## VECTOR BUNDLE VALUED HARMONIC FORMS AND IMMERSIONS OF RIEMANNIAN MANIFOLDS

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## VECTOR BUNDLE VALUED HARMONIC FORMS AND IMMERSIONS OF RIEMANNIAN MANIFOLDS

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The purpose of this paper is to discuss an application of the theory of vector bundle valued harmonic forms on a Riemannian manifold to the study of immersions.

Let M be a Riemannian manifold and E a Riemannian vector bundle over M. Then we can define in a natural way the Laplacian  $\Box$  operating on E-valued differential forms and we can express the scalar product  $\langle \Box \theta, \theta \rangle$ , where  $\theta$  is an E-valued p-form, in terms of curvature and covariant differentials. Moreover, if M is compact, we obtain, by integrating over M, a formula analogous to Bochner's for ordinary (i.e. real valued) differential forms.

Let f be an immersion of M into a Riemannian manifold M'. We may regard the second fundamental form  $\alpha$  of (M, f) as a Hom (T(M), N(M))-valued 1-form. Assuming that M' is of constant sectional curvature, we shall prove that the second fundamental form  $\alpha$  is harmonic, i.e.  $\Box \alpha = 0$ , if the mean curvature normal of (M, f) is parallel. In particular, if the immersion f is a minimal immersion, then  $\alpha$  is harmonic. Conversely, if M is compact and if  $\alpha$  is harmonic, then the mean curvature normal is parallel. We obtain from this result together with the formula of Bochner type the results of Simons [5], Chern [1], Nomizu-Smyth [4] and Erbacher [2] proved by them in different ways. In a future paper we shall discuss the case where M is a Kähler manifold.

1. Let M be an *n*-dimensional Riemannian manifold and E a vector bundle over M with a metric along the fibers and a covariant differentiation  $D_X$  satisfying

$$X\langle arphi,\psi
angle=\langle D_X,arphi
angle+\langle arphi,D_X\psi
angle$$

for any vector field X and any sections  $\varphi$  and  $\psi$  of E. A vector bundle E with these properties will be called a *Riemannian vector bundle*.

We shall denote  $C^{p}(E)$  the real vector space of all *E*-valued differential *p*-forms on *M*. We define an operator

$$\partial : C^{p}(E) \rightarrow C^{p+1}(E), (p = 0, 1, \cdots)$$

by the formula

$$(\partial\theta)(X_{1},\dots,X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} D_{X_{i}}(\theta(X_{1},\dots,\hat{X}_{i},\dots,X_{p+1})) + \sum_{i< j} (-1)^{i+j} \theta([X_{i},X_{j}],X_{1},\dots,\hat{X}_{i},\dots,\hat{X}_{j},\dots,X_{p+1}),$$

where  $X_i$ 's denote vector fields on M. The covariant derivative  $D_X \theta$  of  $\theta \in C^p(E)$  is an E-valued p-form such that

$$(D_X\theta)(X_1,\cdots,X_p)=D_X(\theta(X_1,\cdots,X_p))-\sum_{i=1}^p\theta(X_1,\cdots,\nabla_XX_i,\cdots,X_p),$$

where  $\nabla_X X_i$  denotes the covariant derivative of the vector field  $X_i$  in the Riemannian manifold M.

For an *E*-valued 1-form  $\theta$  we have the formula

$$(\partial \theta)(X, Y) = (D_X \theta)(Y) - (D_Y \theta)(X)$$

The covariant differential  $D\theta$  of  $\theta$  is an *E*-valued (*p*+1)-tensor defined by

$$(D\theta)(X_1,\cdots,X_p,X) = (D_X\theta)(X_1,\cdots,X_p).$$

We define an operator

$$\partial^* : C^p(E) \rightarrow C^{p-1}(E) \qquad (p > 0)$$

as follows. Let  $x \in M$  and let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of the tangent space  $T_x(M)$  of M at x. For any p-1 tangent vectors  $u_1, \dots, u_{p-1}$  at x, put

$$(\partial^* \eta)_x(u_1, \cdots, u_{p-1}) = -\sum_{k=1}^n (D_{e_k} \eta)_x(e_k, u_1, \cdots, u_{p-1})_x$$

where  $(D_{e_k}\eta)_x$  denotes the value of  $D_X\eta$  at x for any vector field X such that  $X_x = e_k$ . Then  $(\partial^*\eta)_x$  is an alternating (p-1)-linear map of  $T_x(M)$  into  $E_x$ , the fiber of E over x, and the assignment  $x \to (\partial^*\eta)_x$  defines an E-valued (p-1)-form  $\partial^*\theta$ . For any E-valued 0-form  $\theta$ , we define  $\partial^*\theta = 0$ .

