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Introduction

For a pointed finite CW pair (X, A), define as usual the k-dimensional unoriented cobordism group $\mathfrak{N}^k(X, A)$ of (X, A) by

$$\mathfrak{N}^k(X, A) = \varinjlim_n [S^{n-k}(X/A), MO(n)]\,,$$

and denote $\sum_{-\infty<k<\infty} \mathfrak{N}^k(X, A)$ by $\mathfrak{N}^*(X, A)$.

We identify the coeficient ring $\mathfrak{N}^*$ with the unoriented bordism ring $\mathfrak{N}_*$ by the Atiyah-Poincaré duality [2]

$$D: \mathfrak{N}_k \to \mathfrak{N}^{-k}\,.$$

Let P_n be the n-dimensional real projective space and η_n be the canonical line bundle over P_n. Define

$$\mathfrak{N}^*(BO(1)) = \varprojlim_n \mathfrak{N}^*(P_n) \cong \mathfrak{N}_*[[W_1]]\,,$$

where $W_1 = \varprojlim_n W_1(\eta_n)$ is the cobordism first Stiefel-Whitney class [4]. On account of the Kunneth formula, the homomorphism

$$\mu^*_{m,n}: \mathfrak{N}^*(P_{m+n}) \to \mathfrak{N}^*(P_m \times P_n)$$

induced by a continuous map $\mu_{m,n}$ satisfying $\mu^*_{m,n}\eta_{m+n} \cong \pi_1^*\eta_m \otimes \pi_2^*\eta_n$ gives rise to the comultiplication

$$\mu^*: \mathfrak{N}^*(BO(1)) \to \mathfrak{N}^*(BO(1)) \underset{\mathfrak{N}_*}{\otimes} \mathfrak{N}^*(BO(1))\,.$$

Let

$$P = W_1 + z_2 W_1^3 + z_4 W_1^5 + z_5 W_1^6 + z_6 W_1^7 + z_7 W_1^8 + \cdots \quad (z_i \in \mathfrak{N}_i)$$

be a primitive element in $\mathfrak{N}^*(BO(1))$ with respect to this comultiplication. Such elements exist ([3]). Fix once and for all a primitive element P of such kind.

Following Novikov [8, appendix II], we define in section 1 a cobordism stable operation Φ_P which is a multiplicative projection characterised by the formula

$$\Phi_P(W_1) = P.$$

The restriction of the natural transformation

$$\mu \mid \text{Image } \Phi_P : \text{Image } \Phi_P \to H^*(X, A; Z_2)$$

is a natural ring isomorphism in the category of finite CW pairs. And this induces a natural $\mathfrak{N}_*$-algebra isomorphism

$$\mathfrak{N}^*(X, A) \cong \mathfrak{N}_* \hat{\otimes} H^*(X, A; Z_2).$$

Conversely, any such natural isomorphism, commuting with suspensions, is induced by Φ_P for some choice of a primitive element P.

In section 2, we study the relation between the operations S_ω and $\bar{S}_\omega$ defined in [8]. The result is applied in section 3 to prove that the coefficient z_{2k} of a primitive element P is the bordism class $[P_{2k}]$ of the real projective space for each $k \geqq 0$.

And the coefficient z_{4k+1} is shown to be the class $[P(1, 2k)]$ of Dold manifold [5] in section 4.

The coefficients z_i of dimensions i other than $2k$ and $4k+1$ are expressed as very complicated polynomials in the generators of Dold [5] or of Milnor [7].

The present paper is motivated by the following classification theorem stated in the proof of Theorem 8.1 in [3].

Theorem. P. (Boardman [3])

For an arbitrary family of decomposable elements $\{y_{2^i-1}; y_{2^i-1} \in \mathfrak{N}_{2^i-1}, i \geqq 1\}$, *there exists one and the only one primitive element*

$$P = W_1 + z_2 W_1^3 + z_4 W_1^5 + z_5 W_1^6 + z_6 W_1^7 + z_7 W_1^8 + \cdots.$$

in $\mathfrak{N}^*(BO(1))$, *satisfying*

$$z_{2^i-1} = y_{2^i-1} \quad (i \geqq 1).$$

The coefficients z_{k-1} *with* k *not a power of* 2 *are a set of polynomial generators for* $\mathfrak{N}_*$.

Moreover, if $z_{2^i-1} = z'_{2^i-1}$ *for* $1 \leqq i \leqq n$ *for primitive elements*

$$P = W_1 + z_2 W_1^3 + z_4 W_1^5 + z_5 W_1^6 + z_6 W_1^7 + z_7 W_1^8 + \cdots$$

and

$$P' = W_1 + z'_2 W_1^3 + z'_4 W_1^5 + z'_5 W_1^6 + z'_6 W_1^7 + z'_7 W_1^8 + \cdots$$

then $z_{k-1}=z'_{k-1}$ *for* k *not a multiple of* 2^{n+1}.

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1. Operation Φ_P

Let $\mathcal{A}^*(0)=\sum_{-\infty<i<\infty}\mathcal{A}^i(0)$ denote the ring of stable operations in the unoriented cobordism theory. There is an isomorphism of $\mathfrak{N}_*$-modules ([6], [8])

$$\Psi : \mathcal{A}^*(0) \to \mathfrak{N}_* \hat{\otimes} Z_2[[W_1, W_2, \cdots, W_k, \cdots]],$$

where $\mathfrak{N}_*$ is identified with $\mathfrak{N}^*$ by the duality and $\hat{\otimes}$ denotes the complete tensor product.

