

The homotopy groups of a spectrum whose  $\{ \mathrm{BP} \}_{*}^{\wedge 2}$ -homology is  $\mathrm{vs} \{ -1 \}_{*}^{\wedge 2} \{ \mathrm{BP} \}_{*}^{\wedge 2} / (2, \mathrm{vs} 1 \text{ infty}) [1] \otimes \Lambda (2)$

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# THE HOMOTOPY GROUPS OF A SPECTRUM WHOSE $BP_*$ -HOMOLOGY IS $v_2^{-1} BP_* / (2, v_1^\infty) [t_1] \otimes \Lambda(t_2)$

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## 1. Introduction

In [5], Mahowald gave some examples of ring spectra obtained as Thom spectra. One of them is  $X_2$  in [5], which is a Thom spectrum associated to  $\omega: \Omega S^2 \rightarrow BO$ , where  $\omega$  is a mapping corresponding to the generator of  $\pi_1(BO)$ . Let  $BP$  denote the Brown-Peterson spectrum at the prime 2. Then the spectrum  $X_2$  is also characterized by the  $BP_*$ -homology  $BP_*(X_2) = BP_*/(2)[t_1]$  as a sub-comodule algebra of  $BP_*(BP)/(2) = BP_*/(2)[t_1, t_2, \dots]$ , where  $BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \dots]$  over Hazewinkel's generators  $v_i$  (cf. [14]).

Relating to  $X_2$ , consider a spectrum  $X$  constructed as follows: Let  $C$  be a cofiber of the Bousfield localization map  $X_2 \rightarrow L_1 X_2$  with respect to the Johnson-Wilson spectrum  $E(1)$  with  $\pi_*(E(1)) = \mathbb{Z}_{(2)}[v_1, v_1^{-1}]$ . Then  $C$  is an  $X_2$ -module spectrum since  $X_2$  is a ring spectrum. Consider the element  $h_{20} \in \pi_5(X_2)$ . Now the spectrum  $X$  is a cofiber of a map  $h_{20}: \Sigma^5 C \rightarrow C$ . By this definition, the  $BP_*$ -homology of  $X$  is  $BP_*(X) = BP_*/(2, v_1^\infty)[t_1] \otimes \Lambda(t_2)$ . Once we determined the homotopy groups  $\pi_*(L_2 X_2)$  in [17], the homotopy groups  $\pi_*(L_2 X)$  can be obtained from it. Here  $L_2$  denotes the Bousfield localization functor with respect to the Johnson-Wilson spectrum  $E(2)$  with  $\pi_*(E(2)) = \mathbb{Z}_{(2)}[v_1, v_2, v_2^{-1}]$  as a subalgebra of  $v_2^{-1} BP_*$ . But, in this paper, we compute, independently of [17], the homotopy groups  $\pi_*(L_2 X)$  of the  $E(2)_*$ -localized spectrum of  $X$  by using the Adams-Novikov spectral sequence. The computation of the  $E_2$ -term is done in the same manner as that of [17], using the  $v_1$ -Bockstein spectral sequence. Different from the odd prime case, there may involve non-trivial differentials of the Adams-Novikov spectral sequence. On the other hand, different from the case for  $X_2$ , this case may support at most one family of non-trivial differentials. In this sense, it is a little easier to determine the homotopy groups of  $L_2 X$  than those of  $L_2 X_2$ . By using the results of [7], we show here that the differentials are all trivial, in a different fashion from that of [17], and have the  $E_\infty$ -term of the spectral sequence. In order to state the result, consider the integers  $A_n$  defined by

$$A_0=1, A_{2n+1}=1+2A_{2n} \text{ and } A_{2n+2}=2A_{2n+1}$$

for  $n \geq 0$ , and use the notations :

$C_\infty\langle x \rangle$  is a  $\mathbf{Z}/2[v_1, v_2, v_2^{-1}]$ -module isomorphic to  
 $\mathbf{Z}/2[v_1, v_1^{-1}, v_2, v_2^{-1}]/\mathbf{Z}/2[v_1, v_2, v_2^{-1}]$   
 generated by elements  $\{x/v_1^j\}_{j>0}$  such that  $v_1(x/v_1^j)=x/v_1^{j-1}$ .  
 $C_j\langle x \rangle$  is a cyclic  $\mathbf{Z}/2[v_1, v_2, v_2^{-1}]$ -module isomorphic to  
 $\mathbf{Z}/2[v_1, v_2, v_2^{-1}]/(v_1^j)$   
 generated by an element  $x/v_1^j$ .

**Theorem.** *The  $E_\infty$ -term of the Adams-Novikov spectral sequence for computing  $\pi_*(L_2X)$  is a  $\mathbf{Z}/2[v_1, v_2, v_2^{-1}]$ -module*

$$M_* \otimes \Lambda(\rho).$$

Here, the graded  $\mathbf{Z}/2[v_1, v_2, v_2^{-1}]$ -module  $M_*$  is given by :

$$\begin{aligned} M_0 &= C_\infty\langle 1 \rangle \oplus \bigoplus_{n,i \geq 0} C_{A_n}\langle v_3^{2^n(2i+1)} \rangle, \\ M_1 &= \bigoplus_{i \geq 0} (C_1\langle v_3^{2^{i+1}} h_{30} \rangle \oplus C_1\langle v_3^{2^{i+1}} h_{31} \rangle \oplus C_3\langle v_3^{4^{i+2}} h_{30} \rangle) \\ &\quad \oplus \bigoplus_{n>0, i \geq 0} C_{A_n}\langle v_3^{2^n(2i+1)+1} h_{21} \rangle \\ &\quad \oplus \bigoplus_{i,k \geq 0} (C_{A_{2k+1}}\langle v_3^{4^k(4i+2)+b_{k+1}} h_{30} \rangle \oplus C_{A_{2k}}\langle v_3^{4^k(2i+1)+b_{k+1}/2} h_{31} \rangle), \\ M_2 &= \bigoplus_{i \geq 0} C_1\langle v_3^{2^{i+1}} h_{30} h_{31} \rangle \\ &\quad \oplus \bigoplus_{i,k \geq 0} (C_{A_{2k+1}}\langle v_3^{4^k(4i+2)+b_{k+1}+1} h_{21} h_{30} \rangle \\ &\quad \oplus C_{A_{2k}}\langle v_3^{4^k(2i+1)+(b_{k+1}/2)+1} h_{21} h_{31} \rangle) \text{ and} \\ M_n &= 0 \text{ for } n > 2. \end{aligned}$$

Furthermore, the generators have the following degrees :

$$|v_3|=14, |h_{20}|=5, |h_{21}|=11, |h_{30}|=13, \text{ and } |h_{31}|=27.$$

In the theorem, an element  $x$  has a degree  $r$  if  $x \in \pi_r(L_2X)$ .

