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COMPACT TRANSFORMATION GROUPS AND FIXED POINT SETS OF RESTRICTED ACTION TO MAXIMAL TORUS

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0. Introduction

Let G be a compact connected Lie group and let T be a maximal torus of G. Define

$$m(G) = \max\,\{\dim H \mid H \text{ is a proper closed subgroup of } G\}\,,$$
$$m_0(G) = \max\,\{\dim H \mid H \text{ is a proper closed subgroup of } G \text{ with rank } H = \text{rank } G\}\,.$$

Let M be a connected manifold with a non-trivial smooth G-action and let H be a closed subgroup of G. Denote by $F(H, M)$ the fixed point set of the restricted action of the given G-action to the subgroup H. Then each connected component F_a $(a \in A)$ of $F(H, M)$ is a regular submanifold of M. Define

$$\dim F(H, M) = \max\,\{\dim F_a \mid a \in A\}$$

if $F(H, M)$ is non-empty and we put

$$\dim F(H, M) = -1$$

if $F(H, M)$ is empty. Then we have the following results.

Theorem 1.

(a) *In general,* $\dim M - \dim F(T, M) \geqslant \dim G - m(G)$.

(b) *If G is semi-simple and*

$$\dim F(G, M) < \dim F(T, M)\,,$$

then

$$\dim M - \dim F(T, M) \geqslant \dim G - m_0(G)\,.$$

Theorem 2. *If*

$$\dim M - \dim F(T, M) = \dim G - m(G)\,,$$

then G is semi-simple, $m(G)=m_0(G)$ and

$$\dim M-\dim F_a=\dim G-m(G)$$

for each connected component F_a of $F(T, M)$. Moreover

$$\dim H=m(G) \quad \textit{and} \quad \operatorname{rank} H=\operatorname{rank} G$$

for a principal isotropy group H.

1. Preliminary lemmas

In this section we prepare several lemmas.

Lemma 1.1. *Let H be a closed subgroup of G and assume $T\subset H$. Then*

$$F(T, G/H)=N(T)H/H\,.$$

In particular, $F(T, G/H)$ is a non-empty finite set.

Proof. It is clear that

$$F(T, G/H)=\{gH\,|\,g^{-1}Tg\subset H\}\,.$$

If $g^{-1}Tg\subset H$, then there is $h\in H$ such that

$$g^{-1}Tg=hTh^{-1}\,,$$

since T is a maximal torus of H^0, the identity component of H. Thus

$$gh\in N(T)\text{: the normalizer of } T \text{ in } G\,.$$

Hence we obtain

$$F(T, G/H)=N(T)H/H\,.$$

Next, there is a natural surjection $N(T)/T\to N(T)H/H$, where $N(T)/T$ is the Weyl group of G which is a finite group. Therefore $F(T,G/H)$ is a non-empty finite set. q.e.d.

In the following, we assume that M is a connected manifold with a non-trivial smooth G-action. It is clear

$$\dim M\geqslant\dim G-m(G)\,. \tag{1.2}$$

Lemma 1.3. $\dim M-\dim F(G, M)>\dim G-m(G)$.

Proof. If $F(G, M)$ is empty, then the inequality is clear from (1.2). If $F(G, M)$ is non-empty, let $n=\dim F(G, M)$ and let F_a be an n-dimensional connected component of $F(G, M)$. For $x\in F_a$,

$$T_xM = T_x(F_a) \oplus N_x$$

as G-vector spaces, where N_x is a normal space of F_a in M. Then there is a non-zero vector $v \in N_x$ with $G_v \neq G$. Thus

$$\dim G - m(G) \leqslant \dim G/G_v < \dim N_x = \dim M - n.$$

q.e.d.

Lemma 1.4. *If*

$$\dim M - \dim F(T, M) \leqslant \dim G - m_0(G)$$

and

$$\dim F(G, M) < \dim F(T, M),$$

then

$$M = G \cdot F(H, M).$$

Here H is a compact connected subgroup of G such that

$$\dim H = m_0(G) \quad \textit{and} \quad \operatorname{rank} H = \operatorname{rank} G.$$

Proof. Let $k = \dim F(T, M)$ and denote by F^k the union of k-dimensional connected components of $F(T, M)$. Then

$$F^k - F(G, M)$$

is non-empty by the assumption. For $x \in F^k - F(G, M)$,

$$T_xM = T_x(G \cdot x) \oplus N_x$$

as G_x-vector spaces, where N_x is a normal space of the orbit $G \cdot x$ in M. Since $T \subset G_x$, $F(T, G \cdot x)$ is a non-empty finite set by Lemma 1.1. Thus

$$\begin{aligned} k &= \dim F(T, T_xM) = \dim F(T, N_x) \\ &\leqslant \dim N_x = \dim M - \dim G/G_x \leqslant \dim M - \dim G + m_0(G). \end{aligned}$$

On the other hand,

$$k \geqslant \dim M - \dim G + m_0(G)$$

by the assumption. Therefore

$$(1) \qquad \dim G_x = m_0(G),$$

$$(2) \qquad F(T, N_x) = N_x.$$

Since the action of G_x on N_x is a slice representation at x, a pricncipal isotropy group H' contains T by (2), and hence

$$\dim H' = m_0(G)$$

by (1). Let H be the identity component of the principal isotropy group H'. Then we have

$$M = G\cdot F(H, M) = \{g\cdot x \mid g\in G,\ x\in F(H, M)\} .$$

q.e.d.

