Normal homogeneous metrics and their spectra

Taniguchi, Y.
Osaka J. Math.
18 (1981), 555-576

NORMAL HOMOGENEOUS METRICS AND THEIR SPECTRA

YOSHIHARU TANIGUCHI

(Received January 31, 1980)

Introduction. Let K be a compact Lie group acting almost effectively and transitively on a compact manifold M. A Riemannian metric on M is called K-normal homogeneous if it is induced canonically from a biinvariant metric on K. We are mainly interested in classifying normal homogeneous metrics on compact homogeneous spaces up to homothetical equivalence.

We put foreward our study by meanss of the spectrum and the eigenspaces of the Laplacian by the following reasons: First, a K-normal homogeneous metric has a remarkable property that the eigenvalues of its Laplacian are expressed explicitely in terms of the representation of the Lie group K and moreover the computations may be carried out in a relatively simple manner. Secondly, the spectrum may give information on the Riemannian manifold which may not be obtained by the curvatures. For example, although flat tori have vanishing curvatures, their spectra considerably distinguish the isometry classes of them. We see indeed that if two Riemannian metrics on a compact manifold have the same spectrum and the eigenspaces, then they are identical, as is shown in **1** (Lemma 1.1).

In the paper [2], Berger has shown that certain normal homogeneous metrics on S^n and $P^n(\boldsymbol{C})$ (n: odd) are not isometric to the ususal ones. In **5**, we shall give some results for compact irreducible symmetric spaces, extending the above results. In fact, we shall compute certain eigenvalues of the Laplacian and prove our theorem, using the work of Oniščik on the classification of transitive compact connected transformation groups on compact manifolds. We see then the following: Let $M=K'/L'$ be a compact irreducible symmetric space given by the symmetric pair (K', L') with a compact simple Lie group K'. Let K be a connected closed subgroup of K' which is transitive on M. Then, a K-normal homogeneous metric on M is isometric to the original symmetric metric if and only if the linear isotropy representation of $L=K\cap L'$ is irreducible. Moreover, while there exist several such subgroups of $SO(n+1)$ acting on S^n (n: odd), many of the normal homogeneous metrics of them are mutually homothetically inequivalent.

In **3**, we shall consider one parameter families of K-normal homogeneous metrics in case K is not simple. We can show that there are uncountably many, homothetically inequivalent, K-normal homogeneous metrics on M in this case, generalizing results of [8], [9] to some extent.

In **4**, we shall prove the following: Let M be a homogeneous space of a compact connected non-semisimple Lie group K'. If the maximal connected semisimple subgroup K of K' acts on M transitively, then any K'-normal homogeneous metric on M never becomes K-normal homogeneous.

For a Lattin letter denoting a Lie group, the corresponding German small letter shall denote its Lie algebra throughout this paper. All inner products used in the paper are always assumed to be positive definite.

The auther wishes to express his hearty thanks to Professor H. Ozeki for his many advices and encouragements and also to Professor M. Takeuchi for his many remarks and comments.

1. A lemma

Let (M, g) be a compact Riemannian manifold. We denote by Δ the Laplacian acting on the space $C^\infty(M)$ of complex valued smooth functions on M. In terms of local coordinates, it is given by

$$(1,1)\qquad \Delta = -\sum_{i,j}\frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}\left(\sqrt{G}g^{i,j}\frac{\partial}{\partial x^j}\right) = -\sum_{i,j}\left(g^{i,j}\frac{\partial^2}{\partial x^i\partial x^j}+\frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}g^{i,j}}{\partial x^i}\frac{\partial}{\partial x^j}\right),$$

where $g=\sum_{i,j}g_{i,j}dx^i dx^j$, $(g^{i,j})=(g_{i,j})^{-1}$ and $G=\det(g_{i,j})$. We denote by Spec (M, g) the spectrum, i.e., the set of eigenvalues of the Laplacian:

$$\text{Spec}\,(M, g) = \{0 = \lambda_0<\lambda_1<\lambda_2<\cdots\}$$

and by E_{λ_i} the eigenspace corresponding to the eigenvalue λ_i.

Lemma 1.1. *Let (N, g') be another compat Riemannian manifold and denote by Δ' and E'_λ the Laplacian of (N, g') and its eigenspace corresponding to the eigenvalue λ respectively. Suppose that $Spec(M, g) = Spec(N, g')$. Then, a diffeomorphism $f\colon M\to N$ is an isometry if and only if*

$$f^*E'_{\lambda_i} = E_{\lambda_i} \qquad (i = 0, 1, 2\cdots).$$

Proof. The necessity is obvious. We prove the converse.

$$\text{Suppose}\quad f^*E'_{\lambda_i} = E_{\lambda_i} \qquad (i = 0, 1, 2\cdots).$$

The differential operator $f^*\Delta' f^{*^{-1}}$ acting on $C^\infty(M)$ coincides with the Laplacian defined by the Riemannian metric f^*g' on M, and its eigenspace E''_{λ_i}

coincides with $f^*E'_{\lambda_i}$ for each i. Therefore, we may assume that $M=N$ and f is the identity map. We know that the algebraic asum $E(M)=\sum_{i=0}^{\infty}E_{\lambda_i}$ is a dense subspace in $C^\infty(M)$ with respect to the C^∞-topology. Our assumption implies that $\Delta=\Delta'$ on $E(M)$, hence we have $\Delta=\Delta'$. From the expression (1.1), we have the lemma.

Here, we remark that if we replace g by cg with a positive constant c, then the corresponding Laplacian is equal to $c^{-1}\Delta$. In particular, we have $\mathrm{Spec}(M, cg)=c^{-1}\mathrm{Spec}(M,g)$.

2. Normal homogeneous metrics and their spectra

We recall the definition of a normal homogeneous metric and a formula to compute its spectrum. Let K be a compact Lie group and L a closed subgroup of K such that K acts *almost effectively* on the homogeneous space $M=K/L$. For an AdK-invariant inner product B on $\mathfrak{k}$, we can define a biinvariant Riemannian metric on K which coincides with B on $T_eK=\mathfrak{k}$. Let $\mathfrak{m}$ be the orthogonal complement to $\mathfrak{l}$ in $\mathfrak{k}$ relative to B, so that $\mathfrak{k}=\mathfrak{l}+\mathfrak{m}$ and $AdL\cdot\mathfrak{m}=\mathfrak{m}$. For X in $\mathfrak{k}$, we denote by X^* the tangent vector to M at $o=L$ induced from the infinitesimal action of X:

$$X^*=\frac{d}{dt}\bigg|_{t=0}\exp tX\cdot o \qquad (X\in\mathfrak{k}).$$

The correspondence $X\mapsto X^*$ gives rise to an L-isomorphism between $\mathfrak{m}$ with the adjoint action of L and T_0M with the linear isotropy action of L, and it gives an L-invariant inner product g_0 on T_0M by

$$g_0(X^*, Y^*)=B(X, Y) \qquad (X, Y\in\mathfrak{m}).$$

g_0 can be uniquely extended to a K-invariant Riemannian metric on M, which we denote by g or $g(K, B)$ specifying K and B. Now, let M be a compact homogeneous manifold. A homogeneous Riemannian metric g on M is said to be *normal homogeneous* if there exist a transitive compact Lie group K acting almost effectively on M and an AdK-invariant inner product B on $\mathfrak{k}$ such that $g=g(K, B)$. If we specify the tranformation group K and the AdK-invariant inner product B, g is said to be *K-normal homogeneous* and is denoted by $g(K, B)$. We denote by $\Delta(K, B)$ the Laplacian of the K-normal homogeneous metric $g(K, B)$.

Let $\mathfrak{k}$ be the Lie algebra of a compact Lie group K. We fix a maximal abelian subaligebra $\mathfrak{t}$ of $\mathfrak{k}$. Since a weight of a finite dimensional representation of K relative to $\mathfrak{t}$ has its values in purely imaginary numbers on $\mathfrak{t}$, we consider a weight as an element of $\sqrt{-1}\mathfrak{t}^*$, where $\mathfrak{t}^*$ denotes the real dual space of $\mathfrak{t}$.

Fixing a lexicographic ordering on $\sqrt{-1}\mathfrak{t}^*$, we denote by $V_w(\mathfrak{k})$ or V_w an irreducible $\mathfrak{k}$-module over $\boldsymbol{C}$ with the highest weight w. In case K is connected, $V_w(\mathfrak{k})$ is often denoted by $V_w(K)$. From an AdK-invariant inner product B on $\mathfrak{k}$, a positive definite inner product on $\sqrt{-1}\mathfrak{t}^*$ is defined in the usual and denoted by the same letter B. We denote by δ_K half the sum of positive roots of $\mathfrak{k}\otimes\boldsymbol{C}$: $\delta_K=\frac{1}{2}\sum_{\text{positive}}\alpha$.