This paper is organized as follows : In the next section, we recall some facts known about the  $v_1$ -Bockstein spectral sequence. In §3, we define elements  $x_n$ , which will play the main role in the computation of the Bockstein spectral sequence. We compute  $E_2$ -terms of the Adams-Novikov spectral sequence computing the homotopy groups  $\pi_*(L_2X)$  in §4, by using the tools given in the previous sections. In section 5, we prepare some lemmas to compute the Adams-Novikov differentials in the last section using the results of [7].

## 2. The Bockstein spectral sequence

Let  $(A, \Gamma)$  denote a Hopf algebroid with  $\Gamma$   $A$ -flat. Then it is known (cf. [14, Ch. A1]) that the category of  $\Gamma$ -comodules has enough injectives and so we can define the Ext groups as a cohomology of an injective resolution. Furthermore it

is given by a cohomology of the cobar resolution. So we can define  $\text{Ext}_F^n(A, M) = H^n(\Omega_F^* M)$  for a  $\Gamma$ -comodule  $M$ , where  $\Omega_F^* M$  is a cobar complex (cf. [14]). The cobar complex  $\Omega_F^* M$  is a differential graded module with

$$\Omega_F^s M = M \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma \quad (s \text{ copies of } \Gamma),$$

and the differentials  $d_r : \Omega_F^r M \rightarrow \Omega_F^{r+1} M$  defined inductively by

$$d_0(m) = \psi(m) - m \otimes 1 \text{ and } d_r(x \otimes y) = d_s(x) \otimes y + (-1)^s x \otimes d_t(y)$$

for  $x \in \Omega_F^s M$  and  $y \in \Omega_F^t A$ . Here  $\psi : M \rightarrow M \otimes_A \Gamma$  denotes the comodule structure map of  $M$ . In the following, every comodule is induced from  $A$  and so we use  $\eta_R$  for  $\psi$ .

Suppose that  $A = \mathbf{Z}_{(2)}[v_1, v_2, \dots]$  and  $\Gamma = A[t_1, t_2, \dots]$ . Consider a Hopf algebroid  $\Phi = A[t_1] \otimes \Lambda(t_2)$  and a coalgebroid  $\Sigma = \Gamma \square_{\Phi} A$  over  $A$ . Then  $\Sigma = A[t_2^2, t_3, \dots]$  and we have the change of rings theorem :

**Lemma 2.1.** *For a comodule  $A$ , there is an isomorphism*

$$\text{Ext}_F^*(A, M \otimes_A \Phi) \cong \text{Ext}_F^*(A, M).$$

*Proof.* Consider a relative injective  $\Gamma$ -resolution of  $M \otimes_A \Phi$  :

$$M \otimes_A \Phi \longrightarrow I_0 \otimes_A \Gamma \longrightarrow I_1 \otimes_A \Gamma \longrightarrow \cdots,$$

which is split as  $A$ -modules. Then apply the cotensor product  $- \square_{\Phi} A$  and we obtain a relative injective  $\Sigma$ -resolution of  $M$  :

$$M \longrightarrow I_0 \otimes_A \Sigma \longrightarrow I_1 \otimes_A \Sigma \longrightarrow \cdots,$$

since  $\Sigma = \Gamma \square_{\Phi} A$ . Thus the both Ext groups are obtained from the same complex  $I_0 \rightarrow I_1 \rightarrow \cdots$ . q.e.d.

In this paper, we will compute  $\text{Ext}_F^*(A, v_2^{-1}A/(2, v_1^{\infty}) \otimes_A \Phi)$ . By virtue of this lemma, we will work in the category of  $\Sigma$ -comodules. In order to compute the Ext groups  $\text{Ext}_F^*(A, v_2^{-1}A/(2, v_1^{\infty}))$ , we adopt the  $v_1$ -Bockstein spectral sequence with  $E_1$ -term

$$\text{Ext}_F^*(A, v_2^{-1}A/(2, v_1)).$$

To compute the  $E_1$ -term we recall [7] the structure

$$(2.2) \quad \text{Ext}_F^*(A, v_2^{-1}A/(2, v_1)[t_1]) = K(2)_*[v_3, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho_2).$$

This is shown by using the change of rings theorems

$$\begin{aligned} \text{Ext}_F^*(A, v_2^{-1}A/(2, v_1)[t_1]) &= \text{Ext}_{K(2)_*K(2)}^*(K(2)_*, K(2)_*[t_1]) \\ &= \text{Ext}_{S(2,2)}^*(\mathbf{Z}/2, \mathbf{Z}/2) \otimes_{K(2)_*} K(2)_*[v_3], \end{aligned}$$

in which  $K(2)_* = \mathbf{Z}/2[v_2, v_2^{-1}]$ ,  $K(2)_*K(2) = K(2)_* \otimes_A \Gamma \otimes_A K(2)_*$  and  $S(2,2) =$

$\mathbb{Z}/2[t_2, t_3, \dots]/(t_i^4 - t_i : i > 1)$ . Note here that the action of  $A$  on  $K(2)_*$  is given by sending  $v_i$  to 0 for  $i \neq 2$  and  $v_2$  to  $v_2$ , and  $(K(2)_*, K(2)_*K(2))$  becomes a Hopf algebroid induced from  $(A, \Gamma)$ . The second equation follows from the  $K(2)_*K(2)$ -comodule structure  $K(2)_*[t_1] = K(2)_*[t_1]/(v_2t_1^4 + v_2^2t_1) \otimes_{K(2)_*} K(2)_*[v_3]$  which is obtained from Landweber's formula  $\eta_R(v_3) \equiv v_3 + v_2t_1^4 + v_2^2t_1 \pmod{(2, v_1)}$ .