Lemma 1.5. *If*

$$\dim M - \dim F(T, M) \leqslant \dim G - m(G) ,$$

then $m(G)=m_0(G)$ *and*

$$M = G\cdot F(H, M) .$$

Here H is a compact connected subgroup of G such that

$$\dim H = m(G) \quad \textit{and} \quad \operatorname{rank} H = \operatorname{rank} G .$$

Proof. Taking account of Lemma 1.3 and using similar arguments as in the proof of Lemma 1.4, we can prove this lemma.

Lemma 1.6. *Let G be a compact connected Lie group and let H be a closed subgroup of G such that*

$$\dim H = m_0(G) \quad \textit{and} \quad \operatorname{rank} H^0 = \operatorname{rank} G .$$

Then $N(H)^0=H^0$, where H^0 is the identity component of H and $N(H)$ is the normalizer of H in G.

Proof. Assume $N(H)^0 \neq H^0$. Then the assumption on H implies $N(H)=G$. Thus H is a normal subgroup of G, and hence

$$\operatorname{rank} G = \operatorname{rank} H^0 + \operatorname{rank} G/H .$$

Then the assumption on H implies $\operatorname{rank} G/H=0$ and hence $G=H$. But this is a contradiction to

$$\dim H = m_0(G) < \dim G .$$

q.e.d.

Lemma 1.7. *Let G be a compact connected semi-simple Lie group and let H be a closed connected subgroup of G such that*

$$\dim H = m_0(G) \quad \textit{and} \quad \operatorname{rank} H = \operatorname{rank} G .$$

Let V be a real G-vector space such that

$$V = G\cdot F(H, V) \quad \textit{and} \quad F(G, V) = \{0\} .$$

Then $S(V)=G/H$ as G-manifolds and $N(H)/H=Z_2$. Here $S(V)$ is a G-invariant unit sphere of V.

Proof. By the assumption on H and V, the identity component of an isotropy subgroup at each point of $S(V)$ is conjugate to H in G. Hence there is an equivariant diffeomorphism

$$S(V) = G/H \underset{N(H)/H}{\times} F(H, S(V))$$

as G-manifolds. Here $F(H, S(V))$ is a unit sphere of $F(H, V)$. Since $N(H)/H$ is a finite group by Lemma 1.6, the natural projection

$$G/H \times F(H, S(V)) \to S(V)$$

is a finite covering as G-manifolds. On the other hand, $S(V)$ is simply connected, because G is simi-simple. Therefore

$$S(V) = G/H$$

as G-manifolds and $F(H,S(V))$ is a zero-sphere S^0. Finally,

$$N(H)/H = F(H, G/H) = F(H, S(V)) = S^0 .$$

Thus $N(H)/H=Z_2$, the cyclic group of order 2. q.e.d.

2. Proof of theorems

Let G be a compact connected Lie group and let T be a maximal torus of G. Let M be a connected manifold with a non-trivial smooth G-action. It is easy to see that

$$F(T, M) = M \quad \text{implies} \quad F(G, M) = M .$$

Thus

$$\dim M - \dim F(T, M) \geqslant 2 ,$$

because

$$\dim M \equiv \dim F_a \pmod 2$$

for each connected component F_a of $F(T, M)$.

If G is not semi-simple, then

$$\dim G - m(G) = 1$$

and hence there is nothing to prove. In particular, if

$$\dim M - \dim F(T, M) = \dim G - m(G) ,$$

then G is semi-simple, and $m(G)=m_0(G)$ by Lemma 1.5.

Now we assume that G is semi-simple and there is a closed connected subgroup H of G such that

$$(*)\quad M=G\cdot F(H, M),\ \dim H=m_0(G)\quad\text{and}\quad\operatorname{rank} H=\operatorname{rank} G\,.$$

Moreover, (i) first suppose that $F(G, M)$ is empty. Then by the assumption (*), the identity component of an isotropy subgroup at each point of M is conjugate to H in G. Hence there is an equivariant diffeomorphism

$$M=G/H\underset{N(H)/H}{\times}F(H, M)$$

as G-manifolds. Since $N(H)/H$ is a finite group by Lemma 1.6, the natural projection

$$p\colon G/H\times F(H, M)\rightarrow M$$

is a finite covering as G-manifolds. Hence we obtain

$$F(T, M)=p(F(T, G/H)\times F(H, M))\,.$$

Here $F(T, G/H)$ is a non-empty finite set by Lemma 1.1. Therefore

$$\begin{aligned}\dim M-\dim F_a&=\dim M-\dim F(H, M)\\&=\dim G/H=\dim G-m_0(G)\,,\end{aligned}$$

for each connected component F_a of $F(T, M)$.