The Laplacian  $\Box$  for E-valued differential forms is defined as

$$\Box = \partial \partial^* + \partial^* \partial.$$

The curvature  $\tilde{R}$  of the covariant differentiation D in E is a Hom (E, E)-valued 2-forms given by

$$\tilde{R}(X, Y)\varphi = D_X(D_Y\varphi) - D_Y(D_X\varphi) - D_{[X,Y]}\varphi$$

for any section  $\varphi$  of E and for any vector fields X and Y in M. We shall denote by  $\langle \theta, \eta \rangle$  the scalar product of two E-valued p-forms, that is,  $\langle \theta, \eta \rangle$  is the smooth function on M given by

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$$\langle \theta, \eta \rangle (x) = \sum_{i_1, \ldots, i_p=1}^n \langle \theta(e_{i_1}, \cdots, e_{i_p}), \eta(e_{i_1}, \cdots, e_{i_p}) \rangle,$$

where  $\{e_1, \dots, e_n\}$  denotes an orthonormal basis of  $T_x(M)$ .

Now we prove the following

**Theorem 1.** Let  $\theta$  be an *E*-valued 1-form. Then

$$\langle \Box heta, heta 
angle = rac{1}{2} \Delta \langle heta, heta 
angle + \langle D heta, D heta 
angle + A$$
,

where  $\Delta$  denotes the Laplacian of the Riemannian manifold M and A denotes a smooth function in M defined as follows:

$$A(x) = \sum_{i,j} \langle (\tilde{R}(e_j, e_i)\theta(e_j), \theta(e_i) \rangle + \sum_i \langle \theta(S(e_i)), \theta(e_i) \rangle,$$

where  $\{e_1, \dots, e_n\}$  is an orthonormal basis of  $T_x(M)$  and S denotes the endomorphism of  $T_x(M)$  defined by the Ricci tensor S of M, i.e.  $S(e_i) = \sum S_{ki} e_k$ .

Proof. Fix a point  $x \in M$  and let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of  $T_x(M)$ . We can choose *n* vector fields  $E_1, \dots, E_n$  in *M* such that  $E_i(x) = e_i$  and  $(\nabla_{E_k} E_i)_x = 0$  for  $i, k = 1, \dots, n$ . Then, because  $\nabla_{e_s} E_i$  are zero for  $i, s = 1, \dots, n$ , we have

$$\begin{aligned} (\partial^* \partial \theta)(e_i) &= -\sum_s \left( D_{e_s} \partial \theta \right)(e_s, e_i) = -\sum_s D_{e_s}((\partial \theta)(E_s, E_i)) \\ &= -\sum_s D_{e_s}((D_{E_s} \theta)(E_i) - (D_{E_i} \theta)(E_s)) \\ &= \sum_s \left( D_{E_s} D_{E_i} \theta \right)(e_i) - \sum_s \left( D_{E_s} D_{E_s} \theta \right)(e_i). \end{aligned}$$

On the other hand,  $\partial^* \theta = -\sum_{s,t} g^{st}(D_{E_t}\theta)(E_s)$  where  $(g^{st})$  is the inverse matrix of the matrix  $(g(E_s, E_t))$ , we have

$$\begin{aligned} (\partial\partial^*\theta)(e_i) &= D_{e_i}(\partial^*\theta) = -\sum_{s,t} (e_i g^{st})(D_{e_i}\theta)(e_s) - \sum_{s,t} \delta^{st} e_i((D_{E_t}\theta)(E_s)) \\ &= -\sum_s e_i((D_{E_s}\theta)(E_s)) = -\sum_s (D_{E_i}D_{E_s}\theta)(e_s), \end{aligned}$$

because  $\nabla_{e_i} E_k = 0$  at x.

Therefore we obtain

$$(\Box\theta)(e_i) = \sum_{s} ((D_{Es}D_{Ei} - D_{Ei}D_{Es})\theta)(e_s) - \sum_{s} (D_{Es}D_{Es}\theta)(e_i).$$

Since  $[E_s, E_i] = 0$  at x, we have

$$\begin{aligned} ((D_{E_s}D_{E_i}-D_{E_i}D_{E_s})\theta)(e_s) &= (([D_{E_s},D_{E_i}]-D_{[E_s,E_i]})\theta) \ (e_s) \\ &= \tilde{R}(e_s,e_i)(\theta(e_s)) - \theta(R(e_s,e_i)e_s). \end{aligned}$$

Therefore

$$egin{aligned} &\langle \Box heta, heta 
angle = \sum\limits_{i} \langle (\Box heta) heta(e_i), (e_i) 
angle \ &= \sum\limits_{i,i} \langle \widetilde{R}(e_s, e_i) heta(e_s), heta(e_i) 
angle + \sum\limits_{i} heta(S(e_i), heta(e_i)) \ &- \sum\limits_{s,i} \langle (D_{Es} D_{Es} heta)(e_i), heta(e_i) 
angle. \end{aligned}$$

Now by a local computation we see that

$$-\sum_{s,i} \langle (D_{E_s} D_{E_s} \theta)(e_i), \theta(e_i) \rangle$$
$$= \langle D\theta, D\theta \rangle(x) + \frac{1}{2} (\Delta \langle \theta, \theta \rangle)(x)$$

Thus we have proved that

$$\langle \Box \theta, \theta \rangle = \frac{1}{2} \Delta \langle \theta, \theta \rangle + \langle D \theta, D \theta \rangle + A.$$

**Corollary 1.** Let  $\theta$  be an *E*-valued 1-form. Assume that  $\Box \theta = 0$  and  $\Delta \langle \theta, \theta \rangle = 0$ . Then we have  $A \leq 0$  everywhere on M.

