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A CHARACTERIZATION OF CONDITIONAL EXPECTATIONS FOR $L_\infty(X)$-VALUED FUNGTIONS

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Introduction. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, (S, X, λ) a measure space and E a Banach space. We consider constant-preserving contractive projections of $L_1(\Omega, \mathcal{A}, \mu E)$ into itself. If $E=R$ or E is a strictlyconvex Banach space, then it is known (Ando [1], Douglas [2] and Landers and Rogge [4]) that such operators coincide precisely with the conditional expectation operators. If $E=L_1(X, S, \lambda)$, the author [5] proved that such operators which are translation invariant coincide with the conditional expectation operators. In this paper we deal with the case when $E=L_\infty(X, S, \lambda)$. If $E=R^2$ with the norm $||(x, y)||_{R^2}=|x| \vee |y|$, then such operators can be expressed as a linear combination of two conditional expectation operators. On the other hand if $E=L_\infty(X, S, \lambda)$ and $E \not\cong R^2$ with the norm $||(x, y)||_{R^2}=|x| \vee |y|$, then such operators coincides with the conditional expectation given some σ-subalgebra.

1. Definitions and lemmas. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and (X, S, λ) a measure space. Let $S^+=\{K: K \in S \text{ and } \lambda(K)>0\}$ and $L_\infty(X, S, \lambda)$ the class of essentially bounded measurable functions on (X, S, λ). Let $E=L_\infty(X, S, \lambda)$, then E is a Banach space with the norm defined by $||a||_E=\operatorname{esssup}_{x\in X}|a(x)|$ for each $a \in L_\infty(X, S, \lambda)$. Let $L_1(\Omega, \mathcal{A}, \mu, E)$ be the class of E-valued Bochner integrable functions on $(\Omega, \mathcal{A}, \mu)$ with the norm defined by

$$||f||_L=\int ||f(\omega)||_E d\mu(\omega) \qquad \text{for each} \quad f \in L_1(\Omega, \mathcal{A}, \mu, E).$$

For the definitions and properties of Bochner integral, see Hille and Phillips [3].

DEFINITION 1. For a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$, a function g is called the conditional expectation of f given $\mathcal{B}$ if g is weakly measurable with respect to $\mathcal{B}$, and

$$\int_B g d\mu=\int_B f d\mu \qquad \text{for each} \quad B \in \mathcal{B},$$

where the integral is the Bochner integral. We denote by $f^{\mathcal{B}}$ the conditional expectation of f given $\mathcal{B}$. We shall denote by R the class of real numbers. For each $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ we define $(\varphi \cdot a) = \varphi(\omega) \cdot a$ for each $\omega \ni \Omega$.

DEFINITION 2. Let P be a linear operator of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself. P is said to be *contractive* if

$$\|P\| = \sup\{\|P(f)\|_L : f \in L_1(\Omega, \mathcal{A}, \mu, E) \quad \text{and} \quad \|f\|_L = 1\} = 1,$$

P is *constant-preserving* if $P(l_\Omega \cdot a) = l_\Omega \cdot a$ for each $a \in E$ and P is called a *projection* if $P \circ P = P$, where l_Ω is the characteristic function of E.

Lemma 1.1. *For each $f \in L_1(\Omega, \mathcal{A}, \mu, E)$ the conditional expectation $f^{\mathcal{B}}$ of f given $\mathcal{B}$ exists uniquely up to almost every-where and the conditional expectation operator $(\cdot)^{\mathcal{B}}$ is a constant-preserving contractive projection for each σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$.*

For the proof see Schwartz [6].

By the definition of conditional expectation $(\varphi \cdot a)^{\mathcal{B}} = \varphi^{\mathcal{B}} \cdot a$ for each $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ and $a \in E$.

Lemma 1.2. *If Q is a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself, then for each $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ with $0 \leqq \varphi \leqq 1$ and $a \in E$ there exists a μ-null set N such that*

$$\|a\|_E - \|Q(\varphi \cdot a)(\omega)\|_E = \|a - Q(\varphi \cdot a)(\omega)\|_E \qquad \textit{for each} \quad \omega \in \Omega - N.$$

For the proof see Miyadera [5].