For a partition $\omega=(i_1, i_2, \cdots, i_r)$, denote W_ω the symmetrized monomial of the W_k and the operation $S_\omega\in\mathcal{A}^*(0)$ is defined by $S_\omega=\Psi^{-1}(W_\omega)$.

For a primitive element

$$P = W_1+z_2W_1^3+z_4W_1^5+z_5W_1^6+z_6W_1^7+z_7W_1^8+\cdots$$

in $\mathfrak{N}^*(BO(1))$ and for a partition $\omega=(i_1, i_2, \cdots, i_r)$, we denote the product $z_{i_1}\cdot z_{i_2}\cdots z_{i_r}$ as $z_\omega^{(P)}$.

Following the line of Novikov [8; appendix II], we define an operation $\Phi_P\in\mathcal{A}^0(0)$ by

$$\Phi_P = \sum_\omega z_\omega^{(P)} S_\omega,$$

where the summation runs through all the partitions.

Lemma 1.1.

(1) $\Phi_P(x\cdot y) = \Phi_P(x)\cdot\Phi_P(y)$.

(2) $\Phi_P(z_0) = z_0$ *for* $z_0\in\mathfrak{N}_0$ *and*

$\Phi_P(y) = 0$ *for* $y\in\mathfrak{N}_i$ $(i>0)$.

(3) $(\Phi_P)^2 = \Phi_P$.

Proof.

(1). By the definition of Φ_P and from the Cartan formula for S_ω ([6], [8]), part (1) is easily derived.

(2). It is obvious by definition that $\Phi_P(z_0)=z_0$ for $z_0\in\mathfrak{N}_0$.

It is known that $S_\omega(W_1)=W_1^{k+1}$ if $\omega=(k)$ for some $k\geqq 0$ and that $S_\omega(W_1)=0$ otherwise ([6], [8]). Thus $\Phi_P(W_1)=P$. By the naturality of Φ_P, $(\Phi_P)^2(W_1)=\Phi_P(P)$ is also a primitive element with the leading term W_1. So it follows from Theorem P in the introduction together with the fact that $\mathfrak{N}_1\cong\mathfrak{N}_3\cong\{0\}$ that

$$(\Phi_P)^2(W_1)-\Phi_P(W_1)=\sum_{j\geqq 1} y_{8j-1}W_1^{8j}$$

for some decomposable elements $y_{8j-1}\in\mathfrak{N}_{8j-1}$.

On the other hand,

$$\begin{aligned}(\Phi_P)^2(W_1)-\Phi_P(W_1) &= \Phi_P(W_1+\sum_{k\geqq 3} z_{k-1}W_1^k)-\Phi_P(W_1)\\ &= \sum_{k\geqq 3}\Phi_P(z_{k-1})(W_1+\sum_{l\geqq 3} z_{l-1}W_1^l)^k .\end{aligned}$$

Comparing both formulas, we see that $\Phi_P(z_{k-1})=0$ for $k\leqq 7$. So $\Phi_P(z_{8-1})=0$ since z_7 is decomposable. So $y_7=0$ and it follows Theorem P that $y_{16j+7}=0$ for all $j\geqq 0$. Repeting this procedure, we can inductively deduce that $\Phi_P(z_{k-1})=0$ for all $k\geqq 3$. At the same time we have proved that $(\Phi_P)^2(W_1)=\Phi_P(W_1)$.

Now $(\Phi_P)^2$ is also a multiplicative operation. As in the weakly complex case ([8]), a multiplicative operation of the unoriented cobordism theory is easily seen to be uniquely determined by its value on W_1. Therefore $(\Phi_P)^2=\Phi_P$. This completes the proof of Lemma 1.1.

Notation. For a partition $\omega=(i_1, i_2, \cdots, i_r)$, let $||\omega||=i_1+i_2+\cdots+i_r$ be its degree and $|\omega|=r$ its length. And we call ω *non-dyadic* if none of the component i_k of ω is of the form 2^m-1.

Theorem 1.2. *On the category of finite pointed CW pairs and continuous maps, there is a natural direct sum splitting as a graded Z_2-vector space*

$$\mathfrak{N}^*(X, A)=\bigoplus_{\omega;\ \text{non-dyadic}} z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A)),$$

where (1) *the restriction*

$$\mu\,|\,\text{Image }\Phi_P : \Phi_P(\mathfrak{N}^*(X, A))\to H^*(X, A\,;\,Z_2)$$

is a natural Z_2-algebra isomorphism, and (2) *the scalar multiplication*

$$z_\omega^{(P)}\cup : \Phi_P(\mathfrak{N}^*(X, A))\to z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A))$$

is a graded Z_2-module isomorphism of degree $-||\omega||$ if ω is non-dyadic.

Therefore we obtain a natural equivalence of graded $\mathfrak{N}_*$-algebras

$$\mathfrak{N}^*(X, A)\xrightarrow[\cong]{}\mathfrak{N}^*\hat{\otimes} H^*(X, A\,;\,Z_2)$$

which commutes with suspension. (Suspension S and a bordism element x act on the right by $S(y\hat{\otimes}a)=y\hat{\otimes}S(a)$ and $x(y\hat{\otimes}a)=x\cdot y\hat{\otimes}a$, respectively.)