Assume now that M is compact and oriented. Then we can define the inner product  $(\theta, \eta)$  of two E-valued p-forms by

$$(\theta,\eta) = \int_M \langle \theta,\eta \rangle * 1.$$

Then we obtain from Theorem 1 the following corollary.

**Corollary 2.** Let  $\theta$  be an *E*-valued 1-form such that  $\Box \theta = 0$ . Then we have

$$(D\theta, D\theta) + \int_{M} A * 1 = 0.$$

If  $A \ge 0$  everywhere on M, then we have  $A \equiv 0$  and  $D\theta = 0$ .

We remark that the operator  $\partial^*$  is the adjoint operator of  $\partial$ , i.e.

$$(\partial \theta, \eta) = (\theta, \partial^* \eta)$$

for any  $\theta \in C^{p}(E)$  and  $\eta \in C^{p+1}(E)$  and hence we have

$$(\Box\theta, \theta) = (\partial\theta, \partial\theta) + (\partial^*\theta, \partial^*\theta).$$

Therefore, if M is compact,  $\Box \theta = 0$  if and only if  $\partial \theta = 0$  and  $\partial^* \theta = 0$ .

2. Let M be an n-dimensional Riemannian manifold isometrically immersed in a Riemannian manifold M' of dimension n + p. We shall denote by N(M)and  $\alpha$  the normal bundle and the second fundamental form of M [3]. The second fundamental form  $\alpha$  is an N(M)-valued symmetric 2-form on M.

In the following we put

$$E = \text{Hom}(T(M), N(M)) = T^*(M) \otimes N(M)$$

and we interprete  $\alpha$  as an *E*-valued 1-form  $\beta$  as follows: For any vector field X in M,  $\beta(X)$  is a section of E such that

$$\beta(X) \cdot Y = \alpha(X, Y)$$

for all vector field Y in M. Then we have

$$\beta(X) \cdot Y = \beta(Y) \cdot X.$$

We call also  $\beta$  the second fundamental form of M.

A metric along the fibres of E is defined naturally by the Riemann metrics of M and M' and a covariant derivation  $D_X$  in E is also naturally defined by the covariant differentiation  $\nabla_X$  in M and  $D_X^{\pm}$  in N(M), where for any normal vector  $\xi$  of  $M, D_X^{\pm}\xi$  is defined as the normal component of  $\nabla_X'\xi$ , where  $\nabla_X'$  denote the covariant differentiation in the Riemannian manifold M' (See [3]).

Let  $\varphi$  be a section of E. We may regard  $\varphi$  as an N(M)-valued 1-form on M and we have

$$egin{aligned} & (D_Xarphi)(Y) = D^+_X(arphi(Y)) - arphi(
abla_X Y)\,, \ & \langle D_Xarphi,\psi
angle + \langle arphi, D_X\psi
angle = X\langle arphi,\psi
angle \end{aligned}$$

for any sections  $\varphi$  and  $\psi$  of E.

The following Proposition 1 may be considered as an interpretation of the equation of Codazzi in our formalism.

**Proposition 1.** Assume that M' is a Riemannian manifold of constant sectional curvature. Then the second fundamental form  $\beta$  of M satisfies the equation  $\partial \beta = 0$ .

Proof. By a straightforward computation we see that

$$(\partial\beta(X, Y))(Z) = \{D_X^{\perp}(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)\} - \{D_Y^{\perp}(\alpha(X, Z)) - \alpha(\nabla_X Y, Z) - \alpha(X, \nabla_Y Z)\}$$

and the right hand side is 0 by [3, Vol. II, P. 25, Cor. 4.4].

For each normal vector  $\nu \in N_X(M)$  we define an endomorphism  $A_{\nu}$  of  $T_x(M)$  by the formula

$$\langle A_{
m v}(u),v
angle = \langle eta(u)v,
u
angle$$

for any tangent vectors  $u, v \in T_x(M)$ . The mean curvature normal  $\eta$  of M is a

normal vector field in M such that

$$rac{1}{n} \mathrm{Tr} \; A_{\mathrm{v}} = \langle \mathrm{v}, \eta \left( x 
ight) 
angle$$

for any  $\nu \in N_x(M)$  and  $x \in M$ .

*M* is said to be *minimal* in *M'* if the mean curvature normal vanishes at each point, that is, if Tr  $A_{\nu} = 0$  for any  $\nu \in N_x(M)$  and  $x \in M$ .

We say that M has a constant mean curvature if the mean curvature normal  $\eta$  is parallel, that is,  $D_X^{\perp}\eta = 0$  for any vector field X in M.