Lemma 1.3. *Let Q be a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself. If, for each $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ and for each nonzero element a of E, there exists $\varphi' \in L_1(\Omega, \mathcal{A}, \mu, R)$ such that $Q(\varphi \cdot a) = \varphi' \cdot a$, then there exists a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $Q(f)$ is the conditional expectation of f given $\mathcal{B}$ for each $f \in L_1(\Omega, \mathcal{A}, \mu, E)$. In particular each constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, R)$ into itself is the conditional expectation given some σ-subalgebra.*

For the proof see Miyadera [5].

Lemma 1.4. *Let $T \in S^+$ and $A \in \mathcal{A}$. Then there exists a μ-null set N such that for each $\omega \in \Omega - N$*

$$|Q(l_A \cdot l_T)(\omega)| \leqq 1 \quad \text{(a.e.x.)}$$

and

$$0 \leqq Q(l_A \cdot l_T)(\omega) \cdot l_T \leqq 1 \quad \text{(a.e.x.)}$$

Proof. By Lemma 1.2 there exists a μ-null set N such that

$$||l_T||_E - ||Q(l_A \cdot l_T)(\omega)||_E = ||l_T - Q(l_A \cdot l_T)(\omega)||_E$$

for each $\omega \in \Omega - N$, and hence for each $\omega \in \Omega - N$,

$$||Q(l_A \cdot l_T)(\omega)||_E \leqq 1$$

and

$$||l_T - Q(l_A \cdot l_T)(\omega)||_E \leqq 1 .$$

Therefore by the definition of $||\cdot||_E$ we have for each $\omega \in \Omega - N$,

$$|Q(l_A \cdot l_T)(\omega)| \leqq 1 \qquad \text{(a.e.x)}$$

and

$$0 \leqq Q(l_A \cdot l_T)(\omega) \cdot l_T \leqq 1 \qquad \text{(a.e.x)} .$$

Lemma 1.5. *Let $A \in \mathcal{A}$ and $T, T' \in S^+$ and $T \cap T' = \phi$. Then*

$$\int Q(l_A \cdot l_T) \cdot l_{T'} / d\mu = 0 \qquad \text{(a.e.x)} .$$

Proof. Since Q is constant-preserving and contractive,

$$1 = \int ||l_A \cdot l_T + l_\Omega \cdot l_{T'}||_E d\mu \geqq \int ||Q(l_A \cdot l_T + l_\Omega l_{T'})||_E d\mu$$
$$= \int ||Q(l_A \cdot l_T) + l_\Omega \cdot l_{T'}||_E d\mu = \int ||Q(l_A \cdot l_T) \cdot l_{T'} + l_\Omega \cdot l_{T'}||_E d\mu$$
$$\geqq || \int (Q(l_A \cdot l_T) \cdot l_{T'} + l_\Omega \cdot l_{T'}) d\mu ||_E .$$

Similarly

$$1 = \int ||l_A \cdot l_T - l_\Omega \cdot l_{T'}||_E d\mu$$
$$\geqq \int ||Q(l_A \cdot l_T - l_\Omega \cdot l_{T'})||_E d\mu = \int ||Q(l_A \cdot l_T) - l_\Omega \cdot l_{T'}||_E d\mu$$
$$\geqq \int ||Q(l_A \cdot l_T) \cdot l_{T'} - l_\Omega \cdot l_{T'}||_E d\mu \geqq || \int (Q(l_A \cdot l_T) \cdot l_{T'} - l_\Omega \cdot l_{T'}) d\mu ||_E .$$

Therefore we have proved that

$$1 \geqq || \int (Q(l_A \cdot l_T) \cdot l_{T'} + l_\Omega \cdot l_{T'}) d\mu ||_E$$

and

$$1 \geqq || \int (Q(l_A \cdot l_T) \cdot l_{T'} - l_\Omega \cdot l_{T'}) d\mu ||_E .$$