Moreover, the converse holds; such an equivalence is induced by $\bigoplus_{\omega;\ \text{non-dyadic}} z_\omega^{(P)}\Phi_P$ for some choice of a primitive element P.

For the proof of the above theorem, we need the following operations which are just the unoriented analogue of those defined in [8].

Lemma 1.3.

For an indecomposable element $y_i \in \mathfrak{N}_i$, *define an operation* $\Delta_{y_i} = \sum_{k\geqq 1} y_i^{k-1} S_{(i)^k}$. $((i)^k = (i, i, \cdots, i)$; *the k copies of i*)

Then

$$\Delta_{y_i}(a \cdot b) = \Delta_{y_i}(a) \cdot b + a \cdot \Delta_{y_i}(b) + y_i \cdot \Delta_{y_i}(a) \cdot \Delta_{y_i}(b)$$

and, in particular,

$$\Delta_{y_i}(y_i \cdot a) = a .$$

The proof of the lemma is straightforward from the definition of Δ_{y_i} and the fact that $S_{(i)}(y_i) = 1 \in Z_2$.

Proof of Theorem 1.2.

First we prove property (1). By (2) of Lemma 1.1, property (1) holds for $(X, A) = (S^0, P)$. Since Φ_P commutes with suspensions, (1) also holds for $(X, A) = (S^n, P)$ for $n \geqq 1$. Since Φ_P is a projection, $\Phi_P(\mathfrak{N}^*(,))$ is also a cohomology theory. So the general cases are proved by induction on the number of cells in $X - A$, using the five lemma.

Next we prove property (2). The multiplication

$$z_\omega^{(P)} \cup : \Phi_P(\mathfrak{N}^*(X, A)) \to z_\omega^{(P)} \Phi_P(\mathfrak{N}^*(X, A))$$

is obviously a graded Z_2-module epimorphism of degree $-||\omega||$.

Suppose $z_\omega^{(P)} \cdot a = 0$ for $a \in \Phi_P(\mathfrak{N}^*(X, A))$ and for a non-dyadic ω. Order the components of $\omega = (i_1, i_2, \cdots, i_r)$ as $i_1 \leqq i_2 \leqq \cdots \leqq i_r$ and define the operation $\Delta_{z\omega}^{(P)}$ by

$$\Delta_{z\omega}^{(P)} = \Delta_{z_{i_1}} \circ \Delta_{z_{i_2}} \circ \cdots \circ \Delta_{z_{i_r}} .$$

Then $a = \Delta_{z\omega}^{(P)}(z_\omega^{(P)} \cdot a) = \Delta_{z\omega}^{(P)}(0) = 0$ by Lemma 1.3. This proves property (2).

Totally order the set of all non-dyadic partitions by $\omega' < \omega$ if (*a*) $||\omega'|| < ||\omega||$ or (*b*) $||\omega'|| = ||\omega||$ and $i_r = j_s, \cdots, i_{r-m+1} = j_{s-m+1}, i_{r-m} > j_{s-m}$ for some $m \geqq 0$, where $\omega' = (i_1, i_2, \cdots, i_r)$ and $\omega = (j_1, j_2, \cdots, j_s)$ with $i_1 \leqq i_2 \leqq \cdots \leqq i_r$ and $j_1 \leqq j_2 \leqq \cdots \leqq j_s$.

We show that

$$\Phi_P \Delta_{z\omega'}^{(P)}(z_\omega^{(P)} \Phi_P(y)) = 0$$

for any homogeneous element y if $\omega' < \omega$. In case $||\omega'|| < ||\omega||$, Lemma 1.3 implies that

$$\Phi_P \Delta_{z\omega'}^{(P)}(z_\omega^{(P)} \Phi_P(y)) = \Phi_P(\sum_i u_i \cdot y_i)$$

for some elements $u_i \in \mathfrak{N}_*$ and $y_i \in \Phi_P(\mathfrak{N}^*(X, A))$ with $\dim u_i \geqq ||\omega|| - ||\omega'|| > 0$. Thus, by Lemma 1.1 (1), (2),

$$\Phi_P(\sum_i u_i y_i) = \sum_i \Phi_P(u_i)\Phi_P(y_i) = 0\,.$$

In case $||\omega'||=||\omega||$ and $i_r=j_s, \cdots, i_{r-m}>j_{s-m}$,

$$\begin{aligned}&\Phi_P\Delta_{z\omega'}^{(P)}(z_\omega^{(P)}\Phi_P(y))\\ &= \Phi_P\Delta_{z(i_1,\cdots,i_{r-m-1})}^{(P)}(z_{j_1}\cdots z_{j_{s-m}}\Delta_{zi_{r-m}}\Phi_P(y)) = 0\,.\end{aligned}$$

The last equality follows from the preceding case.

Let $\sum_{\omega'<\omega} z_{\omega'}^{(P)}\Phi_P(\mathfrak{N}^*(X, A))$ be the graded vector space spanned by all $z_{\omega'}^{(P)}\Phi_P(\mathfrak{N}^*(X, A))$ with $\omega'<\omega$.