(ii) Next suppose that $F(G, M)$ is non-empty. Then each fibre N_x of the normal G-vector bundle of $F(G, M)$ in M satisfies the hypothesis of Lemma 1.7, and hence

$$N(H)/H=Z_2\quad\text{and}\quad S(N_x)=G/H\,.$$

Let U be a G-invariant closed tubular neighborhood of $F(G, M)$ in M. Then there is an equivariant diffeomorphism

$$M=\partial(D(V)\times F(H, M-\operatorname{int} U))/Z_2$$

as G-manifolds. Here V is a real G-vector space (unique up to G-isomorphism) with $S(V)=G/H$, Z_2 acts on the unit disk $D(V)$ as antipodal involution, and G acts naturally on $D(V)$ and trivially on $F(H, M-\operatorname{int} U)$. Hence we obtain

$$\begin{aligned}F(T, M)&=\partial(F(T, D(V))\times F(H, M-\operatorname{int} U))/Z_2\\&=\partial([-1, 1]\times F(H, M-\operatorname{int} U))/Z_2\,.\end{aligned}$$

Therefore

$$\begin{aligned}\dim M-\dim F_a&=\dim M-\dim F(H, M-\operatorname{int} U)\\&=\dim D(V)-1\\&=\dim G/H\\&=\dim G-m_0(G)\,,\end{aligned}$$

for each connected component F_a of $F(T, M)$.

Now the proofs of Theorem 1 and Theorem 2 are completed by Lemma 1.4 and Lemma 1.5.

3. Integers $m(G)$ and $m_0(G)$

In this section we show certain properties of $m(G)$ and $m_0(G)$. It is easy to see that

$$(3.1) \qquad m(G_1\times G_2)\geqslant \max\,(m(G_1)+\dim G_2,\ \dim G_1+m(G_2))\,,$$

and

$$(3.2) \qquad m(G)\geqslant 1\,, \qquad \text{if} \quad G\neq S^1\,.$$

Lemma 3.3. *Let G_1 and G_2 be compact connected Lie groups. Suppose that G_1 is simple and $G_1\neq S^1$. Let H be a closed connected subgroup of $G_1\times G_2$ with* $\dim H=m(G_1\times G_2)$. *Then*

$$H = H_1\times G_2 \qquad \text{or} \quad H = G_1\times H_2$$

where H_a is a closed subgroup of G_a $(a=1, 2)$ with $\dim H_a=m(G_a)$.

Proof. Let p_a: $G_1\times G_2\to G_a$ $(a=1, 2)$ be natural projections, and let i_a: $G_a\to G_1\times G_2$ be natural injections defined by

$$i_1(g) = (g, e_2),\ g\in G_1$$
$$i_2(g) = (e_1, g),\ g\in G_2$$

where e_a is the identity element of G_a $(a=1, 2)$. Define

$$H_a = p_a(H) \quad \text{and} \quad H_a' = i_a^{-1}(H)\,.$$

Then H_a' is a normal subgroup of H_a $(a=1, 2)$ and $H_1'\times H_2'$ is a normal subgroup of H, and $H\subset H_1\times H_2$. Moreover the projection p_a induces an isomorphism

$$p_a'\colon H/H_1'\times H_2' \to H_a/H_a' \ (a = 1, 2)\,.$$

(i) First suppose $H_1\neq G_1$. Then

$$H\subset p_1^{-1}(H_1) = H_1\times G_2\neq G_1\times G_2\,.$$

Hence we obtain

$$H = H_1\times G_2 \quad \text{and} \quad \dim H_1 = m(G_1)$$

from the assumption $\dim H = m(G_1\times G_2)$.

(ii) Next suppose $H_1=G_1$. Then H_1' is a normal subgroup of the simple Lie group G_1 and hence $H_1'=G_1$ or H_1' is a finite group. Since $m(G_1)\geqslant 1$ and there is an isomorphism

$$H/i_1(H_1') = H_2\,,$$

we obtain

$$\begin{aligned} m(G_1\times G_2) &= \dim H = \dim H_1' + \dim H_2 \\ &< \dim H_1' + m(G_1) + \dim G_2 \leqslant \dim H_1' + m(G_1\times G_2)\,. \end{aligned}$$

Thus $\dim H_1' \neq 0$, and hence

$$H_1' = H_1 = G_1\,.$$

Therefore

$$H = G_1\times H_2 \quad \text{and} \quad \dim H_2 = m(G_2)$$

from the assumption $\dim H = m(G_1\times G_2)$. q.e.d.

