϕ_Q on (M, η_m) associated to the holomorphic vector field Q is expressed in the form

$$\phi_Q = \frac{\sum_{\alpha=0}^{N_m} \hat{\gamma}_{\alpha\alpha}(Q)|\tilde{s}_\alpha|^2}{m \sum_{\alpha=0}^{N_m} |\tilde{s}_\alpha|^2}.$$

We define $\hat{\gamma}_Q := (\hat{\gamma}_{\alpha\beta}(Q)) \in \mathcal{H}_m$ by setting $\hat{\gamma}_{\alpha\beta}(Q) := \{\gamma_{\alpha\alpha}(Q) + mc_Q\}\delta_{\alpha\beta}$ for Kronecker's delta $\delta_{\alpha\beta}$. As in deducing (3.16) from $\hat{\gamma}_\alpha(X'') = \gamma_\alpha(X'') + mc_{X''}$, we easily see that

$$\|\gamma_Q\|_m^2 \le \|\hat{\gamma}_Q\|_m^2.$$

Recall that η_m is expressible as $\omega_0 + (\sqrt{-1}/2\pi)q\,\partial\bar{\partial}\xi_m$ for some real-valued smooth function ξ_m on M such that

$$\|\xi_m\|_{C^3(M)} \le C_{21}, \tag{3.34}$$

where all C_j's in this remark are positive constants independent of the choice of m, X and Q. We now observe that

$$\phi_Q = u_Q + q(Q\xi_m). \tag{3.35}$$

Note that $Q \in \Sigma(\mathfrak{p}'_m) \subset \Sigma(\mathfrak{p})$. Since Q sits in the compact set $\Sigma(\mathfrak{p})$, and since $\Sigma(\mathfrak{p})$ is independent of the choice of m, there exist C_{22} and C_{23} such that

$$0 < C_{22} \le \int_M u_Q^2\, \omega_0^n \left(= \int_M \phi_Q^2\, \eta_m^n \right) \le C_{23}.$$

Note that both η_m and $\omega(\ell)$ converge to ω_0 as $m \to \infty$ (see (3.3) and the statement at the beginning of Step 8). Note also that, by the Remark right after (3.2), the function $(n!/m^n) \sum_{\alpha=0}^{N_m} |\tilde{s}_\alpha|^2_{h(\ell)}$ on M converges uniformly to 1, as $m \to \infty$. Again by the

Remark right after (3.2), it now follows from the Cauchy-Schwarz inequality that, for $m \gg 1$,

$$
\begin{aligned}
\|\gamma_Q\|_m^2 &= \sum_{\alpha=0}^{N_m} \gamma_{\alpha\alpha}(Q)^2 \geq C_{24} m^n \int_M \frac{\sum_{\alpha=0}^{N_m} \gamma_{\alpha\alpha}(Q)^2 |\tilde{s}_\alpha|^2}{\sum_{\alpha=0}^{N_m} |\tilde{s}_\alpha|^2}\, \omega(\ell)^n \\
&\geq C_{24} m^{n+2} \int_M (\Phi_m^* \varphi_Q)^2\, \omega(\ell)^n \geq C_{25} m^{n+2} \int_M (\Phi_m^* \varphi_Q)^2\, \eta_m^n
\end{aligned}
\tag{3.36}
$$

for some C_{24} and C_{25}, where $\Phi_m^* \varphi_Q$ is as in (3.4). Then for $m \gg 1$,

$$
\begin{aligned}
C_{26} &= \max_{J \in \Sigma(\mathfrak{p})} \int_M |J|_{\omega_0}^2\, \omega_0^n \geq \int_M |Q|_{\omega_0}^2\, \omega_0^n \geq C_{27} \int_M |Q|_{\eta_m}^2\, \eta_m^n \\
&= C_{27} m \int_M \left\{ \frac{\sum_{\alpha=0}^{N_m} \hat{\gamma}_{\alpha\alpha}(Q)^2 |\tilde{s}_\alpha|_{h(\ell)}^2}{m^2 \sum_{\alpha=0}^{N_m} |\tilde{s}_\alpha|_{h(\ell)}^2} - \phi_Q^2 \right\} \eta_m^n \\
&\geq \frac{C_{28}}{m^{n+1}} \left\{ \int_M \sum_{\alpha=0}^{N_m} \hat{\gamma}_{\alpha\alpha}(Q)^2 |\tilde{s}_\alpha|_{h(\ell)}^2\, \eta_m^n \right\} - C_{29} m \\
&\geq \frac{C_{30}}{m^{n+1}} \left\{ \int_M \sum_{\alpha=0}^{N_m} \hat{\gamma}_{\alpha\alpha}(Q)^2 |\tilde{s}_\alpha|_{h(\ell)}^2\, \omega(\ell)^n \right\} - C_{29} m \\
&\geq C_{31} \frac{\|\hat{\gamma}_Q\|_m^2}{m^{n+1}} - C_{29} m,
\end{aligned}
$$

for some C_{26}, C_{27}, C_{28}, C_{29}, C_{30} and C_{31}. Hence, if $m \gg 1$, then

$$
\|\gamma_Q\|_m^2 \leq \|\hat{\gamma}_Q\|_m^2 \leq C_{32} m^{n+2}. \tag{3.37}
$$