It follows from the above fact that

$$\sum_{\omega'<\omega} z_{\omega'}^{(P)}\Phi_P(\mathfrak{N}^*X, A))\cap z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A)) = 0$$

for each ω, so that there is a direct sum splitting

$$\sum_{\omega;\,\text{non-dyadic}} z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A)) = \bigoplus_{\omega;\,\text{non-dyadic}} z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A))\,.$$

Since it can be proved similarly as above that Image $(\Phi_P\circ\Delta_{z\omega}^{(P)})$ $=$Image Φ_P for each non-dyadic ω, we have proved that there is a natural linear endomorphism of degree zero

$$\sum_{\omega;\,\text{non-dyadic}} z_\omega^{(P)}\Phi_P\Delta_{z\omega}^{(P)}:\mathfrak{N}^*(X, A) \to \bigoplus_{\omega;\,\text{non-dyadic}} z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A))\subset\mathfrak{N}^*(X, A)\,.$$

It is clearly an automorphism for $(X, A)=(S^0, P)$ and therefore an automorphism for every finite CW pair by the effect of suspensions and of the five lemma. Thus

$$\bigoplus_{\omega;\,\text{non-dyadic}} z_\omega^{(P)}\Phi_P(\mathfrak{N}^*(X, A)) = \mathfrak{N}^*(X, A)\,.$$

Since $z_\omega^{(P)}\Phi_P(y)\cdot z_{\omega'}^{(P)}\Phi_P(y') = z_{\omega\omega'}^{(P)}\Phi_P(y\cdot y')$, we have obtained a natural equivalence of graded $\mathfrak{N}_*$-algebras

$$\Theta_P:\mathfrak{N}^*(X, A) \cong \mathfrak{N}^*\mathbin{\hat{\otimes}} H^*(X, A\,;Z_2)$$

which commutes with suspension.

Conversely, each such equivalence Θ induces a natural monomorphism of a graded Z_2-algebra

$$\lambda = \Theta^{-1}|H^*(X, A\,;Z_2): H^*(X, A\,;Z_2) \to \mathfrak{N}^*(X, A)\,.$$

Then the composition $\lambda\circ\mu$ is a stable miltiplicative operation in $\mathcal{A}^*(0)$ and $\lambda\circ\mu(W_1)=\lambda(w_1)=P$ is a primitive element in $\mathfrak{N}^*(BO(1))$. And the element P has the leading term W_1 since

$$\Theta:\mathfrak{N}_*[[W_1]] \to \mathfrak{N}_*\mathbin{\hat{\otimes}} Z_2[[w_1]]$$

is an $\mathfrak{N}_*$-algebra isomorphism. Therefore

$$\Theta = \bigoplus_{\omega;\,\text{non-dyadic}} \{1 \hat{\otimes} (\mu \mid \text{Image } \Phi_P)\}:$$
$$\mathfrak{N}^*(X, A) = \bigoplus_{\omega;\,\text{non-dyadic}} z_\omega^{(P)} \Phi_P(\mathfrak{N}^*(X, A)) \to \hat{\bigoplus_{\omega;\,\text{non-dyadic}}} \{z_\omega^{(P)} \hat{\otimes} H^*(X, A; Z_2)\}$$

This completes the proof of Theorem 1.2.

2. Operations $\bar{S}_\omega$

Let $\bar{W}_\omega$ denote the symmetrized monomial of the cobordism normal characteristic classes $\bar{W}_k$. ($\bar{W}_\omega(\xi)=W_\omega(-\xi)$ for every stable vector bundle ξ.) The operation $\bar{S}_\omega$ is defined in [8] by $\bar{S}_\omega=\Psi^{-1}(\bar{W}_\omega)$, where Ψ is the additive isomorphism mentioned in section 1.

Notation 2.1. (Landweber [6])

For a partition $\omega=(i_1, \cdots, i_r)$ let $r_\omega(i)$ denote the occurrences of the integer i in ω. And define

$$\binom{n}{\omega}=\begin{cases} 0 \quad \text{if} \quad n<|\omega|=r \\ \dfrac{n!}{r_\omega(1)!\, r_\omega(2)! \cdots (n-|\omega|)!} \quad \text{if} \quad n \geqq |\omega|. \end{cases}$$

The modulo 2 reduction of $\binom{n}{\omega}$ is denoted by $\binom{n}{\omega}_2$.

Similarly to the weakly complex case [8], we can easily determine the value $\bar{S}_\omega[P_k]$.

Lemma 2.2.

(1) $\bar{S}_\omega[P_k] = \binom{k+1}{\omega}_2 [P_{k-||\omega||}]$.

(2) $S_\omega[P_k] = \binom{2^p-k-1}{\omega}_2 [P_{k-||\omega||}]$ for p such that $2^p>k+1$.

Proof. By the geometric interpretation of the action of $\mathcal{A}^*(0)$ on $\mathfrak{N}_*$ given in [6], [8], $\bar{S}_\omega[P_k]=\varepsilon W_\omega(\tau_{P_k})=\varepsilon\binom{k+1}{\omega}_2 W_1^{||\omega||}=\binom{k+1}{\omega}_2[P_{k-||\omega||}]$. Part (2) is proved similarly. Now we give some relations between S_ω and $\bar{S}_\omega$.

Lemma 2.3.