Corollary 3.4. *Let G_1 and G_2 be compact connected Lie groups. Suppose that G_1 is simple. Then*

$$\dim(G_1\times G_2) - m(G_1\times G_2) = \min(\dim G_1 - m(G_1), \dim G_2 - m(G_2))\,.$$

Proof. If $G_1 \neq S^1$, Then the equation follows from Lemma 3.3. If $G_1 = S^1$, then $m(G_1\times G_2) = \dim G_2$ and hence the equation holds. q.e.d.

Theorem 3.5. *Let G_1 and G_2 be compact connected Lie groups. Then*

$$\dim(G_1\times G_2) - m(G_1\times G_2) = \min(\dim G_1 - m(G_1), \dim G_2 - m(G_2))\,.$$

Proof. Let G^* be a compact connected covering group of G. Then it is easy to see that

$$m(G^*) = m(G)\,.$$

There are covering groups $G_a{}^*$ of G_a ($a=1, 2$) such that

$$\begin{aligned} G_1{}^* &= H_1\times\cdots\times H_r\times T^m \\ G_2{}^* &= K_1\times\cdots\times K_s\times T^n \end{aligned}$$

where H_i, K_j are compact connected non-abelian simple Lie groups, and T^m, T^n are tori. If m or n is non-zero, then

$$\begin{aligned} &\dim(G_1\times G_2) - m(G_1\times G_2) = 1 \\ &\min(\dim G_1 - m(G_1), \dim G_2 - m(G_2)) = 1\,. \end{aligned}$$

Next, if $m=n=0$, then

$$\begin{aligned} \dim(G_1\times G_2) - m(G_1\times G_2) &= \min_{i,j}(\dim H_i - m(H_i), \dim K_j - m(K_j)) \\ &= \min(\dim G_1 - m(G_1), \dim G_2 - m(G_2)) \end{aligned}$$

be Corollary 3.4. q.e.d.

REMARK 3.6. The integer $m_0(G)$ can be defined only when G is non-abelian (*i.e.* G does not coincide with its maximal torus).

Theorem 3.7. *Let G_1 and G_2 be compact connected non-abelian Lie groups. Then*

$$\dim(G_1\times G_2)-m_0(G_1\times G_2) = \min(\dim G_1-m_0(G_1), \dim G_2-m_0(G_2)) .$$

Proof. Let H be a closed connected subgroup of $G_1\times G_2$ such that

$$\dim H = m_0(G_1\times G_2) \quad \text{and} \quad \operatorname{rank} H = \operatorname{rank}(G_1\times G_2) .$$

Then there are closed connected subgroups H_a of G_a ($a=1, 2$) such that

$$H = H_1\times H_2 \quad \text{and} \quad \operatorname{rank} H_a = \operatorname{rank} G_a \ (a = 1, 2)$$

from the assumption rank H=rank $(G_1\times G_2)$. Moreover

$$\dim H = m_0(G_1\times G_2)$$

implies that

$$H_1 = G_1 \quad \text{and} \quad \dim H_2 = m_0(G_2)$$

or

$$H_2 = G_2 \quad \text{and} \quad \dim H_1 = m_0(G_1) .$$

q.e.d.

Table of $m(G)$ and $m_0(G)$ for simple Lie group G (cf. [1], [2])

G	$\dim G$	$m(G)$	H	$m_0(G)$	U
$SU(n)$, $n\neq 4$	n^2-1	$(n-1)^2$	$S(U(n-1)\times U(1))$	$(n-1)^2$	$S(U(n-1)\times U(1))$
$SU(4)$	15	10	$Sp(2)$	9	$S(U(3)\times U(1))$
$SO(2n+1)$	$2n^2+n$	$2n^2-n$	$SO(2n)$	$2n^2-n$	$SO(2n)$
$Sp(n)$	$2n^2+n$	$2n^2-3n+4$	$Sp(n-1)\times Sp(1)$	$2n^2-3n+4$	$Sp(n-1)\times Sp(1)$
$SO(2n)$, $n>3$	$2n^2-n$	$2n^2-3n+1$	$SO(2n-1)$	$2n^2-5n+4$	$SO(2n-2)\times SO(2)$
G_2	14	8	$SU(3)$	8	$SU(3)$
F_4	52	36	$Spin(9)$	36	$Spin(9)$
E_6	78	52	F_4	46	
E_7	133	79		79	
E_8	248	136		136	

Here H, U are closed connected subgroups of G with $\dim H=m(G)$, $\dim U=m_0(G)$ and rank U=rank G

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