Let  $\nu$  be a normal vector field. Then we have  $\operatorname{Tr} A_{\nu} = n \langle \nu, \eta \rangle$  and hence  $X \cdot \operatorname{Tr} A_{\nu} = n \{\langle D_X^{\perp} \nu, \eta \rangle + \langle \nu, D_X^{\perp} \eta \rangle\}$ . Therefore M has a constant mean curvature, if and only if

$$X \cdot \mathrm{Tr} A_{\nu} = \mathrm{Tr} A_{D^{\perp}_{\boldsymbol{\tau}} \nu}$$

for any normal vector field  $\nu$  and any vector field X in M.

**Proposition 2.** Let M' be a Riemmanian manifold of constant sectional curvature. Then the second fundamental form  $\beta$  of M satisfies the equation  $\partial^*\beta = 0$  if and only if M has a constant mean curvature.

Proof. Let x be a point in M and let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of  $T_x(M)$ . Let  $E_1, \dots, E_n$  be vector fields in a neighborhood of x such that  $(E_i)_x = e_i$  and  $\nabla_{E_i}E_k = 0$  at x for  $i, k = 1, \dots, n$ . Let $(g^{s_t})$  the inverse matrix of the matrix  $(\langle E_s, E_t \rangle)$ . Then  $\partial^*\beta = -g^{s_t}(D_{E_t}\beta)(E_s)^{1}$  and  $(\partial^*\beta) \cdot E_k = -g^{s_t}(D_{E_t}\beta)(E_s) \cdot E_k$ . Since $(D_{E_t}\beta)(E_s)E_k = D_{E_t}^{\perp}(\alpha(E_s, E_k)) - \alpha(\nabla_{E_t}E_s, E_k) - \alpha(E_s, \nabla_{E_t}E_k)$  and since  $\alpha$  is symmetric, we get $(D_{E_t}\beta)(E_s)E_k = (D_{E_t}\beta)(E_s)E_s$ . On the other hand, by Proposition 1, we have  $\partial\beta = 0$  and hence  $(D_{E_t}\beta)(E_k) = (D_{E_k}\beta)(E_t)$ , hence  $(D_{E_t}\beta)(E_s)E_k = (D_{E_k}\beta)(E_t) \cdot E_s$ . Therefore, for any normal vector field  $\nu$ , we have

$$egin{aligned} &\langle (\partial^*eta)\cdot E_{m{k}},
u
angle &= -g^{st}\langle (D_{E_{m{k}}}eta)(E_t)E_s,
u
angle &= -g^{st}\{\langle D_{E_{m{k}}}^{\perp}(lpha(E_t,E_s)),
u
angle - \langle lpha(
abla_{E_{m{k}}}E_t,E_s),
u
angle &- \langle lpha(E_t,
abla_{E_{m{k}}}E_s),
u
angle \}. \end{aligned}$$

Now

$$\begin{split} g^{st} \langle D_{E_k}^{\perp}(\alpha(E_t, E_s), \nu) \\ &= g^{st} \{ E_k \langle \alpha(E_t, E_s), \nu \rangle - \langle \alpha(E_t, E_s), D_{E_k}^{\perp} \nu \rangle \\ &= E_k (g^{st} \langle \alpha(E_t, E_s), \nu \rangle) - (E_k g^{st}) \langle \alpha(E_t, E_s), \nu \rangle - g^{st} \langle \alpha(E_t, E_s), D_{E_k}^{\perp} \nu \rangle \\ &= E_k (T_r A_{\nu}) - T_r A_{D_{E_k}^{\perp \nu}} - E_k g^{st} \cdot \langle \alpha(E_t, E_s), \nu \rangle. \end{split}$$

<sup>1)</sup> We omit here the summation signs.

Since  $\nabla_{E_k} E_i = 0$  at x, we have  $E_k g^{st} = 0$  at x. Therefore we get from the above that

$$\langle (\partial^* \beta) E_k, \nu \rangle (x) = \operatorname{Tr} A_D \frac{1}{k} \nu - E_k (\operatorname{Tr} A_\nu)$$

at x for  $k = 1, 2, \dots, n$  and hence for any vector field X we have  $\langle (\partial^* \beta) X, \nu \rangle \langle x \rangle$ =  $\operatorname{Tr} A_{D_{X}^{\perp} \nu} - X(\operatorname{Tr} A_{\nu})$  at x. Since x is an arbitrary point of M and  $\nu$  is an arbitrary normal vector field, we see from the above equation that  $\partial^* \beta = 0$  if and only if M has a constant mean curvature.

From Propositions 1 and 2 we get the following

**Theorem 2.** Let M be a Riemannian manifold immersed isometrically into a Riemannian manifold M' of constant sectional curvature. Let  $\beta$  be the second fundamental form of M regarded as a Hom(T(M), N(M))-valued 1-form. Then  $\beta$  satisfies the equation  $\Box \beta = 0$ , if M has a constant mean curvature. Conversely, if M is compact and orientable and  $\Box \beta = 0$ , then M has a constant mean curvature.

