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L^p-L^q ESTIMATES FOR WAVE EQUATIONS AND THE KIRCHHOFF EQUATION

Dedicated to Professor Minoru Murata on his sixtieth birthday

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Abstract

The aim of this paper is to derive L^p-L^q estimates for strictly hyperbolic equations with time-dependent coefficients which are of Lipschitz class. Furthermore, L^p-L^q estimates for Kirchhoff equation can be obtained by applying the Schauder-Tychonoff fixed point theorem.

1. Introduction

Let us consider the following strictly hyperbolic Cauchy problem of second order:

$$\text{(L)} \qquad \begin{cases} (\partial_t^2 - c(t)^2\Delta)u(x, t) = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x, 0) = u_0(x), \ \partial_t u(x, 0) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

where $c(t)$ is positive on $\mathbb{R}$, $\partial_t = \partial/\partial t$ and Δ is Laplacian in $\mathbb{R}^n$ defined by $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$. In this paper we shall derive L^p-L^q decay estimates of solutions both to (L) and to the Cauchy problem of Kirchhoff equation.

In the case when $c(t) \equiv$ const., L^p-L^q estimates are well-known in [16, 17] (cf. [3, 9, 14, 18]), while the treatment of time-dependent case is very delicate. In fact, Reissig and Smith obtained the L^p-L^q estimates for (L) in the case when $c(t)$ is bounded, sufficiently smooth and oscillating (see [15]). The core of their argument is to gain the WKB representation of solutions to the ordinary differential equation corresponding to (L) through the Fourier transform, and apply the stationary phase method to the Fourier images. But then, the method in [15] is not effective in the case when $c(t) \in C^1$, since we cannot construct, in general, the WKB representation of solutions. Fortunately, the another representation formulae have been obtained through the theory of asymptotic integrations of ordinary differential equations (see [11, 12]), and an application of the stationary phase method to these representation formulae gives L^p-L^q estimates. But we note that there is a quite difference between these two representation formulae. Actually, the amplitude functions in the WKB representation of solutions belong to $S^0_{1,0}$ (Hörmander's class) in the high frequency part, while the ones in asymp-

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totic integrations method belong to $S^0_{0,0}$, which would cause to need a more delicate analysis to gain the decay in t in high frequency region.

As to L^p-L^q estimates for Kirchhoff equation, D'Ancona and Spagnolo obtained them by reducing the principal term of *linear equation* to the D'Alembertian operator through the Liouville transform (see [5]). But this can be done only if the data have compact supports, since their argument is essentially based on the finite propagation property of linear hyperbolic equations. Such a condition is too restrictive, hence our second aim is to remove this condition on data (see §4). On accout of these estimates, we can obtain the global existence theorem of non-linearly perturbed Kirchhoff equation without any compactly supported condition on data, contrary to [5], and will be discussed in a forthcoming paper [13].

We make the following assumption on $c(t)$:

Assumption A. *The function* $c(t)$ *is of class* $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ *and satisfies*
(i) $\inf_{t\in\mathbb{R}} c(t) > 0$,
(ii) $(1 + |t|)^n c'(t) \in L^1(\mathbb{R})$.

In order to state results, we introduce the notation used in this paper. For $s \in \mathbb{R}$ and $1 \le p \le \infty$, let $\dot{H}^{s,p}(\mathbb{R}^n)$ and $H^{s,p}(\mathbb{R}^n)$ be the Riesz and Bessel potential spaces which are the subspaces of $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ (the space of tempered distributions on $\mathbb{R}^n$) with semi-norm or norm

$$\|u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} = \|\mathcal{F}^{-1}[|\xi|^s \hat{u}(\xi)]\|_{L^p(\mathbb{R}^n)} \equiv \||D|^s u\|_{L^p(\mathbb{R}^n)},$$
$$\|u\|_{H^{s,p}(\mathbb{R}^n)} = \|\mathcal{F}^{-1}[\langle\xi\rangle^s \hat{u}(\xi)]\|_{L^p(\mathbb{R}^n)} \equiv \|\langle D\rangle^s u\|_{L^p(\mathbb{R}^n)},$$

respectively. Here $\hat{}$ denotes the Fourier transform, $\mathcal{F}^{-1}$ is its inverse and $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$. Throughout this paper, we fix the notation as follows:

$$\dot{H}^{s,p} = \dot{H}^{s,p}(\mathbb{R}^n), \quad H^{s,p} = H^{s,p}(\mathbb{R}^n), \quad \dot{H}^s = \dot{H}^{s,2}(\mathbb{R}^n), \quad H^s = H^{s,2}(\mathbb{R}^n).$$

We denote by C the various constants changing from line to line.

Our result reads as follows:

Theorem 1.1. *Let* $n \ge 2$. *Suppose that* $c(t)$ *satisfies* Assumption A. *Let* $1 < p \le 2 \le q < +\infty$ *and* $1/p + 1/q = 1$. *Then each solution* $u(x,t)$ *of the problem* (L) *has the following properties*:

$$\|\partial_t^j \partial_x^\alpha u(\,\cdot\,,t)\|_{L^q} \le C(1 + |t|)^{-((n-1)/2)(1/p-1/q)} \sum_{i=0,1} \|u_i\|_{H^{N_p+j+|\alpha|-i,p}}$$

for $j = 0, 1, 2$ *and every multi-index* $\alpha = (\alpha_1, \dots, \alpha_n)$ *with* $j + |\alpha| \ge 1$ *as long as the norms of data are finite, where* $N_p = ((3n+1)/2)(1/p - 1/q)$.

This paper is organized as follows: In §2 we develop the asymptotic integrations of ordinary differential equations. In §3 we will prove Theorem 1.1, and in the last section we will present L^p-L^q estimates for the Kirchhoff equation.

2. Asymptotic integration of ODE

By applying the Fourier transform on $\mathbb{R}^n_x$ to problem (L), we get

$$v'' + c(t)^2|\xi|^2 v = 0, \quad (' = \partial_t). \tag{2.1}$$

In this section we introduce an asymptotic integration of equation (2.1) along the argument of Ascoli [1] and Wintner [19] (see also [11, 12]). In the following we shall use the notation:

$$\vartheta(t) = \int_0^t c(\tau)\, d\tau.$$

We set

$$W(\xi, t) = \begin{pmatrix} v_0(\xi, t) & v_1(\xi, t) \\ v_0'(\xi, t) & v_1'(\xi, t) \end{pmatrix} = \text{fundamental matrix of the ODE (2.1).}$$

This means that $v_0(\xi, t)$ is the solution of (2.1) with $v_0(\xi, 0) = 1$, $v_0'(\xi, 0) = 0$, while $v_1(\xi, t)$ is the solution of (2.1) with $v_1(\xi, 0) = 0$, $v_1'(\xi, 0) = 1$. Hence the solution $v(\xi, t)$ of (2.1) can be written by

$$\begin{pmatrix} v(\xi, t) \\ v'(\xi, t) \end{pmatrix} = W(\xi, t) \begin{pmatrix} v(\xi, 0) \\ v'(\xi, 0) \end{pmatrix}. \tag{2.2}$$

We introduce some notation as follows:

$$Y(\xi, t) = \begin{pmatrix} \cos(\vartheta(t)|\xi|) & \dfrac{\sin(\vartheta(t)|\xi|)}{c(0)|\xi|} \\ -c(t)|\xi| \sin(\vartheta(t)|\xi|) & \dfrac{c(t)\cos(\vartheta(t)|\xi|)}{c(0)} \end{pmatrix}$$

$$= \text{fundamental matrix of the perturbed ODE:}$$

$$w'' - \frac{c'(t)}{c(t)} w' + c(t)^2|\xi|^2 w = 0. \tag{2.3}$$

Hence,

$$Y(\xi, 0) = I, \quad \det Y(\xi, t) = \frac{c(t)}{c(0)},$$

$$Y(\xi, t)^{-1} = \begin{pmatrix} \cos(\vartheta(t)|\xi|) & -\dfrac{\sin(\vartheta(t)|\xi|)}{c(t)|\xi|} \\ c(0)|\xi| \sin(\vartheta(t)|\xi|) & \dfrac{c(0)\cos(\vartheta(t)|\xi|)}{c(t)} \end{pmatrix}.$$