**Lemma 2.3.** *The  $E_1$ -term is given by*

$$\text{Ext}_F^*(A, v_2^{-1}A/(2, v_1)) = K(2)_*[v_3] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho),$$

where  $K(2)_* = \mathbb{Z}/(2)[v_2, v_2^{-1}]$  and  $h_{21}, h_{30}, h_{31}$  and  $\rho$  are the homology classes represented by  $t_2^2, t_3, t_3^2$  and  $v_2^5t_4 + t_4^2$  in the cobar complex, respectively.

*Proof.* Let  $H^*M$  for a  $\Gamma$ -comodule  $M$  denote the Ext group  $\text{Ext}_F^*(A, M)$ , and  $E_*$  and  $D_*$  be  $\Gamma$ -comodules

$$E_* = v_2^{-1}A/(2, v_1)[t_1] \otimes \Lambda(t_2) \text{ and } D_* = v_2^{-1}A/(2, v_1)[t_1].$$

Then the short exact sequence  $0 \rightarrow D_* \subset E_* \rightarrow \Sigma^{-6}D_* \rightarrow 0$  of  $\Gamma$ -comodules yields the long exact sequence

$$\dots \longrightarrow H^{s,t}D_* \longrightarrow H^{s,t}E_* \longrightarrow H^{s,t-6}D_* \xrightarrow{\delta} H^{s+1,t}D_* \longrightarrow \dots$$

with  $\delta(x) = h_{20}x$ . By (2.2),

$$H^*D_* = K(2)_*[v_3, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho_2).$$

This shows that  $h_{20} : H^sD_* \rightarrow H^{s+1}D_*$  is a monomorphism and we have the lemma. q.e.d.

### 3. The elements $x_n$

In this section we will define elements  $x_n$  such that

$$x_n \equiv v_3^{2^n} \pmod{(2, v_1)} \text{ and } d_0(x_n) \equiv v_1^{e_n} g_n,$$

in which  $g_n$  represents a generator of  $\text{Ext}_F^1(A, v_2^{-1}A/(2, v_1))$  and  $e_n$  to be taken as great as possible. These elements play a central role in the Bockstein spectral sequence.

Hereafter we use the following abbreviation :

$$\begin{aligned} \text{Ext}^*(N) &= \text{Ext}_F^*(A, N) \text{ for a comodule } N, \\ M(j) &= v_2^{-1}A/(2, v_1^j) \text{ and } M = \varinjlim_j M(j) = v_2^{-1}A/(2, v_1^\infty). \end{aligned}$$

Then note that

$$BP_*(L_2X) = M \otimes_A \Phi \text{ and } \text{Ext}^*(M) = \text{Ext}_F^*(A, BP_*(L_2X)).$$

In  $v_2^{-1}BP_*/(2)$ , we define elements  $x_n$ , which will be used to define elements of  $\text{Ext}^*(M)$ . From here on, we compute everything with setting  $v_2=1$  for the sake of simplicity. We also write

$$x \equiv y \pmod{(v_i^j)}$$

for  $x, y \in \Omega_2^*M$  if  $x=y$  in the cobar complex  $\Omega_2^*M(j)$ .

We first introduce elements  $c_{3i}$  ( $i=0, 1$ ) and  $\tilde{c}_{31}$  in  $\Sigma = A[t_2^2, t_3, \dots]$  defined by

$$(3.1) \quad \begin{aligned} v_1^2 c_{30} &= d_0(v_4^2 + v_1^2 v_5) + t_2^8 + t_2^2, \\ v_1 c_{31} &= d_0(v_4) + t_2^4 \text{ and} \\ \tilde{c}_{31} &= c_{31} + v_1(v_3^2 c_{31} + v_3 t_2^2). \end{aligned}$$

**Lemma 3.2.** *The cochains  $c_{30}$  and  $c_{31}$  are cocycles of the cobar complex  $\Omega_2^1 M(j)$  for any  $j > 0$ . Furthermore,*

$$c_{30} \equiv t_3 + v_3 t_2^8 \pmod{(v_1)} \text{ and } c_{31} \equiv t_3^2 + v_1 v_3 t_2^2 \pmod{(v_1^4)}.$$

*Proof.* Since  $d_1 d_0 = 0$ ,  $d_1(t_2) = 0$  and  $d_0(v_1) = 0$ , the first part of the lemma follows immediately from the definition, since the multiplication by  $v_1$  on  $\Omega_2^1 M(j)$  is monomorphic. The latter half is shown by the direct computation using

$$(3.3) \quad \begin{aligned} \eta_R(v_1^2) &= v_1^2, \quad \eta_R(v_4) \equiv v_4 + v_2 t_2^4 + v_1 t_3^2 + v_1^2 v_3 t_2^2 \pmod{(v_1^5)}, \\ \eta_R(v_4^2) &\equiv v_4^2 + v_2^2 t_2^8 + v_2^8 t_2^2 + v_1^2 t_3^4 + v_1^4 v_3^2 t_2^4 \pmod{(v_1^{10})}, \text{ and} \\ \eta_R(v_5) &\equiv v_5 + v_3 t_2^8 + v_2 t_3^4 + v_2^8 t_3 \pmod{(v_1)} \end{aligned}$$

in  $\Sigma$ , noticing that  $d_0(x) = \eta_R(x) - x$ . In fact,  $d_0(v_4^2 + v_1^2 v_5) \equiv t_2^8 + t_2^2 + v_1^2 t_3 + v_1^2 v_3 t_2^8 \pmod{(v_1^3)}$ , by setting  $v_2=1$ , which gives  $c_{30}$ . For  $c_{31}$ , follows from  $\eta_R(v_4)$ .  
q.e.d.

**Lemma 3.4.** *Put  $\varphi_1 = v_1 v_3^2(v_4 + v_4^4)$ , and we have*

$$d_0(\varphi_1) \equiv v_1(c_{30}^2 + \tilde{c}_{31}) \pmod{(v_1^3)}$$

in  $v_2^{-1}\Sigma = v_2^{-1}A[t_2^2, t_3, \dots]$ .