Since

$$\int Q(l_A \cdot l_T) l_{T'} d\mu = 0 \qquad \text{on} \quad (T')^C$$

and

$$\int (l_A \cdot l_{T'}) d\mu = l_{T'},$$

we have

$$\int Q(l_A \cdot l_T) \cdot l_{T'}\, d\mu = 0 \qquad \text{(a.e.x)}.$$

Lemma 1.6. *Let $A \in \mathcal{A}$ and $T \in S^+$. Suppose that there exists T', $T'' \in S^+$ such that $T' \cap T'' = \phi$ and $T' \cup T'' = T^c$, then there exists a μ-nullset N such that $Q(l_A \cdot l_T)(\omega) \cdot l_{T^c} = 0$ (a.e.x) for each $\omega \in \Omega - N$.*

Proof. Since Q is constant-preserving and contractive,

$$\begin{aligned}
1 &= \int \|l_A \cdot l_T + l_\Omega \cdot l_{T''} + (-1)^k l_\Omega \cdot l_{T'}\|_E d\mu \\
&\geqq \int \|Q(l_A \cdot l_T + l_\Omega \cdot l_{T''} + (-1)^k l_\Omega \cdot l_{T'})\|_E d\mu \\
&= \int \|Q(l_A \cdot l_T) + l_\Omega \cdot l_{T''} + (-1)^k l_\Omega \cdot l_{T'}\|_E d\mu \\
&\geqq \int \|(Q(l_A \cdot l_T) \cdot l_{T''} + l_\Omega \cdot l_{T''}\|_E \vee \|Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_\Omega \cdot l_{T'}\|_E) d\mu \\
&\geqq \int \|Q(l_A \cdot l_T) \cdot l_{T''} + l_\Omega \cdot l_{T''}\|_E d\mu \wedge \int \|Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_\Omega \cdot l_{T'}\|_E d\mu \\
&\geqq \left\| \int (Q(l_A \cdot l_T) \cdot l_{T''} + l_\Omega \cdot l_{T''}) d\mu \right\|_E \wedge \left\| \int (Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_\Omega \cdot l_{T'}) d\mu \right\|_E \\
&= 1.
\end{aligned}$$

Here we also used Lemma 1.5 that

$$\int Q(l_A \cdot l_T) \cdot l_{T'}\, d\mu = 0$$

and

$$\int Q(l_A \cdot l_T) \cdot l_{T''}\, d\mu = 0.$$

We have proved that

$$1 = \int (\|Q(l_A \cdot l_T) \cdot l_{T''} + l_\Omega \cdot l_{T''}\|_E \vee \|Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_\Omega \cdot l_{T'}\|_E) d\mu$$

and

$$\|Q(l_A \cdot l_T) \cdot l_{T''} + l_\Omega \cdot l_{T''}\|_E = \|Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_\Omega \cdot l_{T'}\|_E \qquad \text{(a.e.x)}.$$

Therefore we have

$$1 = \int (\|Q(l_A \cdot l_T) \cdot l_{T'} + l_\Omega \cdot l_{T'}\|_E \vee \|Q(l_A \cdot l_T) \cdot l_{T'} - l_\Omega \cdot l_{T'}\|_E) d\mu$$

Since

$$\|Q(l_A\cdot l_T)\cdot l_{T'}+l_\Omega\cdot l_{T'}\|_E \vee \|Q(l_A\cdot l_T)\cdot l_{T'}-l_\Omega\cdot l_{T'}\|_E \geqq 1\,,$$
$$\|Q(l_A\cdot l_T)(\omega)\cdot l_{T'}+l_{T'}\|_E = \|Q(l_A\cdot l_T)(\omega)\cdot l_{T'}-l_{T'}\|_E = 1 \qquad \text{(a.e.x)}\,.$$

Similarly we can prove that

$$\|Q(l_A\cdot l_T)(\omega)\cdot l_{T''}+l_{T''}\|_E = \|Q(l_A\cdot l_T)(\omega)\cdot l_{T''}-l_{T''}\|_E = 1 \qquad \text{(a.e.x)}\,.$$