3. We shall discuss in this section some applications of Theorems 1 and 2. Let M be a Riemannian manifold immersed isometrically into a Riemannian manifold M' of constant sectional curvature c. Let  $x \in M$  and let  $\{e_1, \dots, e_n\}$  and  $\{v_1, \dots, v_p\}$  be orthonormal bases of  $T_x(M)$  and  $N_x(M)$  respectively. We shall denote by  $A_a(a = 1, 2, \dots, p)$  the endomorphism of  $T_x(M)$  defined by $\langle A_a u, v \rangle =$  $\langle \beta(u) \cdot v, v_a \rangle$  and put  $A_a \cdot e_i = \sum_j (A_a)_i^j e_j$ . Then we have the following Gauss equation:

(3.1) 
$$R_{klij} = c\{\delta_{ki}\delta_{lj} - \delta_{kj}\delta_{li}\} + \sum_{a} \{(A_a)_i^k (A_a)_i^l - (A_a)_j^k (A_a)_i^l\},$$

where  $R_{klij}$  denote the components of the curvature tensor with respect to the basis  $\{e_1, \dots, e_n\}$  of  $T_x(M)$ . Then the endomorphism S of  $T_x(M)$  defined by  $S(e_j) = \sum_i S_{lj}(e_i)$  with  $S_{ej} = \sum_k R_{klkj}$  is of the form

(3.2) 
$$S = c(n-1)I + \sum_{a} (\operatorname{Tr} A_{a})A_{a} - \sum_{a} A_{a}^{2},$$

where I denotes the identity endomorphism of  $T_x(M)$ .

Let K be the scalar curvature of M. Then  $K(x) = \operatorname{Tr} S = c(n-1)n + \sum_{a} (\operatorname{Tr} A_{a})^{2} - \sum_{a} \operatorname{Tr} A_{a}^{2}$ . The value  $\eta(x)$  at x of the mean curvature normal  $\eta$  is given by  $\eta(x) = \frac{1}{n} \sum_{a} \operatorname{Tr} A_{a} \cdot \nu_{a}$  and hence  $n^{2} \langle \eta, \eta \rangle_{a}(x) = \sum (\operatorname{Tr} A_{a})^{2}$ . Analogously we have  $\langle \beta, \beta \rangle(x) = \sum_{a} \operatorname{Tr} A_{a}^{2}$ . Hence we get

$$(3.3) K = c(n-1)n + n^2 \langle \gamma, \gamma \rangle - \langle \beta, \beta \rangle,$$

where  $\beta$  and  $\eta$  denotes the second fundamental form and the mean curvature normal of M respectively. For any Riemannian vector bundle E over M we have defined the endomorphism  $\tilde{R}(u, v)$  of the fiber  $E_x$ , where  $u, v \in T_x(M)$ . Let E = $\operatorname{Hom}(T(M), N(M) \text{ and let } \varphi \in E_x$ . Then  $\tilde{R}(u, v) \varphi$  is an element of  $E_x =$  Hom  $(T_x(M), N_x(M))$  such that

(3.4) 
$$(\widetilde{R}(u,v)\varphi)(w) = R^{\perp}(u,v)(w)\varphi) - \varphi(R(u,v)w),$$

where  $u, v, w \in T_x(M)$  and  $R^{\perp}$  denotes the curvature of the Riemannian vector bundle N(M).

Let  $\nu$  be a normal vector of M at x and let N be a normal vector field such that  $N_x = \nu$ . Let X and Y be vector fields in M such that  $X_x = u$  and  $Y_x = v$ . Then we have

$$R^{\perp}(u,v)\nu = (D_X^{\perp}D_Y^{\perp} - D_Y^{\perp}D_X^{\perp} - D^{\perp}_{[X,Y]}) N$$

at x.

Denote by  $\nabla'$  the covariant derivation in the ambiant space M'. Then we

have

$$abla_{X}'Y = 
abla_{X}Y + \alpha(X, Y),$$
 $abla_{X}'N = -A_{N}(X) + D_{X}^{\perp}N.$ 

We see from these two equations that the normal component  $(R'(X, Y)N)^{\perp}$  of R'(X, Y)N, where R' denotes the curvature tensor of  $M_i$ , is equal to  $R^{\perp}(X, Y)N - \alpha(A_N(Y), X) + \alpha(A_N(X), Y)$ . Since M' is of constant curvarute  $R'(X, Y)N = c\{\langle N, Y \rangle X - \langle N, X \rangle Y\} = 0$  and hence we get  $R^{\perp}(X, Y)N = -\alpha(A_N(X), Y) + \alpha(A_N(Y), X)$ . Thus we have

$$R^{\perp}(u, v)\nu = -\alpha(A_{\nu}u, v) + \alpha(A_{\nu}v, u).$$

In particular

$$R^{\perp}(u, v)\nu_a = -\alpha(A_a u, v) + \alpha(u, A_a v).$$

Since  $\alpha(A_a u, v) = \sum_b \langle \alpha(A_a u, v), v_b \rangle v_b = \sum_b (A_b A_a u, v) v_b$ and  $\alpha(u, A_a v) = \sum_b \langle A_b u, A_a v \rangle v_b = \sum_b \langle A_a A_b u, v \rangle v_b$ we get