Proposition 2.1. *Let (M,g) be a normal homogeneous Riemannian manifold with $g=g(K,B)$. Then, we have that*

$$\Delta(K,B)=B(w+2\delta_K,w)\cdot id. \qquad on \quad V_w(\mathfrak{k})$$

for each irreducible $\mathfrak{k}$-submodule $V_w(\mathfrak{k})$ in $C^\infty(M)$.

Throughout this paper, we consider $C^\infty(M)$ as a K-module via

$$(k\cdot f)(x)=f(k^{-1}x) \qquad (k\in K,\ x\in M \text{ and } f\in C^\infty(M)),$$

and consider it as a $\mathfrak{k}$ -module via

$$(X\cdot f)(x)=\left.\frac{d}{dt}\right|_{t=0} f(\exp(-tX)\cdot x) \qquad (X\in\mathfrak{k},\ x\in M \text{ and } f\in C^\infty(M)).$$

For a proof, see [9], [13] or [17].

Corollary 2.1. *Under the same assumption, we have*

(1) *$E_\lambda=\sum V_w(\mathfrak{k})$ for each $\lambda\in Spec(M,g(K,B))$, where the summation runs over all the irreducible $\mathfrak{k}$-submodule $V_w(\mathfrak{k})$ of $C^\infty(M)$ with $\lambda=B(w+2\delta_K,w)$.*

(2) *$Spec(M,g(K,B))=\{B(w+2\delta_K,w);\ V_w\subset C^\infty(M)\}$.*

Next, we remark some effects on K-normal homogeneous metrics caused by the change of K-action by a diffeomorphism of $M=K/L$. We assume that the K-action on M is effective, and consider K as a subgroup of the diffeomorphism group Diffeo (M). Suppose that there is given a diffeomorphism f of M with $f(o)=o$. Then, the homomorphism F from K into Diffeo(M) defined by $F(k)=f^{-1}kf(k\in K)$ gives rise to a new transitive action of K on M. We set $\bar{K}=F(K)$. For an AdK-invariant inner product B on $\mathfrak{k}$, we can define an inner product $\bar{B}$ on $\bar{\mathfrak{k}}$ such that $F_*\colon\mathfrak{k}\to\bar{\mathfrak{k}}$ becomes an isometry:

$$\bar{B}(F_*X,F_*Y)=B(X,Y) \qquad (X,Y\in\mathfrak{k}).$$

Then, $\bar{B}$ is $Ad\bar{K}$-invariant.

Lemma 2.1. (1) *Under the above situation, we have*

$$g(\bar{K},\bar{B})=f^*g(K,B).$$

(2) *Moreover, suppose that* $\bar{K}=K$ *and* B *is invariant under any automorphism of* $\mathfrak{k}$. *Then, we have* $g(\bar{K}, \bar{B})=g(K, B)$, *and* f *is an automorphism of* $(M, g(K,B))$.

Proof. We first remark that if we set $\bar{\mathfrak{l}}=F_*(\mathfrak{l})$ and $\bar{\mathfrak{m}}=F_*(\mathfrak{m})$, then $\bar{\mathfrak{l}}$ is the Lie algebra of the isotropy subgroup of $\bar{K}$ at o and $\bar{\mathfrak{m}}$ is the orthogonal complement to $\bar{\mathfrak{l}}$ relative to $\bar{B}$. For an arbitrary $X\in\mathfrak{m}$, we have

$$(F_*X)^* = \frac{d}{dt}\Big|_{t=0} f^{-1} \operatorname{expt} X \cdot f(o)$$
$$= f^{-1}{}_*X^* .$$

Thus, we have

$$\begin{aligned} g(\bar{K}, \bar{B})((F_*X)^*, (F_*Y)^*) &= \bar{B}(F_*X, F_*Y) \\ &= B(X, Y) \\ &= g(K, B)(X, Y) \qquad (X, Y\in\mathfrak{m}) . \end{aligned}$$

Hence, we obtain $g(\bar{K}, \bar{B})=f^*g(K, B)$ at $o\in M$. Since both $g(\bar{K}, \bar{B})$ and $f^*g(K, B)$ are $\bar{K}$-invariant, they coincide on M. Under the assumptions of (2), we have easily $\bar{\mathfrak{l}}=\mathfrak{l}$, $\bar{\mathfrak{m}}=\mathfrak{m}$ and $\bar{B}=B$. Thus, we get $g(\bar{K}, \bar{B})=g(K, B)$.

In general, two different K-invariant metrics on a homogeneous space $M=K/L$ may be isometric to each other. We consider some conditions on K-invariant metrics to avoid such troubles. For a Riemannian manifold (M, g), we denote by $\mathrm{Iso}(M, g)$ and $\mathrm{Iso}^\circ(M, g)$ the isometry group of (M, g) and its identity component respectively.

Let $\{g_t\}$ be a family of K-invariant metrics and g' a K-invariant metric on $M=K/L$.

Condition (A):

$(A,1)$ g_t *is a K-normal homogeneous metric* $g(K, B_t)$ *on* M. *The family* $\{B_t\}$ *is stable under any automorphism of* $\mathfrak{k}$.

$(A,2)$ K *is a connected subgroup of* $K'=Iso(M, g)$, *and for any subgroup* K'' *of* K', *which is transitive on* M *and isomorphic to* K, *there exists an inner automorphism of* K' *which transforms* K'' *to* K'.

Lemma 2.2. *Under the condition* (A), *if* (M, g') *is isometric to* (M, g_t) *for some* t, *then* g' *coincides with* $g_{t'}$ *for some* t' *as tensor fields on* M.

Proof. Let f be an isometry from (M, g') to (M, g_t). Then we see $f^{-1}Kf\subset\mathrm{Iso}(M, g')$. By (A,2), we may assume $f^{-1}Kf=K$ and $f(o)=o$, as is easily seen. Then, by Lemma 2.1, we have $g'=f^*g_t=g_{t'}$ for some t'.

REMARK 2.1. The condition (A,2) is fulfilled in the following cases:

(1) The identity component of the isometry group K' of (M, g') is locally isomorphic to one of the groups $SU(n)$ $(n\geqq2)$, $SO(2n+1)$ and $Sp(n)$.

(2) The isometry group K' of (M, g') contains a subgroup isomorphic to Pin $(2n)/D$ and the identity component of K' is isomorphic to $\mathrm{Spin}(2n)/D$, where D is a central discrete subgroup of $\mathrm{Spin}(2n)$ $(n\geqq 3)$.

(3) K is a connected subgroup of $K'=\mathrm{Iso}(M, g')$ such that

(a) $\mathfrak{k}'=\mathfrak{k}\oplus\mathfrak{k}''$ (direct sum of ideals),

(b) there are no Lie algebra homomorphism from $\mathfrak{k}$ to $\mathfrak{k}''$ except the trivial one.

If the identity component of $\mathrm{Iso}(M, g')$ is compact simple Lie group of exceptional type, then it contains no proper connected closed subgroup K which acts transitively on M.

See [6] and [10]. It is easy to see that the condition (A,2) is fulfilled in the case (3).

REMARK 2.2. Let K be a compact Lie group acting almost effectively and transitively on M. We denote by $\tilde{K}$ the group of transformations of M induced from the elements of K. Then, a Riemannian metric on M is $\tilde{K}$-normal homogeneous if and only if it is K-normal homogeneous. We often consider a $\tilde{K}$-normal homogeneous metric on M as a K-normal homogeneous one.

3. Variations of normal homogeneous metrics

Let K be a compact connected Lie group acting almost effectively and transitively on M. If K is simple, then the K-normal homogeneous metrics are mutually homothetically equivalent, i.e., isometric to each other up to a positive constant multiple. In this section, we assume that K is compact connected and not simple, $\mathfrak{k}=\mathfrak{k}'\oplus\mathfrak{k}''$ ($\mathfrak{k}', \mathfrak{k}''$: non-zero ideals of $\mathfrak{k}$). Let B' and B'' be inner products on $\mathfrak{k}'$ and $\mathfrak{k}''$ respectively invariant under the adjoint representations. Let $B_{s,t}$ $(s, t>0)$ be the AdK-invariant inner product on $\mathfrak{k}$ defined by the following relations;

(1) $B_{s,t}(\mathfrak{k}', \mathfrak{k}'')=0$,

(2) $B_{s,t}|_{\mathfrak{k}'\times\mathfrak{k}'}=s\cdot B'$,

(3) $B_{s,t}|_{\mathfrak{k}''\times\mathfrak{k}''}=t\cdot B''$.