Since $c'(t)$ decays, in general, as $t \to \pm\infty$, we may call (2.3) the perturbed equation of (2.1). Notice that $Y(\xi, t)$ and $W(\xi, t)$ satisfy the equations

$$\partial_t Y(\xi, t) = A_0(\xi, t)Y(\xi, t), \quad \partial_t W(\xi, t) = A(\xi, t)W(\xi, t), \tag{2.4}$$

respectively, where we set

$$A_0(\xi, t) = \begin{pmatrix} 0 & 1 \\ -c(t)^2|\xi|^2 & \dfrac{c'(t)}{c(t)} \end{pmatrix}, \quad A(\xi, t) = \begin{pmatrix} 0 & 1 \\ -c(t)^2|\xi|^2 & 0 \end{pmatrix}.$$

In what follows, for non-negative functions $f(x)$ and $g(x)$, we denote $f(x) \le Cg(x)$ by $f \lesssim g$, where $C > 0$ is a certain constant.

Then we prove

Lemma 2.1 (see Ascoli [1] and Wintner [19]). *Suppose* Assumption A. *Then there exists* $\lim_{t\to\pm\infty}\{Y(\xi, t)^{-1}W(\xi, t)\}$, *which is* C^∞ *in* $\xi \in \mathbb{R}^n \setminus 0$. *Putting*

$$Q(\xi, t) = Y(\xi, t)^{-1}W(\xi, t) = \begin{pmatrix} a_0(\xi, t) & a_1(\xi, t) \\ b_0(\xi, t) & b_1(\xi, t) \end{pmatrix}, \tag{2.5}$$

we have

$$\sup_{t\in\mathbb{R}}|a_l(\xi, t)| \lesssim |\xi|^{-l}, \quad \sup_{t\in\mathbb{R}}|b_l(\xi, t)| \lesssim |\xi|^{1-l}, \quad l = 0, 1. \tag{2.6}$$

Furthermore, $Q(\xi, t)$ *satisfies the following initial value problem*

$$\partial_t Q(\xi, t) = C(\xi, t)Q(\xi, t), \quad Q(\xi, 0) = I, \tag{2.7}$$

where

$$C(\xi, t) = -\frac{c'(t)}{c(t)} \begin{pmatrix} \sin^2(\vartheta(t)|\xi|) & -\dfrac{\sin(2\vartheta(t)|\xi|)}{2c(0)|\xi|} \\ -\dfrac{1}{2}c(0)|\xi| \sin(2\vartheta(t)|\xi|) & \cos^2(\vartheta(t)|\xi|) \end{pmatrix}. \tag{2.8}$$

Proof. The existence of $\lim_{t\to\pm\infty}\{Y(\xi, t)^{-1}W(\xi, t)\}$ follows from the argument of [1, 19], and we may omit its proof. Differentiating (2.5) and using (2.4) we get

$$\partial_t Q(\xi, t) = Y(\xi, t)^{-1}(A(\xi, t) - A_0(\xi, t))Y(\xi, t)Q(\xi, t).$$

It can be readily checked that

$$Y(\xi,t)^{-1}(A(\xi,t)-A_0(\xi,t))Y(\xi,t)=C(\xi,t),$$

which proves (2.7).

Since $W(\xi,t)=Y(\xi,t)Q(\xi,t)$, we can write

$$v_l(\xi,t)=a_l(\xi,t)\cos(\vartheta(t)|\xi|)+b_l(\xi,t)\frac{\sin(\vartheta(t)|\xi|)}{c(0)|\xi|}, \tag{2.9}$$

$$v_l'(\xi,t)=-c(t)|\xi|a_l(\xi,t)\sin(\vartheta(t)|\xi|)+\frac{c(t)}{c(0)}b_l(\xi,t)\cos(\vartheta(t)|\xi|), \tag{2.10}$$

for $l=0,1$. By the standard energy method we have the hyperbolic energy estimates:

$$|v_l'(\xi,t)|^2+c(t)^2|\xi|^2|v_l(\xi,t)|^2\le \mathrm{e}^{\int_{-\infty}^{+\infty}2|c'(\tau)|/c(\tau)\,d\tau}(c(0)|\xi|)^{2(1-l)}. \tag{2.11}$$

On the other hand, multiplying (2.9) by $c(t)|\xi|$ and combining (2.10), we get, for $l=0,1$,

$$|v_l'(\xi,t)|^2+c(t)^2|\xi|^2|v_l(\xi,t)|^2=c(t)^2|\xi|^2|a_l(\xi,t)|^2+\frac{c(t)^2}{c(0)^2}|b_l(\xi,t)|^2.$$

Hence this equation and (2.11) imply that

$$\frac{c(t)^2}{c(0)^2}|a_0(\xi,t)|^2+\frac{c(t)^2}{c(0)^4}|\xi|^{-2}|b_0(\xi,t)|^2\le \mathrm{e}^{\int_{-\infty}^{+\infty}2|c'(\tau)|/c(\tau)\,d\tau}, \tag{2.12}$$

$$c(t)^2|\xi|^2|a_1(\xi,t)|^2+\frac{c(t)^2}{c(0)^2}|b_1(\xi,t)|^2\le \mathrm{e}^{\int_{-\infty}^{+\infty}2l|c'(\tau)|/c(\tau)\,d\tau}. \tag{2.13}$$

Thus the estimates (2.6) follow from (2.12)–(2.13). The proof of Lemma 2.1 is now complete. □

Summarizing the above argument, we conclude that the solution $\boldsymbol{v}(\xi,t)=\binom{v(\xi,t)}{v'(\xi,t)}$ of (2.1) with data $\boldsymbol{v}_0(\xi)=\binom{v(\xi,0)}{v'(\xi,0)}$ is represented by

$$\boldsymbol{v}(\xi,t)=Y(\xi,t)Q(\xi,t)\boldsymbol{v}_0(\xi).$$

Since the solution $u(x,t)$ of our problem (L) is represented by

$$u(x,t)=\mathcal{F}^{-1}[v_0(\xi,t)\hat{u}_0(\xi)+v_1(\xi,t)\hat{u}_1(\xi)](x),$$
$$\partial_t u(x,t)=\mathcal{F}^{-1}[v_0'(\xi,t)\hat{u}_0(\xi)+v_1'(\xi,t)\hat{u}_1(\xi)](x),$$

we arrive at the following:

Proposition 2.2. *Suppose that $c(t)$ satisfies* Assumption A. *Let $u = u(x, t)$ solve the problem* (L). *Then the Fourier transforms $\hat{u}(\xi, t)$ and $\hat{u}'(\xi, t)$ can be represented by*

$$\hat{u}(\xi, t) = a(\xi, t)\cos(\vartheta(t)|\xi|) + b(\xi, t)\sin(\vartheta(t)|\xi|),$$
$$\hat{u}'(\xi, t) = -c(t)a(\xi, t)|\xi|\sin(\vartheta(t)|\xi|) + c(t)b(\xi, t)|\xi|\cos(\vartheta(t)|\xi|)$$

for $t \in \mathbb{R}$, where

$$a(\xi, t) = \sum_{l=0,1} a_l(\xi, t)\hat{u}_l(\xi), \quad b(\xi, t) = \frac{1}{c(0)|\xi|}\sum_{l=0,1} b_l(\xi, t)\hat{u}_l(\xi).$$

The next aim is to gain the estimates of higher order derivatives of amplitude functions with respect to ξ (see Lemma 2.5 below), which will be used to develop the stationary phase method. Go back to the initial value problem (2.7). Then it follows from the theory of ordinary differential equations that $Q(\xi, t)$ can be written by Picard series:

$$Q(\xi, t) = I + \int_0^t C(\xi, \tau_1)\,d\tau_1 + \int_0^t C(\xi, \tau_1)\,d\tau_1 \int_0^{\tau_1} C(\xi, \tau_2)\,d\tau_2 + \cdots, \tag{2.14}$$

where $C(\xi, t)$ is given by (2.8).