Therefore there exists a μ-unllset N such that

$$Q(l_A\cdot l_T)(\omega)\cdot l_{T'\cup T''} = 0 \qquad \text{(a.e.x)}\,.$$

2. A characterization of conditional expectation for $L_\infty(X)$-valued function

Theorem 1. *If there exist pairwise disjoint elements X_1, X_2 and X_3 of S^+ such that $X_1\cup X_2\cup X_3=X$, then a constant-preserving contractive projection Q of $L_1(\Omega, \mathcal{A}\ \mu, E)$ into itself is a conditional expectation operator given some σ-sub-algebra.*

Proof. Let $A\in\mathcal{A}$ and $T\in S^+$ and $T\subset X_i$. Then by Lemma 1.6 there exists a μ-nullset N such that for each $\omega\in\Omega-N$

$$(1)\qquad Q(l_A\cdot l_T)(\omega)\cdot l_{T^c} = 0 \qquad \text{(a.e.x)}$$

and

$$(2)\qquad Q(l_{A^c}\cdot l_T)(\omega)\cdot l_{T^c} = 0 \qquad \text{(a.e.x)}\,.$$

Since Q is constant-preserving

$$(3)\qquad Q(l_A\cdot l_T)+Q(l_{A^c}\cdot l_T) = Q(l_\Omega\cdot l_T) = l_\Omega\cdot l_T\,.$$

Since Q is constant-preserving and contractive

$$\begin{aligned}1=\mu(A)+\mu(A^c) &= \int (\|l_A\cdot l_T\|_E+\|l_{A^c}\cdot l_T\|_E)d\mu\\ &\geqq \int (\|Q(l_A\cdot l_T)\|_E+\|Q(l_{A^c}\cdot l_T)\|_E)d\mu\\ &\geqq \int \|Q(l_A\cdot l_T)+Q(l_{A^c}\cdot l_T)\|_E d\mu = \int \|l_\Omega\cdot l_T\|_E d\mu = 1\,.\end{aligned}$$

Therefore there exists a μ-nullset N' such that for each $\omega\in\Omega-N'$

$$\|Q(l_A\cdot l_T)(\omega)\|_E+\|Q(l_{A^c}\cdot l_T)(\omega)\|_E = 1\,.$$

This together with (1), (2) and (3), implies that for each $\omega\in\Omega-(N\cup N')$ there exists a real number $k(\omega)$ such that $Q(l_A\cdot l_T)(\omega)=k(\omega)\cdot l_T$. Obviously $k(\cdot)\in L_1(\Omega, \mathcal{A}, \mu, R)$. Since Q is linear, $k(\cdot)$ is independent of the choice of T. Let

$E_i=\{a\colon a\in E,\ a(x)=0$ for each $x\in X_i^c\}$ ($i=1$, 2 and 3). Q is linear and continuous, and hence for each $\varphi\in L_1(\Omega, \mathcal{A}, \mu, R)$ and $a\in E_i$, there exists $\varphi'\in L_1(\Omega, \mathcal{A}, \mu, E_i)$ such that $Q(\varphi\cdot a)=\varphi'\cdot a$. Therefore by Lemma 1.3 there exists a σ-subalgebra $\mathcal{B}_i$ of $\mathcal{A}$ such that $Q(f)=f^{\mathcal{B}_i}$ for each $f\in L_1(\Omega, \mathcal{A}, \mu, E_i)$. We shall prove that $\mathcal{B}_1=\mathcal{B}_2=\mathcal{B}_3$. Let $B\in\mathcal{B}_1$. Then

$$\begin{aligned}\int l_B d\mu &= \int l_B\cdot\|l_{X_1}+l_{X_2}\|_E d\mu = \int \|l_B\cdot l_{X_1}+l_B\cdot l_{X_2}\|_E d\mu\\ &\geqq \int \|Q(l_B\cdot l_{X_1}+l_B l_{X_2})\|_E d\mu = \int \|Q(l_B\cdot l_{X_1})+Q(l_B\cdot l_{X_2})\|_E d\mu\\ &= \int \|(l_B)^{\mathcal{B}_1}\cdot l_{X_1}+(l_B)^{\mathcal{B}_2}\cdot l_{X_2}\|_E d\mu = \int \|l_B\cdot l_{X_1}+(l_B)^{\mathcal{B}_2}\cdot l_{X_2}\|_E d\mu\\ &= \int (l_B\vee (l_B)^{\mathcal{B}_2}) d\mu\,.\end{aligned}$$