*Proof.* Since  $d_0(x) = \eta_R(x) - x$  and  $\eta_R$  is a map of algebras, this is verified by Lemma 3.2 and the following facts on  $\eta_R$ :

$$\begin{aligned} \eta_R(v_1) &= v_1, \quad \eta_R(v_2) = v_2, \\ \eta_R(v_3^2) &\equiv v_3^2 \pmod{(v_1^2)}, \\ \eta_R(v_4) &= v_4 + t_2^4 + v_1 c_{31} \text{ and } \eta_R(v_4^4) \equiv v_4^4 + t_2^{16} + t_2^4 \pmod{(v_1^4)} \end{aligned}$$

in  $v_2^{-1}\Sigma$ . In fact, by Lemma 3.2, we see that

$$c_{30}^2 + \tilde{c}_{31} \equiv v_3^2 t_2^{16} + v_1 v_3^2 c_{31}.$$

On the other hand, we compute

$$d_0(\varphi_1) \equiv v_1 v_3^2 d_0(v_4 + v_4^4) \equiv v_1 v_3^2 (v_1 c_{31} + t_2^{16}).$$

q.e.d.

Note that  $v_2^{-1}\Sigma$  is not a Hopf algebroid and so (3.1) does not imply the above lemma. In fact,  $d_0(v_4^2) = d_0(v_4)^2 + t_2^2$ . This with (3.1) yields the following

**Lemma 3.5.** *In  $v_2^{-1}\Sigma$ ,*

$$d_0(v_1^6 v_5) = v_1^6 (c_{31}^2 + c_{30}).$$

**Lemma 3.6.** *There exist elements  $x_i$  of  $v_2^{-1}A$  with  $x_i \equiv v_3^{2^i} \pmod{(2, v_1)}$  such that*

$$\begin{aligned} d_0(x_0) &= v_1 t_2^2, \\ d_0(x_1) &= v_1^3 c_{31}, \\ d_0(x_2) &= v_1^6 c_{30}, \\ d_0(x_{2n+1}) &\equiv v_1^{1+2a_n} v_3^{2b_n} (v_3^2 c_{31} + v_3 t_2^2) \pmod{(v_1^{2+2a_n})} \text{ and} \\ d_0(x_{2n+2}) &\equiv v_1^{a_{n+1}} v_3^{b_{n+1}} c_{30} \pmod{(v_1^{1+a_{n+1}})} \end{aligned}$$

for  $n > 0$ . Here the integers  $a_n$  and  $b_n$  are given by

$$\begin{aligned} a_0 &= 1 \text{ and } a_n = 4a_{n-1} + 2 \ (n > 0) \\ b_0 &= 0, \ b_1 = 0 \text{ and } b_n = 4b_{n-1} + 4 \ (n > 1). \end{aligned}$$

**Proof.** Define the elements  $x_i$  inductively as follows :

$$\begin{aligned} (3.7) \quad x_0 &= v_3, \\ x_1 &= v_3^2 + v_1^2 v_4, \\ x_2 &= x_1^2 + v_1^6 v_5, \\ x_{2n} &= x_{2n-1}^2 + v_1^{a_n} v_3^{b_n} v_5 \text{ and} \\ x_{2n+1} &= x_{2n}^2 + v_1^{2a_{n-1}} v_3^{2b_n} \varphi_1 + v_1^{2a_n-3} v_3^{2b_n} x_1. \end{aligned}$$

Then the lemma will be proved by induction. The first equation follows immediately from the Landweber formula :  $\eta_R(v_3) = v_3 + v_1 t_2^2$ . The second and the third are verified by (3.1). The others are inductively shown by Lemmas 3.4 and 3.5.

q.e.d.

#### 4. The $E_2$ -term

Put  $L = v_2^{-1}BP_*/(2, v_1)$  and  $M = v_2^{-1}BP_*/(2, v_1^\infty)$ . Then we have the short exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{v_1} M \longrightarrow 0,$$

which yields the long exact sequence

$$(4.1) \quad \begin{aligned} 0 \longrightarrow \text{Ext}^0(L) &\xrightarrow{f_*} \text{Ext}^0(M) \xrightarrow{v_1} \text{Ext}^0(M) \xrightarrow{\delta_0} \\ &\dots \xrightarrow{\delta_{n-1}} \text{Ext}^n(L) \xrightarrow{f_*} \text{Ext}^n(M) \xrightarrow{v_1} \text{Ext}^n(M) \longrightarrow \dots. \end{aligned}$$

Here  $f$  is a  $\Sigma$ -comodule map given by  $f(x)=x/v_1$ ,

$$\text{Ext}^n(N)=\text{Ext}_\Sigma^n(A, N)$$

for a  $\Sigma$ -comodule  $N$ , and note that the  $\text{Ext}$  group  $\text{Ext}^*(L)$  is determined in Lemma 2.3.

We here introduce some notations :

$$K(2)_*=\mathbf{Z}/2[v_2, v_2^{-1}], K=K(2)_*[v_1]=\mathbf{Z}/2[v_1, v_2, v_2^{-1}].$$

For an element  $x \in \text{Ext}^*(L)$ ,

$C_n\langle x \rangle$  denotes a cyclic  $K$ -module isomorphic to  $K/(v_1^n)$  generated by  $\{x/v_1^n + z/v_1^{n-1}\} \in \text{Ext}^*(M)$  for some  $z \in \Omega_\Sigma^* v_2^{-1} BP_*/(2)$ .

$C_\infty\langle x \rangle$  denotes a  $K$ -module isomorphic to  $v_1^{-1}K/K$  with basis  $\{x/v_1^i + z/v_1^{i-1}\}_{j>0} \subset \text{Ext}^*(M)$  for some  $z \in \Omega_\Sigma^* v_2^{-1} BP_*/(2)$ .

Note that these  $C_*\langle x \rangle$  are sub- $K$ -module of  $\text{Ext}^*(M)$ .