We prepare the following two lemmas.

Lemma 2.3. *Let $c_{jk}(\xi, t)$, $j, k = 1, 2$, be the entries of matrix $C(\xi, t)$. Then, for every multi-index μ with $|\mu| \geq 1$ and $j, k = 1, 2$, we have*

$$|\partial_\xi^\mu c_{jk}(\xi, t)| \lesssim \Theta_\mu(t)|\xi|^{k-j}, \quad |\xi| \geq 1, \tag{2.15}$$

where we set

$$\Theta_\mu(t) = \sum_{1\leq|\nu|\leq|\mu|} \frac{|\vartheta(t)|^{|\nu|}|c'(t)|}{c(t)}.$$

Proof. Noting

$$\left|\partial_\xi^\mu|\xi|\right| \lesssim \frac{1}{|\xi|^{|\mu|-1}}, \quad \left|\nabla_\xi^\mu\left(\frac{\xi}{|\xi|}\right)\right| \lesssim \frac{1}{|\xi|^{|\mu|}}, \quad \left|\partial_\xi^\mu\left(\frac{1}{|\xi|}\right)\right| \lesssim \frac{1}{|\xi|^{|\mu|+1}} \tag{2.16}$$

for $|\mu| \geq 1$, we get

$$\left|\partial_\xi^\mu \sin(2\vartheta(t)|\xi|)\right|, \left|\partial_\xi^\mu \cos(2\vartheta(t)|\xi|)\right| \lesssim \sum_{1\leq|\nu|\leq|\mu|} |\vartheta(t)|^{|\nu|}|\xi|^{|\nu|-|\mu|}. \tag{2.17}$$

If $|\xi| \geq 1$, then the left-hand side of (2.17) is uniformly bounded in ξ. Hence (2.15) is true for $c_{11}(\xi, t)$ and $c_{22}(\xi, t)$, since

$$\sin^2(\vartheta(t)|\xi|) = \frac{1}{2}(1 - \cos(2\vartheta(t)|\xi|)),$$
$$\cos^2(\vartheta(t)|\xi|) = \frac{1}{2}(1 + \cos(2\vartheta(t)|\xi|)).$$

Writing $f(\xi, t) = |\xi| \sin(2\vartheta(t)|\xi|)$ and $g(\xi, t) = |\xi|^{-1} \sin(2\vartheta(t)|\xi|)$, we prove by induction,

$$|\partial_\xi^\mu f(\xi, t)| \lesssim \sum_{1\leq|\nu|\leq|\mu|} |\vartheta(t)|^{|\nu|}|\xi|, \quad |\partial_\xi^\mu g(\xi, t)| \lesssim \sum_{1\leq|\nu|\leq|\mu|} |\vartheta(t)|^{|\nu|}|\xi|^{-1} \tag{2.18}$$

for every μ with $|\mu| \geq 1$ and $|\xi| \geq 1$. We suppose that (2.18) holds for κ with $|\kappa| = 1, \ldots, |\mu| - 1$. Then we have, by the Leibniz rule,

$$\sum_{0\leq|\nu|\leq|\mu|} C_{\mu,\nu}\left(\partial_\xi^{\mu-\nu}\frac{1}{|\xi|}\right)\partial_\xi^\nu f(\xi, t) = \partial_\xi^\mu \sin(2\vartheta(t)|\xi|), \tag{2.19}$$

$$\sum_{0\leq|\nu|\leq|\mu|} C_{\mu,\nu}(\partial_\xi^{\mu-\nu}|\xi|)\partial_\xi^\nu g(\xi, t) = \partial_\xi^\mu \sin(2\vartheta(t)|\xi|). \tag{2.20}$$

Using (2.16)–(2.17) and (2.19)–(2.20), we conclude that (2.18) is true for $\kappa = \mu$. Thus (2.15) is also true for $c_{12}(\xi, t)$ and $c_{21}(\xi, t)$. The proof is complete. □

The following lemma is well-known.

Lemma 2.4. *Let $f(t) \in \mathcal{B}(\mathbb{R})$ and $F(t, s)$ be satisfied with*

$$F(t, s) = 1 + \int_s^t f(\tau_1)\, d\tau_1 + \int_s^t f(\tau_1)\, d\tau_1 \int_s^{\tau_1} f(\tau_2)\, d\tau_2 + \cdots.$$

Then $F(t, s) = \mathrm{e}^{\int_s^t f(\tau)\, d\tau}$.

Recalling $\Theta_\mu(t)$ in Lemma 2.3, we have the following:

Lemma 2.5. *Suppose that $c(t)$ satisfies* Assumption A. *Then, for $|\xi| \geq 1$ and every multi-index μ with $1 \leq |\mu| \leq n$, the following estimates hold for $l = 0, 1$:*

$$\sup_{t\in\mathbb{R}}|\partial_\xi^\mu a_l(\xi, t)| \lesssim \left(\mathrm{e}^{\int_{-\infty}^\infty \Theta_\mu(\tau)\, d\tau} - 1\right)|\xi|^{-l},$$
$$\sup_{t\in\mathbb{R}}|\partial_\xi^\mu b_l(\xi, t)| \lesssim \left(\mathrm{e}^{\int_{-\infty}^\infty \Theta_\mu(\tau)\, d\tau} - 1\right)|\xi|^{1-l}.$$

Proof. Notice that the behaviour of $(m+1)$-th term ($m \geq 1$) in the right-hand side of (2.14) with respect to ξ is similar to m-th power $C(\xi, t)^m$ of $C(\xi, t)$, and $C(\xi, t)^m$ is given by

$$C(\xi, t)^m = \left(\frac{c'(t)}{c(t)}\right)^m \begin{pmatrix} p_{11}(\xi, t) & p_{12}(\xi, t)|\xi|^{-1} \\ p_{21}(\xi, t)|\xi| & p_{22}(\xi, t) \end{pmatrix}, \quad m \in \mathbb{N},$$

where $p_{jk}(\xi, t)$, $j, k = 1, 2$, are polynomials of $\sin(\vartheta(t)|\xi|)$ and $\cos(\vartheta(t)|\xi|)$. Hence, taking account of this observation and Lemma 2.3, we deduce from (2.14) that $Q(\xi, t) = (q_{jk}(\xi, t))_{j,k=1,2}$ satisfies

$$|\partial_\xi^\mu q_{jk}(\xi, t)| \lesssim |\xi|^{k-j}\left(\int_0^{|t|} \Theta_\mu(\tau_1)\,d\tau_1 + \int_0^{|t|} \Theta_\mu(\tau_1)\,d\tau_1 \int_0^{\tau_1} \Theta_\mu(\tau_2)\,d\tau_2 + \cdots\right) \tag{2.21}$$

for $|\xi| \geq 1$ and $|\mu| \geq 1$. Now, in view of our assumptions, we have $\int_{-\infty}^{\infty} \Theta_\mu(\tau)\,d\tau < +\infty$ for $|\mu| \leq n$, and hence, applying Lemma 2.4 to (2.21), we get, for $|\xi| \geq 1$,

$$|\partial_\xi^\mu q_{jk}(\xi, t)| \lesssim \left(\mathrm{e}^{\int_{-\infty}^{\infty} \Theta_\mu(\tau)\,d\tau} - 1\right)|\xi|^{k-j}, \quad j, k = 1, 2.$$

Taking account of this estimate and recalling equation (2.5) from Lemma 2.1, we arrive at the desired estimates. This ends the proof of Lemma 2.5. □