$$(3.5) R^{\perp}(u,v)\nu_a = \sum_b \langle [A_a, A_b]u, v \rangle \nu_b,$$

Now by Theorem 1, we have

$$\langle \Box \, eta, eta 
angle = rac{1}{2} + \Delta \langle eta, eta 
angle + \langle Deta, Deta 
angle + A,$$

where

(3.6) 
$$A(x) = \sum_{i,j} \langle \tilde{R}(e_j, e_i)\beta(e_j), \beta(e_i) \rangle + \sum_i \langle \beta(S(e_i), \beta(e_i) \rangle.$$

Now

$$\begin{split} &\sum_{i} \langle \beta(S(e_i), \beta(e_i) \rangle = \sum_{i,j} \langle \alpha(S(e_i), e_j), \alpha(e_i, e_j) \rangle \\ &= \sum_{i,j,a} \langle A_a(S(e_i)), e_j \rangle \langle A_a(e_i), e_j \rangle = \sum_{a} \operatorname{Tr}(SA_a^2) \end{split}$$

and by (3.2) we get

(3.7) 
$$\sum_{i} \langle \beta(S(e_{i}), \beta(e_{i}) \rangle$$
$$= c(n-1) \sum_{a} \operatorname{Tr} A_{a}^{2} + \sum_{a,b} \operatorname{Tr} A_{a} \cdot \operatorname{Tr}(A_{a}A_{b}^{2}) - \sum_{a,b} \operatorname{Tr}(A_{a}^{2}A_{b}^{2})$$

On the other hand,

$$\begin{split} &\sum_{i,j} \langle \tilde{R}(e_j, e_i) \beta(e_j), \beta(e_i) \rangle \\ &= \sum_{i,j,k} \langle R^{\perp}(e_j, e_i) \alpha(e_j, e_k), \alpha(e_i, e_k) \rangle - \sum_{i,j,k} \langle \alpha(e_j, R(e_j, e_i) e_k), \alpha(e_i, e_k) \rangle \\ &= \sum_{i,j,k} \sum_{a,b} \langle A_a e_j, e_k \rangle \langle A_b e_i, e_k \rangle \langle R^{\perp}(e_j, e_i) \nu_a \nu_b \rangle \\ &- \sum_{i,j,k} \sum_a \langle A_a e_j, R(e_j, e_i) e_k \rangle \langle A_a e_i, e_k \rangle \end{split}$$

and by (3.5), the first term equals  $\sum_{a,b} \operatorname{Tr}(A_a A_b[A_a, A_b]) = -\sum_{a,b} \operatorname{Tr}(A_a^2 A_a^2) + \sum_{a,b} \operatorname{Tr}(A_a A_b)^2$  and by the Gauss equation (3.1) the second term equals  $-c \sum_a (\operatorname{Tr} A_a)^2 + c \sum_a \operatorname{Tr}(A_a^2) - \sum_{a,b} (\operatorname{Tr}(A_a A_b))^2 + \sum_{a,b} \operatorname{Tr}(A_a A_b)^2$ .

Therefore we have

(3.8) 
$$\sum_{i,j} \langle \hat{R}(e_u, e_i)\beta(e_j), \beta(e_i) \rangle$$
$$= c \sum_a \operatorname{Tr} A_a^2 - c \sum_v (\operatorname{Tr} A_a)^2 - \sum_{a,b} \operatorname{Tr} (A_a^2 A_a^2) - \sum_{a,b} (\operatorname{Tr} (A_a A_b))^2 + 2 \sum_{a,b} \operatorname{Tr} (A_a A_b)^2$$

Then we get from (3.6), (3.7) and (3.8) that

(3.9) 
$$A(x) = cn \sum_{a} \operatorname{Tr} A_{b}^{2} - c \sum_{a} (\operatorname{Tr} A_{a})^{2} - \sum_{a,b} (\operatorname{Tr}(A_{a}A_{b}))^{2} + \sum_{a,b} \operatorname{Tr} A_{a} \cdot \operatorname{Tr}(A_{a}A_{b}^{2}) + \sum_{a,b} \operatorname{Tr}[A_{a}, A_{b}]^{2}.$$

Now let  $\lambda_1^{(a)}, \dots, \lambda_n^{(a)}$  be eigen-values of  $A_a$  and let  $\{e_1^{(a)}, \dots, e_n^{(n)}\}$  be an orthonormal basis of  $T_x(M)$  such that  $A_a e_i^{(a)} = \lambda_i^{(a)} e_i^{(a)} (i=1,\dots,n, a=1,\dots,p)$ .

We shall denote by  $K_{ij}^{(a)}$  the sectional curvature for the 2-plane spanned by  $e_i^{(a)}$  and  $e_j^{(a)}$ ,  $i \neq j$ .