Hence $l_B(\omega)=l_B(\omega)\vee(l_B)^{\mathcal{B}_2}(\omega)$, which implies that $l_B(\omega)=(l_B)^{\mathcal{B}_2}(\omega)$. Since $\|(l_B)^{\mathcal{B}_2}\|_L=\|l_B\|_L$, $l_B=(l_B)^{\mathcal{B}_2}$. Since B is an arbitrary element of $\mathcal{B}_1$, we have proved that $\mathcal{B}_1\subset\mathcal{B}_2$. Similarly we can prove that $\mathcal{B}_2\subset\mathcal{B}_3$ and $\mathcal{B}_3\subset\mathcal{B}_1$, which imply that $\mathcal{B}_1=\mathcal{B}_2=\mathcal{B}_3$. Write $\mathcal{B}=\mathcal{B}_2=\mathcal{B}_2=\mathcal{B}_3$, then $Q(f)=f^{\mathcal{B}}$ for each $f\in L_1(\Omega, \mathcal{A}, \mu, E)$.

3. A characterization of a constant-preserving contractive projection of for R^2-valued functions. If there do not exist pairwise disjoint element X_1, X_2 and X_3 such that $X_1\cup X_2\cup X_3=X$ and X_1, X_2 and $X_3\in S^+$, then $E\cong R^2$ with the norm $\|(x, y)\|_E=|x|\vee|y|$ or $E\cong R$ with the norm $\|x\|_E=|x|$. Douglas [2] showed that a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, R)$ into itself is a conditional expectation given some σ-subalgebra. Therefore our next aim is to consider the case that $E\cong R^2$ with the norm $\|(x, y)\|_E=|x|\vee|y|$. Note that for each $f\in L_1(\Omega, \mathcal{A}, \mu, R^2)$ there exist $f_1, f_2\in L_1(\Omega, \mathcal{A}, \mu, R)$ such that $f(\omega)=(f_1(\omega), f_2(\omega))$.

Lemma 3.1. *Suppose that $E=R^2$ and Q is a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself. If $f\in L_1(\Omega, \mathcal{A}, \mu, R)$ with $0\leqq f(\omega)\leqq 1$ (a.e. ω) and $Q((f, f))=(f_1, f_2)$ and $Q((f, -f))=(g_1, g_1)$, then $f_2=f_2$ and $g_1=-g_2$.*

Proof. By Lemma 1.2 there exists μ-nullsets N_1 and N_2 such that for each $\omega\in\Omega-N_1$

$$\|(1, 1)\|_E-\|Q((f, f))(\omega)\|_E=\|(1, 1)-Q((f, f))(\omega)\|_E$$

and for each $\omega\in\Omega-N_2$

$$\|(1, -1)\|_E-\|Q((f, -f))(\omega)\|_E=\|(1, -1)-Q((f, -f))(\omega)\|_E\,.$$

Therefore we hv have for each $\omega\in\Omega-N_1$

$$1-(|f_1(\omega)|\vee|f_2(\omega)|)=|1-f_1(\omega)|\vee|1-f_2(\omega)|$$

and for each $\omega\in\Omega-N_2$

$$1-(|g_1(\omega)|\vee|g_2(\omega)|)=|1-g_1(\omega)|\vee|1+g_2(\omega)|$$

and hence $f_1(\omega)=f_2(\omega)$ for each $\omega\in\Omega-N_1$ and $g_1(\omega)=-g_2(\omega)$ for each $\omega\in\Omega-N._2$

Theorem 2. *Let $E=R^2$ with the norm $\|(x, y)\|_E=|x|\vee|y|$. Then Q is a constant-preserving contractive projection if and only if there exist σ-subalgebras $\mathcal{B}$ and C of $\mathcal{A}$ such that*

$$Q((f, g))=(1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+f^{C}-g^{C}),\ 1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+g^{C}-f^{C}))\,.$$