3. Proof of Theorem 1.1

The idea of proof is similar to [15] (see also [14]). Proposition 2.2 assures that the solution of the Cauchy problem (L) is of the form

$$u(x, t) = \frac{1}{2}\sum_{l=0,1} \mathcal{F}^{-1}\Bigg[(\mathrm{e}^{i\vartheta(t)|\xi|} + \mathrm{e}^{-i\vartheta(t)|\xi|})a_l(\xi, t)\hat{u}_l(\xi) + (\mathrm{e}^{i\vartheta(t)|\xi|} - \mathrm{e}^{-i\vartheta(t)|\xi|})\frac{b_l(\xi, t)}{ic(0)|\xi|}\hat{u}_l(\xi)\Bigg](x).$$

To simplify the notation we consider the Fourier transform of the model multiplier

$$\mathcal{F}^{-1}[\mathrm{e}^{i\vartheta(t)|\xi|}a(\xi, t)\hat{\varphi}(\xi)](x),$$

where $a(\xi, t)$ is defined by

$$\text{either} \quad a_l(\xi, t)|\xi|^l \quad \text{or} \quad b_l(\xi, t)|\xi|^{-1+l}$$

for $l = 0, 1$. Then taking account of Lemmas 2.1 and 2.5, we may assume that

$$\sup_{t\in\mathbb{R}, \xi\in\mathbb{R}^n} |a(\xi, t)| \leq C_1, \qquad \sup_{t\in\mathbb{R}, |\xi|\geq 1} |\partial_\xi^\mu a(\xi, t)| \leq C_2, \quad 1 \leq |\mu| \leq n, \tag{3.1}$$

for some $C_1, C_2 > 0$.

We often use the Littlewood-Paley theorem.

Lemma 3.1 ([8] (Theorem 1.11)). *Let $f = f(\xi)$ be a tempered distribution on $\mathbb{R}^n_\xi$ $(n \geq 1)$ such that*

$$\sup_{0<l<+\infty} l^b \operatorname{meas}\{\xi; |f(\xi)| \geq l\} < +\infty$$

for some $1 < b < +\infty$. Then the convolution operator with $\mathcal{F}^{-1}[f]$ is L^p-L^q bounded provided that $1 < p \leq 2 \leq q < +\infty$, $1/p - 1/q = 1/b$, i.e., there exists a constant $C > 0$ such that

$$\|\mathcal{F}^{-1}[f] * u\|_{L^q} \leq C\|u\|_{L^p}, \quad u \in L^p.$$

In the proof of Therem 1.1 it suffices to consider the case of $t \geq 0$, because the problem (L) with $c(t)$ replaced by $c(-t)$ can be treated in the same way. Let us choose a non-decreasing function $\psi \in C^\infty$ such that $\psi(\xi) \equiv 0$ for $|\xi| \leq 1/2$, and 1 for $|\xi| \geq 1$. In what follows we set $K(t) = (1+t)^{-1}$ for $t \geq 0$.

Proposition 3.2. *For any $n \geq 1$ and p, q satisfying $1 < p \leq 2 \leq q < +\infty$, the following estimate holds for all $t \geq 0$:*

$$\left\|\mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|}\left(1-\psi\left(\frac{\xi}{K(t)}\right)\right)a(\xi,t)\hat{\varphi}(\xi)\right]\right\|_{L^q} \leq C(1+t)^{-n(1/p-1/q)}\|\varphi\|_{L^p}, \tag{3.2}$$

where C depends on n, p, q, and the norm $\|a\|_{L^\infty}$.

Proof. The estimate (3.2) with $p = q = 2$ follows from the Plancherel theorem. Hence we may prove the case $p \neq q$. Passing to the transformations $\xi = K(t)\eta$ and $y = K(t)x$, we have

$$\begin{aligned}&\left\|\mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|}\left(1-\psi\left(\frac{\xi}{K(t)}\right)\right)a(\xi,t)\hat{\varphi}(\xi)\right]\right\|_{L^q}\\ &= K(t)^{n-n/q}\left\|\mathcal{F}^{-1}[e^{iK(t)\vartheta(t)|\eta|}(1-\psi(\eta))a(K(t)\eta,t)\hat{\varphi}(K(t)\eta)]\right\|_{L^q}.\end{aligned}$$

Defining

$$T_{r,t} := \mathcal{F}^{-1}[e^{iK(t)\vartheta(t)|\eta|}(1-\psi(\eta))|\eta|^{-r}a(K(t)\eta,t)](x)$$

with the parameter $r \geq 0$, we have

$$\left\|\mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|}\left(1-\psi\left(\frac{\xi}{K(t)}\right)\right)a(\xi,t)\hat{\varphi}(\xi)\right]\right\|_{L^q} = K(t)^{n-n/q}\|T_{0,t} * \mathcal{F}^{-1}[\hat{\varphi}(K(t)\eta)]\|_{L^q}.$$

Notice that the support of $1-\psi(\eta)$ is contained in $\{\eta \in \mathbb{R}^n; 0 \le |\eta| \le 1\}$. Then the set $\{\eta; |\mathcal{F}[T_{r,t}]| \ge l\}$ is monotone increasing in r for each $l > 0$, i.e.,

$$\text{(3.3)} \qquad \operatorname{meas}\{\eta; |\mathcal{F}[T_{r,t}]| \ge l\} \le \operatorname{meas}\{\eta; |\mathcal{F}[T_{r_0,t}]| \ge l\}, \quad \text{if} \quad 0 \le r \le r_0.$$

It follows from the estimate (3.1) of $a(\xi, t)$ that

$$\operatorname{meas}\{\eta; |\mathcal{F}[T_{r_0,t}]| \ge l\} \le \operatorname{meas}\{\eta; |\eta| \le C_1^{1/r_0} l^{-1/r_0}\} = CC_1^{n/r_0} l^{-n/r_0}$$

for each $l > 0$. This together with (3.3) implies that

$$\operatorname{meas}\{\eta; |\mathcal{F}[T_{0,t}]| \ge l\} \le CC_1^{n/r_0} l^{-n/r_0},$$

and hence, we can apply Lemma 3.1 to conclude that the convolution operator with $T_{0,t}$ is L^p-L^q bounded provided that $1 < n/r_0 < +\infty$ and $r_0/n = 1/p - 1/q$, i.e., $r_0 = n(1/p - 1/q)$. Hence we have

$$\|T_{0,t} * \mathcal{F}^{-1}[\hat{\varphi}(K(t)\eta)]\|_{L^q} \le C\|\mathcal{F}^{-1}[\hat{\varphi}(K(t)\eta)]\|_{L^p} = CK(t)^{-n+n/p}\|\varphi\|_{L^p}.$$

Thus we conclude that

$$\begin{aligned} &\left\| \mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|}\left(1 - \psi\left(\frac{\xi}{K(t)}\right)\right) a(\xi, t)\hat{\varphi}(\xi)\right]\right\|_{L^q} \\ &\le CK(t)^{(n-n/q)+(-n+n/p)}\|\varphi\|_{L^p} = CK(t)^{n(1/p-1/q)}\|\varphi\|_{L^p}. \end{aligned}$$

The proof of Proposition 3.2 is complete. □

Next we want to estimate the L^q-norms of Fourier transform of multipliers

$$\mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|}\psi\left(\frac{\xi}{K(t)}\right) a(\xi, t)\hat{\varphi}(\xi)\right](x).$$

For this purpose we will develop the stationary phase method, and need some lemmas. The first one is a special version of well-known Littman's lemma as follows:

Lemma 3.3 ([10]). *Let* $n \ge 2$. *Then for* $v \in C_0^\infty$ *with* $\operatorname{supp} v \not\ni 0$,

$$\|\mathcal{F}^{-1}[e^{it|\xi|}v]\|_{L^\infty} \le C|t|^{-(n-1)/2} \sum_{|\alpha| \le n} \|\partial_\xi^\alpha v\|_{L^1}, \quad t \ne 0.$$

In order to state useful lemmas, let us introduce a non-negative function $\Phi(\xi)$ having its compact support in $\{\xi \in \mathbb{R}^n; 1/2 \le |\xi| \le 2\}$ such that $\sum_{k=-\infty}^{+\infty} \Phi(2^{-k}\xi) = 1$ $(\xi \ne 0)$. Let us define $\Phi_k(\xi) = \Phi(2^{-k}\xi)$, $k \in \mathbb{N}$, and $\Phi_0(\xi) = 1 - \sum_{k=1}^{+\infty} \Phi_k(\xi)$. Then the function $\Phi_0(\xi)$ has its support in $\{\xi \in \mathbb{R}^n; |\xi| \le 2\}$.