Lemma 3.1. *Under the above situation, there exist continuous positive functions $s(r)$ and $t(r)$ $(0<r<\infty)$ satisfying*

(1) *$t(r)$ (resp. $s(r)$) is monotone decreasing (resp. increasing) in r,*

(2) *$s(r)/t(r)=r$,*

(3) *the one parameter family $g_r=g(K, B_{s(r),t(r)})$ of K-normal homogeneous metrics is of constant volume in r.*

Proof. We set $M=K/L$. Let $\mathfrak{m}_{s,t}$ be the orthogonal complement to $\mathfrak{l}$ relative to $B_{s,t}$. We put $\mathfrak{m}=\mathfrak{m}_{1,1}$ and $g=g(K, B_{1,1})$. Let $X_1, \cdots, X_n$ be an orthonormal basis of $\mathfrak{m}$ with respect to $B_{1,1}|_{\mathfrak{m}\times\mathfrak{m}}$. We set $X=(X', X'')$ for any $X\in\mathfrak{k}$, where $X=X'+X''$ ($X'\in\mathfrak{k}'$, $X''\in\mathfrak{k}''$). We set $X_{i,s,t}=(tX_i', sX_i'')$ for $i=1, \cdots, n$, where $X_i=(X_i', X_i'')$ ($i=1, \cdots, n$). Then, $\{X_{1,s,t}, \cdots, X_{n,s,t}\}$ becomes a basis of $\mathfrak{m}_{s,t}$, as is easily seen. We compute the volumes $U(s, t)$ and $V(s, t)$ of the rectilinear parallelopiped in T_0M spanned by the vectors $X_{1,s,t}{}^*$, $\cdots$, $X_{n,s,t}{}^*$ relative to the inner products g and $g(K, B_{s,t})$ respectively. We put

$$\begin{aligned} a_{i,j}(s, t) &= B_{1,1}(X_{i,s,t}, X_j) \\ &= t\cdot B'(X_i', X_j')+s\cdot B''(X_i'', X_j''), \qquad (i=1, \cdots, n). \end{aligned}$$

Then, we have

$$X_{i,s,t}=\sum_{j=1}^{n} a_{i,j}(s, t)X_j+Y_i \quad (Y_i\in\mathfrak{l}), \qquad (i=1, \cdots, n).$$

Hence, we get

$$X_{i,st}{}^*=\sum_{j=1}^{n} a_{i,j}(s, t)\cdot X_j^* \qquad (i=1, \cdots, n).$$

Since $\{X_1^*, \cdots, X_n^*\}$ is an orthonormal basis of (T_0M, g), we have

$$U(s, t)=\det(a_{i,j}(s, t)).$$

On the other hand, we have

$$V(s, t)^2=\det(B_{s,t}(X_{i,s,t}, X_{j,s,t})),$$

since $(\mathfrak{m}_{s,t}, B_{s,t}|_{\mathfrak{m}_{s,t}\times\mathfrak{m}_{s,t}})\xrightarrow{*}(T_0M, g(K, B_{s,t}))$ is an isometry. Here, we have

$$B_{s,t}(X_{i,s,t}, X_{j,s,t})=sB'(tX_i', tX_j')+t\cdot B''(sX_i'', sX_j'')=sta_{i,j}(s, t).$$

Therefore, we see that $U(s, t)=V(s, t)$ if and only if

$$(*) \qquad \det(a_{i,j}(s, t))=s^nt^n.$$

Now, we set $b'=(B'(X_i', X_j'))$ and $b''=(B''(X_i'', X_j''))$. By the definition, $b'+b''=1_n$. Take an orthogonal matrix P which diagonalizes b'. Then, P simultaneously diagonalizes b''.

$$\begin{cases} {}^tPb'P=\begin{pmatrix} e_1' & & 0 \\ & \ddots & \\ 0 & & e_n' \end{pmatrix}, \quad {}^tPb''P=\begin{pmatrix} e_1'' & & 0 \\ & \ddots & \\ 0 & & e_n'' \end{pmatrix}, \\ e_i'+e_i''=1, \quad e_i', e_i''\geqq 0 \qquad (i=1, 2, \cdots, n). \end{cases}$$

Devide the both sides of $(*)$ by t^n and we get

$$(**) \qquad \prod_{i=1}^{n}(e_i'+re_i'')=r^nt^n,$$

where $r=s/t$.

Thus, to prove the lemma, it suffices to put

$$t = (\prod_{i=1}^{n} (e'_i + re''_i))^{1/n}/r \qquad \text{and} \quad s = tr\,.$$

It is obvious that the continuous functions $s(r)$ and $t(r)$ satisfy the conditions (2), (3). If we show that one of $e'_1, \cdots, e'_n$ is not zero, we complete the proof of the lemma. Suppose $e'_i=0$ for $i=1, \cdots, n$. Then, $\mathfrak{m}$ must be included in $\mathfrak{k}''$ and hence $\mathfrak{l}$ contains $\mathfrak{k}'$. This contradicts the assumption of the almost-effectivity of the K-action.

By Cor. 2.1, we get easily the following

Theorem 3.1. *Let K be a compact connected non-simple Lie group acting almost effectively and transitively on a compact manifold M. Then, there exists a one parameter family g_r of K-normal homogeneous metrics on M satisfying the following conditions*:

(1) *$vol(M, g_r)$ is constant in r.*

(2) *For each $r_0>0$, there exists a positive number h such that g_r's are not isometric to each other if $|r-r_0|<h$,*

REMARK 3.1. This theorem partially generalizes the results of [8], [9] to some extent. Let K' be a compact connected Lie group acting almost effectively and transitively on a compact manifold $M=K'/L'$. Suppose that the linear isotropy representation of the isotropy subgroup L' contains a trivial representation. Let $\mathfrak{m}'$ be the orthogonal complement to $\mathfrak{l}'$ relative to an AdK'-invariant inner product B' on $\mathfrak{k}'$. Take a linear subspace $\mathfrak{f}'$ of the L'-trivial part of $\mathfrak{m}'$ such that $\mathfrak{f}'$ becomes a subalgebra of $\mathfrak{k}'$ in itself. Let H be the closure of the connected subgroup of K' corresponding to $\mathfrak{f}'$. Then, we see that $K'\times H$ acts on M almost effectively, in the following way since H centralizes L' (cf. [18]).

$$(k', h)\cdot xL' = k'xh^{-1}L' \qquad (k', x\in K' \quad \text{and} \quad h\in H)\,.$$

Then, by the above theorem we see that there are uncountably many, homothetically inequivalent, $K'\times H$-normal homogeneous metrics on M, which are obviously K'-invariant metrics on M. (Cf. [8], [9].)

4. Normal homogeneous metrics on compact homogeneous spaces of compact non-semisimple Lie groups

Let K' be a compact connected non-semisimple Lie group acting effectively and transitively on M. Let $\mathfrak{z}$ be the center of $\mathfrak{k}'$ and $\mathfrak{k}$ the semisimple part of $\mathfrak{k}'$. We denote by K the connected subgroup of K' corresponding to the subalgebra $\mathfrak{k}$ of $\mathfrak{k}'$. Then, the decomposition

$$\mathfrak{k}' = \mathfrak{z} \oplus \mathfrak{k}$$

is orthogonal with respect to every AdK'-invariant inner product B on $\mathfrak{k}'$. As we have shown in **3**, there are many K'-normal homogeneous metrics on M. However, we have the following

Theorem 4.1. *Let K' be a compact connected non-semisimple Lie group acting effectively and transitively on M, $M=K'/L'$, and K be the maximal connected semisimple subgroup of K'. Suppose that the K-action on M is transitive. Then, any K'-normal homogeneous metric on M never becomes K-normal homogeneous.*

Corollary 4.1. *Under the same assumption, if K' coincides with the identity component of the isometry group of a K'-normal homogeneous metric g' then any K-normal homogeneous metric on M is not isometric to g'.*

To prove the above theorem and its corollary, we prepare a lemma. We retain the notation in **2**.