We compute  $\text{Ext}^*(M)=\text{Ext}_\Sigma^*(A, v_2^{-1}A/(2, v_1^\infty))$  from  $\text{Ext}^*(L)=\text{Ext}_\Sigma^*(A, v_2^{-1}A/(2, v_1))$  by using the following

**Lemma 4.2.** ([8, Remark 3.11]) *Let  $\{x_\lambda\}_{\lambda \in A}$  be a set of generators of  $K(2)_*$ -module  $\text{Ext}^i(L)$ , and  $\{\xi_\lambda\}_{\lambda \in A_0}$  and  $\{\xi_{\lambda,j}\}_{\lambda \in A_1}$  subsets of  $\text{Ext}^i(M)$  such that  $A=A_0 \amalg A_1$ ,*

1) *there exists a positive integer  $a(\lambda)$  for each  $\lambda \in A_0$  such that*

$$v_1^{a(\lambda)-1} \xi_\lambda = f_*(x_\lambda) \text{ and } \delta_i(\xi_\lambda) \neq 0,$$

2)  $\xi_{\lambda,1}=f_*(x_\lambda)$ ,  $v_1 \xi_{\lambda,j}=\xi_{\lambda,j-1}$  and  $\delta_i(\xi_{\lambda,j})=0$  for  $\lambda \in A_1$ .

*Suppose that the set  $\{\delta_i(\xi_\lambda)\}_{\lambda \in A_0}$  is linearly independent over  $K(2)_*$ . Then  $\text{Ext}^i(M)=\bigoplus_{\lambda \in A_0} C_{a(\lambda)}\langle x_\lambda \rangle \bigoplus \bigoplus_{\lambda \in A_1} C_\infty\langle x_\lambda \rangle$ .*

In this section, we will use Lemma 4.2 to compute  $\text{Ext}^*(M)$ , which is the  $E_2$ -term of the Adams-Novikov spectral sequence for computing  $\pi_*(L_2X)$ . Let  $\rho$  denote the homology class of  $\text{Ext}^1(L)$  given in Lemma 2.3.

**Lemma 4.3.** *There exist elements  $\rho_i \in \Omega_\Sigma^1 v_2^{-1}A/(2)$  such that*

$$\rho_i \equiv \rho \pmod{(2, v_1)}$$

*up to homology and*

$$d_1(\rho_i) \equiv 0 \pmod{(2, v_1^2)}.$$



Proof. In [9], Moreira constructed an element  $u \in \Omega_{\mathbb{F}}^1 L$  such that

$$\begin{aligned} d_0(u) &= (\bar{\rho} + \zeta) + (\bar{\rho} + \zeta)^2 \\ &= (\bar{\rho} + t_2^2) + \bar{\rho}^2 + t_2^2 + t_2^4 \end{aligned}$$

in the cobar complex  $\Omega_{\mathbb{F}}^2 L$ . Here  $\zeta$  is represented by a cochain  $t_2 + t_2^2$  in  $\Omega_{\mathbb{F}}^1 L$ , and  $\bar{\rho}$  denotes a cocycle which represents the cohomology class  $\rho$ . Since  $t_2^4$  is homologous to 0, so is  $\bar{\rho}$  to  $\bar{\rho}^2$ . Hence define  $\rho_i = \bar{\rho}^{2^i}$  and we have the lemma. q.e.d.

For each  $j$ , there is an integer  $i$  such that  $\rho_i/v_1^j$  is a cocycle. In this case, we write

$$x\rho/v_1^j = x\rho_i/v_1^j.$$

Such an abbreviation would not cause any confusion.

The main lemma of the last section implies

**Lemma 4.4.** *For the connecting homomorphism  $\delta_0$  in (4.1),*

$$\begin{aligned} \delta_0(v_3^{2t+1}/v_1) &= v_3^{2t} h_{21}, \\ \delta_0(v_3^{4t+2}/v_1^3) &= v_3^{4t} h_{31}, \\ \delta_0(v_3^{8t+4}/v_1^6) &= v_3^{8t} h_{30}, \\ \delta_0(v_3^{4^n(4t+2)}/v_1^{1+2an}) &= v_3^{4^{n+1}t+2bn}(v_3^2 h_{31} + v_3 h_{21}) \text{ and} \\ \delta_0(v_3^{4^{n+1}(2t+1)}/v_1^{a_{n+1}}) &= v_3^{2 \cdot 4^{n+1}t + b_{n+1}} h_{30} \end{aligned}$$

for  $t \geq 0$ ,  $n > 0$ .

Here  $v_3^s/v_1^j$  denotes a cocycle of the cobar complex whose leading term is  $v_3^s/v_1^j$ . Therefore, we obtain the lemma by setting  $v_3^{2^s}/v_1^j = x_n^s/v_1^j$  from Lemma 3.6. Now apply Lemma 4.2 to obtain

**Proposition 4.5.** *The Ext group  $\text{Ext}^0(M)$  is a direct sum of  $C_{\infty}\langle 1 \rangle$  and  $C_{A_n}\langle v_3^{2^n(2t+1)} \rangle$  for  $n \geq 0$  and  $t \geq 0$ . Here  $A_{2n} = a_n$  and  $A_{2n+1} = 1 + 2a_n$ .*

These give us the cokernel of  $\delta_0$ :

**Corollary 4.6.** *The cokernel of  $\delta_0: \text{Ext}^0(M) \rightarrow \text{Ext}^1(L)$  is a  $K(2)_*$ -free module generated by*

$$v_3^{2t+1} h_{21}, v_3^{u'} h_{30}, v_3^u h_{31} \text{ and } v_3^i \rho$$

for  $t \geq 0$ ,  $u \notin T$  and  $u' \notin 2T$ . Here  $T$  is a subset of the natural numbers  $N$ :

$$T = \{n : 4|n \text{ or } 4^{i+1} | (n - 2b_i - 2) \text{ for some } i > 0\},$$

for  $b_i = 4(4^{i-1} - 1)/3$ .