Lemma 3.4 ([2], cf. [3] (Lemma 1)). *Let* $1 < p \le 2 \le q < +\infty$ *and* $1/p + 1/q = 1$. *Then*

$$L^p \subset B_2^{0,p}, \quad L^q \supset B_2^{0,q},$$

where $B_r^{s,p}$ *is the Besov space, i.e., the subspace of* $\mathcal{S}'$ *with the norm*

$$\|v\|_{B_r^{s,p}} = \left\{ \sum_{k=0}^{+\infty} (2^{ks} \|\mathcal{F}^{-1}[\Phi_k \hat{v}]\|_{L^p})^r \right\}^{1/r}, \quad r \ge 1, \ s \in \mathbb{R}.$$

Lemma 3.5 ([3] (Lemma 2)). *Let* $a \in L^\infty$ *and assume that*

$$\|\mathcal{F}^{-1}[a\Phi_k \hat{v}]\|_{L^q} \le C\|v\|_{L^p}, \quad k = 0, 1, 2, \ldots.$$

Then there exists a constant A *independent of* a *such that*

$$\|\mathcal{F}^{-1}[a\hat{v}]\|_{B_r^{0,q}} \le AC\|v\|_{B_r^{0,p}} \quad \textit{for} \quad r \ge 1.$$

Now we are in position to estimate our Fourier multipliers.

Proposition 3.6. *Let* $n \ge 2$, $1 < p \le 2 \le q < +\infty$ *and* $1/p + 1/q = 1$. *Then for all* $t \ge 1$, *the following estimate holds*:

$$\left\| \mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|} \psi\left(\frac{\xi}{K(t)}\right) a(\xi, t)\hat{\varphi}(\xi) \right] \right\|_{L^q} \le C(1+t)^{-((n-1)/2)(1/p-1/q)} \|\varphi\|_{H^{N_p,p}}, \tag{3.4}$$

where $N_p = ((3n+1)/2)(1/p - 1/q)$.

Proof. Let $\Phi_k(\xi)$ $(k = 0, 1, \ldots)$ be functions as introduced before, and we consider the following Fourier images of multipliers

$$\mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|} \psi\left(\frac{\xi}{K(t)}\right) \Phi_k\left(\frac{\xi}{K(t)}\right) a(\xi, t)\hat{\varphi}(\xi) \right](x), \quad k = 0, 1, \ldots.$$

We divide the proof into three steps.

FIRST STEP: L^1-L^∞ *continuity*. Notice that

$$\begin{aligned} &\left\| \mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|} \psi\left(\frac{\xi}{K(t)}\right) \Phi_k\left(\frac{\xi}{K(t)}\right) a(\xi, t)\hat{\varphi}(\xi) \right] \right\|_{L^\infty} \\ &= \left\| \mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|} \psi\left(\frac{\xi}{K(t)}\right) \Phi_k\left(\frac{\xi}{K(t)}\right) |\xi|^{-r} a(\xi, t) \right] * |D|^r \varphi \right\|_{L^\infty} \end{aligned} \tag{3.5}$$

for all $r \geq 0$. Passing to the transformation $\xi/(2^k K(t)) = \eta$ we obtain

$$
\begin{aligned}
&\left\| \mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|}\psi\left(\frac{\xi}{K(t)}\right)\Phi_k\left(\frac{\xi}{K(t)}\right)|\xi|^{-r}a(\xi, t)\right]\right\|_{L^\infty} \\
&= 2^{k(n-r)}K(t)^{n-r}\|\mathcal{F}^{-1}[e^{i2^k K(t)\vartheta(t)|\eta|}v_k(\eta, t)|\eta|^{-r}]\|_{L^\infty},
\end{aligned} \tag{3.6}
$$

where we set

$$v_k(\eta, t) = \psi(2^k\eta)\Phi_k(2^k\eta)a(2^k K(t)\eta, t).$$

Notice that the functions $v_k(\eta, t)$ have their supports in $\{\eta \in \mathbb{R}^n; 1/2 \leq |\eta| \leq 2\}$ on account of $\operatorname{supp}\Phi \subset [1/2, 2]$, and in particular, we have

$$v_k(\eta, t) = \begin{cases} \Phi(\eta)a(2^k K(t)\eta, t) & \text{for} \quad k = 1, 2, \dots, \\ \psi(\eta)\Phi_0(\eta)a(K(t)\eta, t) & \text{for} \quad k = 0, \end{cases} \quad \text{on} \quad \operatorname{supp}\Phi.$$

Then we can apply Lemma 3.3 to get, for $t \geq 1$,

$$
\begin{aligned}
&\|\mathcal{F}^{-1}[e^{i2^k K(t)\vartheta(t)|\eta|}v_k(\eta, t)|\eta|^{-r}]\|_{L^\infty} \\
&\leq C(2^k K(t)|\vartheta(t)|)^{-(n-1)/2}\sum_{|\alpha|\leq n}\|\partial_\eta^\alpha(|\eta|^{-r}v_k(\eta, t))\|_{L^1} \\
&\leq C2^{-k(n-1)/2}\sum_{|\alpha|\leq n}\|\partial_\eta^\alpha(|\eta|^{-r}v_k(\eta, t))\|_{L^1}.
\end{aligned} \tag{3.7}
$$

Now using the estimates (3.1) we have, for $k = 1, 2, \dots$,

$$
\begin{aligned}
&\|\partial_\eta^\alpha(|\eta|^{-r}v_k(\eta, t))\|_{L^1} \\
&= \int_{1/2\leq|\eta|\leq 2}\left|\partial_\eta^\alpha(\Phi(\eta)|\eta|^{-r}a(2^k K(t)\eta, t))\right| d\eta \\
&= \sum_{|\beta|\leq|\alpha|}C_{\alpha,\beta}\int_{1/2\leq|\eta|\leq 2}\left|\partial_\eta^{\alpha-\beta}(\Phi(\eta)|\eta|^{-r})\right|(2^k K(t))^{|\beta|}\left|(\partial_\eta^\beta a)(2^k K(t)\eta, t)\right| d\eta \\
&\leq C_\alpha\sum_{|\beta|\leq|\alpha|}(2^k K(t))^{|\beta|}.
\end{aligned} \tag{3.8}
$$

In case of $k = 0$, (3.8) can be obtained for $\Phi(\eta)$ replaced by $\psi(\eta)\Phi_0(\eta)$. Given $t \geq 1$, taking a least integer k_0 such that $2^{k_0}K(t) > 1$, we have

$$\sum_{|\beta|\leq|\alpha|}(2^k K(t))^{|\beta|} \leq \begin{cases} (2^k K(t))^{|\alpha|} & \text{for all} \quad k \geq k_0, \\ C_\alpha & \text{for} \quad k = 0, 1, \dots, k_0 - 1, \end{cases}$$

and hence, we get, by (3.8),

$$\sum_{|\alpha|\le n} \|\partial_\eta^\alpha(|\eta|^{-r} v_k(\eta, t))\|_{L^1} \le \begin{cases} C(2^k K(t))^n & \text{for all} \quad k \ge k_0, \\ C & \text{for} \quad k = 0, 1, \dots, k_0 - 1, \end{cases} \tag{3.9}$$

with a cetain constant independent of k. Thus we have, by (3.7) and (3.9),

$$\|\mathcal{F}^{-1}[e^{i2^k K(t)\vartheta(t)|\eta|} v_k(\eta, t)|\eta|^{-r}]\|_{L^\infty} \le \begin{cases} C2^{k(n+1)/2} K(t)^n, & k \ge k_0, \\ C2^{-k(n-1)/2}, & k = 0, 1, \dots, k_0 - 1 \end{cases}$$

for all $t \ge 1$. Therefore, combining this estimate with (3.5)–(3.6), we arrive at the following estimate for $t \ge 1$ and $k \ge k_0$:

$$\left\|\mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|}\psi\left(\frac{\xi}{K(t)}\right)\Phi_k\left(\frac{\xi}{K(t)}\right)a(\xi, t)\hat{\varphi}(\xi)\right]\right\|_{L^\infty} \le C2^{k((3n+1)/2-r)}K(t)^{2n-r}\||D|^r\varphi\|_{L^1}.$$