Lemma 4.1. *Under the above situation, we have the following*: (1) *Let $V_w(\mathfrak{k}')$ be an irreducible $\mathfrak{k}'$-submodule of $C^\infty(M)$ such that $V_w(\mathfrak{k}')=V_a(\mathfrak{k})\otimes V_b(\mathfrak{z})$. Suppose that the K-action on M is transitive. Then, $w=0$ if and only if $a=0$.*

(2) *Let $V_w(\mathfrak{k}')$ be an irreducible $\mathfrak{k}'$-submodule of $C^\infty(M)$. Then, $C^\infty(M)$ contains an irreducible $\mathfrak{k}'$-submodule isomorphic to $V_{kw}(\mathfrak{k}')$ for each $k=1, 2, \cdots$.*

(3) *There exists an irreducible $\mathfrak{k}'$-submodule $V_w(\mathfrak{k}')=V_a(\mathfrak{k})\otimes V_b(\mathfrak{z})$ with $a\neq 0$ and $b\neq 0$.*

Proof. (1) If the K-action on M is transitive, then the largest trivial K-submodule of $C^\infty(M)$ consists of constant functions on M. From this, we get (1).

(2) Let f be a highest weight vector of $V_w(\mathfrak{k}')$. f does not vanish at some $x\in M$. Then, we see easily that f^k is a non-zero function on M belonging to a $\mathfrak{k}'$-submodule of $C^\infty(M)$ which is isomorphic to $V_{kw}(\mathfrak{k}')$.

(3) Suppose that for any $\mathfrak{k}'$-irreducible submodule $V_w(\mathfrak{k}')=V_a(\mathfrak{k})\otimes V_b(\mathfrak{z})$ of $C^\infty(M)$ we have $b=0$.

Then the $\mathfrak{z}$-action on $C^\infty(M)$ becomes trivial. This contradicts the assumption of the effectivity of the K'-action on M.

Proof of Theorem 4.1. Suppose that a K'-normal homogeneous metric $g'=g(K', B')$ on M coincides with a K-normal homogeneous metric $g=g(K, B)$ on M. Let $V_w(\mathfrak{k}')=V_a(\mathfrak{k})\otimes V_b(\mathfrak{z})$ be an irreducible $\mathfrak{k}'$-submodule of $C^\infty(M)$ with $a\neq 0$ and $b\neq 0$. Seeing the eigenvalues of the Laplacians on $V_{kw}(\mathfrak{k}')$, we have

$$B'(kw+2\delta_K, kw) = B(ka+2\delta_K, ka) \qquad (k = 1, 2, \cdots).$$

By the facts $B'(\mathfrak{z}, \mathfrak{k})=0$ and $\delta_{K'}=\delta_K$, we have

$$(*) \qquad \begin{cases} B'(a, a)+B'(b, b) = B(a, a), \\ B'(\delta_K, a) = B(\delta_K, a). \end{cases}$$

Here, we have $B'(a, a)/B'(\delta_K, a)=B(a, a)/B(\delta_K, a)$. This comes from the following observation. Put $B^1=B'$ and $B^2=B$. Let $\mathfrak{k}_{1,i}, \cdots, \mathfrak{k}_{r,i}$ be the simple factors of $\mathfrak{k}$ such that each pair of them are orthogonal relative to B^i $(i=1, 2)$.

$$\mathfrak{k} = \mathfrak{k}_{1,i}\oplus\cdots\oplus\mathfrak{k}_{r,i} \qquad (i = 1, 2).$$

We set $a=a_{1,i}+\cdots+a_{r,i}$ according to the decomposition, where $V_a(\mathfrak{k})=V_{a_{1,i}}(\mathfrak{k}_{1,i})\otimes\cdots\otimes V_{a_{r,i}}(\mathfrak{k}_{r,i})$ $(i=1, 2)$. We may assume that $\mathfrak{k}_{j,1}$ is isomorphic to $\mathfrak{k}_{j,2}$ $(j=1, \cdots, r)$ and moreover the coefficients of $a_{j,1}$ relative to a fundamental system of weights of $\mathfrak{k}_{j,1}$ coincide with those of $a_{j,2}$. On the other hand, the retriction of B^i to $\mathfrak{k}_{j,i}\times\mathfrak{k}_{j,i}$ coincides with the Killing form of $\mathfrak{k}_{j,i}$ up to a negative constant multiple for $i=1, 2$ and $j=1, \cdots, r$. Then, we get easily our assertion.

Now, from $(*)$, we get $b=0$. This is a contradiction.

Proof of Corollary 4.1. We next assume that K' coincides with the identity component of the isometry group of a K'-normal homogeneous metric g'. Suppose there exists a K-normal homogeneous metric $g=g(K, B)$ on M isometric to g'. Let f be an isometry from (M, g') to (M, g) with $f(o)=o$. From the assumption, we have $\bar{K}=f^{-1}Kf\subset K'$. Thus, $\bar{K}$ coincides with K. Hence, $\bar{B}$ defined in 2 is also AdK-invariant. By Lemma 2.1, f^*g is K-normal homogeneous, which contradicts the above theorem.

REMARK 4.1. If the fundamental group of M is finite, then as is well known, the K-action on M is transitive. (See [10] or [15].)

5. Normal homogeneous metrics on compact irreducible symmetric spaces

Let K' be a compact connected simple semisimple Lie group and $M=K'/L'$ a compact irreducible symmetric space with a compact symmetric pair (K', L'). In this section, we study K-normal homogeneous metrics on M for all the connected subgroups K of K', which are transitive on M.

We mention some results of Oniščik [10], [11] which describe compact connected transformation groups acting transitively on the spaces under our consideration.

Theorem (Oniščik). *Let K' be a compact connected simple semisimple Lie group and $M=K'/L'$ a compact simply connected irreducible symmetric space with* *a compact symmetric pair* (K', L'). *We assume the* K'*-action on* M *is effective.*

(1) *Any compact connected Lie group of transformations of* M *which contains* K', *coincides with* K'.

(2) (a) *case of rank* 1.

Suppose that M *is of rank* 1. *Then, every compact compact connected Lie group of transformations of* M *is conjugate to one of the following by an inner automorphism of Diffeo* (M):

$M=S^n$; $SO(n+1)$ *with the usual action, and its subgroups*

$SU(m)$, $U(m)$ $(n=2m-1,\ m\geqq 2)$,

K_3, K_4, $Sp(m)$ $(n=4m-1,\ m\geqq 2)$,

where K_3 *and* K_4 *denote the quotient groups of* $Sp(m)\times Sp(1)$ *and* $Sp(m)\times U(1)$ *by certain discrete subgroups of them,*

$Spin(9)$ $(n=15)$,

$Spin(7)$ $(n=7)$,

G_2 $(n=6)$.

$M=P^n(\boldsymbol{C})$; $\widetilde{SU(n+1)}$ *with the usual action and its subgroup*

$\widetilde{Sp(m)}$ $(n=2m-1,\ m\geqq 2)$, (*Cf. Remark* 2.2.)

$M=P^n(\boldsymbol{H})$; $\widetilde{Sp(n+1)}$ *with the usual action.* (*Cf. Remark* 2.2.)

$M=$*Cayley projective plane*; F_4 *with the usual action.*

(*b*) *case of rank* $\geqq 2$.

Except the following cases, every subgroup of K' *which acts transitively on* M, *coincides with* K'.

(i) $M=\widetilde{SU(2n)}/\widetilde{Sp(n)}$ $(n\geqq 2)$: *The subgroups* $SU(2n-1)$ *and* $S(\widetilde{U(1)\times U(2n-1)})$ *act on* M *transitively.*

(ii) $M=\widetilde{SO(2n)}/\widetilde{U(n)}$ $(n\geqq 3)$: *The subgroup* $SO(2n-1)$ *acts on* M M *transitively.*

(iii) $M=Q^5(\boldsymbol{C})=SO(7)/SO(2)\times SO(5)$: *The subgroup* G_2 *acts on transitively.*

(iv) $M=SO(8)/SO(3)\times SO(5)$: *The subgroup* $Spin(7)$ *acts on* M *transitively.*

Moreover, in the last four cases, every compact connected proper subgroup of K' *which is transitive on* M *is conjugate to one of the subgroups listed above by an automorphism of* K'.

In the following, we freely use the convention of Remark 2.2.

Our result is the following

Theorem 5.1.

Spheres: (1) *Let* $n=2m-1$ $(m\geqq 2)$. *Let* g_0, g_1 *and* g_2 *be* $SO(n+1)$-, $U(m)$- *and* $SU(m)$*-normal homogeneous metrics on* S^n *respectively. Then, any* *pair of them are mutually homothetically inequivalent. Moreover,* $Iso^{\circ}(S^n, g_2)=U(m)$.