(1) *If the occurrence* $r_\omega(i) \leqq 1$ *in* ω *for all* i, then $S_\omega=\bar{S}_\omega$.

(2) $S_{(i)^k}=\sum_{||\omega||=k}\bar{S}_{i*\omega}$ *and dually*

$\bar{S}_{(i)^k}=\sum_{||\omega||=k}S_{i*\omega}$,

where $i*\omega$ *is meant a partition* $(i\cdot j_1, i\cdot j_2, \cdots, i\cdot j_r)$ *for* $\omega=(j_1, j_2, \cdots, j_r)$.

After Landweber [6] we denote the partition $(i)^k$ by $k\Delta_i$ and the totality of linear combinations of the S_ω by $A^*(0)$. $A^*(0)$ is proved a Hopf algebra over Z_2 ([6], [8]).

Theorem 2.4. (Landweber [6])
The set $\{S_{2^k\Delta_1}, S_{2^k\Delta_2}; k\geqq 0\}$ *provides a minimal set of generators of* $A^*(0)$.

Corollary 2.5.
The set $\{\bar{S}_{2^k\Delta_1}, \bar{S}_{2^k\Delta_2}; k\geqq 0\}$ *provides a minimal set of generators of* $A^*(0)$.

Proof of Lemma 2.3.
By the Whitney product formula, it follows that $\sum_{\omega=\omega_1\omega_2}W_{\omega_1}\cdot\bar{W}_{\omega_2}=0$ if $\omega\neq(0)$. Therefore $W_{(i)}=\bar{W}_{(i)}$ for all $i\geqq 1$ and we see by induction on the lengths of partitions that $W_\omega=\bar{W}_\omega$ if $r_\omega(i)\leqq 1$ for all i. Part (1) follows from this and from the definition of S_ω and $\bar{S}_\omega$.

Put

$$\sum_{0\leqq i\leqq s}\bar{W}_i x^i=\prod_{1\leqq j\leqq s}(1+u_j x)$$

for a sufficiently large s.

Then part (2) of the lemma is proved by induction on k as follows;

$$\begin{aligned}W_{(i)^k}&=\sum_{0\leqq l\leqq k-1}W_{(i)^l}\bar{W}_{(i)^{k-l}}=\sum_{0\leqq l\leqq k-1}\Big(\sum_{||\omega||=l}\bar{W}_{i*\omega}\Big)\cdot\bar{W}_{(i)^{k-l}}\\&=\sum_{0\leqq l\leqq k-1}\Big\{\sum_{j_1+\cdots+j_m=l}\big(\sum(u_1^i)^{j_1}\cdots(u_m^i)^{j_m}\big)\Big\}\big\{\sum(u_1^i)\cdots(u_{k-l}^i)\big\}\\&=\sum_{i_1+\cdots+i_n=k}\big(\sum(u_1^i)^{i_1}\cdots(u_n^i)^{i_n}\big)\Big(\sum_{0\leqq l\leqq k-1}\binom{n}{k-l}_2\Big)\\&=\sum_{||\omega||=k}\bar{W}_{i*\omega}\binom{|\omega|}{0}_2=\sum_{||\omega||=k}\bar{W}_{i*\omega}.\end{aligned}$$

Part (2) follows from this.

Proof of Corollary 2.5.
It follows from Lemma 2.3 and Theorem 2.4 that

$\bar{S}_{\Delta_1}=S_{\Delta_1}$,
$\bar{S}_{2^k\Delta_1}=S_{2^k\Delta_1}+S_{2^{k-1}\Delta_2}+$decomposables in $A^*(0)$, and
$\bar{S}_{2^k\Delta_2}=S_{2^k\Delta_2}+$decomposables in $A^*(0)$.

Thus the corollary follows from Theorem 2.4.

3. Even dimensional coefficients

Following suit of Novikov [8, appendix I], we obtain the following. We omit the proof.

Lemma 3.1.

For a partition ω and for a positive integer $k=2^p(2q+1)$ $(p\geqq 0,\ q\geqq 1)$, the following formula holds if $||\omega||\geqq 2^p$;

$$\sum_{\omega=\omega_1\omega_2} S_{\omega_1}(z_{k-1-||\omega_2||})\binom{k-||\omega_2||}{\omega_2}_2 = 0\,,$$

where the z_i denote the coefficients of a fixed primitive element P as in the introduction.

Now we prove the following theorem.

Theorem 3.2.

The coefficient z_{2k} of a primitive element

$$P = W_1+z_2W_1^3+z_4W_1^5+z_5W_1^6+z_6W_1^7+z_7W_1^8+\cdots$$

in $\mathfrak{N}^*(BO(1))$ is equal to the bordism class $[P_{2k}]$ for all $k\geqq 1$.

Proof. For $k-1$, the theorem is clear since z_2 is indecomposable from Theorem P in the introduction.

Assume that the theorem holds up to dimension $2(k-1)\geqq 2$.