If we set $r = (3n+1)/2$, we get

$$\left\|\mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|}\psi\left(\frac{\xi}{K(t)}\right)\Phi_k\left(\frac{\xi}{K(t)}\right)a(\xi, t)\hat{\varphi}(\xi)\right]\right\|_{L^\infty} \le CK(t)^{(n-1)/2}\|\varphi\|_{H^{(3n+1)/2,1}} \tag{3.10}$$

where we used the relation $H^{s,p} \subset \dot{H}^{s,p}$ for $s > 0$ and $1 \le p \le +\infty$. As for the case $t \ge 1$ and $k = 0, 1, \dots, k_0 - 1$, we have

$$\left\|\mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|}\psi\left(\frac{\xi}{K(t)}\right)\Phi_k\left(\frac{\xi}{K(t)}\right)a(\xi, t)\hat{\varphi}(\xi)\right]\right\|_{L^\infty} \le C2^{k((n+1)/2-r)}K(t)^{n-r}\||D|^r\varphi\|_{L^1}$$

Thus, putting $r = (n+1)/2$ we arrive at

$$\left\|\mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|}\psi\left(\frac{\xi}{K(t)}\right)\Phi_k\left(\frac{\xi}{K(t)}\right)a(\xi, t)\hat{\varphi}(\xi)\right]\right\|_{L^\infty} \le CK(t)^{(n-1)/2}\|\varphi\|_{H^{(n+1)/2,1}} \tag{3.11}$$

for all $t \ge 1$ and $k = 0, 1, \dots, k_0 - 1$. Summarizing (3.10) and (3.11), we conclude that (3.10) holds for all $t \ge 1$ and $k \in \mathbb{N} \cup \{0\}$.

SECOND STEP: L^2-L^2 *continuity*. Noting

$$\left\|e^{i\vartheta(t)|\xi|}\psi\left(\frac{\xi}{K(t)}\right)\Phi_k\left(\frac{\xi}{K(t)}\right)a(\xi, t)\right\|_{L^\infty} \le \sup_{2^{k-1}\le|\xi|/K(t)\le 2^{k+1}} |a(\xi, t)| \le C_1$$

for all $t \ge 0$, we conclude from the Plancherel theorem that

$$\left\|\mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|}\psi\left(\frac{\xi}{K(t)}\right)\Phi_k\left(\frac{\xi}{K(t)}\right)a(\xi, t)\hat{\varphi}(\xi)\right]\right\|_{L^2} \le C\|\varphi\|_{L^2}.$$

THIRD STEP. Interpolating between L^∞ and L^2 (see [2, Bergh and Löfström], we see that, for $t \geq 1$,

(3.12)
$$\left\| \mathcal{F}^{-1}\left[\mathrm{e}^{i\vartheta(t)|\xi|} \psi\left(\frac{\xi}{K(t)}\right) \Phi_k\left(\frac{\xi}{K(t)}\right) a(\xi, t)\hat{\varphi}(\xi) \right] \right\|_{L^q} \leq CK(t)^{((n-1)/2)(1/p-1/q)} \|\varphi\|_{H^{N_p,p}},$$

with $N_p = ((3n+1)/2)(1/p - 1/q)$. Passing to the transformations $\xi = K(t)\eta$ and $y = K(t)x$, we have

$$\begin{aligned} &\left\| \mathcal{F}^{-1}\left[\mathrm{e}^{i\vartheta(t)|\xi|} \psi\left(\frac{\xi}{K(t)}\right) \Phi_k\left(\frac{\xi}{K(t)}\right) a(\xi, t)\hat{\varphi}(\xi) \right] \right\|_{L^q} \\ &= K(t)^{n-n/q} \left\| \mathcal{F}^{-1}[\mathrm{e}^{iK(t)\vartheta(t)|\eta|} \psi(\eta)\Phi_k(\eta) a(K(t)\eta, t)\hat{\varphi}(K(t)\eta)] \right\|_{L^q} \end{aligned}$$

for $k = \mathbb{N} \cup \{0\}$, and hence, applying Lemma 3.5 to (3.12) with this form and returning to the original form through the transformations $\eta = K(t)^{-1}\xi$ and $x = K(t)^{-1}y$, we get

$$\left\| \mathcal{F}^{-1}\left[\mathrm{e}^{i\vartheta(t)|\xi|} \psi\left(\frac{\xi}{K(t)}\right) a(\xi, t)\hat{\varphi}(\xi) \right] \right\|_{B_2^{0,q}} \leq CK(t)^{((n-1)/2)(1/p-1/q)} \|\varphi\|_{B_2^{N_p,p}} \tag{3.13}$$

provided that $1 < p \leq 2 \leq q < +\infty$ and $1/p + 1/q = 1$. Thus, applying Lemma 3.4 to (3.13), we get the desired estimate (3.4). The proof of Proposition 3.6 is now finished. □

It remains to estimate

$$\left\| \mathcal{F}^{-1}\left[\mathrm{e}^{i\vartheta(t)|\xi|} \psi\left(\frac{\xi}{K(t)}\right) a(\xi, t)\hat{\varphi}(\xi) \right] \right\|_{L^q}$$

near $t = 0$.

Proposition 3.7. *Let $n \geq 2$, $1 < p \leq 2 \leq q < +\infty$ and $1/p + 1/q = 1$. Then the following estimate holds for all $t \geq 0$:*

$$\left\| \mathcal{F}^{-1}\left[\mathrm{e}^{i\vartheta(t)|\xi|} \psi\left(\frac{\xi}{K(t)}\right) a(\xi, t)\hat{\varphi}(\xi) \right] \right\|_{L^q} \leq C\|\varphi\|_{H^{\tilde{N}_p,p}}, \tag{3.14}$$

where $\tilde{N}_p = n(1/p - 1/q)$.

Proof. In the following argument we need not Littman's lemma (see Lemma 3.3), and the proof relies only on the Littlewood-Paley theorem as in the proof of Proposition 3.2. It suffices to prove (3.14) for $p \neq q$, since the case $p = q = 2$ follows from the Plancherel theorem. Passing to the transformations $\xi/K(t) = \eta$ and $y = K(t)x$ we obtain

(3.15)

$$\left\| \mathcal{F}^{-1}\left[e^{i\vartheta(t)|\xi|}\psi\left(\frac{\xi}{K(t)}\right)a(\xi,t)\hat{\varphi}(\xi)\right]\right\|_{L^q} = K(t)^{n-n/q}\|S_{r,t} * \mathcal{F}^{-1}[|\eta|^r\hat{\varphi}(K(t)\eta)]\|_{L^q}$$

with the parameter $r > 0$, where we set

$$S_{r,t} = \mathcal{F}^{-1}[e^{iK(t)\vartheta(t)|\eta|}\psi(\eta)|\eta|^{-r}a(K(t)\eta,t)].$$