(2) *Let* $n=4m-1$ $(m\geqq 2)$. *Let* g_3, g_4 *and* g_5 *be* K_3-, K_4-*and* $Sp(m)$-*normal homogeneous metrics on* S^n *respectively. Then, they are not of constant curvature. Moreover, any pair of* $g_0, g_1, \cdots, g_5$ *except the pair* (g_1, g_4) *are mutually homothetically inequivalent. In addition, we have* $Iso^{\circ}(S^n, g_3)=K_3$.

(3) *Any* $Spin(9)$-*normal homogeneous metric on* S^{15} *is not of constant curvature.*

(4) *Every* $Spin(7)$-*normal homogeneous metric on* S^7 *and every* G_2-*normal homogeneous metric on* S^6 *are of constant curvature.*

Complex projective spaces:

(5) *Let* $n=2m-1$ $(m\geqq 2)$. *Any* $Sp(m)$-*normal homogeneous metric on* $P^n(\boldsymbol{C})$ *is not homothetically equivalent to the Fubini-Study metric.*

$\widetilde{SU(2n)}/\widetilde{Sp(n)}$: (6) *Any* $S(U(1)\times U(2n-1))$ *or* $SU(2n-1)$-*normal homogeneous metric on* $M=SU(2n)/Sp(n)$ *is not homothetically equivalent to the symmetric metric on* M.

$\widetilde{SO(2n)}/\widetilde{U(n)}$: (7) *Any* $SO(2n-1)$-*normal homogeneous metric on* $M=SO(2n)/U(n)$ *is homothetically inequivalent to the symmetric metric on* M.

$Q^5(\boldsymbol{C})$: (8) *Any* G_2-*normal homogeneous metric on* $Q^5(\boldsymbol{C})$ *is not homothetically equivalent to the symmetric metric on* $Q^5(\boldsymbol{C})$.

$SO(8)/SO(3)\times SO(5)$: (9) *Any* $Spin(7)$-*normal homogeneous metric on* $M=SO(8)/SO(3)\times SO(5)$ *is not homothetically equivalent to the symmetric metric on* M.

REMARK 5.1. In cases S^{4m-1} and $SU(2n)/Sp(n)$, we can say nothing about whether g_1 and g_4, or an $S(U(1)\times U(2n-1))$-normal homogeneous metric and $SU(2n-1)$-normal homogeneous one are mutually homothetically equivalent or not.

REMARK 5.2. The linear isotropy representation of the isotropy subgroup of Spin(7) (resp. G_2) acting on S^7 (resp. S^6) is irreducible (see [2]). Hence in particular, we see the following fact: Let K be a compact connected subgroup of K' acting transitively on $M=K'/L'$. Then, a K-normal homogeneous metric on M is isometric to the symmetric metric on M up to a positive constant multiple if and only if the linear isotropy representation of the isotropy subgroup of K is irreducible.

Proof of Theorem 5.1. In the following, we adopt the notation of Bourbaki [4] for the terms of the representation theory. However, for the convenience for typing, we write a, w in place of α, ϖ respectively.

Case S^n.

We consider S^n as the unit sphere in the Euclidean space $\boldsymbol{R}^{n+1}$. Let $\mathcal{H}^k$ be the vector space of homogeneous harmonic polynomials on $\boldsymbol{R}^{n+1}$ of degree k with coefficients in $\boldsymbol{C}$ and H^k the space of real polynomials of $\mathcal{H}^k$. We often identify $\mathcal{H}^k$ with the complexication of H^k. We know that $\mathcal{H}^k$ and H^k are irreducible $SO(n+1)$-module over $\boldsymbol{C}$ and $\boldsymbol{R}$ respectively and that the restriction map of $\mathcal{H}^k$ (resp. H^k) to S^n is one to one (cf. [3] or [13]). We frequently identify $\mathcal{H}^k$ and H^k with their images under the above maps. Then, as is well known, the spectrum and the eigenspaces of the Laplacian of the unit sphere S^n are as follow (cf. [3] or [13]):

$$(5.1)\quad \begin{cases} \text{Spec (the unit sphere } S^n = \{k(k+n-1);\ k=0, 1, 2, \cdots\}\ , \\ E_{k(k+n-1)} = \mathcal{H}^k \qquad (k=0,1,2,\cdots)\ , \\ \dim \mathcal{H}^k = \dfrac{k+\nu}{\nu}\dfrac{(k+2\nu-1)!}{k!(2\nu-1)!} \qquad \left(\nu = \dfrac{1}{2}(n-1)\ ,\ k=0,1,2,\cdots\right). \end{cases}$$

Now, let K be a subgroup of $SO(n+1)$ acting transitively on S^n. After decomposing $\mathcal{H}^k$ into K-irreducible sumbodules, we can compute the spectrum and the eigenspaces of the Laplacian of S^n with a K-normal homogeneous metric using Proposotion 2.1.

(i) case $n=4m-1,\quad K=K_3,\ K_4$ or $Sp(m)$.

We consider S^n as the unit sphere in $\boldsymbol{H}^m$ ($\boldsymbol{H}=\{a+bi+cj+dk;\ a, b, c, d\in\boldsymbol{R}\}$ quaternion field) and $Sp(m)$ as the group of matrices $A\in M_m(\boldsymbol{H})$ such that $A^*\cdot A=1_m$. Let $U(1)$ denote exclusively the subgroup of $Sp(1)$ consisting of complex numbers of modules 1. $Sp(m)$ acts on S^n transitively by matrix multiplication and an element $h\in Sp(1)$ transforms each vector in S^n into a vector in S^n by the right multiplication of h^{-1}. Thus, S^n is an almost effective homogeneous space of $Sp(m)\times Sp(1)$. Let $w_1, w_2, \cdots, w_m$ be the fundamental weights of $Sp(m)$ relative to the simple root system $a_1, a_2, \cdots, a_m$ as in Bourbaki [4] ($\underset{a_1}{\circ}$——$\underset{a_2}{\circ}$——$\cdots$——$\underset{a_m}{\circ}\Leftarrow\circ$) and w_1' the fundamental weight of $Sp(1)$ relative to the simple root system a_1'. Let w_0 denote the weight of $U(1)$-module $\boldsymbol{C}$ with the standard action of $U(1)$.

We decompose $\mathcal{H}^1$ and $\mathcal{H}^2$ into K-irreducible submodules as follows. Let I, J be the linear transformations of $\boldsymbol{H}^m$ defined by the right multiplications of i, j respectively ($i, j\in\boldsymbol{H}$). The $Sp(m)$-action on $\boldsymbol{H}^m$ commutes with I and J, and hence in particular, $(\boldsymbol{H}^m, I)$ becomes an $Sp(m)$-module over $\boldsymbol{C}$ by the complex structure I, which is denoted by $\boldsymbol{H}_c^m$. In the decomposition

$$\boldsymbol{H}^m\otimes\boldsymbol{C} = V^+\oplus V^- \qquad (V^\pm:\ \pm\sqrt{-1}\text{ eigenspaces of } I \text{ respectively})\ ,$$

we have a canonical isomorphism $V^+\cong\boldsymbol{H}_c^m$, and since $IJ=-JI$, J gives rise to an

$Sp(m)$-isomorphism between V^+ and V^-. From the definitions, each element $h \in U(1)$ operates on V^+ (resp. V^-) as the scalar multiplication by h (resp. h^{-1}). On the other hand, we have $H^1 = \boldsymbol{H}^m$ as $Sp(m)\times Sp(1)$-modules over $\boldsymbol{R}$, since the $Sp(m)\times Sp(1)$-actions are orthogonal. Thus, we get

$$(*) \qquad \mathcal{H}^1 = V_{w_1+w_1'} \quad \text{as} \quad Sp(m)\times Sp(1)\text{-modules.}$$