We show that

(3.10) 
$$A(x) = \sum_{v} \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 K_{ij}^{(a)} + \frac{1}{2} \sum_{a, b} \operatorname{Tr}[A_a, A_b]^2.$$

We write A(x) in the following form:

(3.11) 
$$A(x) = B(x) + \sum_{a \neq b} \operatorname{Tr} A_a \cdot \operatorname{Tr}(A_a A_b^2) - \sum_{a \neq b} (\operatorname{Tr}(A_a A_b))^2 + \sum_{a,b} \operatorname{Tr}[A_a, A_b]^2,$$

where

(3.12) 
$$B(x) = \sum_{a} \{ cn \operatorname{Tr} A_{a}^{2} - c(\operatorname{Tr} A_{a})^{2} - (\operatorname{Tr} A_{a}^{2})^{2} + \operatorname{Tr} A_{a} \cdot \operatorname{Tr} A_{a}^{3} \}.$$

Now by a lemma of Nomizu-Smyth [4] we have

(3.13) 
$$\operatorname{cn} \operatorname{Tr} A_a^2 - c(\operatorname{Tr} A_a)^2 - (\operatorname{Tr} A_a^2)^2 + \operatorname{Tr} A_a \cdot \operatorname{Tr}(A_a)^3 \\ = \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 (c + \lambda_j^{(a)} \lambda_j^{(a)})$$

for each a. Now fix an index a and let

$$A_{b}e_{i}^{(a)} = \sum_{j} (A_{b})_{i}^{j}e_{j}^{(a)} \quad (b=1,2,\cdots,p)$$

Then we have  $(A_{a})_{j}^{i} = \delta_{j}^{i} \lambda_{j}^{(a)}$  and hence

(3.14) 
$$(A_a A_b)_j^i = \lambda_i^{(a)}{}_b (A_b)_j^i, (A_b A_a)_j^i = (A_b)_j^i \lambda_j^{(a)}.$$

By the equation of Gauss we have

$$\begin{split} K^{(a)}_{ij} &= R(e^{(a)}_i, e^{(a)}_j, e^{(a)}_i, e^{(a)}_j) \\ &= c + \sum_b (A_b)^i_i (A_b)^j_j - \sum_b (A_b)^i_j (A_b)^j_i. \\ &= c + \lambda^{(a)}_i \lambda^{(a)}_j + \sum_{b \neq a} (A_b)^i_i (A_b)^j_j - \sum_b (A_b)^i_j (A_b)^j_i. \end{split}$$

Hence we have

$$\begin{split} & (\lambda_i^{(a)} \lambda - {}_j^{(a)})^2 (c + \lambda_i^{(a)} \lambda_j^{(a)}) \\ &= (\lambda_i^{(a)} - \lambda_j^{(a)})^2 K_{ij}^{(a)} + \sum_b (\lambda_i^{(a)} - \lambda_j^{(a)})^2 (A_b)_j^i (A_b)_j^j \\ &- \sum_{b \neq a} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 (A_b)_i^i (A_b)_j^j. \end{split}$$

This equality holds also for i = j trivially if we define  $K_{ii}^{(a)} = 0$ . Then by (3.14)

$$\sum_{i
$$= \sum_{i$$$$

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$$\begin{split} &-\frac{1}{2}\sum_{b\neq a}\{\sum_{i,j}\left(\lambda_{i}^{(a)}\right)^{2}(A_{a})_{i}^{i}\sum_{j}\left(A_{b}\right)_{j}^{j}-2\sum_{i}\lambda_{i}^{(a)}(A_{b})_{i}^{i}\sum_{j}\lambda_{j}^{(a)}(A_{b})_{j}^{j}\right)\\ &+\sum_{i}\left(A_{b}\right)_{i}^{i}\sum_{j}\lambda_{j}^{(a)}(A_{b})_{j}^{j}\}\\ &=\sum_{i$$

Then we obtain from (3.11), (3.12) and (3.13) the equality (3.10).

Now we cite the following two lemmas from [1].

**Lemma 1.** Let A and B be symmetric  $n \times n$  matrices. Then

 $\operatorname{Tr}[A, B]^2 \geq -2\operatorname{Tr} A^2 \cdot \operatorname{Tr} B^2$ ,

and the equality holds for non-zero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiple of  $\tilde{A}$  and  $\tilde{B}$  respectively, where

(3.15) 
$$\widehat{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{pmatrix}.$$

**Lemma 2.** Let  $A_1$ ,  $A_2$  and  $A_3$  be  $n \times n$  symmetric matrices and if

 $\operatorname{Tr}[A_a, A_b]^2 = -2\operatorname{Tr} A_a^2 \cdot \operatorname{Tr} A_b^2$ 

for  $1 \leq a < b \leq 3$ , then at least one of the matrices  $A_a$  must be zero.

By Lemma 1, we have

$$\frac{1}{2}\sum_{a,b} \operatorname{Tr}[A_a, A_b]^2 \ge -\sum_{a \neq b} \operatorname{Tr} A_a^2 \cdot \operatorname{Tr} A_b^2 = -2\sum_{a < b} \operatorname{Tr} A_a^2 \cdot \operatorname{Tr} A_b^2.$$
  
Put  $S_a = \operatorname{Tr} A_a^2$ . Then  $\sum_a S_a = \langle \beta, \beta \rangle \langle x \rangle$ .