**Lemma 4.7.** *The complement  $U = N - T$  is given as*

$$U = \{n : 2 \nmid n \text{ or } n = 2 \cdot 4^k t + 6 \cdot 4^{k-1} + 2(4^{k-1} - 1)/3 \\ \text{for some } k > 0 \text{ and } t \geq 0\}$$

For the computation of  $\delta_1$ , we introduce other elements :

**Lemma 4.8.** *Consider an element  $\varphi = v_5 + v_3 v_4^2$ . Then there exist elements  $H_{21}$  and  $H_{32}$  in  $\Sigma$  such that*

$$d_0(\varphi) = H_{32} + t_3 + H_{21}, \quad d_1(H_{21}) = 0 = d_1(H_{32}), \\ H_{21} \equiv t_2^2 \quad \text{and} \quad H_{32} \equiv t_3^4 \pmod{v_1}$$

in the cobar complex  $\Omega_{\Sigma}^1 v_2^{-1} A/(2)$ .

*Proof.* For an element  $\psi = v_3^2 + v_1^7 v_3$ , we compute  $d_0(\psi) = v_1^2 t_2^4$  by  $\eta_R(v_3) = v_3 + v_1 t_2^2 + v_1^4 t_2$  in  $BP_*[t_2, t_3, \dots]$ . Now put

$$H_{32} = t_3^4 + v_1^2 \psi t_2^4.$$

Then, the formula  $\Delta(t_3^4) = t_3^4 \otimes 1 + 1 \otimes t_3^4 + v_1^4 t_2^4 \otimes t_2^4$  yields

$$d_1(H_{32}) = 0 \quad \text{and} \quad H_{32} \equiv t_3^4 \pmod{v_1}.$$

Furthermore, we compute

$$d_0(\varphi) \equiv t_3^4 + t_3 + v_3 t_2^2 \pmod{v_1},$$

and so

$$d_0(\varphi) \equiv H_{32} + t_3 + v_3 t_2^2 \pmod{v_1}.$$

Put, then,

$$H_{21} = d_0(\varphi) + H_{32} + t_3$$

and we have

$$d_1(H_{21}) = 0 \quad \text{and} \quad H_{21} \equiv v_3 t_2^2 \pmod{v_1}.$$

q.e.d.

**Lemma 4.9.** *For the connecting homomorphism  $\delta_1 : \text{Ext}^1(M) \longrightarrow \text{Ext}^2(L)$ , we have*

$$\begin{aligned} \delta_1(v_3^{4t+3} h_{21}/v_1^3) &= v_3^{4t+1} h_{21} h_{31}, \\ \delta_1(v_3^{8t+5} h_{21}/v_1^6) &= v_3^{8t+1} h_{21} h_{30}, \\ \delta_1(v_3^{4n(4t+2)+1} h_{21}/v_1^{1+2an}) &= v_3^{4n+1t+2bn+1} h_{21}(v_3^2 h_{31} + v_3 h_{21}), \\ \delta_1(v_3^{4n+1(2t+1)+1} h_{21}/v_1^{an+1}) &= v_3^{2 \cdot 4n+1t+bn+1+1} h_{21} h_{30} \\ \delta_1(v_3^{2t+1} h_{30}/v_1) &= v_3^{2t} h_{21} h_{30}, \end{aligned}$$

$$\begin{aligned}
\delta_1(v_3^{4t+2}h_{30}/v_1^3) &= v_3^{4t}h_{30}h_{31}, \\
\delta_1(v_3^{4k(4t+2)+b_{k+1}}h_{30}/v_1^{1+2a_k}) &= v_3^{4k(4t+2)-2}h_{30}(h_{31}+v_3^{-1}h_{21}), \\
\delta_1(v_3^{2t+1}h_{31}/v_1) &= v_3^{2t}h_{21}h_{31} \quad \text{and} \\
\delta_1(v_3^{4k(2t+1)+b_{k+1}/2}h_{31}/v_1^{a_k}) &= v_3^{4k(2t+1)-2}h_{30}(h_{31}+v_3^{-1}h_{21}).
\end{aligned}$$

Proof. The first four equations follow immediately from Lemmas 4.4 and 4.8 with replacing  $v_3h_{21}$  by  $H_{21}$ . The fifth, sixth and eighth equations follow immediately from Lemmas 3.2 and 3.6. For the other equations, just put

$$\begin{aligned}
v_3^{4k(4t+2)+b_{k+1}}h_{30}/v_1^{1+2a_k} &= v_3^{4k(4t+2)}d_0(x_{2k+2})/v_1^{1+2a_k+a_{k+1}} \quad \text{and} \\
v_3^{4k(2t+1)+b_{k+1}/2}h_{31}/v_1^{a_k} &= v_3^{4k(2t+1)}d_0(x_{2k+1})/v_1^{a_k+1+2a_k},
\end{aligned}$$

and we have the result by Lemma 3.6.

q.e.d.

Now use Lemma 4.2, and we obtain

**Proposition 4.10.**  $\text{Ext}^1(M)$  is a direct sum of  $\rho\text{Ext}^0(M)$  and

$$\begin{aligned}
e^1(M) &= \bigoplus_{t \geq 0} (C_1 \langle v_3^{2t+1}h_{30} \rangle \oplus C_1 \langle v_3^{2t+1}h_{31} \rangle \oplus C_3 \langle v_3^{4t+2}h_{30} \rangle) \\
&\quad \bigoplus_{n > 0, t \geq 0} C_{A_n} \langle v_3^{2^n(2t+1)+1}h_{21} \rangle \\
&\quad \bigoplus_{t, k \geq 0} (C_{1+2a_k} \langle v_3^{4k(4t+2)+b_{k+1}}h_{30} \rangle \oplus C_{a_k} \langle v_3^{4k(2t+1)+b_{k+1}/2}h_{31} \rangle).
\end{aligned}$$

**Corollary 4.11.** The cokernel of  $\delta_1: \text{Ext}^1(M) \rightarrow \text{Ext}^2(L)$  is a direct sum of  $\rho\text{Coker } \delta_0$  and a  $K(2)_*$ -module generated by

$$v_3^{2t+1}h_{30}h_{31}, v_3^{2u+1}h_{21}h_{31} \quad \text{and} \quad v_3^{2u'+1}h_{21}h_{30}$$

for  $t \geq 0, 2u \notin T$  and  $u' \notin 2T$ .