We next decompose $\mathcal{H}^2$. From (5.2), we have

$$S^2(\mathcal{H}^1) = S^2(V^+) \oplus S^2(V^-) \oplus V^+ \cdot V^- ,$$

where S^2 denotes 2-nd symmetric tensor power and $\cdot$ the symmetric tensor product in $\mathcal{H}^1 \otimes \mathcal{H}^1$. This splitting is the decomposition of $S^2(\mathcal{H}^1)$ into the weight spaces of $U(1)$, i.e., an element $h \in U(1)$ acts on each component by h^2, h^{-2}, 1 respectively. Although both $S^2(V^+)$ and $S^2(V^-)$ are irreducible $Sp(m)$-modules (cf. [16]), $V^+ \cdot V^-$ decomposes as follows: Since both V^+ and V^- are isomorphic to $\boldsymbol{H}_c^m$ as $Sp(m)$-modules, we have

$$V^+ \cdot V^- \cong S^2(\boldsymbol{H}_c^m) \oplus \Lambda^2(\boldsymbol{H}_c^m) ,$$

where Λ^2 denotes the 2-nd exterior power. We denote by S and Λ the subspaces of $V^+ \cdot V^-$ corresponding to $S^2(\boldsymbol{H}_c^m)$ and $\Lambda^2(\boldsymbol{H}_c^m)$ respectively. The $Sp(m)$-module Λ splits into the direct sum of its submodule isomorphic to V_{w_2} and the one dimensional trivial submodule spanned by the "symplectic form". Thereby, we see easily that

$$\begin{aligned} S^2(\mathcal{H}^1) &= S^2(V^+) \oplus S^2(V^-) \oplus S \oplus V_{w_2} \oplus \boldsymbol{C} && \text{as } Sp(m)\text{-modules,} \\ &= V_{2w_1+2w_1'} \oplus V_{w_2} \oplus \boldsymbol{C} && \text{as } Sp(m)\times Sp(1)\text{-modules,} \end{aligned}$$

where $V_{2w_1+2w_1'} = S^2(V^+) \oplus S^2(V^-) \oplus S$ and $\boldsymbol{C}$ denotes the trivial module. Since $S^2(\boldsymbol{R}^{n+1}) \otimes \boldsymbol{C} = \mathcal{H}^2 \oplus \boldsymbol{C}$ as $SO(n+1)$-modules, we have

$$(**) \qquad \mathcal{H}^2 = V_{2w_1+2w_1'} \oplus V_{w_2} \quad \text{as} \quad Sp(m)\times Sp(1)\text{-modules.}$$

From $(*)$, $(**)$, we see easily the decompositions of $\mathcal{H}^1$ and $\mathcal{H}^2$ into $Sp(m)\times U(1)$- or $Sp(m)$-irreducible modules. To compute the eigenvalues of the Laplacian $\Delta(K, B)$, we normlize B as follows:

In case $K = K_3$,

$$B(a_1, a_1) = 2 .$$

We put $B(a_1', a_1') = 4t$ $(t > 0)$ and set $B_t = B$.

In case $K = K_4$,

$$B(a_1, a_1) = 2 .$$

We put $B(w_0, w_0) = t$ (>0) and set $B_t = B$.

In case $K=Sp(m)$,

$$B(a_1, a_1) = 2\,.$$

Every AdK-invariant inner product on $\mathfrak{k}$ is obtained from the above B or B_t by a certain positive constant multiple and invariant under any automorphism of $\mathfrak{k}$ for $K=K_3$ or $Sp(m)$. In case $K=K_4$, the family of AdK-invariant inner products on $\mathfrak{k}$ is stable under any automorphism of $\mathfrak{k}$.

In the following table, we list the K-irreducible decompositions of $\mathcal{H}^1$ and $\mathcal{H}^2$ and the eigenvalues of the Laplacian on each component.

Table 1.

$$\begin{array}{lll}
\mathcal{H}^1 = & V_{w_1+w_1'}(K) & \\
 & 2m-1+3t & (K=K_3), \\
= & V_{w_1+w_0}(K)\oplus V_{w_1-w_0}(K) & \\
 & 2m-1+t \quad 2m-1+t & (K=K_4), \\
= & V_{w_1}(K) \quad \oplus \quad V_{w_1}(K) & \\
 & 2m-1 \qquad 2m-1 & (K=Sp(m)). \\
\mathcal{H}^2 = & V_{2w_1+2w_1'}(K)\oplus V_{w_2}(K) & \\
 & 4m+8t \qquad 4m & (K=K_3), \\
= & V_{2w_1+2w_0}(K) \oplus V_{2w_1}(K)\oplus V_{2w_1-2w_0}(K) \oplus V_{w_2}(K) & \\
 & 4(m+1)+4t \quad 4(m+1) \quad 4(m+1)+4t \quad 4m & (K=K_4), \\
= & V_{2w_1}(K)\oplus V_{2w_1}(K)\oplus V_{2w_1}(K)\oplus V_{w_2}(K) & \\
 & 4(m+1) \quad 4(m+1) \quad 4(m+1) \quad 4m & (K=Sp(m)).
\end{array}$$

In this and the forthcoming tables, the number below a K-irreducible module denotes the eigenvalue of $\Delta(K, B)$ on that submodule of $C^\infty(M)$.

(ii) case $n = 2m-1$, $K = U(m)$ or $SU(m)$.

We consider S^n as the unit sphere in $\boldsymbol{C}^m$. The goups $U(m)$ and $SU(m)$ act on S^n transitively in the usual way. Let $w_0, w_1, \cdots, w_{m-1}$ be the fundamental weights of $U(m)$ so that $w_1, \cdots, w_{m-1}$ are the fuudamental weights of $SU(m)$ and the highest weight of $\boldsymbol{C}^m$ with the standard action of $U(m)$ is w_0+w_1. Our AdK-invariant inner product B on $\mathfrak{k}$ is assumed to be normalized as follows:

In case $K=U(m)$,

$$B(a_1, a_1) = 2\,.$$

We put $B(w_0, w_0) = t$ $(t>0)$ and set $B_t = B$.

In case $K=SU(m)$,

$$B(a_1, a_1) = 2\,.$$

In this case, H^1 is isomorphic to $\mathrm{Hom}_{\boldsymbol{R}}(\boldsymbol{C}^m, \boldsymbol{R})$ as $U(m)$-modules over $\boldsymbol{R}$, ($\boldsymbol{R}$ is considered as a trivial module). We have

$$\mathcal{H}^1 = \boldsymbol{C}^m \oplus (\boldsymbol{C}^m)^* \qquad \text{as } U(m)\text{-modules over } \boldsymbol{C}.$$

Since the $U(m)$-module $\boldsymbol{C}^m \otimes (\boldsymbol{C}^m)^*$ is the direct sum of $V_{w_1+w_{m-1}}$ and the one dimensional trivial submodule $\boldsymbol{C}$, we have

$$S^2(\mathcal{H}^1) = S^2(\boldsymbol{C}^m) \oplus S^2((\boldsymbol{C}^m)^*) \oplus V_{w_1+w_{m-1}} \oplus \boldsymbol{C}$$

as $U(m)$-modules. In the right hand side of the above decomposition, each component is $U(m)$-irreducible

The $SU(m)$-irreducible decompositions of $\mathcal{H}^1$ and $\mathcal{H}^2$ are obtained from the above decompositions.

In the same way as before, we get the following table.

Table 2.

$$\begin{array}{rlclclcl}
\mathcal{H}^1 = & V_{w_1+w_0}(K) & \oplus & V_{w_{m-1}-w_0}(K) & & & & \\
& \dfrac{(m-1)(m+1)}{m}+t & & \dfrac{(m-1)(m+1)}{m}+t & & & & (K=U(m)), \\
= & V_{w_1}(K) & \oplus & V_{w_{m-1}}(K) & & & & \\
& \dfrac{(m-1)(m+1)}{m} & & \dfrac{(m-1)(m+1)}{m} & & & & (K=SU(m)), \\
\mathcal{H}^2 = & V_{2w_1+2w_0}(K) & \oplus & V_{2w_{m-1}-2w_0}(K) & \oplus & V_{w_1+w_{m-1}}(K) & & \\
& \dfrac{2(m-1)(m+2)}{m}+4t & & \dfrac{2(m-1)(m+2)}{m}+4t & & 2m & & (K=U(m)), \\
= & V_{2w_1}(K) & \oplus & V_{2w_{m-1}}(K) & \oplus & V_{w_1+w_{m-1}}(K) & & \\
& 2(m-1)(m+2)/m & & 2(m-1)(m+2)/m & & 2m & & (K=SU(m)).
\end{array}$$

(iii) case $n=15$, $K=\mathrm{Spin}(9)$.

The inclusion map of Spin(9) into $SO(16)$ is given by the spinor representation whose complexification is a Spin(9)-irreducible module with the highest weight w_4 (cf. $\underset{a_1}{\circ}$ —— $\circ$ —— $\circ \Rightarrow \underset{a_4}{\circ}$). The weights of $V_{w_4}(\mathrm{Spin}(9))$ may be obtained by transforming w_4 under the Weyl group of Spin(9). Thereby, seeing the weights of $S^2(V_{w_4}(\mathrm{Spin}(9)))$, we get

$$\mathcal{H}^2 = V_{2w_4}(\mathrm{Spin}(9)) \oplus V_{w_1}(\mathrm{Spin}(9)) \quad \text{as} \quad \mathrm{Spin}(9)\text{-modules.}$$

Then, we have

Table 3.

$$\Delta(\mathrm{Spin}(9), B) = \begin{cases} 40B(a_1, a_1) & \text{on} \quad V_{2w_4}(\mathrm{Spin}(9)) \subset C^\infty(M), \\ 16B(a_1, a_1) & \text{on} \quad V_{w_1}(\mathrm{Spin}(9)) \subset C^\infty(M). \end{cases}$$

(iv) case $n=7$, $K=\mathrm{Spin}(7)$.