Recalling the estimate (3.1) of $a(\xi,t)$ we have

$$\text{meas}\{\eta; |\mathcal{F}[S_{r,t}]| \geq l\} \leq \text{meas}\{\eta; |\eta| \leq C_1^{1/r}l^{-1/r}\} = CC_1^{n/r}l^{-n/r}$$

for each $l > 0$. Hence we conclude from Lemma 3.1 that the convolution operator with $S_{r,t}$ is L^p-L^q bounded provided that $1 < n/r < +\infty$ and $r/n = 1/p - 1/q$, i.e., $r = n(1/p - 1/q)$. Therefore we have, putting $\tilde{N}_p(= r) = n(1/p - 1/q)$,

$$\begin{aligned}\|S_{\tilde{N}_p,t} * \mathcal{F}^{-1}[|\eta|^{\tilde{N}_p}\hat{\varphi}(K(t)\eta)]\|_{L^q} &\leq C\|\mathcal{F}^{-1}[|\eta|^{\tilde{N}_p}\hat{\varphi}(K(t)\eta)]\|_{L^p} \\ &= CK(t)^{-n+n/p-\tilde{N}_p}\|\varphi\|_{\dot{H}^{\tilde{N}_p,p}},\end{aligned}$$

where we performed the transformations $K(t)\eta = \xi$ and $x/K(t) = z$ in the last step. If we combine this estimate with (3.15) for $r = \tilde{N}_p$ and use the relation $H^{\tilde{N}_p,p} \subset \dot{H}^{\tilde{N}_p,p}$, then we obtain the required estimate (3.14). The proof of Proposition 3.7 is complete. □

Completion of the proof of Theorem 1.1. Combining Propositions 3.2, 3.6 and 3.7, we get L^p-L^q estimate

$$\|\mathcal{F}^{-1}[e^{i\vartheta(t)|\xi|}a(\xi,t)\hat{\varphi}(\xi)]\|_{L^q} \leq C(1+t)^{-(n-1)/2(1/p-1/q)}\|\varphi\|_{H^{N_p,p}}$$

provided that $1 < p \leq 2 \leq q < +\infty$ and $1/p + 1/q = 1$. We go back to the representations for $\hat{u}(\xi,t)$ and $\hat{u}'(\xi,t)$ obtained in Proposition 2.2. Then applying the estimates obtained now to the Fourier images $\hat{u}(\xi,t)$ and $\hat{u}'(\xi,t)$, we conclude the proof of Theorem 1.1. □

4. L^p-L^q estimates for the Kirchhoff equation

In this section we shall obtain L^p-L^q estimates for the Kirchhhoff equation. Let us consider the Cauchy problem for the Kirchhoff equation:

(K)
$$\begin{cases}\partial_t^2 u - \left(1 + \int_{\mathbb{R}^n}|\nabla u|^2\,dx\right)\Delta u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x), & x \in \mathbb{R}^n.\end{cases}$$

The global-in-time existence theorem is well-known from the following theorem:

Theorem A ([20] (Yamazaki)). *Let $n \geq 1$ and $s_0 \geq 3/2$. If the data u_0, u_1 satisfy $\{\nabla u_0, u_1\} \in (H^{s_0-1})^n \times H^{s_0-1}$, $\{u_0, u_1\} \in Y_k$ for some $k > 1$, and*

$$\varepsilon_0 \equiv \|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 + |\{u_0, u_1\}|_{Y_k} \ll 1 \quad \textit{for some} \quad k > 1, \tag{4.1}$$

then the problem (K) *has a unique solution $u(x, t)$ satisfying*

$$\{\nabla u, \partial_t u\} \in \bigcap_{j=0,1} C^j(\mathbb{R}; (H^{s_0-1})^n) \times C^j(\mathbb{R}; (H^{s_0-1})). \tag{4.2}$$

Here

$$\begin{aligned} Y_k &:= \{\{\phi, \psi\} \in \dot{H}^{3/2} \times H^{1/2}; |\{\phi, \psi\}|_{Y_k} < +\infty\}, \\ |\{\phi, \psi\}|_{Y_k} &:= \sup_{\tau \in \mathbb{R}} (1 + |\tau|)^k \left| \int_{\mathbb{R}^n} \mathrm{e}^{i\tau|\xi|} |\xi|^3 |\hat{\phi}(\xi)|^2 \, d\xi \right| \\ &\quad + \sup_{\tau \in \mathbb{R}} (1 + |\tau|)^k \left| \int_{\mathbb{R}^n} \mathrm{e}^{i\tau|\xi|} |\xi| |\hat{\psi}(\xi)|^2 \, d\xi \right| \\ &\quad + \sup_{\tau \in \mathbb{R}} (1 + |\tau|)^k \left| \int_{\mathbb{R}^n} \mathrm{e}^{i\tau|\xi|} |\xi|^2 \Re\left(\hat{\phi}(\xi)\overline{\hat{\psi}(\xi)}\right) d\xi \right|. \end{aligned}$$

Based on Theorem A, we shall prove here the following:

Theorem 4.1. *Let $n \geq 2$ and p, q, N_p be as in* Theorem 1.1. *Then each solution $u(x, t)$ satisfying* (4.1)–(4.2) *in* Theorem A *with $k = n + 1$ has the following estimate for all $\delta > 0$:*

$$\|\partial_t^j \partial_x^\alpha u(\cdot, t)\|_{L^q} \leq C(1 + |t|)^{-((n-1)/2-\delta)(1/p-1/q)} \sum_{i=0,1} \|u_i\|_{H^{N_p+j+|\alpha|-i,p}},$$

for $j = 0, 1, 2$ and every multi-index α with $j + |\alpha| \geq 1$ as long as the norms of data are finite.

The definition of Y_k is somewhat complicated. We make explicit some examples of spaces contained in Y_k.

EXAMPLE 4.2. (i) Let $n \geq 4$, $1 \leq p < 2(n-1)/(n+1)$ and $1/p + 1/q = 1$. Then it was proved in [12] that

$$H^{n(1/p-1/q)+1,p} \times H^{n(1/p-1/q),p} \subset Y_{k(p)}, \quad \text{where} \quad k(p) = \frac{n-1}{2}\left(\frac{1}{p} - \frac{1}{q}\right) > 1$$

(see also [21]).

(ii) Let $n \geq 1$ and $k \in (1, n+1]$. Then it was proved in [20] (see also [4, 7]) that the space of [5, 6]

$$\{(\phi, \psi) \in H^2 \times H^1; \|\langle x\rangle^k \phi\|_{H^2} + \|\langle x\rangle^k \psi\|_{H^1} < +\infty\}$$

is contained in Y_k.
(iii) It was proved in [12] that

$$\{\langle x\rangle^{-n}, |D|\langle x\rangle^{-n}\} \in Y_{n+2}, \quad \text{and} \quad \{\langle x\rangle^{-l}, |D|\langle x\rangle^{-l}\} \in Y_{n+3} \quad \text{for} \quad \forall l > n.$$

Now go back to the linear problem (L). Let us introduce a wider class for $c(t)$ than that in Assumption A as follows. Given $\Lambda > 1$, $\delta > 0$ and $n \geq 2$ we say that $c(t)$ belongs to $\mathcal{K}(\Lambda, \delta, n)$ if it belongs to $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ and satisfies

$$\begin{gathered} 1 \leq c(t) \leq \Lambda, \\ |c'(t)| \leq \delta(1+|t|)^{-(n+1)} \quad (\text{a.e. } t \in \mathbb{R}). \end{gathered}$$

We shall prove here the following:

Lemma 4.3. *Let $c(t) \in \mathcal{K}(\Lambda, \delta, n)$. Then, for $|\xi| \geq 1$ and every multi-index μ with $1 \leq |\mu| \leq n$, the following estimates hold for $l = 0, 1$:*

$$\begin{aligned} &\sup_{t\in\mathbb{R}}(\mathrm{e}+|t|)^{-\delta}|\partial_\xi^\mu a_l(\xi, t)| \lesssim |\xi|^{-l}, \\ &\sup_{t\in\mathbb{R}}(\mathrm{e}+|t|)^{-\delta}|\partial_\xi^\mu b_l(\xi, t)| \lesssim |\xi|^{1-l}. \end{aligned}$$

For $|\mu| = 0$ we have the estimates from Lemma 2.1.