In order to show that $S_\omega(z_{2k}+[P_{2k}])=\bar{W}_\omega(z_{2k}+[P_{2k}])=0$ for all ω with $||\omega||=2k$, it saffices from Theorem 2.4 to prove

$$S_{2^s\Delta_i}(z_{2k}+[P_{2k}]) = 0 \quad (i = 1, 2)\,.$$

To prove this, we see from Lemma 3.1 and the induction assumption that it is sufficient to show

$$\sum_{m+n=2s} S_{m\Delta_i}[P_{2k-ni}]\binom{2k+1-\mathrm{ni}}{n}_2 = 0 \quad (i = 1, 2)\,.$$

This is obvious in case $2^si>2k$ or $s=0$ since

$$S_{m\Delta_i}[P_{2k-ni}] = \left\{\sum_{||\omega||=m}\binom{2k+1-ni}{\omega}_2\right\}[P_{2k-2^si}]$$

by Lemmas 2.2 (1) and 2.3 (2).

For the remaining cases, it suffices to prove the following lemma.

Lemma 3.3.

(1) $$\sum_{m+n=s}\left(\sum_{||\omega||=m}\binom{k-n}{\omega}\right)\binom{k-n}{n} \equiv 0 \quad (mod\ 2) \quad for \quad k\geqq s\geqq 2\,.$$

(2) $\sum_{m+n=s}\left(\sum_{\|\omega\|=m}\binom{k-2n}{\omega}\right)\binom{k-2n}{n} \equiv 0 \quad (mod\ 2) \quad for \quad k \geqq 2s \geqq 2\,.$

Proof.

(1) Put

$$A(k,\, s) = \sum_{m+n=s}\left(\sum_{\|\omega\|=m}\binom{k-n}{\omega}\right)\binom{k-n}{n} \quad (k \geqq 0,\ s \geqq 0)\,, \quad \text{and}$$

$$B(k,\, s) = \sum_{m+n=s}\left(\sum_{\|\omega\|=m}\binom{k-2n}{\omega}\right)\binom{k-2n}{n} \quad (k \geqq 0,\ s \geqq 0)\,.$$

Then it holds in general that

$$\binom{k-n}{n} = \binom{k-n-1}{n} + \binom{k-n-1}{n-1} \quad \text{and}$$

$$\sum_{\|\omega\|=m}\binom{k-n}{\omega} = \sum_{0 \leqq \|\omega\| \leqq m}\binom{k-n-1}{\omega}\,.$$

So we obtain that

$$(*) \quad A(k,\, s) = \sum_{0 \leqq s' \leqq s} A(k-1,\, s') + \sum_{0 \leqq s'' \leqq s-1} A(k-2,\, s'') \quad \text{and}$$

$$(**) \quad B(k,\, s) = \sum_{0 \leqq s' \leqq s} B(k-1,\, s') + \sum_{0 \leqq s'' \leqq s-1} B(k-3,\, s'')\,.$$

Part (1) clearly holds when $k=s=2$.

Assume, by induction, that (1) holds for such $(k,\, s)$ that $k_0 > k \geqq 2$ and $k \geqq s \geqq 2$.

Thus, for $(k_0,\, s_0)$ with $k_0 > s_0 \geqq 2$,

$$A(k_0,\, s_0) \equiv \sum_{s'=0,1} A(k_0-1,\, s') + \sum_{s''=0,1} A(k_0-2,\, s'') \equiv 0 \quad (\text{mod } 2)$$

by the induction hypothesis and by the fact that $A(k,\, s) \equiv 1$ for $k \geqq s$ and $s=0,\, 1$.

And for $(k_0,\, k_0)$, the iterated application of $(*)$ shows that

$$\begin{aligned} A(k_0,\, k_0) &\equiv A(k_0-1,\, k_0) + A(k_0-2,\, k_0-1) \\ &\equiv A(1,\, k_0) + \sum_{0 \leqq s'' \leqq k_0-1} A(0,\, s'') \equiv 0 \quad (\text{mod } 2)\,. \end{aligned}$$

Part (2) of the lemma is proved similarly, using the formula $(**)$ repeatedly. This completes the proof of Lemma 3.3 and Theorem 3.2.

REMARK 3.4. Theorem 3.2 has been proved independently by F. Uchida [9] by a geometric method.

4. The coefficients of dimensions $4k+1$

A. Dold has defined in [5] manifolds $P(m, n)$ which are the identification spaces of $S^m \times CP_n$ with $(x, z)=(-x, \bar{z})$. He proved that, for $2^p(2q+1)-1$ ($p\geqq 1$, $q\geqq 1$), the bordism class $[P(2^p-1, 2^p q)]$ provides a polynomial generator of $\mathfrak{N}_*$ in the corresponding dimension.

Theorem 4.1.
The coefficient z_{4k+1} of a primitive element

$$P = W_1+z_2W_1^3+z_4W_1^5+z_5W_1^6+z_6W_1^7+z_7W_1^8+\cdots$$

in $\mathfrak{N}^*(BO(1))$ is equal to the bordism class $[P(1, 2k)]$ for all $k\geqq 1$.

For the proof of this theorem, we need the following notations.

Notation 4.2.

(1) Let $c_p(m)$ denote the coefficient of 2^p in the dyadic expansion of the integer m;

$$m = c_0(m)+c_1(m)\cdot 2+c_2(m)\cdot 2^2+\cdots,\ c_i(m) = 0, 1 .$$

(2) For a partition ω, we denote by $\omega(c_p)$ the partition determined by $r_{\omega(c_p)}(i)=c_p(r_\omega(i))$ for all $i\geqq 1$. Thus $\omega=\prod_{0\leqq p}(\omega(c_p))^{2^p}$. For brevity, $\prod_{2\leqq p}(\omega(c_p))^{2^{p-2}}$ and $\omega(c_1)^2\cdot\omega(c_0)$ are denoted as $\bar{\omega}$ and $\bar{\bar{\omega}}$, respectively; $\omega=(\bar{\omega})^4\bar{\bar{\omega}}$.