Proof. Suppose that Q is a constant-preserving contractive projection. Then by Lemma 3.1 we can get two operators Q_1 and Q_2 of $L_1(\Omega, \mathcal{A}, \mu, R)$ into itself such that for each $f\in L_1(\Omega, \mathcal{A}, \mu, R)$,

$$Q((f, f))=(Q_1(f), Q_1(f)) \quad\text{and}\quad Q((f, -f))=(Q_2(f), -Q_2(f))\,.$$

Since Q is a constant-preserving contractive projection, Q_1 and Q_2 are constant-preservnig contractive projections. Therefore by Lemma 1.3 there exist σ-subalgebra $\mathcal{B}$ and C such that

$$Q_1=(\cdot)^{\mathcal{B}} \quad\text{and}\quad Q_2=(\cdot)^{C}\,.$$

Then

$$\begin{aligned}Q((f, g)) &= Q(1/2(f+g)+1/2(f-g),\ 1/2(f+g)-1/2(f-g))\\ &= (Q_1(1/2(f+g))+Q_2(1/2(f-g)),\ Q_1(1/2(f+g))-Q_2(1/2(f-g)))\\ &= ((1/2(f^{\mathcal{B}}+g^{\mathcal{B}}))+(1/2(f^{C}-g^{C})),\ (1/2(f^{\mathcal{B}}+g^{\mathcal{B}}))-(1/2(f^{C}-g^{C})))\\ &= (1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+f^{C}-g^{C}),\ 1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+g^{C}-f^{C}))\,.\end{aligned}$$

On the other hand let $\mathcal{B}$ and C be σ-subalgebras of $\mathcal{A}$ and

$$Q((f, g))=(1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+f^{C}-g^{C},\ 1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+g^{C}-f^{C}))\,.$$

Since $(\cdot)^{\mathcal{B}}$ and $(\cdot)^{C}$ are constant-preserving projections, Q is a constant-preserving projection. In the following we denote

$$\{\omega: f^{\mathcal{B}}+g^{\mathcal{B}}\geqq 0\} \quad\text{by}\quad \{f^{\mathcal{B}}+g^{\mathcal{B}}\geqq 0\}$$

and

$$\{\omega: f^{C}-g^{C}<0\} \quad\text{by}\quad \{f^{C}-g^{C}<0\},\quad\text{etc.}$$

It holds that

$$
\begin{aligned}
\|Q((f,g))\|_L &= \int \|Q((f,g))\|_E d\mu \\
&= \int |(1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+f^C-g^C)| \vee |1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+g^C-f^C)|\,d\mu \\
&= \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}<0\}\cap\{f^C-g^C\geqq 0\}} 1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+f^C-g^C)d\mu + \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}=0\}\cap\{f^C-g^C<0\}} 1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+g^C-f^C)d\mu \\
&\quad + \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}<0\}\cap\{f^C-g^C\geqq 0\}} 1/2(-f^{\mathcal{B}}-g^{\mathcal{B}}+f^C-g^C)d\mu + \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}<0\}\cap\{f^C-g^C<0\}} 1/2(-f^{\mathcal{B}}-g^{\mathcal{B}}+g^C-f^C)d\mu \\
&= \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}\geqq 0\}} 1/2(f^{\mathcal{B}}+g^{\mathcal{B}})d\mu + \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}<0\}} 1/2(-f^{\mathcal{B}}-g^{\mathcal{B}})d\mu \\
&= \int_{\{f^C-g^C\geqq 0\}} 1/2(f^C-g^C)d\mu + \int_{\{g^C-f^C\geqq 0\}} 1/2(g^C-f^C)d\mu \\
&= \int |1/2(f^{\mathcal{B}}+g^{\mathcal{B}})|\,d\mu + \int |1/2(f^C-g^C)|\,d\mu \\
&= \int |1/2(f+g)|\,d\mu + \int |1/2(f-g)|\,d\mu \\
&\leqq \int (|f| \vee |g|)d\mu = \int \|(f,g)\|_E d\mu\,.
\end{aligned}
$$

Therefore Q is contractive.

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