Proof. Since $|\vartheta(\tau)| \leq 1 + |\tau|$, we have, for $|\mu| \leq n$,

$$\int_0^{|t|} \Theta_\mu(\tau)\, d\tau = \int_0^{|t|} \sum_{\nu\leq\mu} \frac{|\vartheta(\tau)|^{|\nu|}|c'(\tau)|}{c(\tau)}\, d\tau \lesssim \delta \log(\mathrm{e}+|t|).$$

Hence the proof can be done along the same line as in the proof of Lemma 2.5. □

Based on Lemma 4.3, we can prove the following theorem by the same argument as in the proof of Theorem 1.1.

Theorem 4.4. *Let $c(t) \in \mathcal{K}(\Lambda, \delta, n)$. Let p, q and N_p be as in* Theorem 1.1. *Then for each solution $u(x, t)$ of* (L) *the following estimate holds*:

$$\|\partial_t^j \partial_x^\alpha u(\cdot, t)\|_{L^q} \leq C(1+|t|)^{-((n-1)/2-\delta)(1/p-1/q)} \sum_{i=0,1} \|u_i\|_{H^{N_p+j+|\alpha|-i,p}} \tag{4.3}$$

for $j = 0, 1, 2$ *and every multi-index* α *with* $j + |\alpha| \geq 1$ *as long as the norms of data are finite.*

Proof. In order to prove Theorem 1.1 we have derived Propositions 3.2, 3.6 and 3.7. But here, Propositions 3.2, 3.7 and L^2-L^2 continuity from Proposition 3.6 hold also for our case, since the higher order derivatives of $a(\xi, t)$ (see Lemma 2.5) was used only in L^1-L^∞ continuity from Proposition 3.6. Hence if we study L^1-L^∞ continuity for $t \geq 1$, we can conclude the proof of Theorem 4.4 for the estimate (4.3).

We prove

$$\left\| \mathcal{F}^{-1}\left[\mathrm{e}^{i\vartheta(t)|\xi|} \psi\left(\frac{\xi}{K(t)}\right) \Phi_k\left(\frac{\xi}{K(t)}\right) a(\xi, t)\hat{\varphi}(\xi) \right] \right\|_{L^\infty} \leq C K(t)^{(n-1)/2-\delta} \|\varphi\|_{H^{(3n+1)/2,1}} \tag{4.4}$$

for $t \geq 1$. Go back to the proof of L^1-L^∞ continuity from Proposition 3.6 and recall (3.5)–(3.6). Then (3.7) becomes

$$\begin{aligned} &\|\mathcal{F}^{-1}[\mathrm{e}^{i2^k K(t)\vartheta(t)|\eta|} v_k(\eta, t)|\eta|^{-r}]\|_{L^\infty} \\ &\leq C 2^{-k(n-1)/2} K(t)^{-\delta} \sum_{|\alpha| \leq n} \|K(t)^\delta \partial_\eta^\alpha (|\eta|^{-r} v_k(\eta, t))\|_{L^\infty}. \end{aligned} \tag{4.5}$$

Noting Lemma 4.3 and recalling the choice of the integer k_0, we get

$$\sum_{|\alpha| \leq n} \|K(t)^\delta \partial_\eta^\alpha (|\eta|^{-r} v_k(\eta, t))\|_{L^\infty} \leq \begin{cases} C(2^k K(t))^n, & \text{if} \quad k \geq k_0, \\ C, & \text{if} \quad k = 0, 1, \dots, k_0 - 1. \end{cases} \tag{4.6}$$

Hence we combine (4.5)–(4.6) to get, for $t \geq 1$,

$$\|\mathcal{F}^{-1}[\mathrm{e}^{i2^k K(t)\vartheta(t)|\eta|} v_k(\eta, t)|\eta|^{-r}]\|_{L^\infty} \leq \begin{cases} C 2^{k(n+1)/2} K(t)^{-\delta}, & \text{if} \quad k \geq k_0, \\ C 2^{-k(n-1)/2} K(t)^{-\delta}, & \text{if} \quad k = 0, 1, \dots, k_0 - 1. \end{cases}$$

Therefore, we arrive at

$$\begin{aligned} &\left\| \mathcal{F}^{-1}\left[\mathrm{e}^{i\vartheta(t)|\xi|} \psi\left(\frac{\xi}{K(t)}\right) \Phi_k\left(\frac{\xi}{K(t)}\right) a(\xi, t)\hat{\varphi}(\xi) \right] \right\|_{L^\infty} \\ &\leq \begin{cases} C 2^{k((3n+1)/2-r)} K(t)^{2n-r-\delta} \|\varphi\|_{H^{r,1}}, & \text{if} \quad k \geq k_0, \\ C 2^{k((n+1)/2-r)} K(t)^{n-r-\delta} \|\varphi\|_{H^{r,1}}, & \text{if} \quad k = 0, 1, \dots, k_0 - 1, \end{cases} \end{aligned}$$

for $t \geq 1$, where we used the relation $H^{r,1} \subset \dot{H}^{r,1}$. Putting $r = (3n+1)/2$ for $k \geq k_0$ and $r = (n+1)/2$ for $k = 0, 1, \dots, k_0 - 1$, we have the required estimate (4.4). The proof of Theorem 4.4 is complete. □

Once we obtain L^p-L^q estimates for linear problem (L), we can also obtain the same estimates for the Kirchhoff equation through the fixed point argument as in [4, 5].

Proof of Theorem 4.1. Fixing the data satisfying the assumptions of Theorem A, we consider the solution $u(x, t)$ of linear problem (L) in Theorem 4.4, and define

$$\tilde{c}(t) = \sqrt{1 + \|\nabla u(\,\cdot\,, t)\|_{L^2}^2}.$$

This defines the mapping $\Theta\colon c \mapsto \tilde{c}$. By using the method of [21], we obtain

Lemma 4.5. *Let $u(x, t)$ be the solution of* (L) *with data satisfying* $\{\nabla u_0, u_1\} \in (H^{s_0-1})^n \times H^{s_0-1}$ *for some* $s_0 \ge 3/2$ *and* $\{u_0, u_1\} \in Y_{n+1}$. *Then, there exist a constant M depending only on n such that*

$$1 \le \tilde{c}(t) \le 1 + \|\nabla u_0\|_{L^2} + M\varepsilon_0,$$
$$|\tilde{c}'(t)| \le M\varepsilon_0(1 + |t|)^{-(n+1)},$$

where $\varepsilon_0 \equiv \varepsilon_0(u_0, u_1)$ *is the size of data* (*see* (4.1)).

It follows from Lemma 4.5 and the Schauder-Tychonoff fixed point theorem that Θ has a fixed point in $\mathcal{K}(\Lambda, \delta, n)$ for suitable Λ and δ, i.e., $\tilde{c}(t) = c(t)$, and hence, (K) has a unique solution $u(x, t)$ as in Theorem A. In conclusion, it follows from Theorem 4.4 that the solution $u(x, t)$ of (L) is a solution of (K) having L^p-L^q estimate (4.3). The proof of Theorem 4.1 is now complete. □

Final Remark. We can remove the constant δ from the decay rate of Theorem 4.1, if $\{u_0, u_1\} \in Y_k$ for some $k > n + 1$ (see Example 4.2 (iii)). In fact, we can obtain Lemma 4.3 with $\delta = 0$, and prove that $|\tilde{c}'(t)| < \delta(1+|t|)^{-k}$ provided that $|\{u_0, u_1\}|_{(\dot{H}^1 \times L^2)\cap Y_k} \ll 1$. Hence, by the fixed point argument, the decay rate of Theorem 4.1 coincides with the one of Theorem 1.1.

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