Lemma 4.3.

$$\binom{n}{\omega}_2=\prod_{0\leqq p}\binom{c_p(n)}{\omega(c_p)}_2. \quad \text{Thus } \binom{n}{\omega}_2=\binom{\frac{n-c_1(n)\cdot 2-c_0(n)}{4}}{\bar{\omega}}_2\binom{c_1(n)\cdot 2+c_0(n)}{\bar{\bar{\omega}}}_2 .$$

Proof. By definition,

$$\binom{n}{\omega}_2=\binom{n}{r_\omega(1)}_2\binom{n-r_\omega(1)}{r_\omega(2)}_2\cdots\binom{n-\sum_{1\leqq i\leqq k-1}r_\omega(i)}{r_\omega(k)}_2\cdots .$$

Then, by Lucus' theorem [1],

$$\prod_{1\leqq k}\binom{n-\sum_{1\leqq i\leqq k-1}r_\omega(i)}{r_\omega(k)}_2=\prod_{1\leqq k}\left(\prod_{0\leqq p}\binom{c_p(n-\sum_{1\leqq i\leqq k-1}r_\omega(i))}{c_p(r_\omega(k))}_2\right)$$
$$=\prod_{0\leqq p}\left(\prod_{1\leqq k}\binom{c_p(\mathrm{n})-\sum_{1\leqq i\leqq k-1}c_p(r_\omega(i))}{c_p(r_\omega(k))}_2\right)=\prod_{0\leqq p}\binom{c_p(n)}{\omega(c_p)}_2 .$$

This completes the proof.

Now we calculate all the normal Stiefel-Whitney numbers of $P(1, 2\mathrm{k})$. It is easily seen that the cobordism Stiefel-Whitney numbers of manifolds agree with the cohomological ones ([6], [8]). So, by abuse of a notation, we denote both Stiefel-Whitney numbers by W_ω (and the normal ones by $\bar{W}_\omega$).

Lemma 4.4.

$$\bar{W}_\omega[P(1, 2k)] = \begin{cases} 0 & \textit{if} \quad |\bar{\bar{\omega}}| \geqq 3 \quad \textit{and} \quad \bar{\bar{\omega}} \neq 3\Delta_1 \quad \textit{or} \quad \bar{\bar{\omega}} = (1)\,, \\ \binom{2^p-1-k}{\bar{\omega}}_2 & \text{if} \quad \bar{\bar{\omega}} = 3\Delta_1 \quad \text{or} \\ & \qquad 2 \geqq |\bar{\bar{\omega}}| \geqq 1 \quad \text{and} \quad \bar{\bar{\omega}} \neq (1)\,, \end{cases}$$

where p is any integer with $2^p > k+1$.

Proof.

According to Dold [5].

$$H^*(P(1, 2k)\,;\, Z_2) \simeq H^*(P_1 \times CP_{2k}\,;\, Z_2)$$

as a ring. Let c and d denote the 1- and 2-dimensional generators of $H^*(P(1, 2k)\,;\, Z_2)$. The total Whitney class is given in [5] by

$$w_* P(1, 2k) = (1+c)(1+c+d)^{2k+1}\,,$$

and thus

$$\bar{w}_* P(1, 2k) = (1+c)(1+t)^{4(2^p-k-1)}(1+t_1)(1+t_2)\,,$$

where p is any integer with $2^p > k+1$ and $t^2 = t_1 \cdot t_2 = d$ and $t_1 + t_2 = c$.

By formula (26) in [5],

$$t_1^{2i} + t_2^{2i} = 0 \quad \text{and} \quad t_1^{2i+1} + t_2^{2i+1} = cd^i\,.$$

The lemma follows from these facts and the preceding lemma.

Proof of Theorem 4.1.

Theorem P in the introduction asserts that $z_{4+1} = [P(1, 2)]$. Assume, by induction, that $z_{4k'+1} = [P(1, 2k')]$ for $k' \leqq k-1$.

By Lemma 3.1 and Theorem 3.2, together with Lemma 2.2 (2), 4.3 and 4.4,

$$\begin{aligned} S_\omega(z_{4k+1}) = & \sum_{\substack{\omega = \omega_1\omega_2 \\ ||\omega_2|| = 4m \neq 0}} S_{\omega_1}(z_{4(k-m)+1}) \binom{k-m}{\bar{\omega}}_2 \binom{2}{\bar{\bar{\omega}}_2}_2 \\ & + \sum_{\substack{\omega = \omega_1\omega_2 \\ ||\omega_2|| = 2n+1}} \binom{2^p-1-4||\bar{\omega}_1|| - ||\bar{\bar{\omega}}_1||}{(\bar{\omega}_1)^4 \bar{\bar{\omega}}_1}_2 \binom{4||\bar{\omega}_1|| + ||\bar{\bar{\omega}}_1|| + 1}{(\bar{\omega}_2)^4 \bar{\bar{\omega}}_2}_2 \end{aligned}$$

for ω such that $||\omega|| = 4k+1$. (The terms with $||\omega_2|| \underset{(4)}{\equiv} 2$ vanish by Lemma 4.3.)