for some C_{32}. Now by $\zeta(0) = 1$, we define a real analytic function $\tilde{\zeta} = \tilde{\zeta}(x)$ on $\mathbb{R}$ satisfying $\tilde{\zeta}(0) = 0$ by

$$
\tilde{\zeta}(x) := \zeta(x) - 1.
$$

For $X \in \mathfrak{p}_m''$ above, by choosing an orthonormal basis for $(V_m, \rho_{h(\ell)})$ possibly distinct from the original one, we may assume that the representation matrix γ_X of X is a real diagonal matrix. Recall that $\tilde{X} = tX$, where $|t| < \delta_0 := q^{1/2+\ell_0}/\|\hat{\gamma}_{X''}\|_m$. Put $\tilde{X}'' := tX''$. Then by (3.18),

$$
\sqrt{\frac{q}{C_{11}}}\, \|\gamma_{\tilde{X}}\|_m \leq \|\hat{\gamma}_{\tilde{X}''}\|_m = |t| \cdot \|\hat{\gamma}_{X''}\|_m \leq q^{1/2+\ell_0},
$$

i.e., $\|\gamma_{\tilde{X}}\|_m \leq \sqrt{C_{11}}\, q^{\ell_0}$. Hence, if $m \gg 1$,

$$
\|\gamma_{\tilde{\zeta}(\mathrm{ad}\,\tilde{X})Q}\|_m \leq C_{33} q^{\ell_0} \|\gamma_Q\|_m, \tag{3.38}
$$

for some C_{33}. Now by the same argument as in (3.36), we see that, for some C_{34},

$$\|\gamma_{\tilde{\zeta}(\mathrm{ad}\,\tilde{X})Q}\|_m^2 \geq C_{34} m^{n+2} \int_M \Phi_m^* \varphi^2_{\tilde{\zeta}(\mathrm{ad}\,\tilde{X})Q}\, \eta_m^n, \quad \text{if} \quad m \gg 1. \tag{3.39}$$

Put $a_m := \sqrt{\int_M \Phi_m^* \varphi^2_{\tilde{\zeta}(\mathrm{ad}\,\tilde{X})Q}\, \eta_m^n}$. Then for $m \gg 1$, by (3.37), (3.38) and (3.39),

$$a_m \leq C_{35} q^{\ell_0} \tag{3.40}$$

for some C_{35}. Consider the Laplacians $\Box_{\eta_m}$ and $\Box_{\omega_0}$ on functions for the Kähler manifolds (M, η_m) and (M, ω_0), respectively. Note that $\zeta(\mathrm{ad}\,\tilde{X})Q = Q + \tilde{\zeta}(\mathrm{ad}\,\tilde{X})Q$. Then for $m \gg 1$, by (3.35), we obtain

$$\begin{aligned}
&\left| \int_M (\Box_{\eta_m}\phi_Q)\phi_{\zeta(\mathrm{ad}\,\tilde{X})Q}\, \eta_m^n \right| \\
&= \left| \int_M (\Box_{\eta_m}\phi_Q)(\phi_Q + \Phi_m^*\varphi_{\tilde{\zeta}(\mathrm{ad}\,\tilde{X})Q})\, \eta_m^n \right| \\
&\geq \|\bar{\partial}\phi_Q\|^2_{L^2(M,\eta_m)} - \left| \int_M (\Box_{\eta_m}\phi_Q)(\Phi_m^*\varphi_{\tilde{\zeta}(\mathrm{ad}\,\tilde{X})Q})\, \eta_m^n \right| \\
&\geq \|\bar{\partial}\{u_Q + q(Q\xi_m)\}\|^2_{L^2(M,\eta_m)} - a_m \|\Box_{\eta_m}\{u_Q + q(Q\xi_m)\}\|_{L^2(M,\eta_m)} \\
&\geq (1-\epsilon)R_m - (1+\epsilon)a_m S_m,
\end{aligned} \tag{3.41}$$

where we put $R_m := \|\bar{\partial}\{u_Q + q(Q\xi_m)\}\|^2_{L^2(M,\omega_0)}$ and $S_m := \|\Box_{\omega_0}\{u_Q + q(Q\xi_m)\}\|_{L^2(M,\omega_0)}$, and $\epsilon \ll 1$ is a positive constant independent of the choice of m, X and Q. Since Q belongs to the compact set $\Sigma(\mathfrak{p})$, by (3.34) and the equality $\|\bar{\partial}u_Q\|_{L^2(M,\omega_0)} = 1$, we obtain constants C_{36} and C_{37} such that

$$\begin{cases} R_m \geq 1 - 2q\|\bar{\partial}(Q\xi_m)\|_{L^2(M,\omega_0)} \geq 1 - C_{36}q, \\ S_m \leq \|\Box_{\omega_0}u_Q\|_{L^2(M,\omega_0)} + q\|\Box_{\omega_0}(Q\xi_m)\|_{L^2(M,\omega_0)} \leq C_{37}. \end{cases} \tag{3.42}$$

Then for $m \gg 1$, by (3.40), (3.41) and (3.42), we finally obtain

$$\left| \int_M (\Box_{\eta_m}\phi_Q)\phi_{\zeta(\mathrm{ad}\,\tilde{X})Q}\, \eta_m^n \right| \geq (1-\epsilon)(1 - C_{36}q) - (1+\epsilon)C_{35}C_{37}q^{\ell_0} > 0,$$

which implies (3.33), as required.

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