**Lemma 4.12.** For the connecting homomorphism  $\delta_2: \text{Ext}^1(M) \rightarrow \text{Ext}^2(L)$ , we have

$$\begin{aligned}
\delta_2(v_3^{2t+1}h_{30}h_{31}/v_1) &= v_3^{2t}h_{21}h_{30}h_{31}, \\
\delta_2(v_3^{4t+3}h_{21}h_{30}/v_1^3) &= v_3^{4t+1}h_{21}h_{30}h_{31}, \\
\delta_2(v_3^{4k(4t+2)+b_{k+1}+1}h_{21}h_{30}/v_1^{1+2a_k}) &= v_3^{4k(4t+2)-1}h_{21}h_{30}h_{31}, \\
\delta_2(v_3^{4k(2t+1)+(b_{k+1}/2)+1}h_{21}h_{31}/v_1^{a_k}) &= v_3^{4k(2t+1)-1}h_{21}h_{30}h_{31}.
\end{aligned}$$

Proof. Note that  $\delta_2(v_3^{2t+1}h_{30}h_{31}/v_1) = \delta_0(v_3^{2t+1}/v_1)h_{30}h_{31}$  since  $h_{3i} = c_{3i}$ 's are cocycles by Lemma 3.2. Now the first equation follows from Lemmas 4.4 and 4.9. For the other equations, use Lemmas 4.8 and 4.9 since  $\delta_2(v_3^{2t+1}h_{21}h_{3i}/v_1^i) = \delta_1(v_3^{2t}h_{3i}/v_1^i)v_3h_{21}$  if we use the representative  $H_{21}$  for the cohomology class  $v_3h_{21}$ .

Again by Lemma 4.2, we obtain

q.e.d.

**Proposition 4.13.**  $\text{Ext}^2(M)$  is a direct sum of  $\rho e^1(M)$  and

$$\begin{aligned}
e^2(M) &= \bigoplus_{t, k \geq 0} (C_{1+2a_k} \langle v_3^{4k(4t+2)+b_{k+1}+1}h_{21}h_{30} \rangle \\
&\quad \oplus C_{a_k} \langle v_3^{4k(2t+1)+(b_{k+1}/2)+1}h_{21}h_{31} \rangle) \oplus C_1 \langle v_3^{2t+1}h_{30}h_{31} \rangle.
\end{aligned}$$

**Corollary 4.14.** *The cokernel of  $\delta_2: \text{Ext}^2(M) \rightarrow \text{Ext}^3(L)$  is a  $K(2)_*$ -module  $\rho \text{Coker } \delta_1$ .*

Now the following proposition follows immediately, by the same argument as above.

**Proposition 4.15.** *For  $n > 3$ ,  $\text{Ext}^n(M) = 0$ , and*

$$\text{Ext}^3(M) = \rho e^2(M).$$

## 5. On the map $j_*: E_2(X) \rightarrow E_2(C)$

As is stated in the introduction,  $C$  denotes the cofiber of  $X_2 \rightarrow L_2 X_2$ . Then it is an  $X_2$ -module spectrum and  $h_{20} \in \pi_5(X_2)$  induces a map  $h_{20}: C \rightarrow C$ . In fact, it is the composition

$$C = S^0 \wedge C \xrightarrow{h_{20} \wedge C} X_2 \wedge C \xrightarrow{\nu} C,$$

in which  $\nu$  denotes the  $X_2$ -module structure. Then we have a cofiber sequence

$$\Sigma^5 C \xrightarrow{h_{20}} C \xrightarrow{i} X \xrightarrow{j} \Sigma^6 C.$$

Let  $E_r^*(Y)$  denote the  $E_r$ -term of the Adams-Novikov spectral sequence converging to  $\pi_*(L_2 Y)$  for a spectrum  $Y$ , and  $d_r^{AN}$ , its differentials. Then this gives rise to the exact sequence

$$0 \longrightarrow E_2^{0,t}(C) \xrightarrow{i_*} E_2^{0,t}(X) \xrightarrow{j_*} E_2^{0,t-6}(C) \xrightarrow{\delta} E_2^{1,t}(C) \longrightarrow \dots$$

Here  $E_2^{s,t}(X) = \text{Ext}^{s,t}(M)$ , whose structure is given in the previous section. We further consider a cofiber  $E$  of  $h_{20}: C \rightarrow C$ . Then we have a commutative diagram

$$(5.1) \quad \begin{array}{ccccccc} C & \xrightarrow{h_{20}} & C & \xrightarrow{i} & X & \xrightarrow{j} & \Sigma C \\ \downarrow v_1 & & \downarrow v_1 & & \downarrow v_1 & & \downarrow v_1 \\ C & \xrightarrow{h_{20}} & C & \xrightarrow{i} & X & \xrightarrow{j} & \Sigma C, \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ \Sigma D & \xrightarrow{h_{20}} & \Sigma D & \xrightarrow{i} & \Sigma E & \xrightarrow{j} & \Sigma^2 D \end{array}$$

in which rows and columns are cofibrations.

**Lemma 5.2.** *Let  $v_3^t/v_1^A$  denote a generator of  $E_2(X)$  as a  $\mathbb{Z}/2[v_1, v_2, v_2^{-1}]$ -module. Then*

$$j_*(v_3^t/v_1^{A-1}) = 0.$$

*Proof.* If  $t = 2^n(2s+1)$  for some  $n, s \geq 0$ , then  $v_3^t/v_1^A$  is a homology class represented by  $x_n^{2s+1}/v_1^{An}$ . For  $n=0$ , the lemma is trivial. Now suppose that  $j_*(x_n^{2s+1}/v_1^{An}) = 0$  for even  $n=2m$ . Then squaring this, we obtain

$$j_*(x_{n+1}^{2s+1}/v_1^{An+1}) = v_3^w/v_1$$

for some  $w \geq 0$ . Consider the diagram

$$\begin{array}{ccccc} E_2^0(X) & \xrightarrow{j_*} & E_2^0(C) \\ \downarrow \delta & & \downarrow \delta \\ E_2^1(D) & \xrightarrow{i_*} & E_2^1(E) & \xrightarrow{j_*} & E_2^1(D) \end{array}$$