Therefore, by the induction hypothesis and by Lemma 4.3, together with the fact that $|\bar{\bar{\omega}}_1| + |\bar{\bar{\omega}}_2| = |\bar{\bar{\omega}}| + 4l$ $(l \geqq 0)$, it can be shown that

$$S_\omega(z_{4k+1}) = \sum 0 + \sum 0 = 0 \quad \text{if} \quad |\bar{\bar{\omega}}| \geqq 5\,.$$

In case $\bar{\bar{\omega}}=(2i, 2i, 4j, 4(k-||\varpi||-i-j)+1)$,

$$S_\omega(z_{4k+1}) = \sum_{\bar\varpi=\bar\varpi_1\bar\varpi_2} \left\{ \binom{2^{p'}-1-(k-||\varpi_2||-i)}{\varpi_1}_2 \binom{k-||\varpi_2||-i}{\varpi_2}_2 + \binom{2^{p'}-1-(||\varpi||+i+j-||\varpi_2||)}{\varpi_1}_2 \binom{||\varpi||+i+j-||\varpi_2||}{\varpi_2}_2 \right\}$$

by the induction hypothesis and by Lemma 4.3.

Suppose $\binom{2^{p'}-1-(k-||\varpi_2||-i)}{\varpi_1}_2 \binom{k-||\varpi_2||-i}{\varpi_2}_2 = 1$ for some separation $\varpi_1\varpi_2$ of ϖ.

Since $c_p(2^{p'}-1-(k-||\varpi_2||-i)) \neq c_p(k-||\varpi_2||-i)$ for each p, there is at most one $i \geqq 1$ such that $c_p(r_{\bar\varpi}(i)) \neq 0$. Let r be the number of such odd integers $2i+1 \geqq 1$ that satisfy $r_{\bar\varpi}(2i+1)>0$. Then, by Lemma 4.3, the numbers of such separations $\varpi_1\varpi_2=\varpi$ and $\varpi_1{}'\varpi_2{}'=\varpi$ that satisfy

$$\binom{2^{p'}-1-(k-||\varpi_2||-i)}{\varpi_1}_2 \binom{k-||\varpi_2||-i}{\varpi_2}_2 = 1 \quad \text{and}$$

$$\binom{2^{p'}-1-(||\varpi||+j+k-||\varpi_2{}'||)}{\varpi_1{}'}_2 \binom{||\varpi||+j+k-||\varpi_2{}'||}{\varpi_2{}'}_2 = 1\,,$$

respectively, are both 2^r.

The situation is the same if we suppose

$$\binom{2^{p'}-1-(||\varpi||+j+k-||\varpi_2||)}{\varpi_1}_2 \binom{||\varpi||+j+k-||\varpi_2||}{\varpi_2}_2 = 1$$

for some separation $\varpi_1 \cdot \varpi_2 = \varpi$.

Therefore $S_\omega(z_{4k+1})=0+0=0$ or $=1+1=0$ if $\bar{\bar{\omega}}=(2j, 2j, 4k, 4(s-||\varpi||-j-k)+1)$.

We can prove analogously in other cases when $\bar{\bar{\omega}}=(1)$ or $|\bar{\bar{\omega}}| \geqq 3$ and $\bar{\bar{\omega}} \neq 3\Delta_1$ that $S_\omega(z_{4k+1})=0$.

When $|\bar{\bar{\omega}}|=2$, from dimensional reasons, $\bar{\bar{\omega}}=(2j, 4(s-||\varpi||)-2j+1)$ for some $j \geqq 1$. In this case

$$\begin{aligned} S_\omega(z_{4k+1}) &= \sum_{\substack{\bar\varpi=\bar\varpi_1\bar\varpi_2 \\ \bar\varpi_2 \neq (0)}} \binom{2^p-1-(k-||\varpi_2||)}{\varpi_1}_2 \binom{k-||\varpi_2||}{\varpi_2}_2 \\ &\quad + \sum_{\bar\varpi=\bar\varpi_1\bar\varpi_2} \binom{2^p-1-(2||\varpi||-2||\varpi_2||+j)}{\varpi_1}_2 \binom{2||\varpi||-2||\varpi_2||+j}{\varpi_2}_2 \\ &= \sum_{\bar\varpi=\bar\varpi\cdot(0)} \binom{2^p-1-(k-||\varpi_2||)}{\varpi_1}_2 \binom{k-||\varpi_2||}{\varpi_2}_2 \\ &= \binom{2^p-1-k}{\varpi}_2 \end{aligned}$$

as required.

When $\bar{\bar{\omega}}=3\Delta_1$ or $\bar{\bar{\omega}}=\Delta_{4n+1}$ $(n\geqq 1)$, analogous arguments show that $S_\omega(z_{4k+1})$ $=\binom{2^p-1-k}{\bar{\omega}}$.

Comparing these facts with Lemma 4.4, we deduce that $S_\omega(z_{4k+1})$ $=\bar{W}_\omega(z_{4k+1})=\bar{W}_\omega(P(1, 2k))$ for all ω with $||\omega||=4k+1$. This completes the proof of Theorem 4.1.

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