The inclusion map of Spin(7) into $SO(8)$ is given by the spinor representation whose complexification is an irreducible Spin(7)-module with the highest weight w_3 (cf. $\underset{a_1}{\circ}\!\!-\!\!\!-\!\!\!-\circ\!\Rightarrow\!\underset{a_3}{\circ}$). Therefore, we have $\mathcal{H}^1 = V_{w_3}(\mathrm{Spin}(7))$ as Spin(7)-modules. For each k, $\mathcal{H}^k$ includes $V_{kw_3}(\mathrm{Spin}(7))$. From (5.1) and the formula of Weyl, comparing the dimensions of $\mathcal{H}^k$ and $V_{kw_3}(\mathrm{Spin}(7))$, we have $\mathcal{H}^k = V_{kw_3}$ (Spin(7)).

Table 4.

$$\mathcal{H}^k = V_{kw_3}(\mathrm{Spin}\,(7))$$
$$\frac{3}{8}B(a_1, a_2)k(k+6) \qquad (k = 0, 1, 2, \cdots).$$

(v) case $n = 6$, $K = G_2$.

The inclusion map of G_1 into $SO(7)$ is given by the real representation of G_2, whose complexification is an irreducible G_2-module with the highest weight w_1 (cf. $\underset{a_1}{\circ}\!\Lleftarrow\!\underset{a_2}{\circ}$). In this case, we get similarly the following

Table 5.

$$\mathcal{H}^k = V_{kw_1}(G_2)$$
$$B(a_1, a_1)k(k+5) \qquad (k = 0, 1, 2, \cdots).$$

REMARK 5.1. In the last two cases, the linear isotropy representations of the isotropy subgroups of Spin(7) or G_2 are both irreducible, according to [2]. From these facts, the assertion (4) of Theorem 5.1 can be obtained. In the all cases above, the family of AdK-invariant inner products on $\mathfrak{k}$ is stable under any automorphism of $\mathfrak{k}$.

Now, from Table 1–3, we see by Lemma 2.2 and Remark 2.1 that $g_1, g_2, \cdots, g_5$ are not of constant curvature. We shall observe $g_1, \cdots, g_5$ in more details. Note that if a Riemannian metric on a sphere S^n has the isometry group of dimension $\frac{1}{2}n(n+1)$, then g is of constant curvature. Thus, we have $Iso^\circ(S^n, g_1) = U(m)$ $(n=2m-1)$ and $Iso^\circ(S^n, g_3) = K_3$ $(n=4m-1)$ by the theorem of Oniščik. In particular, we see that g_1 and g_2 are homothetically inequivalent to each other by Cor.4.1. In case $n=4m-1$, the inclusion relations among our groups are given as follows:

$$\begin{array}{ccccc} & & U(2m) & \supset & SU(2m) \\ & & \cup & & \cup \\ K_3(\underset{\text{loc.}}{\cong} Sp(m)\times Sp(1)) & \supset & K_4 \underset{\text{loc.}}{\cong} (Sp(m)\times U(1)) & \supset & Sp(m) \end{array}$$

From this, $Iso^\circ(S^n, g_4)$ must be isomorphic to one of the groups K_3, K_4 and $U(2m)$. In any case, g_4 and g_5 fulfill the condition (A). They are homothetically inequivalent by Table 1 and Lemma 2.2. The same assertions are valid for the pairs (g_3, g_4) and (g_3, g_5) by the possibilities of $Iso^\circ(S^n, g_4)$ and $Iso^\circ(S^n, g_5)$. Table 2 shows that $V_{w_1+w_{m-1}}$ is contained in E_{2m} for g_1, g_2. On the other hand, this space splits into two subspaces contained in different eigenspaces for g_5. (We have $U(2m)$-module $V_{w_1+w_{m-1}}=Sp(m)$-module $V_{2w_1}\oplus Sp(m)$-module V_{w_2}, see (i), (ii).) Hence, we see that g_5 is not homothetically equivalent to g_1 or g_2. From the inclusion relations listed above, $Iso^\circ(S^n, g_2)$ must be isomorphic to $SU(2m)$ or $U(2m)$. If g_2 and g_4 are mutually homothetically equivalent, their isometry groups must be isomorphic to $U(2m)$. But then, we see easily that g_4 and g_2 fulfill the condition (A). From Tables 1–2, they must be mutually homothetically inequivalent. g_1 and g_3 are not mutually homothetically equivalent, since their isometry groups are not isomorphic to each other. The same assertion is valid for g_2 and g_3, since their isometry groups cannot be isomorphic to each other by the inclusion relations listed above.

Thus, we have proved (1) and (2). Table 3 implies (3). Tables 4–5 together with Lemma 1.1 show (4).

Case $P^n(\boldsymbol{C})$ $(n=2m-1)$.

$Sp(m)$ acts transitively on the projective space $P^n(\boldsymbol{C})$ of $\boldsymbol{H}_c^m$ (cf. (i)). The eigenspace E corresponding to the non-zero first eigenvalue of the Laplacian of the Fubini-Study space $P^n(\boldsymbol{C})$ is isomorphic to $V_{w_1+w_{2m-1}}(SU(n+1))$ (see [3] or [13]).

As we have seen in case (i) of S^n, we know

$$E = V_{2w_1}\oplus V_{w_2} \quad \text{as} \quad Sp(m)\text{-modules.}$$

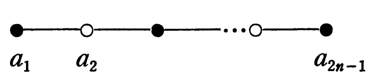
We get easily

$$\Delta(Sp(m), B) = \begin{cases} (4m+4)B(a_1, a_1)/2 & \text{on} \quad V_{2w_1}(Sp(m))\subset C^\infty(P^n(\boldsymbol{C}))\,, \\ 4m\cdot B(a_1, a_1)/2 & \text{on} \quad V_{w_2}(Sp(m))\subset C^\infty(P^n(\boldsymbol{C}))\,. \end{cases}$$

Thus, we see by Lemma 2.2 and Remark 2.1 that any $Sp(m)$-mormal homogeneous metric on $P^n(\boldsymbol{C})$ is not homothetically equivalent to the Fubini-Study metric.

Case $SU(2n)/Sp(n)$.

The Satake diagram of the symmetric space $M=SU(2n)/Sp(n)$ is given as follows:



(see [1]).

We can read from the diagram that $V_{w_2}(SU(2n))$ is a spherical representation of $SU(2n)$ on M (see [13]). Hence, it is a subspace of a certain eigenspace of the Laplacian of the symmetric metric. The subgroup $S(U(1)\times U(2n-1))$ of $SU(2n)$ acts transitively on M. Put $K=S(U(1)\times U(2n-1))$. Let $w_0, w_1, \cdots, w_l$

($l=2n-2$) be the fundamental weights of K so that

(1) w_0 is the fundamental weight of the center of K,

(2) $w_1, \cdots, w_l$ are the fundamental weights of the maximal connected semi-simple subgroup $SU(2n-1)$ of K,

(3) $C^{2n}=V_{w_0+w_1}(K)\oplus V_{w_0}(K)$ as K-modules.

Let $a_1, \cdots, a_l$ be the simple system of roots of K. Every AdK-invariant inner product B on $\mathfrak{k}$ is invariant under any automorphism of $\mathfrak{k}$. Normalizing B, we can assume $B(a_1, a_1)=2$ and $B(w_0, w_0)=t$ ($t>0$). We have

$$\begin{aligned}\Lambda^2 C^{2n} &= \Lambda^2 V_{w_0+w_1}(K)\oplus V_{w_0+w_1}(K)\\ &= V_{2w_0+w_2}(K)\oplus V_{2w_0+w_1}(K) \qquad \text{as} \quad K\text{-modules.}\end{aligned}$$

The $SU(2n-1)$-irreducible decomposition of $\Lambda^2 C^{2n}$ is obtained from the above one by setting $w_0=0$ formally. We get

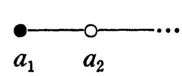
$$\Delta(K, B)=\begin{cases}\dfrac{2l^2+l-3}{l+1}+4t & \text{on} \quad V_{2w_0+w_2}(K)\subset C^\infty(M)\,,\\[2ex] \dfrac{l^2+2l}{l+1}+4t & \text{on} \quad V_{2w_0+w_1}(K)\subset C^\infty(M),\ (K=S(U(1)\times U(2n-1))),\end{cases}$$

$$\Delta(K, B)=\begin{cases}\dfrac{2l^2+l-3}{l+1} & \text{on} \quad V_{w_2}(K)\subset C^\infty(M)\,,\\[2ex] \dfrac{l^2+2l}{l+1} & \text{on} \quad V_{w_1}(K)\subset C^\infty(M)\,, \quad (K=SU(2n-1))\,.\end{cases}$$

Thus, we get (6) by Remark 2.1 and Lemma 2.2.