induced from (5.1). Since  $\delta(x_{n+1}^{2s+1}/v_1^{An+1})$  is in the image of  $i_*$  by Lemma 4.4,  $\delta(v_3^w/v_1)=0$  in  $E_2^1(D)$  by the above diagram, and so  $2|w$  since  $\delta(v_3^w/v_1)=wv_3^{w-1}h_{21}$  by Landweber's formula  $d_0(v_3)=v_1t_2^2+v_1^4t_2$  in  $BP_*[t_2, t_3, \dots]$ . Thus we have

$$j_*(x_{n+1}^{2s+1}/v_1^{An+1})=v_3^{2u}/v_1.$$

Square this, and we have

$$j_*(x_{n+2}^{2s+1}/v_1^{An+2})=v_3^{4u}/v_1^2.$$

Notice that  $j_*(x)=y$  if  $d_0(x)=yt_2$ , where  $d_0(x)=\eta_R(x)-x$ . A direct computation shows us  $d_0(v_3^{4u}x_1/v_1^4)=v_3^{4u}t_2/v_1^2$  in the cobar complex  $\Omega_2^*M$ . Thus we have shown inductively that  $j_*(v_3^{2^{n(2s+1)}}/v_1^{An})$  equals to 0 if  $n$  is even, and to  $v_3^{2s}/v_1$  for some  $u$  if  $n$  is odd. q.e.d.

## 6. The Adams-Novikov differential

Consider the cofiber  $E$  of  $h_{20}: \Sigma^5 D \rightarrow D$ . Then by [7, Th. 7.1], we immediately obtain the following

**Proposition 6.1.** *The Adams-Novikov spectral sequence for computing  $\pi_*(L_2E)$  collapses from the  $E_2$ -term.*

Note that the  $E_2$ -term for our  $X$  is

$$E_2^*(X)=\text{Ext}^*(A, v_2^{-1}BP_*(X))=\text{Ext}^*(M).$$

**Lemma 6.2.** *For the Adams-Novikov differential  $d_3^{AN}: E_2^0(X) \rightarrow E_2^3(X)$ ,  $d_3^{AN}(v_3^t/v_1^A)$  is a sum of the elements of the form  $v_3^{2u+1}h_{21}h_{31}\rho/v_1^k$  for  $i=0, 1$  and  $k>1$ . Here  $v_3^t/v_1^A$  is a generator of the  $\mathbb{Z}/2[v_1, v_2, v_2^{-1}]$ -module  $M_0$ .*

*Proof.* Consider the diagram (5.1). The third column induces the long exact sequence

$$\cdots \longrightarrow \text{Ext}^3(M) \xrightarrow{v_1} \text{Ext}^3(M) \xrightarrow{\delta_3} \text{Ext}^4(L) \longrightarrow \cdots$$

of the  $E_2$ -terms. If the  $\delta_0$  image of  $v_3^t/v_1^A$  is  $x \neq 0$ , then  $\delta_3(d_3^{AN}(v_3^t/v_1^A))=d_3^{AN}(x)=0$  by Proposition 6.1. Thus  $d_3^{AN}(v_3^t/v_1^A)$  is divisible by  $v_1$ . Furthermore it implies that  $v_3^{2t+1}h_{30}h_{31}\rho/v_1$  cannot be a target of  $d_3^{AN}$ . In fact, it is not divisible by  $v_1$  by Proposition 4.15. Now the lemma follows from Lemma 4.15. q.e.d.

**Theorem 6.3.** *The Adams-Novikov spectral sequence for computing  $\pi_*(L_2X)$  collapses from the  $E_2$ -term.*

*Proof.* By proposition 4.15, the Adams-Novikov differentials are all trivial except for  $d_3^{AN} : E_2^0(X) \rightarrow E_2^3(X)$ . So it is sufficient to show that  $d_3^{AN}(v_3^t/v_1^j) = 0$  for each  $v_3^t/v_1^j \in E_2^0(X)$ . By Lemma 6.2,

$$(6.4) \quad d_3^{AN}(v_3^t/v_1^{A-k}) = \sum_{u,i} \lambda_{u,i} v_3^{2u+1} h_{21} h_{3i} \rho / v_1^2$$

for some  $k \geq 0$ , where  $\lambda_{u,i} \in \mathbb{Z}/2$ . Since

$$d_3(v_3^{2u+1} h_{21} h_{3i} \rho / v_1^2) = v_3^{2u} h_{20}^2 h_{3i} \rho / v_1 \neq 0$$

in the cobar complex  $\Omega_r^4 BP_*(C)$ , we see that

$$(6.5) \quad j_*\left(\sum_{u,i} \lambda_{u,i} v_3^{2u+1} h_{21} h_{3i} \rho / v_1^2\right) = \sum_{u,i} \lambda_{u,i} v_3^{2u} h_{20} h_{3i} \rho / v_1 \neq 0.$$

Now send (6.4) by  $j_*$  and we have a contradiction to Lemma 5.2, which says  $j_*(v_3^t/v_1^{A-k}) = 0$  if  $k > 0$ . If  $k = 0$  and  $j_*(v_3^t/v_1^A) \neq 0$ , then

$$j_*(v_3^t/v_1^A) = v_3^{2u} / v_1$$

for some  $u \geq 0$  as is seen in the proof of Lemma 5.2. Therefore, (6.4) and (6.5) yield

$$d_3^{AN}(v_3^{2u}/v_1) = \sum_{u,i} \lambda_{u,i} v_3^{2u} h_{20} h_{3i} \rho / v_1 \neq 0$$

in  $E_2^*(C)$  for some  $\lambda_{u,i} \in \mathbb{Z}/2$ . Now pull this back to  $E_2^*(D)$  under the map  $i_* : E_2^*(D) \rightarrow E_2^*(C)$  to obtain the non-trivial differential

$$d_3^{AN}(v_3^{2u}) = \sum_{u,i} \lambda_{u,i} v_3^{2u} h_{20} h_{3i} \rho \neq 0$$

in  $E_2^*(D)$ , which again contradicts to a result of [7] which says  $d_3^{AN}(v_3^{4k}) = 0$  and  $d_3^{AN}(v_3^{4k+2}) = v_3^{4k} h_{20}^3$  for  $k > 0$ . q.e.d.

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