Since

$$\begin{split} 0 &\leq \sum_{a < b} (S_a - S_b)^2 = \sum_{a < b} (S_a^2 + S_b^2) - 2 \sum_{a < b} S_a S_b \\ &= (p - 1) \sum_a S_a^2 - 2 \sum_{a < b} S_a S_b \\ &= (p - 1) \{ (\sum_a S_a)^2 - 2 \sum_{a < b} S_a S_b \} - 2 \sum_{a < b} S_a S_b \\ &= (p - 1) \langle \beta, \beta \rangle^2 (x) - 2p \sum_{a < b} S_a S_b \end{split}$$

we have

$$-2\sum_{a < b} S_a S_b \ge -\frac{(p-1)}{p} \langle \beta, \beta \rangle^2(x)$$

ane here the equality holds if and only if  $S_a = S_b$  for  $a, b=1, \dots, p$ . Therefore we get

(3.16) 
$$\frac{1}{2}\sum_{a,b} \operatorname{Tr}[A_a, A_b]^2 \ge -\frac{(p-1)}{p} \langle \beta, \beta \rangle^2(x)$$

and the equality holds if and only if  $\operatorname{Tr} A_a^2 = \operatorname{Tr} A_b^2 = \operatorname{Tr} A_b^2$  for  $a, b = 1, \dots, p$  and either  $A_a$  are all zero except possibly one of them or  $A_a$  are all zero except two of them, say  $A_1$  and  $A_2$ , and they can be transformed simultaneously by an orthogonal matrix into scalar multiple of the matrices of the form (3.15). Thus we obtain from (3.10) the inequality

(3.17) 
$$A(x) \geq \sum_{a} \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 K_{ij}^{(a)} - \frac{p-1}{p} \langle \beta, \beta \rangle^2 (x).$$

Assume now that the scalar curvatures of M are bounded below by a positive constant d. Then

$$\sum_{a} \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 K_{ij}^{(a)} \ge d \sum_{a} \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2$$

and

$$\sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 = (n-1) \operatorname{Tr} A_a^2 - 2 \sum_{i < j} \lambda_i^{(a)} \lambda_j^{(a)}$$
$$-2 \sum_{i < j} \lambda_i^{(a)} \lambda_j^{(a)} = \operatorname{Tr} A_a^2 - (\operatorname{Tr} A_a)^2$$

and hence

$$\sum_{a} \sum_{i < j} (\lambda_i^{(a)} - \lambda_j^{(a)})^2 = n \langle \beta, \beta \rangle (x) - n^2 \langle \eta, \eta \rangle (x),$$

where  $\eta$  denotes the mean curvature normal of M. Thus we get the following inequality

$$(3.17) A \ge \left(dn - \frac{p-1}{p} \langle \beta, \beta \rangle\right) \langle \beta, \beta \rangle - dn^2 \langle \eta, \eta \rangle$$

at each point of M.

We obtain from Corollaries 1 and 2 of Theorem 1 and Theorem 2 the following

**Theorem 3.** Let M be an n-dimensional, Riemannian manifold with sectional curvatures bounded below by a positive constant d. Assume that M is immersed in a Riemannian manifold M' of constant sectional curvature of dimension n+p and that M has a constant mean curvature. Then, if M is compact and orientable or if the length of the second fundamental form  $\beta$  of M is constant, then we have

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$$(3.18) 0 \ge A \ge \left\{ dn - \frac{p-1}{p} \langle \beta, \beta \rangle \right\} \langle \beta, \beta \rangle - dn^2 \langle \eta, \eta \rangle$$

at each point of M, where  $\eta$  denotes the mean curvature normal of M which is parallel and  $\langle \eta, \eta \rangle$  is a constant.

Now assume *M* is compact and oriented and let  $k = \langle \eta, \eta \rangle$ . Then integrating both sides of the inequality (3.18) we obtain

$$dn^2k \int_M *1 \ge \int_M \left\{ dn - \frac{p-1}{p} \langle \beta, \beta \rangle \right\} \langle \beta, \beta \rangle *1$$

and we have the equality here if and only if

$$dn^2k=\left\{dn{-}rac{p{-}1}{p}\!\!\left<\!eta,eta
ight>
ight\}\!\!\left<\!eta,eta
ight>
ight\}\!\!\left<\!eta,eta
ight>
ight\}\!\!\left<\!eta,eta
ight>
ight\}$$

and this implies also that A = 0 and that  $\beta$  is parallel by Theorem 1. Then  $\langle \beta, \beta \rangle$  must satisfy the quadratic equation  $(p-1)x^2 - p \, dn \, x + pn^2 k = 0$  and since the discriminant of this equation should be positive we should have the inequality

$$d \geq \frac{4k(p-1)}{p}.$$

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