Case $M=SO(2n)/U(n)$.

The Satake diagram of the symmetric space M is given as follows:



(see [1]).

From the diagram, we see that the $SO(2n)$-irreducible representation $V_{w_2}(SO(2n)) =\Lambda^2 R^{2n}\otimes C$ is a spherical representation of $SO(2n)$ on M. The subgroup $SO(2n-1)$ acts on M transitively. By the decomposition

$$R^{2n}=R^{2n-1}\oplus R \qquad \text{as} \quad SO(2n-1)\text{-modules,}$$

we have

$$\Lambda^2 R^{2n}\otimes C=\Lambda^2 R^{2n-1}\otimes C\oplus R^{2n-1}\otimes C \qquad \text{as} \quad SO(2n-1)\text{-modules.}$$

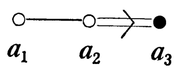
Then, we get easily

$$\Delta(SO(2n-1), B)=\begin{cases}(2n-1)B(a_1, a_1)/2 & \text{on} \quad V_{w_1}(SO(2n-1))\subset C^\infty(M)\,,\\ (4n-4)B(a_1, a_1)/2 & \text{on} \quad V_{w_2}(SO(2n-1))\subset C^\infty(M)\,.\end{cases}$$

We see (7) by the same reason as before.

Case $Q^5(\boldsymbol{C}) = SO(7)/SO(2)\times SO(5)$.

The Satake diagram of the symmetric space M is given as follows:



(see [1]).

From this diagram, we see that the $SO(2n)$-irreducible module $\Lambda^2\boldsymbol{R}^7\otimes\boldsymbol{C}$ is a spherical representation of $SO(7)$ on $Q^5(\boldsymbol{C})$. The inclusion map of G_2 into $SO(7)$ is given as in case (v) of S^n. The G_2-irreducible decomposition of $\Lambda^2\boldsymbol{R}^7\otimes\boldsymbol{C}$can be given as follows: From the formula of Weyl, we have

$$\dim V_{w_1}(G_2) = 7\,, \quad \dim V_{w_2}(G_2) = 14 \quad \text{and} \quad \dim V_w > 14$$

for the other w ($w \neq 0$). Since the G_2-action on M is transitive, $\Lambda^2\boldsymbol{R}^7\otimes\boldsymbol{C}$ contains no G_2-trivial submodule. We see

$$\Lambda^2\boldsymbol{R}^7\otimes\boldsymbol{C} = V_{w_1}(G_2)\oplus V_{w_2}(G_2) \qquad \text{as} \quad G_2\text{-modules,}$$

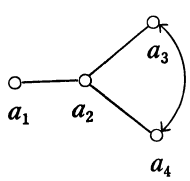
since $\dim\Lambda^2\boldsymbol{R}^7\otimes\boldsymbol{C} = 21$. We get easily

$$\Delta(G_2, B) = \begin{cases} 6B(a_1, a_1) & \text{on} \quad V_{w_1}(G_2)\subset C^\infty(M)\,, \\ 12B(a_1, a_1) & \text{on} \quad V_{w_2}(G_2)\subset C(^\infty M)\,. \end{cases}$$

Therefore, we get (8) by the same reason as before.

Case $M = SO(8)/SO(3)\times SO(5)$.

The Satake diagram of the symmtric space M is given as follows:



(see [1]).

The $SO(8)$-module $\Lambda^3\boldsymbol{R}^8\otimes\boldsymbol{C}$ is an irreducible module with the highest weight w_3+w_4 of dimension 56, and hence a spherical representation of $SO(8)$ on M. The inclusion map of Spin(7) into $SO(8)$ is given as in case (iv) of S^n. The Spin(7)-irreducible decomposition of $\Lambda^3\boldsymbol{R}^8\otimes\boldsymbol{C}$ is given as follows: The wieghts of Spin(7)-irreducible module $\boldsymbol{R}^8\otimes\boldsymbol{C}$ are $\frac{1}{2}(\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3)$ (cf. [4]). Hence, the Spin(7)-module $\Lambda^3\boldsymbol{R}^8\otimes\boldsymbol{C}$ contains an irreducible Spin(7)-submodule with the highest weight $\frac{3}{2}\varepsilon_1+\frac{1}{2}\varepsilon_2+\frac{1}{2}\varepsilon_3 = w_1+w_3$ of dimension 48. From the formula of Weyl, we see that $V_{w_3}(\mathrm{Spin}(7))$ is the only irreducible Spin(7)-module of dimension 8. Hence, we get the decomposition

$$\Lambda^3 \boldsymbol{R}^8 \otimes \boldsymbol{C} = V_{w_1+w_3}(\mathrm{Spin}\ 7)) \oplus V_{w_3}(\mathrm{Spin}\ (7)), \qquad \text{as} \quad \mathrm{Spin}\ (7)\text{-modules.}$$

We get easily

$$\Delta(\mathrm{Spin}\ (7), B) = \begin{cases} \dfrac{49}{8} \cdot B(a_1, a_1) & \text{on} \quad V_{w_1+w_3}(\mathrm{Spin}\ (7)) \subset C^\infty(M), \\ \dfrac{21}{8} \cdot B(a_1, a_1) & \text{on} \quad V_{w_3}(\mathrm{Spin}\ (7)) \subset C^\infty(M). \end{cases}$$

Therefore, we have (9) by the same reason as before.

References

[1] S. Araki: *On root systems and an infinitesimal classification of irreducible symmetric spaces,* J. Math. Osaka City Univ. **13** (1962), 1–34.

[2] M. Berger: *Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive,* Ann. Scuola Norm. Sup. Pisa **15** (1961), 179–246.

[3] M. Berger, P. Gauduchon et E. Mazet: Le spectre d'une variété riemannienne, Lecture notes in Mathematics 194. Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[4] N. Bourbaki; Groupes et algèbres de Lie, Chap. 4, 5 et 6, Hermann, Paris, 1968.

[5] N. Bourbaki: Groupes et algèbres de Lie, Chap. 7 et 8, Hermann, Paris, 1975.

[6] E. B. Dynkin: *Semisimple subalgebras of semisimple Lie algebras,* Amer. Math. Soc. Transl. (2) **6** (1957), 111–244.

[7] S. Helgason: Differential geometry and symmetric spaces, Academic Press, New York, 1962.

[8] G.W. Lukesh: *Variations of metrics on homogeneous manifolds,* J. Math. Soc. Japan **31** (1979), 655–667.

[9] H. Muto and H. Urakawa: *On the least positive eigenvalue of Laplacian for compact homogeneous spaces,* (preprint).

[10] A.L. Oniščik: *Inclusion relations among transitive compact transformation groups,* Amer. Math. Soc. Transl. (2) **50** (1962), 5–58.

[11] A.L. Oniščik: *Transitive compact transformation groups,* Amer. Math. Soc. Transl. (2) **55** (1966), 153–194.

[12] H. Ozeki: *On a transitive transformation group of a compact group manifold,* Osaka J. Math. **14** (1977), 519–531.

[13] M. Takeuchi: Modern theory of spherical functions, Iwanami Shoten, Tokyo, 1974, (in Japanese).

[14] N. Wallach: Harmonic analysis on homogeneous spaces, Marcel Dekker, New York, 1973.

[15] H.C. Wang: *Closed manifolds with a homogeneous complex structure,* Amer. J. Math. **76** (1954), 1–32.

[16] H. Weyl: The classical groups, Princeton University Press, Princeton, New

Jersey, 1939.
[17] S. Yamaguchi: *Spectra of flag manifolds,* Mem. Fac. Sci. Kyushu Univ. Ser. A **33** (1979), 95–112.
[18] W. Ziller: *The Jacobi equation on naturally reductive compact homogeneous spaces,* Comm. Math. Helv. **52** (1977), 573–590.

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan